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{kronburg|deiss|eschbach}@informatik.uni-kl.de

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Sonderforschungsbereich 501

Universität Kaiserslautern
Erwin-Schrödinger Straße
D-67653 Kaiserslautern

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Martin Kronenburg Thomas Deiß Robert Eschbach

University of Kaiserslautern, Erwin Schrödinger Straße,
D-67663 Kaiserslautern, Germany
{kronburg|deiss|eschbach}@informatik.uni-kl.de

Abstract

The notion of *formal description techniques for timed systems* (T-FDTs) has been introduced in [EDK98a] to provide a unifying framework for description techniques that are formal and that allow to describe the ongoing behavior of systems. In this paper we show that three well known temporal logics, MTL, MTL-$\int$, and CTL*, can be embedded in this framework. Moreover, we provide evidence that a large number of different kinds of temporal logics can be considered as T-FDTs.

1 Introduction

Formal description techniques (FDTs) are widely recommended and used tools for the development and maintenance of software and hardware systems. Nevertheless, there is usually the problem what FDTs should be used in a specific project. To base such a selection of suitable FDTs on a more objective foundation our long term objective is the establishment of an approach to the formal analysis of FDTs. To achieve this goal the following three tasks have to be performed.

Characterization of the Description Techniques to be investigated

It has to be explained, which conditions allow a description technique to be considered a *formal* one. Since an FDT cannot describe arbitrary aspects of arbitrary classes of systems, this general notion of FDT has to be instantiated to the application area one is interested in. This means to specify precisely the general models that can be represented in descriptions written in an FDT. Thus, the result of the first task is a refinement of the general notion of FDT. It defines the class of description techniques that are interesting w.r.t. the specific application domain considered.

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Definition of Criteria

To investigate concrete FDTs belonging to this class of FDTs, criteria examining a certain aspect of them have to be defined precisely. This includes the definition, whether or to which degree a specific FDT satisfies a criterion. Consequently, the result of this second task is a list of criteria. Since there is a very broad range of criteria including aspects like the difficulty of learning an FDT, it cannot be expected, that all of them can be defined formally, even less, that this is done in a uniform way.

Application of Criteria

To apply the criteria to concrete FDTs, it is necessary at first to embed them in the result of the first task, i.e. in the definition of the class of FDTs one is interested in. Next, the criteria have to be applied to these FDTs in the context of the specific project for which they are intended to be used. The relevance of the criteria in this context must be given, such that the attributes of the FDTs can be combined in a useful way. As the result of this step one gets for each considered concrete FDT a list containing the attributes assigned to this FDT w.r.t. the applied criteria.

In [EDK98a] we dealt with the first task. There we introduced the notion of *formal description techniques for timed systems*, T-FDTs for short. This is intended to be a unifying framework for a class of description techniques that are *formal* and allow to describe *timing* aspects of systems.

In this paper we want to provide evidence that temporal logics, a widely used and accepted class of description techniques, are actually T-FDTs. Therefore, we show that three well known temporal logics, namely MTL [Koy92], MTL-$\int$ [LH95], and CTL* [Eme90], can be embedded in the framework of T-FDTs. Moreover, we present the embedding process in such a way that it becomes obvious that probably all kinds of temporal logics can be interpreted as instantiations of T-FDTs. By this we do not only provide evidence that T-FDTs can be used as a unifying framework for those description techniques we are interested in. We also do the first step of the third task, the application of criteria, for the three considered temporal logics, namely their embedding in the framework of T-FDTs.

At first we repeat in the following section the definition of T-FDTs and the underlying models introduced in [EDK98a]. The main part of this paper is Section 3. There we show how temporal logics in general and MTL, MTL-$\int$, and CTL* in particular can be embedded in the framework of T-FDTs. We finish with some concluding remarks. The appendix contains the complete proofs of the three considered temporal logics being T-FDTs.

2 FDTs for Timed Systems

In this section we define *FDTs for timed systems*. We only mention the main ideas concerning formality and timing aspects that are necessary to understand the embedding of the temporal logics in the general framework presented in the following section. For

a complete and detailed introduction of FDTs for timed systems see [EDK98a]. For some details it is necessary to have a basic knowledge of category theory. Nevertheless, we believe that the main ideas can be understood without this.
For a category $\mathbb{C}$ let $\mid \mathbb{C} \mid$ denote its class of objects. For the sake of simplicity we sometimes omit the bars and write $A \in \mathbb{C}$ for an object A in $\mathbb{C}$. Analogously, we write $f : A \to B \in \mathbb{C}$ for a morphism $f : A \to B$ of $\mathbb{C}$. For an introduction to category theory see [Lan71].

2.1 Formality

To our opinion a description technique is formal if it provides a *formal syntax* and a *formal semantics*. The formal syntax is usually given by a formal language providing *sentences* that can be built. Formal semantics consists of *structures* and a so called *satisfaction relation* between sentences and structures. This relation between sentences and structures determines whether a sentence is valid in a structure or not.
These concepts are captured by the notion of *institutions*, cf. [GB92]. An institution provides a category SIG of *signatures* which form the basic vocabulary to describe a system. Formal syntax and semantics are given by functors *Sen* and *Str* from the category SIG to sets of sentences and categories of structures over a signature, respectively. These notions are later used when defining FDTs for timed systems.

2.2 Timing Aspects

We now concentrate on the models needed to describe *timing* aspects of a system. These models are based on the notions *signature* and *structure* as they are common in many–sorted first order logic.

2.2.1 Timed Signatures

A *signature* $\Sigma = (S, F, P, D)$ consists of a finite set S of sorts, sets F and P of function and predicate symbols, and a set D of declarations $f : \overline{s} \to s$ and $p : \overline{s}$, describing the sorts of the arguments and the range of function and predicate symbols. A *signature morphism* $\sigma : \Sigma \to \Sigma'$ consists of three functions $\sigma = (\sigma_S, \sigma_F, \sigma_P)$, mapping sorts, function and predicate symbols compatible with the declarations of Σ and Σ'. $\tilde{\Sigma} = (\tilde{S}, \tilde{F}, \tilde{P}, \tilde{D})$ is an enrichment of a signature Σ if the component-wise union $(\Sigma \uplus \tilde{\Sigma})$ is a signature.
Since we want to describe timing aspects we are only interested in a special kind of signatures, called *timed signatures*. They provide a distinguished subsignature to represent the time model used. This subsignature has to introduce at least a distinguished sort T representing the time domain and a relation symbol $<$ to denote a precedence between points of time. In addition, since all functions and predicates can depend on time, we require, that the declaration of each function and relation symbol contains the sort T.

Definition 1 (Timed Signature) *A* timed signature $\Sigma = (S, F, P, D)$ *is a signature which satisfies the following conditions:*

1. *There exists a distinguished subsignature* $\Sigma_T \subseteq \Sigma$ *with* $\Sigma_T = (S_T, F_T, P_T, D_T)$ *called* time signature. *The complement* $\Sigma_D = (S_D, F_D, P_D, D_D)$ *with* $\Sigma_D = \Sigma \setminus \Sigma_T$ *is called the* defined enrichment *of* Σ_T.
2. *There exist a distinguished sort* $T \in S_T$ *and a distinguished relation symbol* $< \in P_T$ *with declaration* $<: T, T \in D_T$.
3. *Each function symbol* $f \in F_D$ *and each relation symbol* $p \in P_D$ *has a declaration* $f : \overline{s}, T \to s$ *and* $p : \tilde{s}, T$, *respectively, where* $\overline{s} \in S_D^*$, $\tilde{s} \in S_D^+$, *and* $s \in S_D$.

Given two timed signatures Σ *and* Σ' *a signature morphism* $\sigma : \Sigma \to \Sigma'$ *is called a* timed signature morphism *if it satisfies* $\sigma(T) = T'$, $\sigma(<) = <'$, *and* $\sigma|_{\Sigma_T} : \Sigma_T \to \Sigma'_T$ *is a signature morphism itself.*
The category of all timed signatures with timed signature morphisms as morphisms is denoted by TSIG.

If nothing else is stated Σ and Σ' always denote timed signatures and $\sigma : \Sigma \to \Sigma'$ a timed signature morphism.

2.2.2 Models for Timed Systems

To represent the various timing aspects of a systems adequately, to our opinion, three kinds of models are needed. The so called *behavior models* are the most important ones. A behavior model represents one single evolution of the functions and predicates, i.e. of the system that is considered, over time. *State models* are reductions of a behavior model. Using a state model we consider the values of functions and predicates at a specific point of time. The third kind of models, the so called *system models*, is needed to express the *overall* behavior of a system, i.e. a system model is a set of several behavior models. By this a system model represents the various reactions of a system to different sequences of inputs.

Behavior Models

To define behavior models we need the definition of *structures* in a category theoretical setting. As usual a *structure* $\mathfrak{A}$ over a signature Σ consists of a family $A = (A_s \mid s \in S)$ of non-empty carrier sets and a mapping $\_^{\mathfrak{A}}$ from the function and predicate symbols to functions and predicates on A compatible with the declarations of Σ. A *homomorphism* h between two structures $\mathfrak{A}, \mathfrak{B}$ over Σ is a family $h = (h_s \mid s \in S)$ of mappings $h_s : A_s \to B_s$ that is compatible with the functions and predicates. STR(Σ) denotes the category of all structures over Σ with homomorphisms as morphisms.
For a signature morphism $\sigma : \Sigma \to \Sigma'$ the structures over these signatures are related by the (contravariant) *forgetful functor* $\_|_\sigma :$ STR$(\Sigma') \to$ STR(Σ). This functor assigns to a structure $\mathfrak{A}' \in$ STR(Σ') a structure $\mathfrak{A}'|_\sigma = \mathfrak{A} \in$ STR(Σ) such that the carrier sets are given by $A_s = A'_{\sigma(s)}$ for $s \in S$ and the functions and relations by $f^{\mathfrak{A}} = \sigma(f)^{\mathfrak{A}'}$ and $p^{\mathfrak{A}} = \sigma(p)^{\mathfrak{A}'}$. A homomorphism $h' : \mathfrak{A}' \to \mathfrak{B}'$ is mapped to a homomorphism $h'|_\sigma : \mathfrak{A}'|_\sigma \to \mathfrak{B}'|_\sigma$ in an obvious way. In the following we denote by *Str* the contravariant

functor Str : SIG $\rightarrow$ CAT[1] which maps a signature Σ to $\text{STR}(\Sigma)$ and a signature morphism σ to the forgetful functor $\_|\sigma$.
Behavior models are structures over timed signatures that interpret the time sort and the relation symbol on it as a partial ordering.

Definition 2 (Behavior Model) *A* behavior model $\mathfrak{B}$ *over* Σ *is a structure over* Σ *where* $(B_T, <^{\mathfrak{B}})$ *is a partial ordering.*
The category of all behavior models over Σ *as objects and homomorphisms as morphisms is denoted by* $\text{BEH}(\Sigma)$.

The set B_T will usually be referred to as the set of all *time points* of $\mathfrak{B}$. The restriction of $\mathfrak{B}$ to Σ_T, denoted by $\mathfrak{B}|_{\Sigma_T}$, is called the *time model* of $\mathfrak{B}$. The behavior models over a timed signature Σ form a subcategory of the category $\text{STR}(\Sigma)$ of structures.

State Models

To define the state models associated with a behavior model we first have to remove time from time signatures. Given a timed signature $\Sigma = \Sigma_T \cup \Sigma_D$ the corresponding *state signature* $\Sigma_{St} = (S_{St}, F_{St}, P_{St}, D_{St})$ is defined by: $S_{St} := S_D$, $F_{St} := F_D$, $P_{St} := P_D$, and $D_{St} := \{f : \overline{s} \rightarrow s \mid (f : \overline{s}, T \rightarrow s) \in D_D\} \cup \{p : \overline{s} \mid (p : \overline{s}, T) \in D_D\}$,
The state models over a state signature are derived from a behavior model by fixing the point of time.

Definition 3 (State Model) *Let* $\mathfrak{B} \in \text{BEH}(\Sigma)$ *be a behavior model over* Σ, *and* $t \in B_T$ *a time point. The* state model $\mathfrak{B}_{St,t} \in Str(\Sigma_{St})$ *of* $\mathfrak{B}$ *at the time point* t *is defined by*

1. $B_{St,t} := (B_s \mid s \in S_D)$,
2. $f^{\mathfrak{B}_{St,t}}(\overline{b}) := f^{\mathfrak{B}}(\overline{b}, t)$ *for each* $f : \overline{s} \rightarrow s \in \Sigma_{St}$ *and all* $\overline{b} \in B_{\overline{s}}$,
3. $p^{\mathfrak{B}_{St,t}}(\overline{b}) := p^{\mathfrak{B}}(\overline{b}, t)$ *for each* $p : \overline{s} \in \Sigma_{St}$ *and all* $\overline{b} \in B_{\overline{s}}$.

The category of all state models over $\Sigma \in$ SSIG *as objects and homomorphisms as morphisms is denoted by* $\text{STATE}(\Sigma)$.

System Models

As *system models*, i.e. as the representation of the overall behavior of a system, we do not allow an arbitrary collection of behavior models. Instead we require the behavior models to have the same carrier sets and the same time model.

Definition 4 (System Model) *Let* $\mathfrak{S} \subseteq \text{BEH}(\Sigma)$ *be a non-empty set of behavior models.* $\mathfrak{S}$ *is called a* system model *over* Σ *if the following holds for all* $\mathfrak{B}, \mathfrak{B}' \in \mathfrak{S}$:

1. $B_s = B'_s$ *for all* $s \in S_D$,
2. $\mathfrak{B}|_{\Sigma_T} = \mathfrak{B}'|_{\Sigma_T}$.

The class of all system models over Σ *is denoted by* $\text{SYS}(\Sigma)$.

[1] CAT is the category having categories as objects and functors as morphisms, see [Lan71] for details.

It is possible to define *system homomorphisms* between system models using so called *system products*, see [EDK98a]. Here we only need system products. A system product is a special behavior model that compromises the behavior models of a system model in a single behavior model.

Definition 5 (System Product) *Let* $\mathfrak{S} = \{\mathfrak{B}_i \mid i \in I\} \in \mathrm{SYS}(\Sigma)$ *be a system model over* Σ *and* I *an index set. The* system product $\prod_S(\mathfrak{B}_i \mid i \in I) \in \mathrm{BEH}(\Sigma)$ *of* $\mathfrak{S}$ *is the behavior model* $\mathfrak{A} \in \mathrm{BEH}(\Sigma)$ *defined by*

1. $\mathfrak{A}|_{\Sigma_T} := \mathfrak{B}|_{\Sigma_T}$, *where* $\mathfrak{B} \in \mathfrak{S}$,
2. $A_s := \prod(B_{i,s} \mid i \in I)$ *for all* $s \in S_D$,
3. $f^{\mathfrak{A}}(\overline{\alpha}, t) := \prod(f^{\mathfrak{B}_i}(\overline{\alpha}(i), t) \mid i \in I)$ *for all* $f : \overline{s}, T \to s \in \Sigma_D$, $\overline{\alpha} \in A_{\overline{s}}$, *and* $t \in A_T$,
4. $p^{\mathfrak{A}}(\overline{\alpha}, t)$ *:iff* $p^{\mathfrak{B}_i}(\overline{\alpha}(i), t)$ *for all* $i \in I$, *for all* $p : \overline{s}, T \in \Sigma_D$, $\overline{\alpha} \in A_{\overline{s}}$, *and* $t \in A_T$,

where $\alpha \in A_s$ *denotes a function* $\alpha : I \to \biguplus\{B_{i,s} \mid i \in I\}$ *with* $\alpha(i) \in B_{i,s}$ *for all* $i \in I$.

FDTs for Timed Systems

Combining the idea of formality expressed by institutions with the models introduced to express timing aspects of a system leads to the definition of the notion of *FDTs for timed systems*. For a deeper motivation of the several conditions stated in the following definition we refer to [EDK98a]. Here we only mention that the coincidence and isomorphism condition are required for each kind of FDT. They express that *truth is invariant under change of notation* (see [GB92]) and under isomorphisms of structures, respectively. The other three conditions result from the specific timing models we consider here.

In the following definition SET denotes the category of all sets as objects and mappings between sets as morphisms. Taking categories themselves as objects and functors as morphisms, the category CAT is obtained.

Based on a satisfaction relation $\models$ the theory of a set $\mathfrak{K}$ of sentences, $Th(\mathfrak{K})$, is defined as for first order logic.

Definition 6 (FDTs for Timed Systems) *An* FDT $\mathcal{F}$ *consists of*

1. *a non-empty category* $\mathrm{SIG}_{\mathcal{F}}$, *whose objects are called* signatures,
2. *a functor* $Sen_{\mathcal{F}} : \mathrm{SIG}_{\mathcal{F}} \to \mathrm{SET}$,
3. *a contravariant functor* $Str_{\mathcal{F}} : \mathrm{SIG}_{\mathcal{F}} \to \mathrm{CAT}$[2],
4. *a family of relations* $\models_{\mathcal{F},\Sigma} \subseteq |Str_{\mathcal{F}}(\Sigma)| \times Sen_{\mathcal{F}}(\Sigma)$[3] *called* satisfaction relations,

such that the following conditions hold for all $\mathfrak{A}, \mathfrak{B} \in Str_{\mathcal{F}}(\Sigma)$, $\mathfrak{A}' \in Str_{\mathcal{F}}(\Sigma')$, $\varphi \in Sen_{\mathcal{F}}(\Sigma)$, *and* $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{\mathcal{F}}$:

- coincidence condition: $\mathfrak{A}' \models_{\mathcal{F},\Sigma'} Sen_{\mathcal{F}}(\sigma)(\varphi)$ *iff* $Str_{\mathcal{F}}(\sigma)(\mathfrak{A}') \models_{\mathcal{F},\Sigma} \varphi$,

[2]Strictly speaking CAT needs to include 'large' categories. As in [GB92] we refer to the 'hierarchy of universes' discussed in [Lan71].

[3]Instead of $(\mathfrak{A}, \varphi) \in \models_{\mathcal{F},\Sigma}$ we write as usual $\mathfrak{A} \models_{\mathcal{F},\Sigma} \varphi$.

- isomorphism condition: $\mathfrak{A} \cong \mathfrak{B}$ *and* $\mathfrak{A} \models_{\mathcal{F},\Sigma} \varphi$ *implies* $\mathfrak{B} \models_{\mathcal{F},\Sigma} \varphi$.

An FDT for timed systems $\mathcal{F}$ *(T-FDT for short) is an FDT fulfilling*

- signature condition: $\mathrm{S}\textsc{ig}_{\mathcal{F}} = \mathrm{TS}\textsc{ig}_{\mathcal{F}} \uplus \mathrm{SS}\textsc{ig}_{\mathcal{F}}$, *where* $\mathrm{TS}\textsc{ig}_{\mathcal{F}} \subseteq \mathrm{TS}\textsc{ig}$, $\mathrm{SS}\textsc{ig}_{\mathcal{F}} \subseteq \mathrm{SS}\textsc{ig}$, *and* $\forall \Sigma \in \mathrm{SS}\textsc{ig}_{\mathcal{F}}\ \exists \Sigma' \in \mathrm{TS}\textsc{ig}_{\mathcal{F}}\ \Sigma = \Sigma'_{St}$,
- structure condition: $Str_{\mathcal{F}}(\Sigma) \subseteq \begin{cases} \mathrm{B}\textsc{eh}(\Sigma) & \text{if } \Sigma \in \mathrm{TS}\textsc{ig}_{\mathcal{F}} \\ \mathrm{S}\textsc{tate}(\Sigma) & \text{if } \Sigma \in \mathrm{SS}\textsc{ig}_{\mathcal{F}} \end{cases}$
and $Str_{\mathcal{F}}(\sigma) = Str(\sigma)$ *for all* $\sigma : \Sigma \to \Sigma' \in \mathrm{S}\textsc{ig}_{\mathcal{F}}$,
- system product condition: $\forall \Sigma \in \mathrm{TS}\textsc{ig}_{\mathcal{F}}$ *and each system model* $\{\mathfrak{B}_i \mid i \in I\} \subseteq |Str_{\mathcal{F}}(\Sigma)|$ *with* $\prod_S \mathfrak{B}_i \in |Str_{\mathcal{F}}(\Sigma)|$ *it holds that* $Th(\{\mathfrak{B}_i \mid i \in I\}) = Th(\prod_S \mathfrak{B}_i)$.

3 Temporal Logics as T-FDTs

In this section we provide evidence that the description techniques usually called temporal logics belong to the class of T-FDTs. Temporal logics are widely used and accepted description techniques for the specification of systems where timing aspects are important. Temporal logics are special cases of modal logics where the relation on the set of possible worlds is required to be a partial ordering.

From the way T-FDTs are defined it could be expected that temporal logics fit nicely into this framework. Since there exists a large variety of different temporal logics it is not possible to prove this for each single temporal logic. In the following we show that three well known representatives are actually T-FDTs, namely MTL [Koy92], MTL-$\int$ [LH95], and CTL* [Eme90]. During the presentations of the embedding process we explain why we believe that this process can easily be transferred to other temporal logics. This is an experience we made by adopting the proof for MTL, the temporal logic we have considered at first, to CTL* and MTL-$\int$. Moreover, the three selected temporal logics cover many different aspects. This is emphasized in 3.1 where all three temporal logics are briefly introduced.

To show that a description technique $\mathcal{F}$ is a T-FDT it is necessary to instantiate the definition of a T-FDT. This means to provide definitions of the categories of signatures $\mathrm{SS}\textsc{ig}_{\mathcal{F}}$ and $\mathrm{TS}\textsc{ig}_{\mathcal{F}}$, the functors $Sen_{\mathcal{F}}$ and $Str_{\mathcal{F}}$, and the family of satisfaction relations $\models_{\mathcal{F},\Sigma}$. It has to be justified that this characterization as a T-FDT corresponds to the original definition of $\mathcal{F}$. Furthermore, it has to be proven that the conditions of Definition 6 for FDTs in general and for T-FDTs in particular are satisfied. The instantiations are presented in 3.2 and the proofs are outlined in 3.3. The full and detailed proofs can be found in the appendix.

