Portfolio Optimization with Risk Constraints in the View of Stochastic Interest Rates

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Abstract

We discuss the portfolio selection problem of an investor/portfolio manager in an arbitrage-free financial market where a money market account, coupon bonds and a stock are traded continuously. We allow for stochastic interest rates and in particular consider one and two-factor Vasiček models for the instantaneous short rates. In both cases we consider a complete and an incomplete market setting by adding a suitable number of bonds.

The goal of an investor is to find a portfolio which maximizes expected utility from terminal wealth under budget and present expected short-fall (PESF) risk constraints. We analyze this portfolio optimization problem in both complete and incomplete financial markets in three different cases: (a) when the PESF risk is minimum, (b) when the PESF risk is between minimum and maximum and (c) without risk constraints. (a) corresponds to the portfolio insurer problem, in (b) the risk constraint is binding, i.e., it is satisfied with equality, and (c) corresponds to the unconstrained Merton investment.

In all cases we find the optimal terminal wealth and portfolio process using the martingale method and Malliavin calculus respectively. In particular we solve in the incomplete market settings the dual problem explicitly. We compare the optimal terminal wealth in the cases mentioned using numerical examples. Without risk constraints, we further compare the investment strategies for complete and incomplete market numerically.

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Frequently Used Notations

Abbreviations	
max	maximize
min	minimize
lim	limits
sup	supremum
inf	infimum
SDE	stochastic differential equation
Symbols	
$(\Omega, \mathbb{F}, \mathbb{P})$	complete probability space with sample space Ω, $\sigma-$field $\mathbb{F}$ and natural probability measure $\mathbb{P}$
$\mathcal{F}$	augmented filtration
$W^{\mathbb{P}}$	Brownian motion with respect to $\mathbb{P}$
$\mathbb{E}(X)$	expectation of the random variable X with respect to $\mathbb{P}$
$\mathbb{E}^{\mathbb{Q}}(X)$	expectation of the random variable X with respect to $\mathbb{Q}$
$\mathbb{E}[X \mid \mathcal{F}](X)$	conditional expectation of the random variable X with respect to $\mathcal{F}$
$\mathcal{M}^e$	set of equivalent martingale measure with respect to $\mathbb{P}$
$\exp(x)$	$= e^x$
$x^\top$	transpose of the vector x
$[0, T]$	time interval from 0 to T
$\mathbb{1}_A$	indicator function of A
$\Phi(x)$	standard normal probability distribution function of x
$\frac{\partial f}{\partial x}$	partial derivative of f with respect to x
$D_t F$	Malliavin derivative of F
$\mathbb{R}$	set of real numbers
$U(x)$	utility function of x

1. Introduction

1.1. Problem formulation

Financial markets, in which financial securities are traded, have been vastly growing during the last decades. So over the last years the question of optimal investment has become more and more important. The financial securities, sometimes called financial assets, can be divided into two main classes: risk-less assets (for example money market account) and risk-bearing assets (for instance bonds and stocks).The money market account is an interest-paying bank account. A bond is a security that promises its holder regular coupon payments and/or repayment of the principal, at a fixed maturity date. A stock is a security that represents the ownership in a corporation and thereby is linked to the corporation's profits or losses.

We consider a financial market in which a money market account, n bonds and a stock are traded. One of the problems of mathematical finance is the problem of an investor or a portfolio manager, endowed with an initial wealth x, who wants to invest in the financial assets provided at the market so as to maximize his/her expected utility from terminal wealth $X(T)$. The problem can be formulated in mathematical terms as follows:

$$\max_{\varphi \in \mathcal{A}(x)} \mathbb{E}\left(U\left(X\left(T\right)\right)\right). \tag{1.1}$$

Here, φ refers to the vector of the fractions of wealth invested in different securities (it describes the trading strategy of an investor), $X(T) = X^{\varphi}(T)$ is the portfolio value at T when following φ and starting with $X(0) = x$, $\mathcal{A}(x)$ defines the admissible set of trading strategies when the initial wealth is x, $\mathbb{E}$ denotes the expectation with to respect the physical probability measure $\mathbb{P}$ and U is a utility function. We will consider a power utility function:

$$U(x) = \frac{x^{\gamma}}{\gamma} \text{ for } \gamma \in \{(-\infty, 1) \setminus \{0\}\} \text{ and } x \geq 0,$$

where we set $U(0) = -\infty$ for $\gamma < 0$.