3.1 Introduction into MTL, MTL-$\int$, and CTL*

Metric Temporal Logic (MTL) is a propositional temporal logic introduced by Koymans in [Koy92]. After that many authors have used this term to denote temporal logics using temporal operators with time bounds (see e.g. [AH90]). In contrast to the traditional temporal operators MTL provides so called *bounded operators* to express quantitative

timing properties. While for example the formula $\varphi\mathcal{U}\psi$ only states that eventually ψ will become true and that up to this time point φ is valid, the formula $\varphi\mathcal{U}_{\leq\tau}\psi$ requires additionally that ψ has to become valid within the next τ time units. In the next sections we follow mainly the original approach of Koymans, but from the large set of possible temporal operators we consider only the future operator $\mathcal{U}$ (*until*). Based on it the other future temporal operators can be defined in the usual way. Therefore this restriction is not a substantial one. Adding the past operator $\mathcal{S}$ (*since*) is conceptually not difficult, but would blow up the proofs.
MTL-$\int$, introduced by Lakhneche and Hooman in [LH95], is an extension of MTL that allows the *specification of properties about duration of system's states* [LH95]. The term $\int^r \varphi$, for example, represents the entire time the formula φ is valid during the next r time units. Then the formula $\int^r \varphi \leq 11$ expresses the property that this time does not exceeds eleven time units. In contrast to MTL, MTL-$\int$ also allows existential quantification over variables ranging over real numbers. Thus, MTL-$\int$ is that what Emerson calls an *interpreted First-order temporal logic* [Eme90]. Another well known description technique that allows to express properties about durations is the *duration calculus* [CHR91, HC91]. In [LH95] it is proven that every formula of the duration calculus can be translated into an equivalent MTL-$\int$ formula. Furthermore, in [LH95] it is shown that MTL-$\int$ is more expressive than the duration calculus.
As MTL, the *Computational Tree Logic* (CTL*), introduced by Emerson in [Eme90] is a propositional temporal logic. While MTL and MTL-$\int$ provide a linear time model CTL* is based on a branching one. Furthermore, CTL* provides no means to express quantitative timing properties. We follow mainly [Eme90] but use the same basic operators for the propositional part as for MTL and MTL-$\int$. Furthermore, we consider only the universal path-quantifier A and omit the existential one, E, which can be defined in the usual way using A, viz. $\mathsf{E}\varphi \equiv \neg\mathsf{A}\neg\varphi$. Both changes are syntactical ones and do not alter CTL* substantially. They are made in order to simplify the proof that CTL* is a T-FDT. In the following we expect the reader to be familiar with the considered temporal logics.

3.2 Instantiating the Categories and the Functors

According to the definition of a T-FDT the following entities have to be instantiated for each temporal logic $\mathcal{F}$:

- the category $\text{SIG}_{\mathcal{F}}$, i.e. the categories $\text{TSIG}_{\mathcal{F}}$ and $\text{SSIG}_{\mathcal{F}}$,
- the functor $Sen_{\mathcal{F}} : \text{SIG}_{\mathcal{F}} \to \text{SET}$,
- the functor $Str_{\mathcal{F}} : \text{SIG}_{\mathcal{F}} \to \text{CAT}$, and
- the family of satisfaction relations $\models_{\Sigma,\mathcal{F}} \subseteq |Str_{\mathcal{F}}(\Sigma)| \times Sen_{\mathcal{F}}(\Sigma)$.

We refer to all three temporal logics by *TL*.

3.2.1 Instantiation of $\text{SIG}_{\mathcal{F}}$

With regard to the signature condition of Definition 6 it is naturally to define TSIG_{TL} as a subcategory of TSIG by requiring additional conditions on timed signatures and on the signature morphisms contributing to TSIG_{TL}.
We have made the experience that the sorts as well as the function and predicate symbols of a signature in TSIG_{TL} can be classified into three groups. The first group is summed up in a time signature Σ_T. It consists of all entities needed to define the time model of the temporal logic and is different for each temporal logic. The other two groups are special enrichments of such a signature and belong to the defined enrichment Σ_D. The first kind of enrichment is the so called *propositional enrichment*. It provides the entities for the definition of the basic (untimed) propositional calculus, which can be defined in the same way for all three temporal logics. The entities of the second, so called *term enrichment* are used for the construction of special terms needed by a temporal logic. In the case of MTL-$\int$ this term enrichment is needed to construct duration and first-order terms. Concerning MTL we use the term enrichment to construct the terms that can occur as indices of the until operator. While Σ_T and the propositional enrichment have to be non-empty, the term enrichment can be empty (see e.g. the definition of CTL*).

Time Signatures
Since all three temporal logics use different time models they differ in their time signatures.
MTL as defined by Koymans provides an abstract linear time model allowing to handle quantitative temporal properties by measuring the distance between two time points. Therefore, in addition to the sort T and the predicate symbol $<$ we need a function symbol d to express this distance. Furthermore, the range of d has to be determined using a sort Δ. To express later in the definition of Str_{MTL}, i.e. in the definition of the MTL behavior models, the conditions on the domain associated with Δ the two function symbols $+$ and 0 are needed additionally.

Definition 7 (MTL Time Signature) *A time signature* $\Sigma_T = (S_T, F_T, P_T, D_T)$ *is an* MTL time signature *if it fulfills the following conditions:*
- $S_T = \{T, \Delta\}$
- $F_T = \{d, +, 0\}$
- $P_T = \{<\}$
- $D_T = \{d : T, T \to \Delta; + : \Delta, \Delta \to \Delta; 0 :\to \Delta; <: T, T\}$

The time model of **MTL-$\int$** is a specialization of the abstract MTL time model. In the case of MTL-$\int$ the domains associated on the one hand with the time points and on the other hand with the metrical space are both the nonnegative reals $\mathbb{R}_{\geq 0}$. Therefore, the sort Δ and the symbols $d, +, 0$ for stating properties of the domain associated with Δ are not needed in MTL-$\int$. As an extension of MTL time signatures in MTL-$\int$ the length of the interval that is considered can be bounded. To express this length a function symbol δ is needed ranging over the domain assigned to the sort L. We need an additional sort because the range assigned to L contains in addition to the nonnegative reals also a new element ∞. This element is needed to express unbounded intervals.

Definition 8 (MTL-$\int$ Time Signature) *A time signature $\Sigma_T = (S_T, F_T, P_T, D_T)$ is an* MTL-$\int$ time signature *if it fulfills the following conditions:*
- $S_T = \{T, L\}$
- $F_T = \{\delta\}$
- $P_T = \{<\}$
- $D_T = \{\delta :\to L; <: T, T\}$

CTL* provides a branching time model with the only restriction that all branches have to start at a common time point. To express this later in the definition of Str_{CTL^*} only the additional function symbol 0, representing this starting point, is needed.

Definition 9 (CTL* Time Signature) *A time signature $\Sigma_T = (S_T, F_T, P_T, D_T)$ is a* CTL* time signature *if it fulfills the following conditions:*
- $S_T = \{T\}$
- $F_T = \{0\}$
- $P_T = \{<\}$
- $D_T = \{0 :\to T; <: T, T\}$

Propositional Enrichment
Each timed signature of each considered temporal logic has to contain a propositional enrichment. It consists of the sort B, the two function symbols *true* and *false*, and a finite set $PL = \{p_1, \ldots, p_n\}$ of function symbols, so called *propositional letters* (or observables). Such observables are usually used to model conditions that can change over time, e.g. an observable `occupied` can represent the situation whether a person is in a particular room or not. Based on these entities we define later the functor Str_{TL} so that the interpretation of them results in the usual boolean structure.

Definition 10 (Propositional Enrichment) *Let Σ_T be a time signature. A* propositional enrichment $\Sigma_P = (S_P, F_P, P_P, D_P)$ *of Σ_T is given by*
- $S_P = \{B\}$
- $F_P = \{true, false\} \uplus PL$ *with* $PL = \{p_1, \ldots, p_n\}$ *for* $n \in \mathbb{N}$
- $P_P = \emptyset$
- $D_P = \{f : T \to B | f \in F_P\}$

Note, that for the sake of minimality it would be sufficient to require only one of the two function symbols *true* or *false* to be in F_P. Furthermore, we want to point out that within our framework it is not possible to introduce *true* and *false* as predicate symbols with the declaration $true, false : T$. Defining them this way would result in 0-ary predicate symbols when constructing the corresponding state signature which are not allowed.

Term Enrichment
Concerning CTL* all components needed for the definition of the CTL* specific timed signatures are introduced so far. In the case of the two metrical temporal logics a further term enrichment is needed. For **MTL** this is a set F_{Term} of function symbols to construct the terms that can occur as indices of the until-operator $\mathcal{U}$ as for example in the formula $\varphi\, \mathcal{U}_{\prec t_1 + t_2} \psi$. Here, $t_1 + t_2$ is the index term of $\mathcal{U}$.

Definition 11 (MTL Term Enrichment) *Let Σ_T be an MTL time signature. An* MTL term enrichment *of Σ_T is an enrichment $\Sigma^{MTL}_{Term} = (S_{Term}, F_{Term}, P_{Term}, D_{Term})$ defined by:*
- $S_{Term} = \emptyset$
- $F_{Term} = \{f_1, \ldots, f_m\}, m \in \mathbb{N}$
- $P_{Term} = \emptyset$
- $D_{Term} = \{f : \Delta^k, T \to \Delta \mid f \in F_{Term}, k \in \mathbb{N}\}$

Since **MTL-$\int$** allows only nonnegative real numbers and the infinity-symbol ∞ to occur in the index of the until operator $\mathcal{U}$ or the since operator $\mathcal{S}$, e.g. $\mathcal{U}_{\prec\infty}$ or $\mathcal{S}_{\prec 121}$, we do not need a similar term enrichment for the indices in MTL-$\int$. Nevertheless, MTL-$\int$ is a first-order temporal logic that allows to compute values over the time domain and to relate them to one another. To construct these terms and formulae a term enrichment $\Sigma^{MTL-\int}_{Term}$ is needed. For example in the MTL-$\int$-formula $\int^{f(4)} p(y) \leq g(x)$ we have that $f(4)$, $g(x)$, and $\int^{f(4)} p(y)$ are such terms that when evaluated represent values over the time domain. In this example f and g are *term function symbols*, x and y are *term variables*, and p and $\leq$ are *term relation symbols.*

Definition 12 (MTL-$\int$ Term Enrichment) *Let Σ_T be an MTL-$\int$ time signature. An* MTL-$\int$ term enrichment *of Σ_T is an enrichment*
$\Sigma^{MTL\text{-}\int}_{Term} = (S_{Term}, F_{Term}, P_{Term}, D_{Term})$ *defined by:*

- $S_{Term} = \{R\}$
- $F_{Term} = Var \uplus Func$*, where Var and Func are sets of* term variables *and* term function symbols*, respectively.*
- P_{Term} *is a set of* term relation symbols.
- $D_{Term} = D_{Var} \uplus D_{Func} \uplus D_{Rel}$ *with* $D_{Var} = \{x : T \to R | x \in Var\}$, $D_{Func} = \{f : R^k, T \to R \mid f \in F_{Term}, k \geq 0\}$, *and* $D_{Rel} = \{p : R^k, T \mid p \in P_{Term}, k \geq 1\}$

Timed Signatures

Summing up the three groups of entities for each temporal logic leads to the following definition of the classes of timed signatures. The union of the signatures and enrichments is done component wise.

Definition 13 (Timed signatures) *A timed signature $\Sigma = (S, F, P, D)$ is a timed signature of the corresponding temporal logic if it is the disjoint union of the mentioned components.*

1. MTL: *an MTL time signature Σ_T^{MTL}, a propositional enrichment Σ_P, and an MTL term enrichment Σ^{MTL}_{Term};*
2. MTL-$\int$: *an MTL-$\int$ time signature $\Sigma_T^{MTL\text{-}\int}$, a propositional enrichment Σ_P, and an MTL-$\int$ term enrichment $\Sigma^{MTL\text{-}\int}_{Term}$;*
3. CTL*: *a CTL* time signature $\Sigma_T^{CTL^*}$ and a propositional enrichment Σ_P.*

Based on the timed signatures we can now define the corresponding subcategories of TSIG. In addition to the restrictions stated in Definition 1 we require for the morphisms of each category that they leave the boolean function symbols *true* and *false* unchanged.

Definition 14 (Categories TSIG_{MTL},$\text{TSIG}_{MTL\text{-}\int}$,$\text{TSIG}_{CTL^*}$**)** *The category with MTL/ MTL-$\int$/CTL* timed signatures as objects and the corresponding signature morphisms $\sigma : \Sigma \to \Sigma'$ with $\sigma(false) = false$ and $\sigma(true) = true$ as morphisms is the category* $\text{TSIG}_{MTL}/\text{TSIG}_{MTL\text{-}\int}/\text{TSIG}_{CTL^*}$

State Signatures SSIG_{TL}
Since the main purpose of a temporal logic is to describe how the behavior of a system *evolves* over time all three considered temporal logics do not support to refer directly to single states. Therefore, the category SSIG_{TL} is for all three temporal logics the empty category. We expect that this can be done for all temporal logics when showing that they are T-FDTs. Consequently we have $\text{SIG}_{TL} = \text{TSIG}_{TL}$.

3.2.2 Instantiation of $Str_{\mathcal{F}}$

Since we require SSIG_{TL} to be the empty category we have to define the functor Str_{TL} only for the timed signatures. With regard to the structure condition of Definition 6 we define Str_{TL} as a restriction of the functor Str. The interpretation of the propositional enrichment is the same for all three temporal logics.

Definition 15 (Propositional Behavior Model) *Let Σ be a timed signature that can be split into three disjoint parts $\Sigma = \Sigma_T \uplus \Sigma_P \uplus \Sigma_r$ where Σ_T is a time signature, Σ_P a propositional enrichment of Σ_T, and Σ_r an enrichment of Σ_T. A behavior model $\mathfrak{A} = (A, \_^{\mathfrak{A}})$ over Σ, $\mathfrak{A} \in Beh(\Sigma)$, is a* propositional behavior model *over Σ if it fulfills the following conditions*

1. $A_B = \{TRUE, FALSE\}$
2. $\forall t \in A_T : (true^{\mathfrak{A}}(t) = TRUE \text{ and } false^{\mathfrak{A}}(t) = FALSE)$

The notion *rigid*, defined below, is used to simplify the remaining conditions, especially those required for the interpretation of the symbols of the term enrichments. Informally spoken, a function or predicate is called *rigid* if it is invariant under time. For instance the interpretation of the symbols of the term enrichments are usually required to be rigid. This means that for example the usual arithmetic operations $+, -, *, /$ have to be time independent and have to compute always the same result for the same (non-time) arguments.

Definition 16 (Rigid) *Let Σ be a timed signature and $\mathfrak{A} \in Str(\Sigma)$ a structure over Σ. Further let f and p be a function and predicate symbol of Σ_D, respectively, with the declaration $f : S_1, \ldots, S_n, T \to S$ and $p : S_1, \ldots, S_m, T$, $n \geq 0$, $m > 0$.*
$f^{\mathfrak{A}}$ is rigid *iff $\forall t, t' \in A_T, \forall \bar{\alpha} \in A_{S_1} \times \ldots \times A_{S_n} : f^{\mathfrak{A}}(\bar{\alpha}, t) = f^{\mathfrak{A}}(\bar{\alpha}, t')$.*
$p^{\mathfrak{A}}$ is rigid *iff $\forall t, t' \in A_T$, $\forall \bar{\alpha} \in A_{S_1} \times \ldots \times A_{S_m} : (\bar{\alpha}, t) \in p^{\mathfrak{A}}$ iff $(\bar{\alpha}, t') \in p^{\mathfrak{A}}$.*

Based on propositional behavior models we now define for each temporal logic a specific category of behavior models.

Definition 17 (MTL Behavior Model) *Let $\Sigma \in \text{SIG}_{MTL}$ be an MTL timed signature. A propositional behavior model $\mathfrak{A} = (A, \_^{\mathfrak{A}})$, $\mathfrak{A} \in Str(\Sigma)$, is called an* MTL behavior model *over Σ if it fulfills the following conditions:*

1. $\forall f \in F_{Term}$ $f^{\mathfrak{A}}$ *is rigid.*
2. $(A_T, <^{\mathfrak{A}})$ *is a total ordering.*

3. $d^{\mathfrak{A}}$ *is surjective.*

4. $\forall t, t', t'' \in A_T$:
   - $d^{\mathfrak{A}}(t,t') = 0^{\mathfrak{A}}$ *iff* $t = t'$
   - $d^{\mathfrak{A}}(t,t') = d^{\mathfrak{A}}(t',t)$
   - *if* $t <^{\mathfrak{A}} t' <^{\mathfrak{A}} t''$ *then* $d^{\mathfrak{A}}(t,t'') = +^{\mathfrak{A}}(d^{\mathfrak{A}}(t,t'), d^{\mathfrak{A}}(t',t''))$

5. $\forall \delta, \delta', \delta'' \in A_\Delta$:
   - $+^{\mathfrak{A}}(\delta,\delta') = +^{\mathfrak{A}}(\delta',\delta)$
   - $+^{\mathfrak{A}}(\delta, 0^{\mathfrak{A}}) = \delta$
   - $+^{\mathfrak{A}}(\delta, +^{\mathfrak{A}}(\delta',\delta'')) = +^{\mathfrak{A}}(+^{\mathfrak{A}}(\delta,\delta'),\delta'')$
   - $+^{\mathfrak{A}}(\delta,\delta') = +^{\mathfrak{A}}(\delta,\delta'') \leadsto \delta' = \delta''$
   - $+^{\mathfrak{A}}(\delta,\delta') = 0^{\mathfrak{A}} \leadsto \delta = 0^{\mathfrak{A}}$ *and* $\delta' = 0^{\mathfrak{A}}$
   - $\exists \tilde{\delta} \in A_\Delta : +^{\mathfrak{A}}(\delta',\tilde{\delta}) = \delta$ *or* $+^{\mathfrak{A}}(\delta,\tilde{\delta}) = \delta'$

The category with MTL behavior models over an MTL timed signature Σ *as objects and the corresponding homomorphisms as morphisms is denoted by* $\mathrm{BEH}_{MTL}(\Sigma)$.

Condition 1 requires all functions symbols of the term enrichment to be rigid. Concerning the time model the conditions are directly taken from [Koy92]. At first, Koymans requires that $(A_T, <^{\mathfrak{A}})$ is a total ordering. According to [Koy92] the *surjectivity of d is demanded to get a nice correspondence between T and* Δ, i.e. between A_T and A_Δ. Condition 4 demands the usual conditions of a metric apart from the replacement of the triangular inequality by a conditional equality. Condition 5 requires several properties of the function $+^{\mathfrak{A}}$, e.g. that it is commutative and associative.
The relations $\le^{\mathfrak{A}}, >^{\mathfrak{A}}$, and $\ge^{\mathfrak{A}}$ are derived from $<^{\mathfrak{A}}$ as usual. Furthermore we use the relations $\preceq^{\mathfrak{A}}, \prec^{\mathfrak{A}}, \succ^{\mathfrak{A}} \subseteq A_\Delta \times A_\Delta$. $\preceq^{\mathfrak{A}}$ is defined by: $\delta \preceq^{\mathfrak{A}} \delta'$:iff $\exists \delta'' : \delta' = \delta +^{\mathfrak{A}} \delta''$. The relations $\prec^{\mathfrak{A}}$ and $\succ^{\mathfrak{A}}$ are derived from $\preceq^{\mathfrak{A}}$ as usual.

Definition 18 (MTL-$\int$ Behavior Model) *Let* $\Sigma \in \mathrm{SIG}_{MTL\text{-}\int}$ *be an MTL-$\int$ timed signature. A propositional behavior model* $\mathfrak{A} = (A, \_^{\mathfrak{A}})$, $\mathfrak{A} \in Str(\Sigma)$, *is called an* MTL-$\int$ behavior model *over* Σ *if it fulfills the following conditions:*

1. $A_T = \mathbb{R}_{\geq 0}$, $A_R = \mathbb{R}$, *and* $A_L = \mathbb{R}_{\geq 0} \uplus \{\infty\}$
2. $<^{\mathfrak{A}}$ *is the usual ordering on* $\mathbb{R}$.
3. *The elements of* F_{Term} *and* P_{Term} *are rigid.*
4. finite variability*:*
   $\forall p \in F_P$, $\forall t \in \mathbb{R}_{\geq 0}$ *there is a finite partition* $[t_0,t_1),[t_1,t_2),\ldots,[t_{n-1},t_n)$, $n \in \mathbb{N}$, $t_n = t$, $t_0 = 0$, *of* $[0,t)$ *with* $\forall i, 0 \le i < n$ $((\forall \bar{t} \in [t_i,t_{i+1}) : p^{\mathfrak{A}}(\bar{t}) = TRUE)$ *or* $(\forall \bar{t} \in [t_i,t_{i+1}) : p^{\mathfrak{A}}(\bar{t}) = FALSE))$

The category with MTL-$\int$ behavior models over an MTL-$\int$ timed signature Σ *as objects and the corresponding homomorphisms leaving the carrier sets invariant, i.e.* $\forall s \in \{T,R,L\}$ $\forall r \in A_s$: $h_s(r) = r$, *as morphisms is denoted by* $\mathrm{BEH}_{MTL\text{-}\int}(\Sigma)$.

Similar to MTL we require the structures in $Str_{MTL\text{-}\int}(\Sigma)$ to interpret the symbols of the term enrichment rigidly. The rest of the conditions dealing with the time model mainly determine the domains assigned to the sorts. Additionally, the finite variability condition excludes zeno behavior models in $Str_{MTL\text{-}\int}(\Sigma)$ and is important for the integrability of boolean formulae (see [LH95]). The restriction concerning the morphisms is necessary since MTL-$\int$ fixes the carrier sets to be exactly those sets stated in the definition and nothing else; even isomorphic carrier sets are not allowed.

To define the time model of CTL* formally we use the following terms. Let $(S,<)$ be a partial ordering. $s \in S$ is the *minimal element* of S iff $\forall s' \in S : s' \neq s \leadsto s < s'$. A subset $C \subseteq S$ of S is a *path* of $(S,<)$ iff $(C,<')$, $<' = < \cap (C \times C)$, is a total ordering. D is a *fullpath* of $(S,<)$ if it is a *maximal* path, i.e. $\forall s \in S \setminus D, \exists d \in D : s \not< d \wedge d \not< s$. Let $\mathfrak{A}$ be a behavior model. Then $dsucc : A_T \to 2^{A_T}$, the function assigning the set of all direct successors of a time point t to t, is defined by $dsucc(t) = \{t' | t' >^{\mathfrak{A}} t \wedge \neg\exists t'' : t <^{\mathfrak{A}} t'' <^{\mathfrak{A}} t'\}$.

Definition 19 (CTL* Behavior Model) *Let $\Sigma \in$ SIG_{CTL^*} be a CTL* timed signature. A propositional behavior model $\mathfrak{A} = (A, \_^{\mathfrak{A}})$, $\mathfrak{A} \in Str(\Sigma)$ is called a* CTL* behavior model *over Σ if it fulfills the following conditions:*

1. *$(A_T, <^{\mathfrak{A}})$ is a partial ordering.*
2. *$0^{\mathfrak{A}}$ is the minimal element of A_T.*
3. *For every fullpath C of $(A_T, <^{\mathfrak{A}}) : (C, <^{\mathfrak{A}} \cap (C \times C))$ is isomorphic to $(\mathbb{N}, <^{\mathbb{N}})$.*
4. *$dsucc(t) \neq \emptyset$ for all $t \in A_T$.*
5. *$\forall t, t' : t \neq t' \leadsto dsucc(t) \cap dsucc(t') = \emptyset$*

The category with CTL behavior models over a CTL* timed signature Σ as objects and the corresponding homomorphisms as morphisms is denoted by* $\text{BEH}_{CTL^*}(\Sigma)$.

Since a CTL* timed signature does not contain a term enrichment all conditions deal with the time model. Condition 2 requires a common start point for all branches. Condition 3 determines the time model of a single branch. Conditions 4 and 5 restrict the way in which the time model can branch.
For each CTL* behavior model $\mathfrak{A} \in Str_{CTL^*}(\Sigma)$ we use $FP(\mathfrak{A})$ to denote the class of all fullpaths of $(A_T, <^{\mathfrak{A}})$. Since every fullpath in $FP(\mathfrak{A})$ is isomorphic to $\mathbb{N}$ according to the previous definition we use the convention that $x = (t_0, t_1, t_2, \dots) \in FP(\mathfrak{A})$ denotes a fullpath and that x^i denotes the i-th suffix path $(t_i, t_{i+1}, \dots)$ of x. Note, that for each fullpath $x \in FP(\mathfrak{A})$ we have $t_0 = 0^{\mathfrak{A}}$. The *suffix closure* $FP_{sc}(\mathfrak{A})$ of $FP(\mathfrak{A})$ is given by $y \in FP_{sc}(\mathfrak{A})$ iff $\exists x \in FP(\mathfrak{A})$ and $\exists i \in \mathbb{N}$ with $y = x^i$. We refer to the elements of $FP_{sc}(\mathfrak{A})$ as *tracks* (of $\mathfrak{A}$). $hd(y)$ denotes the first element of a track y.

Based on these categories of behavior models and the general functor Str for each of the three temporal logics the functor Str_{TL} is defined as follows:

Definition 20 (Functor Str_{TL}) *The functor Str_{TL}: $\text{SIG}_{TL} \to \text{CAT}$ is given by $Str_{TL}(\Sigma) := \text{BEH}_{TL}(\Sigma)$ and $Str_{TL}(\sigma) := Str(\sigma)$.*

That this is actually a well defined functor is shown by the proof of the structure condition (see Theorem 38). The problem is the assignment $Str_{TL}(\sigma) := Str(\sigma)$. By this we assign to a signature morphism $\sigma : \Sigma \to \Sigma'$ the forgetful functor $|_\sigma : Str(\Sigma') \to Str(\Sigma)$. That this is well defined we have to show that given a TL behavior model $\mathfrak{A}'$ over Σ', i.e. $\mathfrak{A}' \in Str_{TL}(\Sigma') = \text{BEH}_{TL}(\Sigma')$, then also $\mathfrak{A} := \mathfrak{A}'|_\sigma$ is a TL behavior model over Σ, i.e. $\mathfrak{A} \in Str_{TL}(\Sigma) = \text{BEH}_{TL}(\Sigma)$. Otherwise, we can not assign $|_\sigma$ to σ by the functor Str_{TL}.

3.2.3 Instantiation of $Sen_{\mathcal{F}}$

Concerning the objects for all three temporal logics the functor Sen_{TL} can easily be defined based on the usual syntax definitions given for the temporal logics. To emphasize the common principal construction of sentences of temporal logics we split the object part of the definition of Sen_{TL} into four parts. We expect that for almost all temporal logics these four parts can be identified.

- A functor $Term_{TL}$ specifying special terms that are not considered as sentences on their own, but are needed to construct the set of sentences. Usually this functor is based on the term enrichment of a timed signature.
- A set Γ_{TL} of so called *connector symbols* that allow the combination of sentences to more complex ones.
- A functor $AtSen_{TL}$ specifying the set of the so called *atomic sentences*. An atomic sentence is a sentence that contains no connector symbol.
- A functor Sen_{TL} specifying the set of all sentences. Sen_{TL} is always defined recursively starting with the atomic sentences and using the connector symbols to build more complex sentences.

The morphism part of the functor Sen_{TL} extends in all cases the underlying signature morphism in a natural way. Since this construction is very similar for all three temporal logics we present the morphism part only for MTL.
In the following we give the definitions of these four parts for each temporal logic separately.

MTL

At first we introduce a functor $Term_{MTL}$ specifying the so called *index terms* that can be used as indices of the operator $\mathcal{U}$. These terms are not considered as sentences on their own but are necessary to construct the set of connector symbols $\Gamma_{MTL,\Sigma}$ associated with a signature. Usually these terms are "small" terms, i.e. typical examples are constants τ or the addition of two constants, $\tau_1 + \tau_2$. Nevertheless, more complex terms are possible.
Since all functions assigned to function symbols of the term enrichment are required to be rigid the time sort is omitted in the following definition. This means that in an index term a function symbol of the term enrichment is used with an arity that is one less than the arity given in the declaration of the term enrichment. Instead of having for example the connector symbol $\mathcal{U}_{\prec\tau(t)}$ with $\forall t_1, t_2 : \tau(t_1) = \tau(t_2)$ the connector symbol $\mathcal{U}_{\prec\tau}$ is used.

Definition 21 (Functor $Term_{MTL}$, Index Terms)
The functor $Term_{MTL} : \mathrm{SIG}_{MTL} \to \mathrm{SET}$ *is given by:*
$Term_{MTL}(\Sigma)$ *is the least set fulfilling the following conditions:*

1. $f \in Term_{MTL}(\Sigma)$ *for all* $f \in F_{Term}$ *with* $f : T \to \Delta \in D_{Term}$,
2. *if* $t_1, \ldots, t_k \in Term_{MTL}(\Sigma)$ *and* $f \in F_{Term}$ *with* $f : \Delta^k, T \to \Delta \in D_{Term}$ *then* $f(t_1, \ldots, t_k) \in Term_{MTL}(\Sigma)$.

$c \in Term_{MTL}(\Sigma)$ *is called an* index term *over* Σ.
On the morphisms of SIG_{MTL} *we define* $Term_{MTL}$ *as follows. Let* $\sigma : \Sigma \to \Sigma'$ *be a signature morphism. For* $t \in Term_{MTL}(\Sigma)$ *we define* $Term_{MTL}(\sigma) : Term_{MTL}(\Sigma) \to Term_{MTL}(\Sigma')$ *by:*

1. *if* $t \equiv f$ *with* $f : T \to \Delta \in D_{Term}$ *then* $Term_{MTL}(\sigma)(f) := \sigma(f)$
2. *if* $t \equiv f(t_1, \dots t_n), n \geq 1$ *then*
   $Term_{MTL}(\sigma)(f(t_1, \dots, t_n)) := \sigma(f)(Term_{MTL}(\sigma)(t_1), \dots, Term_{MTL}(\sigma)(t_n))$

The set of the connector symbols consists of the implication symbol $\to$ for the propositional basis and the temporal operator $\mathcal{U}$ extended with index terms. To focus on the main ideas, especially when proving the conditions required for a T-FDT in the next section, we consider here only the future part of MTL. As already mentioned adding the past operator $\mathcal{S}$ is conceptually not difficult, but would blow up the proofs. In the case of MTL-$\int$ we consider the operator $\mathcal{S}$, too.

Definition 22 (Connector Symbols) *Let* $\Sigma \in \mathrm{SIG}_{MTL}$ *be an MTL-signature. The alphabet* $\Gamma_{MTL,\Sigma}$ *of* connector symbols *over* Σ *is given by:*

$$\Gamma_{MTL,\Sigma} := \{\to\} \cup \{\mathcal{U}_{\sim c} \mid \sim \in \{\prec, \succ, =\}, c \in Term_{MTL}(\Sigma)\}.$$

The set of the atomic sentences corresponds with the function symbols of the propositional enrichment.

Definition 23 (Atomic Sentences) *The functor* $AtSen_{MTL} : \mathrm{SIG}_{MTL} \to \mathrm{SET}$ *is given by* $AtSen_{MTL}(\Sigma) := F_P$.
On the morphisms of SIG_{MTL} *we define* $AtSen_{MTL}$ *as follows. Let* $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL}$ *be a signature morphism. For* $\varphi \in AtSen_{MTL}(\Sigma)$ *we define* $AtSen_{MTL}(\sigma) : AtSen_{MTL}(\Sigma) \to AtSen_{MTL}(\Sigma')$ *by* $AtSen_{MTL}(\sigma)(\varphi) := \sigma(\varphi)$.

The functor Sen_{MTL} is now the natural extension of $AtSen_{MTL}$ using the set of connector symbols.

Definition 24 (Sentences) *The functor* $Sen_{MTL} : \mathrm{SIG}_{MTL} \to \mathrm{SET}$ *is given by:* $Sen_{MTL}(\Sigma)$ *is the least set fulfilling the following conditions:*

1. $AtSen_{MTL}(\Sigma) \subseteq Sen_{MTL}(\Sigma)$,
2. *if* $\varphi, \psi \in Sen_{MTL}(\Sigma)$ *and* $\oplus \in \Gamma_{MTL,\Sigma}$ *then* $\varphi \oplus \psi \in Sen_{MTL}(\Sigma)$.

On the morphisms of SIG_{MTL} *we define* Sen_{MTL} *as follows. Let* $\sigma : \Sigma \to \Sigma'$ *be a signature morphism. For* $\varphi \in Sen_{MTL}(\Sigma)$ *we define* $Sen_{MTL}(\sigma) : Sen_{MTL}(\Sigma) \to Sen_{MTL}(\Sigma')$ *by:*

1. *if* $\varphi \in AtSen_{MTL}(\Sigma)$ *then* $Sen_{MTL}(\sigma)(\varphi) := AtSen_{MTL}(\sigma)(\varphi)$,
2. *if* $\varphi \equiv \psi_1 \to \psi_2$ *then* $Sen_{MTL}(\sigma)(\psi_1 \to \psi_2) = Sen_{MTL}(\sigma)(\psi_1) \to Sen_{MTL}(\sigma)(\psi_2)$,
3. *if* $\varphi \equiv \psi_1 \mathcal{U}_{\sim c} \psi_2$, $\sim \in \{\prec, \succ, =\}, c \in Term_{MTL}(\Sigma)$ *then*
   $Sen_{MTL}(\sigma)(\psi_1 \mathcal{U}_{\sim c} \psi_2) = Sen_{MTL}(\sigma)(\psi_1) \mathcal{U}_{\sim Term_{MTL}(\sigma)(c)} Sen_{MTL}(\sigma)(\psi_2)$.

MTL-$\int$

In the case of MTL-$\int$ so called *duration terms* are needed to express duration properties. Basic duration terms are the real numbers $\mathbb{R}$ and the term variables *Var* of an MTL-$\int$ term enrichment. Besides the usual recursively constructed terms using the term function symbols of an MTL-$\int$ term enrichment MTL-$\int$ provides duration terms of the form $\int^\tau \varphi$ where τ is a duration term and φ an MTL-$\int$ sentence, i.e. a common formula. As defined later in Definition 34 $\int^\tau \varphi$ represents the time φ is valid during the next τ time units.
Analogously to MTL we omit in a duration term the time argument. Since MTL-$\int$ sentences are needed to construct duration terms of the form $\int^\tau \varphi$ we have already to refer to the functor $Sen_{MTL\text{-}\int}$ defined below.

Definition 25 (Duration Terms) *The functor* $Term_{MTL\text{-}\int} : \mathrm{SIG}_{MTL\text{-}\int} \to \mathrm{SET}$ *is given by:*
$Term_{MTL\text{-}\int}(\Sigma)$ *is the least set fulfilling the following conditions:*

1. $\mathbb{R} \subseteq Term_{MTL\text{-}\int}(\Sigma)$,
2. $Var \subseteq Term_{MTL\text{-}\int}(\Sigma)$,
3. *if* $t_1, \ldots t_k \in Term_{MTL\text{-}\int}(\Sigma)$ *and* $g \in F_{Term}$ *with* $g : R^k, T \to R \in D_{Term}$ *then also* $g(t_1, \ldots t_k) \in Term_{MTL\text{-}\int}(\Sigma)$,
4. *if* $\varphi \in Sen_{MTL\text{-}\int}(\Sigma)$ *and* $t \in Term_{MTL\text{-}\int}(\Sigma)$ *then also* $\int^t \varphi \in Term_{MTL\text{-}\int}(\Sigma)$.

$c \in Term_{MTL\text{-}\int}(\Sigma)$ *is called a* duration term *over* Σ.
On the morphisms of $\mathrm{SIG}_{MTL\text{-}\int}$ *we define* $Term_{MTL\text{-}\int}$ *as follows. Let* $\sigma : \Sigma \to \Sigma'$ *be a signature morphism. For* $\varphi \in Term_{MTL\text{-}\int}(\Sigma)$ *we define* $Term_{MTL\text{-}\int}(\sigma) : Term_{MTL\text{-}\int}(\Sigma) \to Term_{MTL\text{-}\int}(\Sigma')$ *by:*

1. *if* $\varphi \in \mathbb{R}$ *then* $Term_{MTL\text{-}\int}(\sigma)(\varphi) := \varphi$,
2. *if* $\varphi \in Var$ *then* $Term_{MTL\text{-}\int}(\sigma)(\varphi) := \sigma(\varphi)$,
3. *if* $\varphi \equiv f(t_1, \ldots t_n), n \geq 1$ *then*
   $Term_{MTL\text{-}\int}(\sigma)(f(t_1, \ldots, t_n)) := \sigma(f)(Term_{MTL\text{-}\int}(\sigma)(t_1), \ldots, Term_{MTL\text{-}\int}(\sigma)(t_n))$,
4. *if* $\varphi \equiv \int^t \psi$ *then* $Term_{MTL\text{-}\int}(\sigma)(\int^t \psi) := \int^{Term_{MTL\text{-}\int}(\sigma)(t)} Sen_{MTL\text{-}\int}(\sigma)(\psi)$.

Besides the common logical connectives $\neg$, $\vee$, and $\exists$ and the classical indexed temporal operators $\mathcal{U}$ and $\mathcal{S}$, MTL-$\int$ also provides the operator $\mathcal{C}$. This operator allows to combine two time periods and by this the formulation of a compositional proof rule (see [LH95] for details). The satisfaction relation $\models_{MTL\text{-}\int,\Sigma}$ is later defined so that a formula $\varphi_1 \mathcal{C} \varphi_2$ holds if the future can be divided into two periods: a first period that satisfies φ_1 and a second one satisfying φ_2.

Definition 26 (Connector Symbols) *Let* $\Sigma \in \mathrm{SIG}_{MTL\text{-}\int}$ *be an MTL-*$\int$ *timed signature. The alphabet* $\Gamma_{MTL\text{-}\int,\Sigma}$ *of* connector symbols *is given by*
$\Gamma_{MTL\text{-}\int,\Sigma} := \{\neg, \vee, \mathcal{C}\} \cup \{\exists x | x \in Var\} \cup \{\mathcal{U}_{\prec\tau}, \mathcal{S}_{\prec\tau} | \tau \in \mathbb{R}_{\geq 0} \cup \{\infty\}\} \cup \{\mathcal{U}_{=\tau}, \mathcal{S}_{=\tau} | \tau \in \mathbb{R}_{\geq 0}\}$

As usual for first order temporal logics the set of basic formulae, i.e. the atomic sentences in our framework, associated with an MTL-$\int$ timed signature Σ consist of the set of observables F_P of the propositional enrichment of Σ and the expressions using relation symbols and duration terms as their arguments. The final set of sentences are defined recursively as usual.

Definition 27 (Atomic Sentences) *The functor* $AtSen_{MTL\text{-}\int} : \mathrm{SIG}_{MTL\text{-}\int} \to \mathrm{SET}$ *is given by* $AtSen_{MTL\text{-}\int}(\Sigma) := F_P \cup \{r(t_1, \ldots, t_k) | r \in P_{Term}$ *with* $r : R^k, T \in D_{Rel}, t_i \in Term_{MTL\text{-}\int}(\Sigma)\}$.

Definition 28 (Sentences) *The functor* $Sen_{MTL\text{-}\int} : \mathrm{SIG}_{MTL\text{-}\int} \to \mathrm{SET}$ *is given by:* $Sen_{MTL\text{-}\int}(\Sigma)$ *is the least set fulfilling the following conditions:*

1. $AtSen_{MTL\text{-}\int}(\Sigma) \subseteq Sen_{MTL\text{-}\int}(\Sigma)$,
2. *if* $\varphi \in Sen_{MTL\text{-}\int}(\Sigma)$ *and* $\oplus \in \{\neg\} \cup \{\exists x \mid x \in Var\}$ *then* $\oplus\varphi \in Sen_{MTL\text{-}\int}(\Sigma)$,
3. *if* $\varphi, \psi \in Sen_{MTL\text{-}\int}(\Sigma)$ *and* $\oplus \in \Gamma_{MTL\text{-}\int,\Sigma} \setminus (\{\neg\} \cup \{\exists x \mid x \in Var\})$ *then* $\varphi \oplus \psi \in Sen_{MTL\text{-}\int}(\Sigma)$.

CTL*

In the case of CTL* the timed signatures provide no symbols to construct special terms. Therefore, only the connector symbols and the two functors $AtSen_{CTL^*}$ and Sen_{CTL^*} have to be defined. In contrast to the two metrical temporal logics the set of connector symbols of CTL* is independent of a specific signature.

Definition 29 (Connector Symbols) *Let* $\Sigma \in \mathrm{SIG}_{CTL^*}$ *be a CTL*-timed signature. The alphabet* Γ_{CTL^*} *of* connector symbols *is given by* $\Gamma_{CTL^*} := \{\to \mathsf{A}, \mathsf{X}, \mathsf{U}\}$.

Definition 30 (Atomic Sentences) *The functor* $AtSen_{CTL^*} : \mathrm{SIG}_{CTL^*} \to \mathrm{SET}$ *is given by* $AtSen_{CTL^*}(\Sigma) := F_P$.

In conformity with [Eme90] we inductively define besides the functor Sen_{CTL^*} an additional functor $PSen_{CTL^*} : \mathrm{SIG}_{CTL^*} \to \mathrm{SET}$. $Sen_{CTL^*}(\Sigma)$ is the language associated with a CTL*-signature and contains the so called *state formulae*. $PSen_{CTL^*}(\Sigma)$ consists of the so called *path formulae*. They are only needed to construct state formulae but do not contribute directly to the set of CTL* sentences. The difference between these two groups of formulae results from the way in which a formula is interpreted: a state formula over single *states*, represented by a specific time point of a CTL*-structure, and a path formula over paths of a CTL*-structure, see Definition 36.

Definition 31 (Sentences) *The functors* $Sen_{CTL^*} : \mathrm{SIG}_{CTL^*} \to \mathrm{SET}$ *and* $PSen_{CTL^*} : \mathrm{SIG}_{CTL^*} \to \mathrm{SET}$ *are given by:*
$Sen_{CTL^*}(\Sigma)$ *and* $PSen_{CTL^*}(\Sigma)$ *are the least sets fulfilling the following conditions:*

1. $AtSen_{CTL^*}(\Sigma) \subseteq Sen_{CTL^*}(\Sigma)$,
2. *if* $\varphi_1, \varphi_2 \in Sen_{CTL^*}(\Sigma)$ *then* $\varphi_1 \to \varphi_2 \in Sen_{CTL^*}(\Sigma)$,
3. *if* $\psi \in PSen_{CTL^*}(\Sigma)$ *then* $\mathsf{A}\psi \in Sen_{CTL^*}(\Sigma)$,

4. $Sen_{CTL^*}(\Sigma) \subseteq PSen_{CTL^*}(\Sigma)$,
5. *if* $\psi_1, \psi_2 \in PSen_{CTL^*}(\Sigma)$ *then* $\psi_1 \to \psi_2 \in PSen_{CTL^*}(\Sigma)$,
6. *if* $\psi_1, \psi_2 \in PSen_{CTL^*}(\Sigma)$ *then* $\mathsf{X}\psi_1 \in PSen_{CTL^*}(\Sigma)$ *and* $\psi_1 \mathsf{U} \psi_2 \in PSen_{CTL^*}(\Sigma)$.