Consider a portfolio manager who is not willing to take on arbitrary risk . Instead, he/she chooses the trading strategy only among strategies that allow an acceptable risk. We measure risk by a so-called expected short-fall risk defined by

$$\mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,(X(T)-q)^{-}\right],$$

where $\mathbb{Q} \in \mathcal{M}^e$, hereby $\mathcal{M}^e$ denotes the set of equivalent martingale measures with respect to $\mathbb{P}$, q is the pre-set short fall level and $\beta(T)$ is the discount factor. Then the risk is incorporated in the optimization problem as a constraint and Problem (1.1) is modified as:

$$\begin{aligned} &\max_{\varphi\in\mathcal{A}(x)} \mathbb{E}\left[U(X(T))\right] \\ &\quad\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,(X(T)-q)^{-}\right] \leq \delta \quad \text{(risk constraint)}, \end{aligned} \tag{1.2}$$

where $\delta \in \mathbb{R}^{+} = [0,\infty)$ is the upper limit of the investor's risk. Let us call the agent following this strategy Expected Shortfall Portfolio Manager (ES-PM).

Note that $\delta = \infty$ corresponds to the optimization problem (1.1), let us call this agent Merton Portfolio Manager (M-PM), while the case $\delta = 0$ corresponds to the so-called portfolio insurer problem, i.e. the risk constraint is $X(T) \geq q$ and we call this agent Portfolio Insurer Portfolio Manager (PI-PM).

1.2. Background on portfolio optimization

The modern theory of optimal portfolio management began with Markowitz's work [46] in 1952. This theory tries to understand how financial markets work, how they should be made more efficient and regulated to facilitate the economic activities. The theory of optimal portfolio management has become an increasingly mathematical area of research. Markowitz's idea, in his work [46], was to consider the trade-offs between the mean returns and the covariance of stocks in the one period model to judge investment strategies on stocks. His approach became famous because of being simple and thus being applicable to high dimension, and it is still used. Sharp (see [58]) used Markowitz's ideas and determined the correlation of each stock and the market.

One-period models are static models: The investment strategies are fixed at the beginning of the investment period and the results are observed at the end of the investment period. Merton (see [48], [49]) introduced a way how to solve the continuous-time portfolio optimization problem. He managed to reduce the problem to a control problem which could be solved by stochastic control methodology. Merton considered also the personal preferences of the investor, which is characterized by the utility function, and under some assumptions including constant market parameters he derived a closed form solution of this problem.

The martingale method to portfolio management in complete markets was introduced by Harrison & Kreps [31]. It was later developed by Harrison & Pliska (see [32], [33]) in the context of option pricing. The martingale approach on the utility maximization problem in a complete market was first studied by Pliska [54] and it was extended by Karatzas et al. [40], Cox & Huang (see [10], [11]), Cvitanic & Karatzas [13], to mention but a few. Xu [64] introduced the application of duality methods to the portfolio optimization problem. This method goes back to Bismut [8]. In the case of an incomplete market, the utility maximization problem was treated by Karatzas et al. [41], Kramkov & Schachermayer [44], Cvitanić et al. [14] among others, using the martingale approach and duality methods. Ocone & Karatzas [53] introduced the techniques of Malliavin calculus to derive optimal trading strategies.

The portfolio optimization problem in the view of stochastic term structures for the interest rates was first inferred by Karatzas et al. [40] and Karatzas [39]. In these papers the stochastic process of interest rates is not specified, hence leading to the general non explicit results for the optimal portfolio process. Later on, one-factor term structure models for the interest rates were used to obtain a closed form solution for optimal policies, e.g. the Vasiček model by Bajeux-Besnainou & Portain [4], Sørensen [62], Korn & Kraft [42], Bajeux-Besnainou et al. [3], Horsky [36]and Hainaut [30], and Cox-Ingersoll-Ross (CIR) model by Deelstra & Koehl [17, 16] and Kraft [43].