Note, that $PSen_{CTL^*}(\sigma)$ restricted to state formulae, i.e. to $Sen_{CTL^*}(\Sigma)$, coincides with $Sen_{CTL^*}(\sigma)$.

3.2.4 Instantiation of $\models_{\mathcal{F},\Sigma}$

Analogously to the previous instantiations there is a scheme for the specification of the family of satisfaction relations which is valid for all three temporal logics considered here. Again, we are convinced that this scheme can be transferred to other temporal logics, too.
If there are special terms as in the case of MTL and MTL-$\int$ we first have to define their meaning w.r.t. a given behavior model. This means, given a set $\mathcal{T}$ of terms and a behavior model $\mathfrak{A} = (A, -^{\mathfrak{A}})$ we have to define an *interpretation* $\mathfrak{I}_{\mathfrak{A}} : A_T \times \mathcal{T} \to A$.
The actual satisfaction relations are defined in two steps. At first an additional satisfaction relation $\Vert\!\models_{TL}$ is introduced. This relation interprets a sentence w.r.t. a given behavior model and a specific time point. Based on this, there are two different approaches for defining $\models_{TL}$, i.e. the interpretation of a sentence under a given behavior model. The first approach, called the *floating version*, requires a sentence to be valid under $\Vert\!\models_{TL}$ at all time points. This is the way MTL is defined. The second approach, called the *anchored version*, demands the validity of a sentence under $\Vert\!\models_{TL}$ at a distinguished time point, usually the starting point of a behavior model. This is the way MTL-$\int$ and CTL* are defined.
Since each temporal logic has its specific features we have to consider each temporal logic separately.

MTL
The interpretation of the index terms is defined as usual. Note that for the interpretation of the terms we have to add again the omitted time argument.

Definition 32 (Interpretation $\mathfrak{I}^{MTL}_{\mathfrak{A}}$) *Let $\Sigma \in$ SIG$_{MTL}$ be an MTL timed signature and $\mathfrak{A} \in Str_{MTL}(\Sigma)$ an MTL behavior model over Σ. $\mathfrak{A}$ induces the* interpretation *$\mathfrak{I}^{MTL}_{\mathfrak{A}} : A_T \times Term_{MTL}(\Sigma) \to A_\Delta$ defined as follows. Let $t \in A_T$ be an arbitrary time point.*

1. *if $f \in Term_{MTL}(\Sigma)$ with $f : T \to \Delta \in D_{Term}$ then $\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, f) := f^{\mathfrak{A}}(t)$*
2. *if $f(s_1, \ldots, s_k) \in Term_{MTL}(\Sigma)$, $k \geq 1$, then*
   $\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, f(s_1, \ldots, s_k)) := f^{\mathfrak{A}}(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, s_1), \ldots, \mathfrak{I}^{MTL}_{\mathfrak{A}}(t, s_k), t)$

The definition of the interpretation of the MTL-sentences is the usual one.

Definition 33 (Satisfaction Relation $\models_{MTL,\Sigma}$) *Let $\Sigma \in$ SIG$_{MTL}$ be an MTL timed signature, $\mathfrak{A} \in Str_{MTL}(\Sigma)$ an MTL behavior model, and $t \in A_T$ a time point.*
$\Vert\!\models_{MTL,\Sigma} \subseteq |Str_{MTL}(\Sigma)| \times A_T \times Sen_{MTL}(\Sigma)$ is defined by (we use infix notation, i.e. $(\mathfrak{A}, t) \Vert\!\models_{MTL,\Sigma} \varphi$ iff $(\mathfrak{A}, t, \varphi) \in \Vert\!\models_{MTL,\Sigma}$).

1. $\forall p \in F_D : (\mathfrak{A}, t) \Vdash_{MTL,\Sigma} p$ *iff* $p^{\mathfrak{A}}(t) = TRUE$
2. $(\mathfrak{A}, t) \Vdash_{MTL,\Sigma} \varphi \rightarrow \psi$ *iff* $(\mathfrak{A}, t) \Vdash_{MTL,\Sigma} \psi$ *or* $(\mathfrak{A}, t) \nVdash_{MTL,\Sigma} \varphi$
3. $(\mathfrak{A}, t) \Vdash_{MTL,\Sigma} \varphi \mathcal{U}_{\sim c} \psi$ *iff* $\exists t', t' \geq^{\mathfrak{A}} t, d^{\mathfrak{A}}(t', t) \sim^{\mathfrak{A}} \mathfrak{I}_{\mathfrak{A}}(c)$: $(\mathfrak{A}, t') \Vdash_{MTL,\Sigma} \psi$ *and* $\forall t'', t \leq^{\mathfrak{A}} t'' <^{\mathfrak{A}} t' : (\mathfrak{A}, t'') \Vdash_{MTL,\Sigma} \varphi$, $\sim \in \{\prec, \succ, =\}$

$\models_{MTL,\Sigma} \subseteq |Str_{MTL}(\Sigma)| \times Sen_{MTL}(\Sigma)$ *is defined as follows:*

$$\mathfrak{A} \models_{MTL,\Sigma} \varphi \text{ :iff } \forall t \in A_T : (\mathfrak{A}, t) \Vdash_{MTL,\Sigma} \varphi$$

MTL-$\int$

Due to the special feature of MTL-$\int$ allowing to integrate over sentences (i.e. formulae), the definition of the interpretation of the terms is more complicated than for MTL. Corresponding to [LH95] we define for each sentence φ and each behavior model $\mathfrak{A}$ a function $f_\varphi^{\mathfrak{A}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ whose value is 1 at a time point t if and only if φ is evaluated to $TRUE$ at t by $\mathfrak{A}$, i.e.

$$f_\varphi^{\mathfrak{A}}(t) = \begin{cases} 1 & \text{if } (\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi \\ 0 & \text{otherwise} \end{cases}$$

In the following the meaning of the term $\int^\tau \varphi$ is defined by means of the Riemann integral of the function associated with φ. It is required that the *Dirichlet condition* is satisfied for $f_\varphi^{\mathfrak{A}}$. We refer to [LH95] for a deeper discussion of the relations between the Dirichlet condition, the Riemann integrability, and the finite variability condition required for MTL-$\int$ behavior models.

In the following definitions $+, -$ denote the usual operations and $\leq, <$ the usual relations on $\mathbb{R}$.

Definition 34 (Interpretation $\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}$) *Let $\Sigma \in$ SIG$_{MTL\text{-}\int}$ be an MTL-$\int$ timed signature and $\mathfrak{A} \in Str_{MTL\text{-}\int}(\Sigma)$ an MTL-$\int$ behavior model over Σ. $\mathfrak{A}$ induces an* interpretation $\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int} : A_T \times Term_{MTL\text{-}\int}(\Sigma) \rightarrow \mathbb{R}$ *defined as follows. Let $t \in A_T$ be an arbitrary time point.*

1. *if* $r \in \mathbb{R}$ *then* $\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, r) = r$
2. *if* $x \in Var$ *then* $\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, x) = x^{\mathfrak{A}}(t)$
3. *if* $s_1, \ldots, s_k \in Term_{MTL\text{-}\int}(\Sigma)$ *and* $g \in F_{Term}$ *with* $g : R^k, T \rightarrow R \in D_{Term}$ *then* $\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, g(s_1, \ldots, s_k)) = g^{\mathfrak{A}}(\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, s_1), \ldots, \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, s_k), t)$
4. $\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, \int^\tau \varphi) = \begin{cases} \int_t^{t+\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t,\tau)} f_\varphi^{\mathfrak{A}}(t')\,dt' & \textit{if } f_\varphi^{\mathfrak{A}} \textit{ satisfies the Dirichlet condition and } \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t,\tau) \geq 0 \\ 0 & \textit{otherwise} \end{cases}$

For the definition of the satisfaction relation we need the following two notions. Let Σ be an MTL-$\int$ timed signature. The *concatenation* of two MTL-$\int$ behavior models $\mathfrak{A}_1, \mathfrak{A}_2 \in$ BEH$_{MTL\text{-}\int}(\Sigma)$ with $\mathfrak{A}_1|_{\Sigma_{Term}} = \mathfrak{A}_2|_{\Sigma_{Term}}$ is the MTL-$\int$ behavior model $\mathfrak{A}_1\mathfrak{A}_2 \in$ BEH$_{MTL\text{-}\int}(\Sigma)$ with

- $\mathfrak{A}_1\mathfrak{A}_2|_{\Sigma_{Term}} = \mathfrak{A}_1|_{\Sigma_{Term}}$
- $\delta^{\mathfrak{A}_1\mathfrak{A}_2} = \delta^{\mathfrak{A}_1} + \delta^{\mathfrak{A}_2}$
- $\forall p \in PL : p^{\mathfrak{A}_1\mathfrak{A}_2}(t) = \begin{cases} p^{\mathfrak{A}_1}(t) & 0 \leq t < \delta^{\mathfrak{A}_1} \\ p^{\mathfrak{A}_2}(t - \delta^{\mathfrak{A}_1}) & \delta^{\mathfrak{A}_1} \leq t < \delta^{\mathfrak{A}_1} + \delta^{\mathfrak{A}_2} \end{cases}$

Let $\tau \in \mathbb{R}$ be a real number and $x \in Var$ a variable. The *variant* $\mathfrak{A}_x^\tau \in \mathrm{BEH}_{MTL\text{-}\int}(\Sigma)$ of $\mathfrak{A}$ w.r.t. τ and x is given by

- $\mathfrak{A}_x^\tau|_{\Sigma\setminus\{x\}} = \mathfrak{A}|_{\Sigma\setminus\{x\}}$
- $\forall t \in \mathbb{R} : x^{\mathfrak{A}_x^\tau}(t) = \tau$

Definition 35 (Satisfaction Relation) *Let $\Sigma \in \mathrm{SIG}_{MTL\text{-}\int}$ be an MTL-$\int$ timed signature, $\mathfrak{A} \in Str_{MTL\text{-}\int}(\Sigma)$ an MTL-$\int$ behavior model, and $t \in A_T$ a time point.*
The relation $\Vdash_{MTL\text{-}\int,\Sigma} \subseteq |Str_{MTL\text{-}\int}(\Sigma)| \times A_T \times Sen_{MTL\text{-}\int}(\Sigma)$ is defined by (we use infix notation, i.e. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi$ iff $(\mathfrak{A}, t, \varphi) \in \Vdash_{MTL\text{-}\int,\Sigma}$).

1. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} true$ *and* $(\mathfrak{A}, t) \nVdash_{MTL\text{-}\int,\Sigma} false$
2. $\forall p \in F_D : (\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} p$ *iff* $p^{\mathfrak{A}}(t) = TRUE$ *and* $t < \delta^{\mathfrak{A}}$
3. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} r(s_1, \ldots, s_k)$ *iff* $(\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, s_1), \ldots, \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, s_k), t) \in r^{\mathfrak{A}}$
4. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \neg\varphi$ *iff* $(\mathfrak{A}, t) \nVdash_{MTL\text{-}\int,\Sigma} \varphi$
5. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_1 \vee \varphi_2$ *iff* $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_1$ *or* $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_2$
6. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi\mathcal{U}_{<\tau}\psi$ *iff* $\exists t' \in \mathbb{R}_{\geq 0}, t \leq t' < t + \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, \tau) :$
   $(\mathfrak{A}, t') \Vdash_{MTL\text{-}\int,\Sigma} \psi$ *and*
   $\forall t'' \in \mathbb{R}_{\geq 0}, t \leq t'' < t' : (\mathfrak{A}, t'') \Vdash_{MTL\text{-}\int,\Sigma} \varphi$
7. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi\mathcal{U}_{=\tau}\psi$ *iff* $(\mathfrak{A}, t + \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, \tau)) \Vdash_{MTL\text{-}\int,\Sigma} \psi$ *and*
   $\forall t' \in \mathbb{R}_{\geq 0}, t \leq t' < t + \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, \tau) : (\mathfrak{A}, t') \Vdash_{MTL\text{-}\int,\Sigma} \varphi$
8. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi\mathcal{S}_{<\tau}\psi$ *iff* $\exists t' \in \mathbb{R}_{\geq 0}, \max(0, t - \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, \tau)) < t' \leq t :$
   $(\mathfrak{A}, t') \Vdash_{MTL\text{-}\int,\Sigma} \psi$ *and*
   $\forall t'' \in \mathbb{R}_{\geq 0}, t' < t'' \leq t : (\mathfrak{A}, t'') \Vdash_{MTL\text{-}\int,\Sigma} \varphi$
9. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi\mathcal{S}_{=\tau}\psi$ *iff* $(\mathfrak{A}, \max(0, t - \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, \tau))) \Vdash_{MTL\text{-}\int,\Sigma} \psi$ *and*
   $\forall t' \in \mathbb{R}_{\geq 0}, t - \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, \tau) \leq t' < t : (\mathfrak{A}, t') \Vdash_{MTL\text{-}\int,\Sigma} \varphi$
10. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_1\mathcal{C}\varphi_2$ *iff there exist behavior models* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *over* Σ *with*
    $\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2$, $\delta^{\mathfrak{A}_1} \geq t$, $(\mathfrak{A}_1, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_1$,
    and $(\mathfrak{A}_2, 0) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_2$
11. $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \exists x\varphi$ *iff there exists a value* $\tau \in \mathbb{R}$ *with* $(\mathfrak{A}_x^\tau, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi$

$\models_{MTL\text{-}\int,\Sigma} \subseteq |Str_{MTL\text{-}\int}(\Sigma)| \times Sen_{MTL\text{-}\int}(\Sigma)$ is defined as follows:

$$\mathfrak{A} \models_{MTL\text{-}\int,\Sigma} \varphi \;:\!\mathit{iff}\; (\mathfrak{A}, 0) \Vdash_{MTL\text{-}\int,\Sigma} \varphi$$

CTL*

In contrast to the previous two metrical temporal logics no interpretation of special terms has to be defined for CTL*. But, due to the way the sentences are defined in CTL* we have to split the satisfaction relation $\Vdash_{CTL^*,\Sigma}$ into two relations:

- $\Vdash^s_{CTL^*,\Sigma}$ for the interpretation of a state formula
- $\Vdash^p_{CTL^*,\Sigma}$ for the interpretation of a path formula

Definition 36 (Satisfaction Relation) *Let $\Sigma \in \mathrm{SIG}_{CTL^*}$ be a CTL* timed signature, $\mathfrak{A} \in Str_{CTL^*}(\Sigma)$ a CTL* behavior model, $t \in A_T$ a time point, and $x \in FP_{sc}(\mathfrak{A})$ a track over $\mathfrak{A}$. The relations $\Vdash^s_{CTL^*,\Sigma} \subseteq |Str_{CTL^*}(\Sigma)| \times A_T \times Sen_{CTL^*}(\Sigma)$, $\Vdash^p_{CTL^*,\Sigma} \subseteq |Str_{CTL^*}(\Sigma)| \times FP_{sc}(\mathfrak{A}) \times PSen_{CTL^*}(\Sigma)$ are defined by (we use infix notations with the sentences on the right side of the relation symbols):*

1. $\forall p \in F_D : (\mathfrak{A}, t) \Vdash^s_{CTL^*,\Sigma} p$ *iff* $p^{\mathfrak{A}}(t) = TRUE$
2. $(\mathfrak{A}, t) \Vdash^s_{CTL^*,\Sigma} \varphi_1 \to \varphi_2$ *iff* $(\mathfrak{A}, t) \Vdash^s_{CTL^*,\Sigma} \varphi_2$ *or* $(\mathfrak{A}, t) \not\Vdash^s_{CTL^*,\Sigma} \varphi_1$
3. $(\mathfrak{A}, t) \Vdash^s_{CTL^*,\Sigma} \mathsf{A}\psi$ *iff for all* $y \in FP_{sc}(\mathfrak{A}), hd(y) = t$, $(\mathfrak{A}, y) \Vdash^p_{CTL^*,\Sigma} \psi$
4. *if* $\psi \in Sen_{CTL^*}(\Sigma)$ *then* $(\mathfrak{A}, x) \Vdash^p_{CTL^*,\Sigma} \psi$ *iff* $(\mathfrak{A}, t) \Vdash^s_{CTL^*,\Sigma} \psi$ *and* $hd(x) = t$
5. $(\mathfrak{A}, x) \Vdash^p_{CTL^*,\Sigma} \psi_1 \to \psi_2$ *iff* $(\mathfrak{A}, x) \Vdash^p_{CTL^*,\Sigma} \psi_2$ *or* $(\mathfrak{A}, x) \not\Vdash^p_{CTL^*,\Sigma} \psi_1$
6. $(\mathfrak{A}, x) \Vdash^p_{CTL^*,\Sigma} \mathsf{X}\psi$ *iff* $(\mathfrak{A}, x^1) \Vdash^p_{CTL^*,\Sigma} \psi$
7. $(\mathfrak{A}, x) \Vdash^p_{CTL^*,\Sigma} \psi_1 \mathsf{U} \psi_2$ *iff* $\exists i : (\mathfrak{A}, x^i) \Vdash^p_{CTL^*,\Sigma} \psi_2$ *and* $\forall j, j < i : (\mathfrak{A}, x^j) \Vdash^p_{CTL^*,\Sigma} \psi_1$

$\models_{CTL^*,\Sigma} \subseteq |Str_{CTL^*}(\Sigma)| \times Sen_{CTL^*}(\Sigma)$ is defined as follows:

$$\mathfrak{A} \models_{CTL^*,\Sigma} \varphi \ \text{:iff} \ (\mathfrak{A}, 0^{\mathfrak{A}}) \Vdash^s_{CTL^*,\Sigma} \varphi$$

3.3 Proofs of the Conditions

For all three considered temporal logics most of the proofs of the T-FDT specific conditions are straightforward. The signature condition and the structure condition concerning the object part of Str_{TL} are direct consequences of the definitions of SIG_{TL} and $Str_{TL}(\Sigma)$ because in both cases we consider a subcategory of TSIG and $Beh(\Sigma)$, respectively. Moreover, the only system products in $Str_{TL}(\Sigma)$ are the ones consisting of only one behavior. For these system products the system product condition is trivially fulfilled. Technically this restriction of the system products is manifested by the condition that the sort assigned to the (boolean) sort B is fixed to $\{TRUE, FALSE\}$. Nevertheless, it is possible to consider for example the model class $Mod(\varphi)$ of a sentence φ.

The proofs of the morphism part of the structure condition as well as the coincidence and isomorphism condition are the most difficult tasks. Nevertheless, the proofs have the same structure for all three temporal logics. Therefore the proofs concerning CTL* and MTL-$\int$, which were the second and third temporal logic we considered, could be performed more easily and more quickly than the proofs concerning MTL. Only

some technical details had to be changed. We expect that the proof skeletons can be transferred to further temporal logics, too.
In the following sections we present the proof skeletons for the three conditions. The complete and detailed proofs for each of the three temporal logics can be found in the appendix.
In this section Σ and Σ' are always TL timed signatures and $\sigma : \Sigma \to \Sigma'$ is a timed signature morphism.

3.3.1 Proof of the Structure Condition

The structure condition determines several properties of the functor Str_{TL}. Concerning the object part of Str_{TL} the condition $Str_{TL}(\Sigma) \subseteq Beh(\Sigma)$ is a direct consequence of the way Str_{TL} is defined. That Str_{TL} fulfills additionally the condition for the morphism part, i.e. $Str_{TL}(\sigma) = Str(\sigma)$, is not obvious. As already mentioned in the remark made on the Definition 20 the main task is to show that Str_{TL} is a well defined functor.
To prove this we have to show that given a structure $\mathfrak{A}' \in Str_{TL}(\Sigma')$ and a signature morphism $\sigma : \Sigma \to \Sigma'$ then also the structure $\mathfrak{A} = \mathfrak{A}'|_\sigma$ is a TL behavior model over Σ, i.e $\mathfrak{A} \in Str_{TL}(\Sigma)$. If this property is fulfilled by Str_{TL} we say that Str_{TL} is *closed under signature morphisms*. If the functor Str_{TL} is not closed under signature morphisms it cannot behave as Str.
To prove this closure property we need a so called *signature invariance lemma* which is also needed to prove the coincidence condition. It states that the time signature and the propositional enrichment of a TL timed signature are invariant under signature morphisms. The general version of this lemma is:

Lemma 37 (Signature Invariance) *A timed signature morphism $\sigma : \Sigma \to \Sigma' \in$ SIG_{TL} has the following properties. (We use the primed versions of the symbols to denote the elements of Σ'.)*

$\sigma(\Sigma_T) = \Sigma'_T$, *i.e.*
- $\forall f \in F_T : \sigma(f) = f'$
- $\forall p \in P_T : \sigma(p) = p'$
- $\forall s \in S_T : \sigma(s) = s'$

$\sigma(\Sigma_P) = \Sigma'_P$, *i.e.*
- $\sigma(B) = B'$
- $\sigma(true) = true$
- $\sigma(false) = false$
- $\sigma(PL) \subseteq PL'$

This lemma guarantees a kind of syntactical invariance. It is now possible to derive from this lemma the following *Structure Closure Theorem*. This theorem states the property we need to show that Str_{TL} is a well defined functor and that the structure condition holds. Its general version is as follows.