The starting point of portfolio optimization problems that incorporate risk constraints was the work by Markowitz [46]. He studied an optimization problem in a discrete time framework. He considered the variance of assets as a risk measure. One of the drawbacks of this measure is the consideration of positive variations of assets as a risk rather than a profit. In a continuous-time Black-Scholes framework Basak & Shapiro [6] investigated the utility maximization problem with Value at risk (VaR) and expected short-fall (ESF) as risk constraints. The literature dealing with portfolio optimization problem with risk constraints in a Black-Scholes market include the papers by Emmer et al. [20], Lakner & Nygren [45], Grecksch et al. [24], Basak et al. [5] and Damitrasinovic-Vidovic et al. [19]. Hainaut [30] discussed the same problem with interest rates modeled by one-factor Vasiček, and VaR as a risk measure in a complete market using the martingale approach. For the incomplete market case Gundel & Weber [27] investigated the utility maximization problem with risk constraint. They prove the existence and uniqueness of a probability measure that solves part of the dual problem.

1.3. Main results

There are two main approaches for solving portfolio optimization problems: Stochastic control and martingale methods. We used the martingale method. The martingale approach is mainly based on two steps. The first step is the static optimization problem which deals with the determination of optimal terminal wealth. The second step is the dynamic problem which deals with the derivation of the admissible trading strategy that generates the optimal terminal wealth. The static optimization problem can be solved using a Lagrangian approach. This approach is a two-step optimization process. The first step is the primal optimization problem that deals with the computation of the optimal terminal wealth candidate which still depends on the Lagrange multipliers and $\mathbb{Q} \in \mathcal{M}^e$. The second step is the dual optimization problem that is concerned with the determination of the Lagrange multiplier/s and $\mathbb{Q} \in \mathcal{M}^e$.

Most literature on portfolio optimization assumes that the interest rates are deterministic. This is very far from the reality, more specifically from a long term investment point of view, the interest rates are stochastic. The models for the interest rates can be classified into forward rate and short rate models. For the details about interest rate models we refer for example to Brigo & Mercurio [15]. We consider short rate models and we have listed some of the popular ones in Section 2.1. To overcome the restriction of the deterministic interest rates many attempts have been made to solve portfolio optimization problems in a framework that allows interest rates to be stochastic. Using stochastic control methods Korn and Kraft [42] have studied the case where short rates follow a one-factor Ho & Lee and a one-factor Vasiček model. Deelstra et al. [17] and [16] investigated a portfolio optimization problem using the martingale approach in a complete market and considered the short rates to be modeled by a one-factor CIR. Bäuerle & Rieder [7] investigated the portfolio problem by allowing the drift and the interest rates to depend on an external continuous-time Markov-chain process. Hainaut [30] studied a portfolio optimization problem that incorporates risk constraints measured in terms of VaR and the short rates following a one-factor Vasiček model.

VaR measures the probability of a loss and ignores completely the magnitude of a loss. As a consequence, although the probability of extreme losses is pre-specified, a large loss may exceed the loss for the unconstrained optimization problem. One of the risk measures that take the magnitude of a loss into account is the expected short-fall (ESF) also called expected loss (EL). First, in the context of a complete and arbitrage-free financial market, i.e. when the number of uncertainties is equal to the number of risk-bearing traded assets, we discuss the portfolio optimization problem with an ESF constraint under a pre-set level δ and allowing the interest rates to be stochastic. In particular, interest rates are modeled by a one-factor Vasiček model. The problem of M-PM was studied for example Korn & Kraft [42] and [30]. We derive the trading strategies for the cases of ES-PM and PI-PM using Malliavin calculus.