Theorem 38 (Structure Closure) *Let $\mathfrak{A}' \in Str(\Sigma')$ be a structure over Σ' and $\mathfrak{A} = \mathfrak{A}'|_\sigma$, $\mathfrak{A} \in Str(\Sigma)$, a structure over Σ. Then $\mathfrak{A}'$ is a TL behavior model over Σ' if and only if $\mathfrak{A}$ is a TL behavior model over Σ, i.e.*

$$\mathfrak{A} \in \mathrm{SIG}_{TL}(\Sigma) \text{ iff } \mathfrak{A}' \in \mathrm{SIG}_{TL}(\Sigma')$$

A direct consequence of this theorem is the following corollary stating that the domain, functions, and predicates assigned to the time signatures and the propositional enrichment excluding those assigned to the function symbols of PL are the same. This corollary is used to show the coincidence condition.

Corollary 39 *Let $\mathfrak{A}' \in \mathrm{SIG}_{TL}(\Sigma')$ be a TL behavior model over Σ' and $\mathfrak{A} = \mathfrak{A}'|_\sigma$, $\mathfrak{A} \in \mathrm{SIG}_{TL}(\Sigma)$, a TL behavior model over Σ. Then the following holds:*

- $\forall s \in S_T : A_s = A'_{s'}$
- $\forall f \in F_T : f^{\mathfrak{A}} = f'^{\mathfrak{A}'}$
- $\forall p \in P_T : p^{\mathfrak{A}} = p'^{\mathfrak{A}'}$
- $A_B = A'_{B'}$
- $true^{\mathfrak{A}} = true^{\mathfrak{A}'}$
- $false^{\mathfrak{A}} = false^{\mathfrak{A}'}$

3.3.2 Proof of the Coincidence Condition

To prove the coincidence condition for a temporal logic two properties are needed. The first one, stated by the *term coincidence lemma* is needed if special terms are used by a temporal logic. It expresses the corresponding coincidence condition w.r.t. these special terms. By this, it represents a first part of the *semantical invariance* required by the coincidence condition.
In this lemma and in Lemma 44 which is the corresponding one of the proof of the isomorphism condition $Term_{TL} : \mathrm{SIG}_{TL} \to \mathrm{SET}$ is the functor specifying the terms in TL and $\mathfrak{I}^{TL}_{\mathfrak{A}}$ is the interpretation of these terms induced by a TL behavior model $\mathfrak{A}$. Furthermore, in this section $\mathfrak{A}' \in Str_{TL}(\Sigma')$ is always an arbitrary TL behavior model over Σ' and $\mathfrak{A} = \mathfrak{A}'|_\sigma$ is its reduct according to σ.

Lemma 40 (Term Coincidence) *For all terms $c \in Term_{TL}(\Sigma)$ and all time points $t \in A_T (= A'_{T'})$ it holds:*

$$\mathfrak{I}^{TL}_{\mathfrak{A}}(t, c) = \mathfrak{I}^{TL}_{\mathfrak{A}'}(t, Term_{TL}(\sigma)(c))$$

This lemma is proven by induction over the structure of $c \in Term_{TL}(\Sigma)$.
The second lemma guarantees a corresponding version of the coincidence condition for the satisfaction relation $\Vdash_{TL}$:

Lemma 41 ($\Vdash_{TL}$-coincidence) *Let $t \in A_T (= A'_{T'})$ be an arbitrary time point. Then the following is valid for all sentences $\varphi \in Sen_{TL}(\Sigma)$:*

$$(\mathfrak{A}', t) \Vdash_{TL,\Sigma'} Sen_{TL}(\sigma)(\varphi) \textit{ iff } (Str_{TL}(\sigma)(\mathfrak{A}'), t) \Vdash_{TL,\Sigma} \varphi$$

This lemma is proven by induction over the structure of the sentence $\varphi \in Sen_{TL}(\Sigma)$ and by using Corollary 39 and Lemma 40.
Based on Lemma 41 it is now quite easy to show that the coincidence condition is satisfied by TL, i.e. the following theorem holds:

Theorem 42 (Coincidence) *For all sentences $\varphi \in Sen_{TL}(\Sigma)$ over Σ it holds:*

$$\mathfrak{A}' \models_{TL,\Sigma'} Sen_{TL}(\sigma)(\varphi) \textit{ iff } Str_{TL}(\sigma)(\mathfrak{A}') \models_{TL,\Sigma} \varphi$$

Besides the mentioned lemmata further lemmata depending on specific features of a TL can be necessary to complete the proof of the coincidence condition. For example in the case of CTL* it is necessary to consider the fact that the satisfaction relation $\Vdash_{CTL^*}$ is split into two satisfaction relations.

3.3.3 Proof of the Isomorphism Condition

The scheme of the proof of the isomorphism condition consists essentially of three properties. As in the previous sections we present here only the general versions of the lemmata and the final theorem. Their instantiations and proofs of each of the three considered temporal logics can be found in the appendix. Moreover, technical details are only considered in the appendix.
In this section $\mathfrak{A}, \mathfrak{B} \in Str_{TL}(\Sigma)$ are always arbitrary isomorphic TL behavior models over Σ, i.e. $\mathfrak{A} \cong \mathfrak{B}$. $h = (h_s | s \in S)$ always denotes an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. The three lemmata are special versions of the isomorphism condition considering the boolean part of a TL behavior model (Lemma 43), the terms of a temporal logic (Lemma 44), and the satisfaction relation $\models_{TL,\Sigma}$ (Lemma 45). Since Lemma 43 concerns only the propositional enrichment of a time signature we can proof this lemma for all three temporal logics together.

Lemma 43 (Boolean Isomorphism) *It holds:*

1. $h_B(FALSE) = FALSE$ *and* $h_B(TRUE) = TRUE$
2. $\forall t \in A_T\ \forall p \in F_P$: $p^{\mathfrak{A}}(t) = TRUE$ *iff* $p^{\mathfrak{B}}(h_T(t)) = TRUE$

Proof:

ad 1.: Since $\mathfrak{A}$ and $\mathfrak{B}$ are propositional behavior models we have according to Definition 15

$$\forall t^a \in A_T\colon false^{\mathfrak{A}}(t^a) = FALSE \quad \text{(i)}$$
$$\forall t^b \in B_T\colon false^{\mathfrak{B}}(t^b) = FALSE \quad \text{(ii)}$$

Since h is a homomorphism we have:

$$\forall t^a \in A_T\colon h_B(false^{\mathfrak{A}}(t^a)) = false^{\mathfrak{B}}(h_T(t^a)) \quad \text{(iii)}$$

It follows for each $t^a \in A_T$ with $t^b := h_T(t^a)$:

$h_B(false^{\mathfrak{A}}(t^a)) \overset{(i)}{=} h_B(FALSE)$ and

$h_B(false^{\mathfrak{A}}(t^a)) \overset{(iii)}{=} false^{\mathfrak{B}}(h_T(t^a)) = false^{\mathfrak{B}}(t^b) \overset{(ii)}{=} FALSE$

$\rightsquigarrow h_B(FALSE) = FALSE$

Since h_B is bijective on $\{TRUE, FALSE\}$ it follows $h_B(TRUE) = TRUE$.

ad 2.: Case A) $p^{\mathfrak{A}}(t) = TRUE$:

$$p^{\mathfrak{B}}(h_T(t)) \overset{h \text{ homom.}}{=} h_B(p^{\mathfrak{A}}(t)) = h_B(TRUE) \overset{1.}{=} TRUE$$

Case B) $p^{\mathfrak{A}}(t) = FALSE$:

$$p^{\mathfrak{B}}(h_T(t)) \overset{h \text{ homom.}}{=} h_B(p^{\mathfrak{A}}(t)) = h_B(FALSE) \overset{1.}{=} FALSE$$

q.e.d.

Lemma 44 (Term Isomorphism) *For all terms $c \in Term_{TL}(\Sigma)$ and all time points $t \in A_T$ it holds:*

$$\mathfrak{I}^{TL}_{\mathfrak{B}}(h(t), c) = h(\mathfrak{I}^{TL}_{\mathfrak{A}}(t, c))$$

This lemma is proven by induction over the structure of c.

Lemma 45 ($\Vert\!\models_{TL,\Sigma}$-Isomorphism) *For all time points $t \in A_T$ and all sentences $\varphi \in Sen_{TL}$ it holds:*

$$(\mathfrak{A}, t) \Vert\!\models_{TL,\Sigma} \varphi \textit{ iff } (\mathfrak{B}, h(t)) \Vert\!\models_{TL,\Sigma} \varphi$$

This lemma is proven by induction over the structure of φ and by using Lemmata 43 and 44.
Using Lemma 45 it is quite easy to prove that the isomorphism condition itself holds.

Theorem 46 (Isomorphism) *For all sentences $\varphi \in Sen_{TL}$ it holds:*

$$\mathfrak{A} \models_{TL,\Sigma} \varphi \textit{ iff } \mathfrak{B} \models_{TL,\Sigma} \varphi$$

This completes the proof that the three considered temporal logics are actually T-FDTs. We want to state again, that we are confident that the instantiation process as well as the presented proof schemes can be transferred to a large number of further temporal logics, too. Moreover, we believe that the main ideas can also be used to show for other classes of description techniques that they are T-FDTs.

3.4 Justification of the Characterization

In the introduction of Section 3 we mentioned three tasks to be done to show that a specific description technique is a T-FDT:

- instantiation of the definition of a T-FDT
- proof of the conditions required of a T-FDT
- justification that the characterization as a T-FDT corresponds to the original definition of the description technique

Up to now we have dealt with the first two tasks. Both have been performed formally. Concerning the third task we only want to provide some informal arguments why we are convinced that our characterization of the three considered temporal logics as T-FDTs corresponds to their original definitions.
Looking at the original definitions of the temporal logics in the cited papers one can easily see that these definitions are given in a very similar way compared with our T-FDT characterization. For all three temporal logics the original definition is divided into two parts: syntax and semantics. The syntax part corresponds exactly to the definition of the functor Sen_{TL}. The semantics part comprises the functor Str_{TL} and the family of satisfaction relations $\models_{TL,\Sigma}$. Especially the definitions of Sen_{TL} and $\models_{TL,\Sigma}$ are nearly copies of the corresponding original definitions.
The main, but nevertheless small differences between the original and the T-FDT definitions are the following ones. We explicitly introduce signatures which is not the case in the original papers. We embed what is often called *underlying computational* or *semantical model* in the definition of the functor Str_{TL}. Note that Koymans as well as Emerson use the term *structure* for defining the semantic model nearly in the same way we use this term. In addition to all three original definitions we consider morphisms between the objects of the several classes contributing to a T-FDT.
Summing up, we can state that the main parts of the original definitions have been copied. Only slight modifications concerning terminology have been made which to our opinion do not change the essence of the considered temporal logics.

4 Related Work

We want to discuss briefly two fields of work related to the topics considered in this paper. The first one deals with our long term objective, the formal analysis of description techniques. The second point of the discussion addresses other approaches investigating and classifying temporal logics.

A similar approach to investigate FDTs is considered in [AR97], where a *pattern for analyzing a formal specification activity* is presented. Notions such as e.g. *end product*, *formal model*, and *rationale* are introduced and applied to *formal methods*. This is done in the framework of institutions, too. Compared to this work, we intended to be more formal and hence more precise when defining what constitutes an FDT and the models we consider. Also, in [AR97] methodological aspects are considered, whereas we restricted ourselves to characterize FDTs for timed systems more precisely.

In [Bro96] several models, as e.g. *data models* or *system–component models*, are presented. The models are oriented towards components of distributed systems. The mathematical model of ongoing behavior is based on streams of messages exchanged between the components. In contrast, our definitions are more oriented towards sequences of states. Although these approaches are interchangeable to some extent, the technical expositions are different. As a second point, [Bro96] aims at a comprehensive set of mathematical models as a formal foundation of software engineering, whereas we are interested in the formal analysis of T-FDTs.

In [BCN95] the emphasis of the analysis is on tools for real-time software specification. The authors describe the evolution and the state of the art of such tools. They classify description techniques in operational, descriptive, or dual ones and analyze a large number of description techniques according to several aspects. In contrast to our approach this analysis is not based on a formal basis and a common framework.

Concerning temporal logics Emerson introduces in [Eme90] not only CTL*, but he also provides some informal classification of temporal logics. He considers six pairs of attributes: propositional versus first-order, global versus compositional, branching versus linear time, points versus intervals, discrete versus continuous, and past versus future.

In [AH90] Alur and Henzinger present a framework which allows to analyze and classify real-time logics according to their complexity and expressiveness. Especially, expressiveness is an aspect we also want to investigate by several appropriate criteria. In [AH91] the same authors give a survey of logic- and automata-based description techniques. They present a semantical framework for real-time systems and summarize several results about expressive power, algorithmic finite state verification, and deductive verification. Whereas in contrast to the classification of Emerson the approaches of Alur and Henzinger are also formal, they are mainly intended to analyze temporal logics, especially their expressiveness. Our approach is intended to cover a larger class of description techniques and to analyze a larger class of aspects.

5 Concluding Remarks

We have shown that the three well known temporal logics MTL, MTL-$\int$, and CTL* fit well into the general framework of formal description techniques for timed systems. Furthermore, we have presented a general scheme for the instantiation as well as for the proofs of the several conditions that have to be fulfilled by each T-FDT. This general scheme and its relatively easy instantiation for the three considered temporal logics provide evidence that at least a large number, presumably all temporal logics can be embedded in the framework provided by T-FDTs.

That this framework is not only suitable for temporal logics but also for other kinds of description techniques is shown in [EDK98b]. In this paper we consider *abstract state machines* (ASM) [Gur95, Gur97], a description technique completely different than temporal logics, and show that ASM can be embedded in the framework of T-FDTs, too.

The embedding of such description techniques is also the first step of the third task in our approach to the formal analysis of FDTs, i.e. the application of criteria. Currently we are formulating criteria for this analysis and applying them to the description techniques we have already proven to be T-FDTs. An initial set of criteria investigating such different aspects as expressiveness, properties of the time model, and compositionality exists. We have made the experience that for the criteria established up to now our framework of T-FDTs is quite adequate.

In the future we want to enlarge this set of criteria, embed further description techniques in the framework of T-FDTs, and apply the criteria to all these T-FDTs.

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A Proofs for MTL

In this section we present the complete and detailed proofs that the structure, coincidence, and isomorphism condition holds for the temporal logic MTL defined as a T-FDT in Section 3.2.

A.1 Proof of the Structure Condition

As stated in the presentation of the general scheme the main task is to show that $\mathfrak{A} := \mathfrak{A}'|_\sigma$ is an MTL behavior model over Σ if $\mathfrak{A}'$ is an MTL behavior model over Σ'. We first show the instantiation of Lemma 37, i.e. the signature invariance of MTL timed signatures under MTL timed signature morphisms.

Lemma 47 (MTL Signature Invariance) *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL}$ be MTL timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL}$ an MTL timed signature morphism. Then σ has the following properties. (We use the primed versions of the symbols to denote the elements of Σ'.)*

- $\sigma(T) = T'$, $\sigma(\Delta) = \Delta'$, $\sigma(d) = d'$, $\sigma(0) = 0'$, $\sigma(+) = +'$, $\sigma(<) =<'$
- $\sigma(B) = B'$, $\sigma(\mathit{true}) = \mathit{true}$, $\sigma(\mathit{false}) = \mathit{false}$, $\sigma(PL) \subseteq PL'$

Proof:
$\sigma(<) =<'$ and $\sigma(T) = T'$ is already required for each timed signature morphism by Definition 1 and $\sigma(\mathit{true}) = \mathit{true}$ and $\sigma(\mathit{false}) = \mathit{false}$ by Definition 14.
Since σ has to map the sorts, function and predicate symbols such that the mapping is compatible with the declarations of Σ and Σ' and since $\sigma|_{\Sigma_T} : \Sigma_T \to \Sigma'_T$ has to be a signature morphism itself (see Definition 1) we can deduce from $\sigma(T) = T'$ directly $\sigma(\Delta) = \Delta'$ and then $\sigma(B) = B'$. The mapping of the function symbols $\sigma(f) = f'$, $f \in \{d, +, 0\}$ and of the propositional letters, i.e. $\sigma(PL) \subseteq PL'$ is also a consequence of the compatibility property of σ. q.e.d.

We can now prove the MTL version of the structure closure theorem:

Theorem 48 (MTL Structure Closure) *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL}$ be MTL timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL}$ an MTL timed signature morphism. Let further $\mathfrak{A}' \in \mathit{Str}(\Sigma')$ be a structure over Σ' and $\mathfrak{A} = \mathfrak{A}'|_\sigma$, $\mathfrak{A} \in \mathit{Str}(\Sigma)$, a structure over Σ. Then $\mathfrak{A}'$ is an MTL behavior model over Σ' if and only if $\mathfrak{A}$ is an MTL behavior model over Σ, i.e.*

$$\mathfrak{A} \in \mathrm{SIG}_{MTL}(\Sigma) \text{ iff } \mathfrak{A}' \in \mathrm{SIG}_{MTL}(\Sigma')$$

Proof:
To show that a structure $\mathfrak{B}$ is an MTL behavior model we have to show that the conditions of the Definitions 15 and 17 are satisfied.
According to the definition of the forgetful functor $|\sigma$ we have for all carrier sets $A_s = A'_{\sigma(s)}$, $s \in S$, and for all functions and relations symbols $f \in F \uplus P$: $f^{\mathfrak{A}} = \sigma(f)^{\mathfrak{A}'}$.
Since σ is an MTL timed signature morphism we have according to Lemma 47 $\sigma(s) = s'$ for $s \in \{T, \Delta, B\}$ and $\sigma(f) = f'$ for $f \in \{d, 0, +, <, \mathit{true}, \mathit{false}\}$. It follows $A_s = A'_{\sigma(s)} =$

$A'_{s'}$, $s \in \{T, \Delta, B\}$ and $f^{\mathfrak{A}} = \sigma(f)^{\mathfrak{A}'} = f'^{\mathfrak{A}'}$ for $f \in \{d, 0, +, <, true, false\}$. This means that all carrier sets and all functions and relations are equal for $\mathfrak{A}$ and $\mathfrak{A}'$. Consequently, the conditions of the Definitions 15 and 17 are valid for $\mathfrak{A}$ if and only if they are valid for $\mathfrak{A}'$. q.e.d.

The proof of the MTL structure closure theorem already includes the proof of the MTL version of Corollary 39.

Corollary 49 *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL}$ be MTL timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL}$ an MTL timed signature morphism. Let $\mathfrak{A}' \in \mathrm{SIG}_{MTL}(\Sigma')$ be an MTL behavior model over Σ' and $\mathfrak{A} = \mathfrak{A}'|_\sigma$, $\mathfrak{A} \in \mathrm{SIG}_{MTL}(\Sigma)$, an MTL behavior model over Σ. Then the following holds:*

- $\forall s \in \{T, \Delta, B\} : A_s = A'_{s'}$
- $\forall f \in \{d, 0, +, <\} : f^{\mathfrak{A}} = f'^{\mathfrak{A}'}$
- $true^{\mathfrak{A}} = true^{\mathfrak{A}'}$
- $false^{\mathfrak{A}} = false^{\mathfrak{A}'}$

A.2 Proof of the Coincidence Condition

The main parts of the proof of the coincidence condition are the two lemmata stating a corresponding property for the interpretation of the index terms and for the satisfaction relation $\Vdash_{MTL,\Sigma}$.

Lemma 50 (MTL Term Coincidence) *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL}$ be MTL timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL}$ an MTL timed signature morphism, $\mathfrak{A}' \in Str_{MTL}(\Sigma')$ an MTL behavior model and $\mathfrak{A} = \mathfrak{A}'|\sigma$. Then the following equation is valid for all terms $c \in Term_{MTL}(\Sigma)$ and all time points $t \in A_T (= A'_{T'})$*

$$\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, c) = \mathfrak{I}^{MTL}_{\mathfrak{A}'}(t, Term_{MTL}(\sigma)(c))$$

Proof:
Let $t \in A_T \overset{Cor.\ 49}{=} A'_{T'}$ be an arbitrary time point.
By induction over the structure of $c \in Term_{MTL}(\Sigma)$:

base case: $c \equiv f$ with $f : T \to \Delta \in D_{Term}$

$$\begin{aligned}\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, c) &\overset{Def.\ 32}{=} f^{\mathfrak{A}}(t) \overset{Def.\ |\sigma}{=} \sigma(f)^{\mathfrak{A}'}(t) \overset{Def.\ 32}{=} \mathfrak{I}^{MTL}_{\mathfrak{A}'}(t, \sigma(f)) \\ &\overset{Def.\ 21}{=} \mathfrak{I}^{MTL}_{\mathfrak{A}'}(t, Term_{MTL}(\sigma)(f))\end{aligned}$$

induction step: $c \equiv f(t_1, \ldots, t_n)$

$$\begin{aligned}&\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, f(t_1, \ldots, t_n)) \\ \overset{Def.\ 32}{=}\ & f^{\mathfrak{A}}(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, t_1), \ldots, \mathfrak{I}^{MTL}_{\mathfrak{A}}(t, t_n), t) \\ \overset{Def.\ |\sigma}{=}\ & \sigma(f)^{\mathfrak{A}'}(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, t_1), \ldots, \mathfrak{I}^{MTL}_{\mathfrak{A}}(t, t_n), t) \\ \overset{ind.\ hyp.}{=}\ & \sigma(f)^{\mathfrak{A}'}(\mathfrak{I}^{MTL}_{\mathfrak{A}'}(t, Term_{MTL}(\sigma)(t_1)), \ldots, \mathfrak{I}^{MTL}_{\mathfrak{A}'}(t, Term_{MTL}(\sigma)(t_n)), t) \\ \overset{Def.\ 32}{=}\ & \mathfrak{I}^{MTL}_{\mathfrak{A}'}(t, \sigma(f)(Term_{MTL}(\sigma)(t_1), \ldots, Term_{MTL}(\sigma)(t_n))) \\ \overset{Def.\ 21}{=}\ & \mathfrak{I}^{MTL}_{\mathfrak{A}'}(t, Term_{MTL}(\sigma)(f(t_1, \ldots, t_n)))\end{aligned}$$

q.e.d.

Lemma 51 **($\Vert\!\models_{MTL,\Sigma}$ -coincidence)** *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL}$ be MTL timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL}$ an MTL timed signature morphism, $\mathfrak{A}' \in Str_{MTL}(\Sigma')$ an MTL behavior model and $\mathfrak{A} = \mathfrak{A}'|\sigma$. Let $t \in A_T$ $(= A'_{T'})$ be an arbitrary time point. Then the following is valid for all sentences $\varphi \in Sen_{MTL}(\Sigma)$:*

$$(\mathfrak{A}', t)\Vert\!\models_{MTL,\Sigma'} Sen_{MTL}(\sigma)(\varphi) \textit{ iff } (Str_{MTL}(\sigma)(\mathfrak{A}'), t)\Vert\!\models_{MTL,\Sigma} \varphi$$

Proof:
Let $t \in A_T$ be an arbitrary time point.
By induction over the structure of $\varphi \in Sen_{MTL}(\Sigma)$:

base case: $\varphi \equiv p$, $p \in F_P$:

$$
\begin{aligned}
(Str_{MTL}(\sigma)(\mathfrak{A}'), t)\Vert\!\models_{MTL,\Sigma} p &\text{ iff } (\mathfrak{A}'|\sigma, t)\Vert\!\models_{MTL,\Sigma} p && \text{(Def. } Str_{MTL}(\sigma)\text{)}\\
&\text{ iff } (\mathfrak{A}, t)\Vert\!\models_{MTL,\Sigma} p && (\mathfrak{A} = \mathfrak{A}'|\sigma)\\
&\text{ iff } p^{\mathfrak{A}}(t) = TRUE && \text{(Def. 33)}\\
&\text{ iff } \sigma(p)^{\mathfrak{A}'}(t) = TRUE && \text{(Def. } |\sigma\text{)}\\
&\text{ iff } (\mathfrak{A}', t)\Vert\!\models_{MTL,\Sigma'} \sigma(p) && \text{(Def. 33)}\\
&\text{ iff } (\mathfrak{A}', t)\Vert\!\models_{MTL,\Sigma'} Sen_{MTL}(\sigma)(p) && \text{(Def. 24+23)}
\end{aligned}
$$

induction step:

(i) $\varphi \equiv \varphi_1 \to \varphi_2$:

$$
\begin{aligned}
&(\mathfrak{A}', t)\Vert\!\models_{MTL,\Sigma'} Sen_{MTL}(\sigma)(\varphi_1 \to \varphi_2)\\
&\quad\text{iff } (\mathfrak{A}', t)\Vert\!\models_{MTL,\Sigma'} Sen_{MTL}(\sigma)(\varphi_1) \to Sen_{MTL}(\sigma)(\varphi_2) && \text{(Def. 24)}\\
&\quad\text{iff } (\mathfrak{A}', t)\Vert\!\models_{MTL,\Sigma'} Sen_{MTL}(\sigma)(\varphi_2) \text{ or } (\mathfrak{A}', t)\Vert\!\not\models_{MTL\text{-}f,\Sigma'} Sen_{MTL}(\sigma)(\varphi_1) && \text{(Def. 33)}\\
&\quad\text{iff } (Str_{MTL}(\sigma)(\mathfrak{A}'), t)\Vert\!\models_{MTL,\Sigma} \varphi_2 \text{ or } (Str_{MTL}(\sigma)(\mathfrak{A}'), t)\Vert\!\not\models_{MTL\text{-}f,\Sigma} \varphi_1 && \text{(ind. hyp.)}\\
&\quad\text{iff } (Str_{MTL}(\sigma)(\mathfrak{A}'), t)\Vert\!\models_{MTL,\Sigma} \varphi_1 \to \varphi_2 && \text{(Def. 33)}
\end{aligned}
$$

(ii) $\varphi \equiv \varphi_1 \mathcal{U}_{\sim c} \varphi_2$, $\sim \in \{\prec, =, \succ\}$, $c \in Term_{MTL}(\Sigma)$:

$$
\begin{aligned}
&(\mathfrak{A}', t)\Vert\!\models_{MTL,\Sigma'} Sen_{MTL}(\sigma)(\varphi_1 \mathcal{U}_{\sim c} \varphi_2)\\
&\quad\text{iff } (\mathfrak{A}', t)\Vert\!\models_{MTL,\Sigma'} Sen_{MTL}(\sigma)(\varphi_1) \mathcal{U}_{\sim Term_{MTL}(\sigma)(c)} Sen_{MTL}(\sigma)(\varphi_2) && \text{(Def. 24)}\\
&\quad\text{iff } \exists t', t' \geq'^{\mathfrak{A}'} t, t' \sim'^{\mathfrak{A}'} t +'^{\mathfrak{A}'} \mathfrak{I}^{MTL}_{\mathfrak{A}'}(t, Term_{MTL}(\sigma)(c)) : (\mathfrak{A}', t')\Vert\!\models_{MTL,\Sigma'} \varphi_2\\
&\qquad\text{and } \forall t'', t \leq'^{\mathfrak{A}'} t'' <'^{\mathfrak{A}'} t' : (\mathfrak{A}', t'')\Vert\!\models_{MTL,\Sigma'} \varphi_1 && \text{(Def. 33)}\\
&\quad\text{iff } \exists t', t' \geq^{\mathfrak{A}} t, t' \sim^{\mathfrak{A}} t +^{\mathfrak{A}} \mathfrak{I}^{MTL}_{\mathfrak{A}}(t, c) : (\mathfrak{A}', t')\Vert\!\models_{MTL,\Sigma'} \varphi_2\\
&\qquad\text{and } \forall t'', t \leq^{\mathfrak{A}} t'' <^{\mathfrak{A}} t' : (\mathfrak{A}', t'')\Vert\!\models_{MTL,\Sigma'} \varphi_1 && \text{(Lem. 48, 50, and } A_T = A'_T\text{)}\\
&\quad\text{iff } \exists t', t' \geq^{\mathfrak{A}} t, t' \sim^{\mathfrak{A}} t +^{\mathfrak{A}} \mathfrak{I}^{MTL}_{\mathfrak{A}}(t, c) :\\
&\qquad(Str_{MTL}(\sigma)(\mathfrak{A}'), t)\Vert\!\models_{MTL,\Sigma'} \varphi_2\\
&\qquad\text{and } \forall t'', t \leq^{\mathfrak{A}} t'' <^{\mathfrak{A}} t' : (\mathfrak{A}', t'')\Vert\!\models_{MTL,\Sigma'} \varphi_1 && \text{(ind. hyp.)}\\
&\quad\text{iff } (Str_{MTL}(\sigma)(\mathfrak{A}'), t)\Vert\!\models_{MTL,\Sigma} \varphi_1 \mathcal{U}_{\sim c} \varphi_2 && \text{(Def. 33)}
\end{aligned}
$$

q.e.d.

Theorem 52 (MTL Coincidence) *For MTL defined as a T-FDT the coincidence condition holds, i.e. for all $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL}$, $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL}$, $\mathfrak{A}' \in Str_{MTL}(\Sigma')$, and $\varphi \in Sen_{MTL}(\Sigma)$ it holds:*

$$\mathfrak{A}' \models_{MTL,\Sigma'} Sen_{MTL}(\sigma)(\varphi) \textit{ iff } Str_{MTL}(\sigma)(\mathfrak{A}') \models_{MTL,\Sigma} \varphi$$

Proof:

$$\begin{aligned}
\mathfrak{A}' \models_{MTL,\Sigma'} Sen_{MTL}(\sigma)(\varphi) &\text{ iff } \forall t \in A'_T(\mathfrak{A}', t) \Vdash_{MTL,\Sigma'} Sen_{MTL}(\sigma)(\varphi) && \text{(Def. 33)}\\
&\text{ iff } \forall t \in A_T(Str_{MTL}(\sigma)(\mathfrak{A}'), t) \Vdash_{MTL,\Sigma} \varphi && \text{(Lem. 51)}\\
&\text{ iff } Str_{MTL}(\sigma)(\mathfrak{A}') \models_{MTL,\Sigma} \varphi && \text{(Def. 33)}
\end{aligned}$$

q.e.d.

A.3 Proof of the Isomorphism Condition

To prove the isomorphism condition three lemmata are needed. Lemma 43 states properties of isomorphic structures concerning the propositional part and has already been proven in Section 3.3.3. Lemma 53 states that interpretations of index terms are invariant w.r.t. isomorphisms. Lemma 54 guarantees the corresponding isomorphism condition for the satisfaction relation $\Vdash_{MTL,\Sigma}$.

Lemma 53 (MTL Term Isomorphism) *Let $\Sigma \in \mathrm{SIG}_{MTL}$ be an MTL timed signature, $\mathfrak{A}, \mathfrak{B} \in Str_{MTL}(\Sigma)$ isomorphic MTL behavior models over Σ, i.e. $\mathfrak{A} \cong \mathfrak{B}$, and $h = (h_s | s \in S)$ an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Then for all index terms $c \in Term_{MTL}(\Sigma)$ and all time points $t \in A_T$ it holds:*

$$\mathfrak{I}^{MTL}_{\mathfrak{B}}(h_T(t), c) = h_\Delta(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, c))$$

Proof:
Let $t \in A_T$ be an arbitrary time point.
By induction over the structure of $c \in Term_{MTL}(\Sigma)$:

base case: $c \equiv f,\ f : T \to \Delta \in D_{Term}$:

$$\mathfrak{I}^{MTL}_{\mathfrak{B}}(h_T(t), f) \overset{Def.\ 32}{=} f^{\mathfrak{B}}(h_T(t)) \overset{h \text{ homom.}}{=} h_\Delta(f^{\mathfrak{A}}(t)) \overset{Def.\ 32}{=} h_\Delta(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, f))$$

induction step: $c \equiv f(t_1, \ldots, t_n) \in Term_{MTL}(\Sigma)$

$$\begin{aligned}
\mathfrak{I}^{MTL}_{\mathfrak{B}}(h_T(t), c) &= \mathfrak{I}^{MTL}_{\mathfrak{B}}(h_T(t), f(t_1, \ldots, t_n))\\
&\overset{Def.\ 32}{=} f^{\mathfrak{B}}(\mathfrak{I}^{MTL}_{\mathfrak{B}}(h_T(t), t_1), \ldots, \mathfrak{I}^{MTL}_{\mathfrak{B}}(h_T(t), t_n), h_T(t))\\
&\overset{ind.\ hyp.}{=} f^{\mathfrak{B}}(h_\Delta(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, t_1)), \ldots, h_\Delta(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, t_n)), h_T(t))\\
&\overset{h \text{ homom.}}{=} h_\Delta(f^{\mathfrak{A}}(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, t_1), \ldots, \mathfrak{I}^{MTL}_{\mathfrak{A}}(t, t_n), t))\\
&\overset{Def.\ 32}{=} h_\Delta(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, f(t, \ldots, t_n)))\\
&= h_\Delta(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, c))
\end{aligned}$$

q.e.d.

Lemma 54 **($\|\!\models_{MTL,\Sigma}$ -Isomorphism)** *Let $\Sigma \in \mathrm{SIG}_{MTL}$ be an MTL timed signature, $\mathfrak{A}, \mathfrak{B} \in Str_{MTL}(\Sigma)$ isomorphic MTL behavior models over Σ, i.e. $\mathfrak{A} \cong \mathfrak{B}$, and $h = (h_s | s \in S)$ an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Then for all time points $t \in A_T$ and all sentences $\varphi \in Sen_{MTL}$ it holds:*

$$(\mathfrak{A}, t) \|\!\models_{MTL,\Sigma} \varphi \text{ iff } (\mathfrak{B}, h_T(t)) \|\!\models_{MTL,\Sigma} \varphi$$

Proof:
By induction over the structure of $\varphi \in Sen_{MTL}$:

Let $t \in A_T$ be an arbitrary time point of MTL behavior model $\mathfrak{A}$.
base case: $\varphi \equiv p, p \in F_P$:

$$
\begin{aligned}
(\mathfrak{A}, t) \|\!\models_{MTL,\Sigma} p \text{ iff } & p^{\mathfrak{A}}(t) = TRUE && \text{(Def. 33)}\\
\text{iff } & p^{\mathfrak{B}}(h_T(t)) = TRUE && \text{(Lem. 43)}\\
\text{iff } & (\mathfrak{B}, h_T(t)) \|\!\models_{MTL,\Sigma} p && \text{(Def. 33)}
\end{aligned}
$$

induction step:
(i) $\varphi \equiv \varphi_1 \to \varphi_2$:

$(\mathfrak{A}, t) \|\!\models_{MTL,\Sigma} \varphi_1 \to \varphi_2$

$$
\begin{aligned}
&\text{iff } (\mathfrak{A}, t) \|\!\not\models_{MTL,\Sigma} \varphi_1 \text{ or } (\mathfrak{A}, t) \|\!\models_{MTL,\Sigma} \varphi_2 && \text{(Def. 33)}\\
&\text{iff } (\mathfrak{B}, h_T(t)) \|\!\not\models_{MTL,\Sigma} \varphi_1 \text{ or } (\mathfrak{B}, h_T(t)) \|\!\models_{MTL,\Sigma} \varphi_2 && \text{(ind. hyp.)}\\
&\text{iff } (\mathfrak{B}, h_T(t)) \|\!\models_{MTL,\Sigma} \varphi_1 \to \varphi_2 && \text{(Def. 33)}
\end{aligned}
$$

(ii) $\varphi \equiv \varphi_1 \mathcal{U}_{\sim c} \varphi_2$, $(\sim \in \{\prec, =, \succ\})$, $c \in Term_{MTL}(\Sigma)$:

$(\mathfrak{A}, t) \|\!\models_{MTL,\Sigma} \varphi_1 \mathcal{U}_{\sim c} \varphi_2$

$$
\begin{aligned}
&\text{iff } \exists t'_A \in A_T, t'_A \geq^{\mathfrak{A}} t, d^{\mathfrak{A}}(t'_A, t) \sim^{\mathfrak{A}} \mathfrak{I}^{MTL}_{\mathfrak{A}}(t, c):\\
&\quad (\mathfrak{A}, t'_A) \|\!\models_{MTL,\Sigma} \varphi_2 \text{ and}\\
&\quad \forall t''_A \in A_T, t \leq^{\mathfrak{A}} t''_A <^{\mathfrak{A}} t'_A\text{: } (\mathfrak{A}, t''_A) \|\!\models_{MTL,\Sigma} \varphi_1 && \text{(Def. 33)}\\
&\text{iff } \exists t'_A \in A_T, t'_A \geq^{\mathfrak{A}} t, d^{\mathfrak{A}}(t'_A, t) \sim^{\mathfrak{A}} \mathfrak{I}^{MTL}_{\mathfrak{A}}(t, c):\\
&\quad (\mathfrak{B}, h_T(t'_A)) \|\!\models_{MTL,\Sigma} \varphi_2 \text{ and}\\
&\quad \forall t''_A \in A_T, t \leq^{\mathfrak{A}} t''_A <^{\mathfrak{A}} t'_A\text{: } (\mathfrak{B}, h_T(t''_A)) \|\!\models_{MTL,\Sigma} \varphi_1 && \text{(ind. hyp.)}\\
&\text{iff } \exists t'_A \in A_T, t'_A \geq^{\mathfrak{A}} t, h_\Delta(d^{\mathfrak{A}}(t'_A, t)) \sim^{\mathfrak{B}} h_\Delta(\mathfrak{I}^{MTL}_{\mathfrak{A}}(t, c)):\\
&\quad (\mathfrak{B}, h_T(t'_A)) \|\!\models_{MTL,\Sigma} \varphi_2 \text{ and}\\
&\quad \forall t''_A \in A_T, t \leq^{\mathfrak{A}} t''_A <^{\mathfrak{A}} t'_A\text{: } (\mathfrak{B}, h_T(t''_A)) \|\!\models_{MTL,\Sigma} \varphi_1 && (h \text{ homom.})\\
&\text{iff } \exists t'_A \in A_T, t'_A \geq^{\mathfrak{A}} t, h_\Delta(d^{\mathfrak{A}}(t'_A, t)) \sim^{\mathfrak{B}} \mathfrak{I}^{MTL}_{\mathfrak{B}}(h_T(t), c)):\\
&\quad (\mathfrak{B}, h_T(t'_A)) \|\!\models_{MTL,\Sigma} \varphi_2 \text{ and}\\
&\quad \forall t''_A \in A_T, t \leq^{\mathfrak{A}} t''_A <^{\mathfrak{A}} t'_A\text{: } (\mathfrak{B}, h_T(t''_A)) \|\!\models_{MTL,\Sigma} \varphi_1 && \text{(Lem. 53)}\\
&\text{iff } \exists t'_A \in A_T, t'_A \geq^{\mathfrak{A}} t, d^{\mathfrak{B}}(h_T(t'_A), h_T(t)) \sim^{\mathfrak{B}} \mathfrak{I}^{MTL}_{\mathfrak{B}}(h_T(t), c)):\\
&\quad (\mathfrak{B}, h_T(t'_A)) \|\!\models_{MTL,\Sigma} \varphi_2 \text{ and}\\
&\quad \forall t''_A \in A_T, t \leq^{\mathfrak{A}} t''_A <^{\mathfrak{A}} t'_A\text{: } (\mathfrak{B}, h_T(t''_A)) \|\!\models_{MTL,\Sigma} \varphi_1 && (h \text{ homom.})\\
&\text{iff } \exists t'_B \in B_T, t'_B \geq^{\mathfrak{B}} h_T(t), d^{\mathfrak{B}}(t'_B, h_T(t)) \sim^{\mathfrak{B}} \mathfrak{I}^{MTL}_{\mathfrak{B}}(h_T(t), c):\\
&\quad (\mathfrak{B}, t'_B) \|\!\models_{MTL,\Sigma} \varphi_2 \text{ and}\\
&\quad \forall t''_B \in B_T, h_T(t) \leq^{\mathfrak{B}} t''_B <^{\mathfrak{B}} t'_B\text{: } (\mathfrak{B}, t''_B) \|\!\models_{MTL,\Sigma} \varphi_1\\
& && (t'_B = h_T(t'_A),\ t''_B = h_T(t''_A),\ h \text{ isomorph.})\\
&\text{iff } (\mathfrak{B}, h_T(t)) \|\!\models_{MTL,\Sigma} \varphi_1 \mathcal{U}_{\sim c} \varphi_2 && \text{(Def. 33)}
\end{aligned}
$$

q.e.d.

Theorem 55 (MTL Isomorphism) *For MTL defined as a T-FDT the isomorphism condition holds, i.e. for all* $\Sigma \in \textsc{Sig}_{MTL}$, $\mathfrak{A}, \mathfrak{B} \in Str_{MTL}(\Sigma)$ *it holds:*

$$\mathfrak{A} \cong \mathfrak{B} \textit{ and } \mathfrak{A} \models_{MTL,\Sigma} \varphi \textit{ then } \mathfrak{B} \models_{MTL,\Sigma} \varphi$$

Proof:

$$\begin{array}{llr}
\mathfrak{A} \models_{MTL,\Sigma} \varphi & \text{iff } \forall t \in A_T : (\mathfrak{A}, t) \Vdash_{MTL,\Sigma} \varphi & \text{(Def. 33)} \\
& \text{iff } \forall t \in A_T : (\mathfrak{B}, h_T(t)) \Vdash_{MTL,\Sigma} \varphi & (h_T \text{ is bijective and Lem. 54)} \\
& \text{iff } \mathfrak{B} \models_{MTL,\Sigma} \varphi & \text{(Def. 33)}
\end{array}$$

q.e.d.

B Proofs for MTL-$\int$

In this section we present the complete and detailed proofs that the structure, coincidence, and isomorphism condition holds for the temporal logic MTL-$\int$ defined as a T-FDT in Section 3.2.

B.1 Proof of the Structure Condition

Analogously to MTL we follow the general proof scheme presented in Section 3.3.1.

Lemma 56 (MTL-$\int$ Signature Invariance) *Let $\Sigma, \Sigma' \in \textsc{Sig}_{MTL\text{-}\int}$ be MTL-$\int$ timed signatures and $\sigma : \Sigma \to \Sigma' \in \textsc{Sig}_{MTL\text{-}\int}$ an MTL-$\int$ timed signature morphism. Then σ has the following properties. (We use the primed versions of the symbols to denote the elements of Σ'.)*

- $\sigma(T) = T'$, $\sigma(L) = L'$, $\sigma(\delta) = \delta'$, $\sigma(<) =<'$
- $\sigma(B) = B'$, $\sigma(true) = true$, $\sigma(false) = false$, $\sigma(PL) \subseteq PL'$

Proof:
$\sigma(<) =<'$ and $\sigma(T) = T'$ is already required for each timed signature morphism by Definition 1 and $\sigma(true) = true$ and $\sigma(false) = false$ by Definition 14. Since $\sigma|_{\Sigma_T} : \Sigma_T \to \Sigma'_T$ has to be a signature morphism itself (see Definition 1) we can deduce from $\sigma(T) = T'$ directly $\sigma(L) = L'$ and $\sigma(\delta) = \delta'$.
Since σ has to map the sorts, function, and predicate symbols so that the mapping is compatible with the declarations of Σ and Σ' we conclude $\sigma(B) = B'$ and from this $\sigma(PL) \subseteq PL'$. q.e.d.

Theorem 57 (MTL-$\int$ Structure Closure) *Let $\Sigma, \Sigma' \in \textsc{Sig}_{MTL\text{-}\int}$ be MTL-$\int$ timed signatures and $\sigma : \Sigma \to \Sigma' \in \textsc{Sig}_{MTL\text{-}\int}$ an MTL-$\int$ timed signature morphism. Let further $\mathfrak{A}' \in Str(\Sigma')$ be a structure over Σ' and $\mathfrak{A} = \mathfrak{A}'|_\sigma$, $\mathfrak{A} \in Str(\Sigma)$, a structure over Σ. Then $\mathfrak{A}'$ is an MTL-$\int$ behavior model over Σ' if and only if $\mathfrak{A}$ is an MTL-$\int$ behavior model over Σ, i.e.*

$$\mathfrak{A} \in \textsc{Sig}_{MTL\text{-}\int}(\Sigma) \text{ iff } \mathfrak{A}' \in \textsc{Sig}_{MTL\text{-}\int}(\Sigma')$$

Proof:
To show that a structure $\mathfrak{B}$ is an MTL-$\int$ behavior model we have to show that the conditions of the Definitions 15 and 18 are satisfied.
Since σ maps sorts to sorts we can conclude from Lemma 56 $\sigma(R) = R'$. Analogously to the proof of the MTL structure closure theorem we derive $A_s = A'_{s'}$, $s \in \{T, L, R, B\}$, $true^{\mathfrak{A}} = true^{\mathfrak{A}'}$, and $false^{\mathfrak{A}} = false^{\mathfrak{A}'}$. Hence, the conditions of Definition 15 and the first condition of Definition 18 are satisfied by $\mathfrak{A}$ iff they are satisfied by $\mathfrak{A}'$.
Because of $\sigma(PL) \subseteq PL'$ (Lemma 56) the finite variability condition of Definition 18 is only satisfied by $\mathfrak{A}$ iff it is satisfied by $\mathfrak{A}'$.
The remaining condition of Definition 18, the rigidness requirement for all functions and predicates assigned to the symbols of an MTL-$\int$ term enrichment is a direct consequence

of the properties of the morphism σ and the forgetful functor $|\sigma$. We have for all $f \in F_{Term}$, $p \in P_{Term}$: $\sigma(f) \in F'_{Term}$, $\sigma(p) \in P'_{Term}$, $f^{\mathfrak{A}} = \sigma(f)^{\mathfrak{A}'}$, and $p^{\mathfrak{A}} = \sigma(p)^{\mathfrak{A}'}$. Hence, $f^{\mathfrak{A}}$ and $p^{\mathfrak{A}}$ are rigid iff $\sigma(f)^{\mathfrak{A}'}$ and $\sigma(p)^{\mathfrak{A}'}$ are rigid. q.e.d.

The proof of the MTL-$\int$ structure closure theorem already includes the proof of the MTL-$\int$ version of Corollary 39.

Corollary 58 *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$ be MTL-$\int$ timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$ an MTL-$\int$ timed signature morphism. Let $\mathfrak{A}' \in \mathrm{SIG}_{MTL\text{-}\int}(\Sigma')$ be an MTL-$\int$ behavior model over Σ' and $\mathfrak{A} = \mathfrak{A}'|_\sigma$, $\mathfrak{A} \in \mathrm{SIG}_{MTL\text{-}\int}(\Sigma)$, an MTL-$\int$ behavior model over Σ. Then the following holds:*

- $\forall s \in \{T, R, L, B\} : A_s = A'_{s'}$
- $<^{\mathfrak{A}} = <'^{\mathfrak{A}'}$
- $\delta^{\mathfrak{A}} = \delta'^{\mathfrak{A}'}$
- $true^{\mathfrak{A}} = true^{\mathfrak{A}'}$
- $false^{\mathfrak{A}} = false^{\mathfrak{A}'}$

B.2 Proof of the Coincidence Condition

As already mentioned the definitions of the duration terms and of the MTL-$\int$ sentences are mutually recursive. To be formally correct we have to consider duration terms and MTL-$\int$ sentences together. But in order to keep the proof clear and manageable we consider both, terms and sentences, separately in two lemmata which is also in accordance with the general scheme. Therefore, we assume for the proof of the MTL-$\int$ term coincidence lemma that the $\Vdash_{MTL\text{-}\int,\Sigma}$-coincidence lemma is already shown and vice versa. Although this seems to be a circular reasoning, this is not the case when combining both proofs together in a single induction proof.
To be more concrete we need the $\Vdash_{MTL\text{-}\int,\Sigma}$-coincidence lemma to proof the following lemma which itself is used in the proof of Lemma 60. The following lemma states a coincidence condition for the functions $f_\varphi^{\mathfrak{A}}$ associated with a sentence φ w.r.t. an MTL-$\int$ behavior model $\mathfrak{A}$.

Lemma 59 *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$ be MTL-$\int$ timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$ an MTL-$\int$ timed signature morphism, $\mathfrak{A}' \in Str_{MTL\text{-}\int}(\Sigma')$ an MTL-$\int$ behavior model and $\mathfrak{A} = \mathfrak{A}'|\sigma$. Then the following is valid for all sentences $\varphi \in Sen_{MTL\text{-}\int}(\Sigma)$:*

$$f_\varphi^{\mathfrak{A}} = f^{\mathfrak{A}'}_{Sen_{MTL\text{-}\int}(\sigma)(\varphi)}$$

Proof:
Let t be an arbitrary time point. Note $A_T \overset{Cor.\ 58}{=} A'_{T'}$. According to the definition of these functions (see page 20) we have

$$f_\varphi^{\mathfrak{A}}(t) = \begin{cases} 1 & \text{if } (\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{\mathfrak{A}'}_{Sen_{MTL\text{-}\int}(\sigma)(\varphi)}(t) = \begin{cases} 1 & \text{if } (\mathfrak{A}', t) \Vdash_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi) \\ 0 & \text{otherwise} \end{cases}$$

Since Lemma 61 states

$$(\mathfrak{A}', t) \Vdash_{MTL\text{-}\int, \Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi) \text{ iff } (Str_{MTL\text{-}\int}(\sigma)(\mathfrak{A}'), t) \Vdash_{MTL\text{-}\int, \Sigma} \varphi$$

with $\mathfrak{A} := Str_{MTL\text{-}\int}(\sigma)(\mathfrak{A}')$ we conclude $f_\varphi^{\mathfrak{A}} = f_{Sen_{MTL\text{-}\int}(\sigma)(\varphi)}^{\mathfrak{A}'}$. q.e.d.

Lemma 60 (MTL-$\int$ Term Coincidence) *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$ be MTL-$\int$ timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$ an MTL-$\int$ timed signature morphism, $\mathfrak{A}' \in Str_{MTL\text{-}\int}(\Sigma')$ an MTL-$\int$ behavior model and $\mathfrak{A} = \mathfrak{A}'|\sigma$. Then the following equation is valid for all terms $c \in Term_{MTL\text{-}\int}(\Sigma)$ and all time points $t \in A_T (= A'_{T'})$.*

$$\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, c) = \mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, Term_{MTL\text{-}\int}(\sigma)(c))$$

Proof:
By induction over the structure of $c \in Term_{MTL\text{-}\int}(\Sigma)$:

base case:

(i) $c \in \mathbb{R}$: $\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, c) \overset{Def.\ 34}{=} c$ and
$\mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, Term_{MTL\text{-}\int}(\sigma)(c)) \overset{Def.\ 25}{=} \mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, c) \overset{Def.\ 34}{=} c$

(ii) $c \equiv x, x \in Var$:

$$\begin{aligned} \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, c) &\overset{Def.\ 34}{=} x^{\mathfrak{A}}(t) \\ &\overset{Def.\ |\sigma}{=} \sigma(x)^{\mathfrak{A}'}(t) \\ &\overset{Def.\ 34}{=} \mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, \sigma(x)) \\ &\overset{Def.\ 25}{=} \mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, Term_{MTL\text{-}\int}(\sigma)(c)) \end{aligned}$$

induction step:
(iii) $c \equiv g(s_1, \ldots, s_k)$:

$$\begin{aligned} &\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, g(s_1, \ldots, s_k)) \\ \overset{Def.\ 34}{=} \quad & g^{\mathfrak{A}}(\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, s_1), \ldots, \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, s_k), t) \\ \overset{Def.\ |\sigma}{=} \quad & \sigma(g)^{\mathfrak{A}'}(\mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, s_1), \ldots, \mathfrak{I}_{\mathfrak{A}}^{MTL\text{-}\int}(t, s_k), t) \\ \overset{ind.\ hyp.}{=} \quad & \sigma(g)^{\mathfrak{A}'}(\mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, Term_{MTL\text{-}\int}(\sigma)(t, s_1)), \ldots, \mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, Term_{MTL\text{-}\int}(\sigma)(t, s_k)), t) \\ \overset{Def.\ 34}{=} \quad & \mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, \sigma(g)(Term_{MTL\text{-}\int}(\sigma)(s_1), \ldots, Term_{MTL\text{-}\int}(\sigma)(s_k))) \\ \overset{Def.\ 25}{=} \quad & \mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, Term_{MTL\text{-}\int}(\sigma)(g(s_1, \ldots, s_k))) \\ = \quad & \mathfrak{I}_{\mathfrak{A}'}^{MTL\text{-}\int}(t, Term_{MTL\text{-}\int}(\sigma)(c)) \end{aligned}$$

(iv) $c = \int^\tau \varphi$, $\varphi \in Sen_{MTL\text{-}\int}(\Sigma)$, $\tau \in Term_{MTL\text{-}\int}(\Sigma)$:

$$
\begin{aligned}
&\mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}}(t, \textstyle\int^\tau \varphi) \\
\overset{Def.\ 34}{=}\ & \begin{cases} \int_t^{t+\mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}}(t,\tau)} f^{\mathfrak{A}}_{\varphi}(t')dt' & \text{if } f^{\mathfrak{A}}_{\varphi} \text{ satisfies the Dirichlet condition and} \\ & \mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}}(t,\tau) \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
\overset{ind.\ hyp.}{=}\ & \begin{cases} \int_t^{t+\mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, Term_{MTL\text{-}\int}(\sigma)(\tau))} f^{\mathfrak{A}}_{\varphi}(t')dt' & \text{if } f^{\mathfrak{A}}_{\varphi} \text{ satisfies the D. cond. and} \\ & \mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, Term_{MTL\text{-}\int}(\sigma)(\tau)) \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
\overset{Lem.\ 59}{=}\ & \begin{cases} \int_t^{t+\mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, Term_{MTL\text{-}\int}(\sigma)(\tau))} f^{\mathfrak{A}'}_{Sen_{MTL\text{-}\int}(\sigma)(\varphi)}(t')dt' & \\ & \text{if } f^{\mathfrak{A}'}_{Sen_{MTL\text{-}\int}(\sigma)(\varphi)} \text{ satisfies the D. cond.} \\ & \text{and } \mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, Term_{MTL\text{-}\int}(\sigma)(\tau)) \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
\overset{Def.\ 34}{=}\ & \mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, \textstyle\int^{Term_{MTL\text{-}\int}(\sigma)(\tau)} Sen_{MTL\text{-}\int}(\sigma)(\varphi)) \\
\overset{Def.\ 25}{=}\ & \mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, Term_{MTL\text{-}\int}(\sigma)(\textstyle\int^\tau \varphi))
\end{aligned}
$$

q.e.d.

Lemma 61 ($\|\!\!=_{MTL\text{-}\int,\Sigma}$-coincidence) *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$ be MTL-$\int$ timed signatures, $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$ an MTL-$\int$ timed signature morphism, $\mathfrak{A}' \in Str_{MTL\text{-}\int}(\Sigma')$ an MTL-$\int$ behavior model and $\mathfrak{A} = \mathfrak{A}'|\sigma$. Let $t \in A_T$ $(= A'_{T'})$ be an arbitrary time point. Then the following is valid for all sentences $\varphi \in Sen_{MTL\text{-}\int}(\Sigma)$:*

$$(\mathfrak{A}', t) \|\!\!=_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi) \text{ iff } (Str_{MTL\text{-}\int}(\sigma)(\mathfrak{A}'), t) \|\!\!=_{MTL\text{-}\int,\Sigma} \varphi$$

Proof:
By induction over the structure of $\varphi \in Sen_{MTL\text{-}\int}(\Sigma)$

base case:
(i) $\varphi \equiv p$, $p \in F_P$: analogously to the proof of Lemma 51.
(ii) $\varphi \equiv r(s_1, \ldots, s_k)$, $r \in P_{Term}$, $s_i \in Term_{MTL\text{-}\int}(\Sigma)$:

$$
\begin{aligned}
& (\mathfrak{A}', t) \|\!\!=_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(r(s_1, \ldots, s_k)) & \\
\text{iff } & (\mathfrak{A}', t) \|\!\!=_{MTL\text{-}\int,\Sigma'} \sigma(r)(Term_{MTL\text{-}\int}(s_1), \ldots, Term_{MTL\text{-}\int}(s_k)) & \text{(Def. 28)} \\
\text{iff } & (\mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, Term_{MTL\text{-}\int}(s_1)), \ldots, \mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, Term_{MTL\text{-}\int}(s_k)), t) \in \sigma(r)^{\mathfrak{A}'} & \text{(Def. 35)} \\
\text{iff } & (\mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, Term_{MTL\text{-}\int}(s_1)), \ldots, \mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}'}(t, Term_{MTL\text{-}\int}(s_k)), t) \in r^{\mathfrak{A}} & \text{(Def. } |\sigma) \\
\text{iff } & (\mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}}(t, s_1), \ldots, \mathfrak{I}^{MTL\text{-}\int}_{\mathfrak{A}}(t, s_k), t) \in r^{\mathfrak{A}} & \text{(Lem. 60)} \\
\text{iff } & (\mathfrak{A}, t) \|\!\!=_{MTL\text{-}\int,\Sigma} r(s_1, \ldots, s_k) & \text{(Def. 35)} \\
\text{iff } & (\mathfrak{A}'|\sigma, t) \|\!\!=_{MTL\text{-}\int,\Sigma} r(s_1, \ldots, s_k) & \text{(Def. } \mathfrak{A}) \\
\text{iff } & (Str_{MTL\text{-}\int}(\sigma)(\mathfrak{A}'), t) \|\!\!=_{MTL\text{-}\int,\Sigma} r(s_1, \ldots s_k) & \text{(Def. } Str_{MTL\text{-}\int})
\end{aligned}
$$

induction step:

(iii) the cases $\varphi \equiv \neg\varphi; \varphi \equiv \varphi_1 \vee \varphi_2; \varphi \equiv \varphi_1 \mathcal{O} \varphi_2, \mathcal{O} \in \{\mathcal{U}_{\prec\tau}, \mathcal{U}_{=\tau}, \mathcal{S}_{\prec\tau}, \mathcal{S}_{=\tau}\}$ are proven analogously to the corresponding cases in the proof of Lemma 51.

(iv) $\varphi \equiv \exists x \varphi$:

To show this case we need the following fact: if $\mathfrak{A} = \mathfrak{A}'|\sigma$ then also

$$\mathfrak{A}^r_x = \mathfrak{A}'^r_{\sigma(x)}|\sigma \qquad (\dagger)$$

To show † we have to consider only the value x:
$x^{\mathfrak{A}^r_x} = r$ and $x^{\mathfrak{A}'^r_{\sigma(x)}|\sigma} = \sigma(x)^{\mathfrak{A}'^r_{\sigma(x)}} = r$.

$(\mathfrak{A}', t) \Vdash_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\exists x \varphi)$
iff $(\mathfrak{A}', t) \Vdash_{MTL\text{-}\int,\Sigma'} \exists \sigma(x) Sen_{MTL\text{-}\int}(\sigma)(\varphi)$ (Def. 28)
iff there exists $r \in \mathbb{R}$ with $(\mathfrak{A}'^r_{\sigma(x)}, t) \Vdash_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi)$ (Def. 35)
iff there exists $r \in \mathbb{R}$ with $(\mathfrak{A}'^r_{\sigma(x)}|\sigma, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi$ (ind. hyp.)
iff there exists $r \in \mathbb{R}$ with $(\mathfrak{A}^r_x, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi$ (†)
iff $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \exists x \varphi$ (Def. 35)
iff $(Str_{MTL\text{-}\int}(\sigma)(\mathfrak{A}'), t) \Vdash_{MTL\text{-}\int,\Sigma} \exists x \varphi$ (Def. $\mathfrak{A}$, $Str_{MTL\text{-}\int}$)

(v) $\varphi \equiv \varphi_1 \mathcal{C} \varphi_2$:

Here we need the obvious fact: $\mathfrak{A}' = \mathfrak{A}'_1\mathfrak{A}'_2$ iff $\mathfrak{A}'|\sigma = \mathfrak{A}'_1|\sigma\mathfrak{A}'_2|\sigma$ (‡)

$(\mathfrak{A}', t) \Vdash_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi_1 \mathcal{C} \varphi_2)$
iff $(\mathfrak{A}', t) \Vdash_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi_1) \mathcal{C} Sen_{MTL\text{-}\int}(\sigma)(\varphi_2)$ (Def. 28)
iff there exists $\mathfrak{A}'_1, \mathfrak{A}'_2$ with $\mathfrak{A}' = \mathfrak{A}'_1\mathfrak{A}'_2$, $\sigma(\delta)^{\mathfrak{A}'_1} \geq t$:
$(\mathfrak{A}'_1, t) \Vdash_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi_1)$ and
$(\mathfrak{A}'_2, 0) \Vdash_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi_2)$ (Def. 35)
iff there exists $\mathfrak{A}_1, \mathfrak{A}_2$ with $\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2$, $\delta^{\mathfrak{A}_1} \geq t$:
$(\mathfrak{A}_1, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_1$ and $(\mathfrak{A}_2, 0) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_2$ (ind. hyp., ‡)
iff $(\mathfrak{A}, t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_1 \mathcal{C} \varphi_2$ (Def. 35)
iff $(Str_{MTL\text{-}\int}(\sigma)(\mathfrak{A}'), t) \Vdash_{MTL\text{-}\int,\Sigma} \varphi_1 \mathcal{C} \varphi_2$ (Def. $\mathfrak{A}$, $Str_{MTL\text{-}\int}$)

q.e.d.

Theorem 62 (MTL-∫ Coincidence) *For MTL-∫ defined as a T-FDT the coincidence condition holds, i.e. for all* $\Sigma, \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$, $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{MTL\text{-}\int}$, $\mathfrak{A}' \in Str_{MTL\text{-}\int}(\Sigma')$, *and* $\varphi \in Sen_{MTL\text{-}\int}(\Sigma)$ *it holds:*

$$\mathfrak{A}' \models_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi) \textit{ iff } Str_{MTL\text{-}\int}(\sigma)(\mathfrak{A}') \models_{MTL\text{-}\int,\Sigma} \varphi$$

Proof:

$\mathfrak{A}' \models_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi)$ iff $(\mathfrak{A}', 0) \Vdash_{MTL\text{-}\int,\Sigma'} Sen_{MTL\text{-}\int}(\sigma)(\varphi)$ (Def. 35)
iff $(Str_{MTL\text{-}\int}(\sigma)(\mathfrak{A}'), 0) \Vdash_{MTL\text{-}\int,\Sigma} \varphi$ (Lem. 61)
iff $Str_{MTL\text{-}\int}(\sigma)(\mathfrak{A}') \models_{MTL\text{-}\int,\Sigma} \varphi$ (Def. 35)

q.e.d.

B.3 Proof of the Isomorphism Condition

In the case of MTL-$\int$ the isomorphism condition is trivially fulfilled. The reason are the restrictions we impose on the homomorphisms of $\text{Beh}_{MTL\text{-}\int}(\Sigma)$ in Definition 18. Since we require there for all carrier sets $A_s, s \in \{T, R, L\}$, that the homomorphism h_s have to be the identities on these carrier sets, two structures $\mathfrak{A}, \mathfrak{B} \in \text{Beh}_{MTL\text{-}\int}(\Sigma)$ are isomorphic if and only if they are equal.

C Proofs for CTL*

In this section we present the complete and detailed proofs that the structure, coincidence, and isomorphism condition holds for the temporal logic CTL* defined as T-FDT in Section 3.2.

C.1 Proof of the Structure Condition

As for the other two considered temporal logics the proof of the structure condition is a direct instantiation of the proof scheme presented in Section 3.3.1.

Lemma 63 (CTL* Signature Invariance) *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{CTL^*}$ be CTL* timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{CTL^*}$ a CTL* timed signature morphism. Then σ has the following properties. (We use the primed versions of the symbols to denote the elements of Σ'.)*

- $\sigma(T) = T'$, $\sigma(0) = 0'$, $\sigma(<) =<'$
- $\sigma(B) = B'$, $\sigma(true) = true$, $\sigma(false) = false$, $\sigma(PL) \subseteq PL'$

Proof:
Analogously to the proofs of the other two temporal logics these properties are direct consequences of the Definitions 1 and 13 and of the compatibility property of σ. q.e.d.

Theorem 64 (CTL* Structure Closure) *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{CTL^*}$ be CTL* timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{CTL^*}$ a CTL* timed signature morphism. Let further $\mathfrak{A}' \in Str(\Sigma')$ be a structure over Σ' and $\mathfrak{A} = \mathfrak{A}'|_\sigma$, $\mathfrak{A} \in Str(\Sigma)$, a structure over Σ. Then $\mathfrak{A}'$ is a CTL* behavior model over Σ' if and only if $\mathfrak{A}$ is a CTL* behavior model over Σ, i.e.*

$$\mathfrak{A} \in \mathrm{SIG}_{CTL^*}(\Sigma) \text{ iff } \mathfrak{A}' \in \mathrm{SIG}_{CTL^*}(\Sigma')$$

Proof:
To show that a structure $\mathfrak{B}$ is a CTL* behavior model we have to show that the conditions of the Definitions 15 and 19 are satisfied.
According to the definition of the forgetful functor $|\sigma$ we have $A_T = A'_{\sigma(T)}$, $A_B = A'_{\sigma(B)}$, and for all functions and relations $f \in F \uplus P$: $f^{\mathfrak{A}} = \sigma(f)^{\mathfrak{A}'}$.
From Lemma 63 we can conclude $A_T = A'_{T'}$, $A_B = A'_{B'}$, and $f^{\mathfrak{A}} = \sigma(f)^{\mathfrak{A}'} = f'^{\mathfrak{A}'}$ for $f \in \{0, <, true, false\}$. This means that all carrier sets and all functions and relations are equal for $\mathfrak{A}$ and $\mathfrak{A}'$. Consequently, the conditions of the Definitions 15 and 19 are valid for $\mathfrak{A}$ if and only if they are valid for $\mathfrak{A}'$. q.e.d.

The following corollary includes an additional property concerning the fullpaths.

Corollary 65 *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{CTL^*}$ be CTL* timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{CTL^*}$ a CTL* timed signature morphism. Let $\mathfrak{A}' \in \mathrm{SIG}_{CTL^*}(\Sigma')$ be a CTL* behavior model over Σ' and $\mathfrak{A} = \mathfrak{A}'|_\sigma$, $\mathfrak{A} \in \mathrm{SIG}_{CTL^*}(\Sigma)$, a CTL* behavior model over Σ. Then the following holds:*

- $A_T = A'_{T'}$
- $A_T = A'_{T'}$
- $0^{\mathfrak{A}} = 0'^{\mathfrak{A}'}$
- $<^{\mathfrak{A}} = <'^{\mathfrak{A}'}$
- $true^{\mathfrak{A}} = true^{\mathfrak{A}'}$
- $false^{\mathfrak{A}} = false^{\mathfrak{A}'}$
- $FP(\mathfrak{A}) = FP(\mathfrak{A}')$

Proof:
We only have to proof the property $FP(\mathfrak{A}) = FP(\mathfrak{A}')$. The other properties are shown in the proof of Theorem 64.
We have $A_T \overset{def.\,|\sigma}{=} A'_{\sigma(T)} \overset{Lem.\,63}{=} A'_T$ and $<^{\mathfrak{A}} \overset{def.\,|\sigma}{=} \sigma(<)^{\mathfrak{A}'} \overset{Lem.\,63}{=} <'^{\mathfrak{A}'}$.
It follows: $(A_T, <^{\mathfrak{A}}) = (A'_T, <'^{\mathfrak{A}'})$ and finally: $FP(\mathfrak{A}) = FP(\mathfrak{A}')$ q.e.d.

C.2 Proof of the Coincidence Condition

The proofs of the coincidence and isomorphism condition for CTL* mainly correspond to the general scheme. The differences are that no terms have to be considered, instead we have to deal with the fact that there are two additional satisfaction relations $\Vdash^s_{CTL^*,\Sigma}$ and $\Vdash^p_{CTL^*,\Sigma}$. This results in two additional lemmata: Lemma 66 for the coincidence condition and Lemma 69 for the isomorphism condition.

Lemma 66 *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{CTL^*}$ be CTL* timed signatures, $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{CTL^*}$ a CTL* timed signature morphism, $\mathfrak{A}' \in Str_{CTL^*}(\Sigma')$ a CTL* behavior model over Σ', $\varphi \in Sen_{CTL^*}(\Sigma)$ a sentence over Σ, $t \in A_T$ a time point, and $x \in FP_{sc}(\mathfrak{A})$ a track with $t = hd(x)$.*
If

$$(\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi) \text{ iff } (Str(\sigma)(\mathfrak{A}'), t) \Vdash^s_{CTL^*,\Sigma} \varphi$$

then also

$$(\mathfrak{A}', x) \Vdash^p_{CTL^*,\Sigma'} PSen_{CTL^*}(\sigma)(\varphi) \text{ iff } (Str(\sigma)(\mathfrak{A}'), x) \Vdash^p_{CTL^*,\Sigma} \varphi$$

Proof:
According to Corollary 65 we have $A_T = A'_{T'}$, and $FP_{sc}(\mathfrak{A}) = FP_{sc}(\mathfrak{A}')$.

$$\begin{array}{ll}
(\mathfrak{A}', x) \Vdash^p_{CTL^*,\Sigma'} PSen_{CTL^*}(\sigma)(\varphi) & \\
\quad \text{iff } (\mathfrak{A}', x) \Vdash^p_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi) & \\
 & (PSen_{CTL^*}(\sigma)(\psi) = Sen_{CTL^*}(\sigma)(\psi) \text{ for } \psi \in Sen_{CTL^*}(\Sigma)) \\
\quad \text{iff } (\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi) \text{ and } t = hd(x) & (\text{Def. } 36) \\
\quad \text{iff } (Str(\sigma)(\mathfrak{A}'), t) \Vdash^s_{CTL^*,\Sigma} \varphi \text{ and } t = hd(x) & (\text{premise}) \\
\quad \text{iff } (Str(\sigma)(\mathfrak{A}'), x) \Vdash^p_{CTL^*,\Sigma} \varphi & (\text{Def. } 36)
\end{array}$$

q.e.d.

Lemma 67 **($\Vdash_{CTL^*,\Sigma}$-coincidence)** *Let $\Sigma, \Sigma' \in \mathrm{SIG}_{CTL^*}$ be CTL* timed signatures and $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{CTL^*}$ a CTL* timed signature morphism, $\mathfrak{A}' \in Str_{CTL^*}(\Sigma')$ a CTL* behavior model and $\mathfrak{A} = \mathfrak{A}'|\sigma$. Let $t \in A_T$ $(= A'_{T'})$ be an arbitrary time point and $x \in FP_{sc}(\mathfrak{A})$ an arbitrary track. Then the following is valid for all $\varphi \in Sen_{CTL^*}(\Sigma)$ and all $\psi \in PSen_{CTL^*}(\Sigma)$:*

$$(\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi) \text{ iff } (Str_{CTL^*}(\sigma)(\mathfrak{A}'), t) \Vdash^s_{CTL^*,\Sigma} \varphi \quad (1)$$

$$(\mathfrak{A}', x) \Vdash^p_{CTL^*,\Sigma'} PSen_{CTL^*}(\sigma)(\psi) \text{ iff } (Str_{CTL^*}(\sigma)(\mathfrak{A}'), x) \Vdash^p_{CTL^*,\Sigma} \psi \quad (2)$$

Proof:
By induction over $\varphi \in Sen_{CTL^*}(\Sigma)$ and $\psi \in PSen_{CTL^*}(\Sigma)$.
Remember that $A_T = A'_{T'}$ and $FP_{sc}(\mathfrak{A}) = FP_{sc}(\mathfrak{A}')$ according to Corollary 65.

base case:
(i) for (1): $\varphi \equiv p$, $p \in F_P$:

$$
\begin{aligned}
(Str_{CTL^*}(\sigma)(\mathfrak{A}'), t) \Vdash^s_{CTL^*,\Sigma} p &\text{ iff } (\mathfrak{A}'|\sigma, t) \Vdash^s_{CTL^*,\Sigma} p && (\text{Def. } Str_{CTL^*})\\
&\text{ iff } (\mathfrak{A}, t) \Vdash^s_{CTL^*,\Sigma} p && (\mathfrak{A} = \mathfrak{A}'|\sigma)\\
&\text{ iff } p^{\mathfrak{A}}(t) = TRUE && (\text{Def. } 36)\\
&\text{ iff } \sigma(p)^{\mathfrak{A}'}(t) = TRUE && (\text{Def. } |\sigma)\\
&\text{ iff } (\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} \sigma(p) && (\text{Def. } 36)\\
&\text{ iff } (\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(p) && (\text{Def. } 31{+}30)
\end{aligned}
$$

(ii) for (2): $\psi \equiv p$, $p \in F_P$:
Together with Lemma 66 we can conclude from (i) the base case for (2).

induction step:
(iii) $\varphi \equiv \varphi_1 \to \varphi_2$, $\varphi_1, \varphi_2 \in Sen_{CTL^*}(\Sigma)$:

$$
\begin{aligned}
&(\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi_1 \to \varphi_2)\\
&\quad\text{iff } (\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi_1) \to Sen_{CTL^*}(\sigma)(\varphi_2) && (\text{Def. } 31)\\
&\quad\text{iff } (\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi_2) \text{ or } (\mathfrak{A}', t) \nVdash^s_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi_1) && (\text{Def. } 36)\\
&\quad\text{iff } (Str_{CTL^*}(\sigma)(\mathfrak{A}'), t) \Vdash^s_{CTL^*,\Sigma} \varphi_2 \text{ or } (Str_{CTL^*}(\sigma)(\mathfrak{A}'), t) \nVdash^s_{CTL^*,\Sigma} \varphi_1 && (\text{ind. hyp.})\\
&\quad\text{iff } (Str_{CTL^*}(\sigma)(\mathfrak{A}'), t) \Vdash^s_{CTL^*,\Sigma} \varphi_1 \to \varphi_2 && (\text{Def. } 36)
\end{aligned}
$$

(iv) $\varphi \equiv \mathsf{A}\psi$, $\psi \in PSen_{CTL^*}(\Sigma)$:

$$
\begin{aligned}
&(\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\mathsf{A}\psi)\\
&\quad\text{iff } (\mathfrak{A}', t) \Vdash^s_{CTL^*,\Sigma'} \mathsf{A}PSen_{CTL^*}(\sigma)(\psi) && (\text{Def. } 31)\\
&\quad\text{iff for all } y \in FP_{sc}(\mathfrak{A}'), t = hd(y) : (\mathfrak{A}', y) \Vdash^p_{CTL^*,\Sigma'} PSen_{CTL^*}(\sigma)(\psi) && (\text{Def. } 36)\\
&\quad\text{iff for all } y \in FP_{sc}(\mathfrak{A}), t = hd(y) : (\mathfrak{A}', y) \Vdash^p_{CTL^*,\Sigma'} PSen_{CTL^*}(\sigma)(\psi) && (\text{Cor. } 65)\\
&\quad\text{iff for all } y \in FP_{sc}(\mathfrak{A}), t = hd(y) : (Str_{CTL^*}(\sigma)(\mathfrak{A}'), y) \Vdash^p_{CTL^*,\Sigma} \psi && (\text{ind. hyp.})\\
&\quad\text{iff } (Str_{CTL^*}(\sigma)(\mathfrak{A}'), t) \Vdash^s_{CTL^*,\Sigma} \mathsf{A}\psi && (\text{Def. } 36)
\end{aligned}
$$

(v) $\psi \equiv \psi_1 \to \psi_2$, $\psi_1, \psi_2 \in PSen_{CTL^*}(\Sigma)$:
analogously to (iii) for $Sen_{CTL^*}(\Sigma)$

(vi) $\psi \equiv \mathsf{X}\psi_1$, $\psi_1 \in PSen_{CTL^*}(\Sigma)$:

$$
\begin{aligned}
&(\mathfrak{A}', x) \Vdash^p_{CTL^*,\Sigma'} PSen_{CTL^*}(\sigma)(\mathsf{X}\psi_1)\\
&\quad\text{iff } (\mathfrak{A}', x) \Vdash^p_{CTL^*,\Sigma'} \mathsf{X}PSen_{CTL^*}(\sigma)(\psi_1) && (\text{Def. } 31)\\
&\quad\text{iff } (\mathfrak{A}', x^1) \Vdash^p_{CTL^*,\Sigma'} PSen_{CTL^*}(\sigma)(\psi_1) && (\text{Def. } 36)\\
&\quad\text{iff } (Str_{CTL^*}(\sigma)(\mathfrak{A}'), x^1) \Vdash^p_{CTL^*,\Sigma} \psi_1 && (\text{ind. hyp.})\\
&\quad\text{iff } (Str_{CTL^*}(\sigma)(\mathfrak{A}'), x) \Vdash^p_{CTL^*,\Sigma} \mathsf{X}\psi_1 && (\text{Def. } 36)
\end{aligned}
$$

(vii) $\psi \equiv \psi_1 \mathsf{U} \psi_2$, $\psi_1, \psi_2 \in PSen_{CTL^*}(\Sigma)$:
analogously to (vi) q.e.d.

Theorem 68 (CTL* Coincidence) *For CTL* defined as a T-FDT the coincidence condition holds, i.e. for all $\Sigma, \Sigma' \in \mathrm{SIG}_{CTL^*}$, $\sigma : \Sigma \to \Sigma' \in \mathrm{SIG}_{CTL^*}$, $\mathfrak{A}' \in Str_{CTL^*}(\Sigma')$, and $\varphi \in Sen_{CTL^*}(\Sigma)$ it holds:*

$$\mathfrak{A}' \models_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi) \textit{ iff } Str_{CTL^*}(\sigma)(\mathfrak{A}') \models_{CTL^*,\Sigma} \varphi$$

Proof:

$$
\begin{aligned}
\mathfrak{A}' \models_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi) & \text{ iff } (\mathfrak{A}', 0^{\mathfrak{A}'}) \Vdash^{s}_{CTL^*,\Sigma'} Sen_{CTL^*}(\sigma)(\varphi) && \text{(Def. 36)}\\
& \text{ iff } (Str_{CTL^*}(\sigma)(\mathfrak{A}'), 0^{\mathfrak{A}}) \Vdash^{s}_{CTL^*,\Sigma} \varphi && \text{(Lem. 67, Cor. 65)}\\
& \text{ iff } Str_{CTL^*}(\sigma)(\mathfrak{A}') \models_{CTL^*,\Sigma} \varphi && \text{(Def. 36)}
\end{aligned}
$$

q.e.d.

C.3 Proof of the Isomorphism Condition

For the following lemma we extend an isomorphism $h_T : A_T \to B_T$ to tracks, i.e. to $h_T : FP_{sc}(\mathfrak{A}) \to FP_{sc}(\mathfrak{B})$ by $h_T(x) = (h_T(t_0), h_T(t_1), \dots)$ if $x = (t_0, t_1, \dots)$. Note, that $h_T(x)$ *is* a path of $\mathfrak{B}$, because $t_i <^{\mathfrak{A}} t_{i+1}$ iff $h_T(t_i) <^{\mathfrak{B}} h_T(t_{i+1}), i \geq 0$.

Lemma 69 *Let $\Sigma \in \text{SIG}_{CTL^*}$, $\mathfrak{A}, \mathfrak{B} \in Str_{CTL^*}(\Sigma)$ with $\mathfrak{A} \cong \mathfrak{B}$ and h an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, $\varphi \in Sen_{CTL^*}(\Sigma)$, $t \in A_T$, and $x \in FP_{sc}(\mathfrak{A})$ with $t = hd(x)$.*
If

$$(\mathfrak{A}, t) \Vdash^{s}_{CTL^*,\Sigma} \varphi \textit{ iff } (\mathfrak{B}, h_T(t)) \Vdash^{s}_{CTL^*,\Sigma} \varphi$$

then also

$$(\mathfrak{A}, x) \Vdash^{p}_{CTL^*,\Sigma} \varphi \textit{ iff } (\mathfrak{B}, h_T(x)) \Vdash^{p}_{CTL^*,\Sigma} \varphi$$

Proof:

$$
\begin{aligned}
(\mathfrak{A}, x) \Vdash^{p}_{CTL^*,\Sigma} \varphi & \text{ iff } (\mathfrak{A}, t) \Vdash^{s}_{CTL^*,\Sigma} \varphi \text{ and } t = hd(x) && \text{(Def. 36)}\\
& \text{ iff } (\mathfrak{B}, t) \Vdash^{s}_{CTL^*,\Sigma} \varphi \text{ and } t = hd(x) && \text{(premise)}\\
& \text{ iff } (\mathfrak{B}, x) \Vdash^{p}_{CTL^*,\Sigma} \varphi && \text{(Def. 36)}
\end{aligned}
$$

q.e.d.

Lemma 70 **($\Vdash_{CTL^*,\Sigma}$-Isomorphism)** *Let $\Sigma \in \text{SIG}_{CTL^*}$ be a CTL* timed signature and $\mathfrak{A}, \mathfrak{B} \in Str_{CTL^*}(\Sigma)$ isomorphic CTL* behavior models over Σ, i.e. $\mathfrak{A} \cong \mathfrak{B}$, and $h = (h_s | s \in S)$ an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Then for all time points $t \in A_T$, all tracks $x \in FP_{sc}(\mathfrak{A})$, all $\varphi \in Sen_{CTL^*}$, and all $\psi \in PSen_{CTL^*}$ it holds:*

$$(\mathfrak{A}, t) \Vdash^{s}_{CTL^*,\Sigma} \varphi \textit{ iff } (\mathfrak{B}, h_T(t)) \Vdash^{s}_{CTL^*,\Sigma} \varphi \tag{3}$$

$$(\mathfrak{A}, x) \Vdash^{p}_{CTL^*,\Sigma} \psi \textit{ iff } (\mathfrak{B}, h_T(x)) \Vdash^{p}_{CTL^*,\Sigma} \psi \tag{4}$$

Proof:
By induction over $\varphi \in Sen_{CTL^*}$ and $\psi \in PSen_{CTL^*}$.

base case:

(i) for (3): $\varphi \equiv p, p \in F_P$:

$$
\begin{aligned}
(\mathfrak{A}, t) \Vdash^{s}_{CTL^*,\Sigma} p & \text{ iff } p^{\mathfrak{A}}(t) = TRUE && \text{(Def. 36)}\\
& \text{ iff } p^{\mathfrak{B}}(h_T(t)) = TRUE && \text{(Lem. 43)}\\
& \text{ iff } (\mathfrak{B}, h_T(t)) \Vdash^{p}_{CTL^*,\Sigma} p && \text{(Def. 36)}
\end{aligned}
$$

(ii) for (4): $\psi \equiv p, p \in F_P$:
Together with Lemma 69 we conclude from (i) the base case for (4).

induction step:

(iii) $\varphi \equiv \varphi_1 \rightarrow \varphi_2$, $\varphi_1, \varphi_2 \in Sen_{CTL^*}(\Sigma)$:

$(\mathfrak{A}, t) \Vdash^s_{CTL^*,\Sigma} \varphi_1 \rightarrow \varphi_2$

iff $(\mathfrak{A}, t) \nVdash^s_{CTL^*,\Sigma} \varphi_1$ or $(\mathfrak{A}, t) \Vdash^s_{CTL^*,\Sigma} \varphi_2$ (Def. 36)

iff $(\mathfrak{B}, h_T(t)) \nVdash^s_{CTL^*,\Sigma} \varphi_1$ or $(\mathfrak{B}, h_T(t)) \Vdash^s_{CTL^*,\Sigma} \varphi_2$ (ind. hyp.)

iff $(\mathfrak{B}, h_T(t)) \Vdash^s_{CTL^*,\Sigma} \varphi_1 \rightarrow \varphi_2$ (Def. 36)

(iv) $\varphi \equiv A\psi$, $\psi \in PSen_{CTL^*}(\Sigma)$:

$(\mathfrak{A}, t) \Vdash^s_{CTL^*,\Sigma} A\psi$

iff $(\mathfrak{A}, x) \Vdash^p_{CTL^*,\Sigma} \psi$ for all $x \in FP_{sc}(\mathfrak{A}), t = hd(x)$ (Def. 36)

iff $(\mathfrak{B}, h_T(x)) \Vdash^p_{CTL^*,\Sigma} \psi$ for all $h_T(x) \in FP_{sc}(\mathfrak{B})$,

$hd(h_T(x)) = h_T(t)$ (ind. hyp.)

iff $(\mathfrak{B}, h_T(t)) \Vdash^s_{CTL^*,\Sigma} A\psi$ (Def. 36)

(v) $\psi \equiv \psi_1 \rightarrow \psi_2$, $\psi_1, \psi_2 \in PSen_{CTL^*}(\Sigma)$:

analogously to (iii) for $PSen_{CTL^*}(\Sigma)$

(vi) $\psi \equiv \mathsf{X}\psi_1$, $\psi_1 \in PSen_{CTL^*}(\Sigma)$:

$(\mathfrak{A}, x) \Vdash^p_{CTL^*,\Sigma} \mathsf{X}\psi_1$

iff $(\mathfrak{A}, x^1) \Vdash^p_{CTL^*,\Sigma} \psi_1$ (Def. 36)

iff $(\mathfrak{B}, h_T(x^1)) \Vdash^p_{CTL^*,\Sigma} \psi_1$ (ind. hyp.)

iff $(\mathfrak{B}, h_T(x)) \Vdash^p_{CTL^*,\Sigma} \mathsf{X}\psi_1$ (Def. 36)

(vii) $\psi \equiv \psi_1 \mathsf{U} \psi_2$, $\psi_1, \psi_2 \in PSen_{CTL^*}(\Sigma)$:

analogously to (vi) q.e.d.

Theorem 71 (CTL* Isomorphism) *For CTL* defined as a T-FDT the isomorphism condition holds, i.e. for all* $\Sigma \in \text{SIG}_{CTL^*}$, $\mathfrak{A}, \mathfrak{B} \in Str_{CTL^*}(\Sigma)$ *it holds:*

$$\mathfrak{A} \cong \mathfrak{B} \textit{ and } \mathfrak{A} \models_{CTL^*,\Sigma} \varphi \textit{ then } \mathfrak{B} \models_{CTL^*,\Sigma} \varphi$$

Proof:

$\mathfrak{A} \models_{CTL^*,\Sigma} \varphi$ iff $(\mathfrak{A}, 0^{\mathfrak{A}}) \Vdash^s_{CTL^*,\Sigma} \varphi$ (Def. 36)

iff $(\mathfrak{B}, h_T(0^{\mathfrak{A}})) \Vdash^s_{CTL^*,\Sigma} \varphi$ (Lem. 70)

iff $(\mathfrak{B}, 0^{\mathfrak{B}}) \Vdash^s_{CTL^*,\Sigma} \varphi$ ($h_T(0^{\mathfrak{A}}) = 0^{\mathfrak{B}}$, since h is an isomorphism)

iff $\mathfrak{B} \models_{CTL^*,\Sigma} \varphi$ (Def. 36)

q.e.d.