Financial markets are generally incomplete, i.e. the number of random sources is greater than the number of risk-bearing traded assets. This implies that the set $\mathcal{M}^e$ is composed of infinitely many elements. So, the portfolio optimization problem in the case of an incomplete market is considerably more difficult. It was studied, in continuous-time and the using martingale approach, for example by Karatzas et al. [41] who used the idea of completing the market by introducing additional fictitious risk-bearing securities. He & Pearson [34] and Kramkov & Schachermayer [44] proved the existence and uniqueness of $\mathbb{Q}^* \in \mathcal{M}^e$ that solves the dual problem for the given Lagrange multiplier. But the question remains, how does $\mathbb{Q}^* \in \mathcal{M}^e$ look like?

Based on the absence of arbitrage and putting some restrictions on the market price of risk, we also address the question above. We consider an incomplete financial market composed of a money market account and a stock with interest rates described by the one-factor Vasiček model. We derive explicitly the measure $\mathbb{Q}^* \in \mathcal{M}^e$ that solves the dual problem and we provide the optimal trading strategy for the case of M-PM. We discuss further the portfolio optimization problem in this incomplete market for the cases of ES-PM and PI-PM: We solve the dual problem and then give optimal portfolio processes.

One-factor models of interest rates have some drawbacks in general, for example a perfect correlation of interest rates of different maturities, however large the difference in maturities is. For more details about interest rate models see [15], [60], [50], among others. We relax this assumption by considering next the interest rates to be described by a two-factor model, more precisely a two-factor Vasiček model.

We examine the portfolio optimization problem in a financial market consisting of a market account, two bonds and a stock, i.e. a complete market, with interest rates modeled by a two-factor Vasiček model. We compute explicitly the optimal trading strategy for the case of M-PM. We investigate moreover the portfolio optimization problem for the cases of ES-PM and PI-PM and derive the investment strategies that generate the optimal terminal wealth.

We extend our findings to an incomplete market with a money market account, one bond, a stock and a two-factor Vasiček model for the interest rates. We calculate the $\mathbb{Q}^* \in \mathcal{M}^e$ that solves the dual problem. After getting $\mathbb{Q}^* \in \mathcal{M}^e$ the techniques are applied as for the case of a complete market to solve the dynamic optimization problem. We then derive the optimal terminal wealth for M-PM, ES-PM and PI-PM, and compute their corresponding trading strategies.

1.4. Thesis outline

We conclude this introductory part by giving the structure of the thesis. This work is divided into 6 chapters:

Chapter 2 starts with the description of the financial market: It gives some definitions, assumptions and theorems that are needed to construct the market models. We discuss briefly the risk measures. Then the portfolio optimization problem for the unlimited and limited risk, in terms of the risk constraint, is stated. To solve this problem we use the martingale approach which is discussed in Section 2.3.

Chapter 3 deals with the review of the utility maximization problem first without a risk constraint and second in the presence of limited expected shortfall risk in the case of a Black-Scholes market, i.e. when all the market parameters are constant (including the interest rates). This problem has been studied and the closed form solutions were obtained, e.g by Basak & Shapiro [6]. We present their results in this chapter for a quick reference and comparison.

In Chapter 4 we investigate the expected utility maximization problem when the interest rates are allowed to be stochastic. In particular, we consider a one-factor Vasiček term structure model for the interest rates. This problem for the case of a complete market and without a risk constraint was addressed for instance by Hainaut [30] and Korn & Kraft [42]. Hainaut [30] further embedded VaR in the portfolio optimization problem. We study the same problem with the limited ESF risk constraint in a complete market. We present the trading strategies using Malliavin calculus. We further discuss in this chapter the corresponding problem in the general case of incomplete financial market. One of the main challenges in this market using the martingale approach, is to solve one part of the dual problem, which is concerned with the determination of $\mathbb{Q}^* \in \mathcal{M}^e$ that minimizes the Lagrangian function. Under a condition on a so-called asymptotic elasticity Kramkov & Schachermayer [44] proved the existence and uniqueness of $\mathbb{Q}^* \in \mathcal{M}^e$ for the case of M-PM but they did not show how it looks like. Under some assumptions on the market price of risk, we provide in this chapter the explicit solution of $\mathbb{Q}^* \in \mathcal{M}^e$. We treat furthermore the cases of ES-PM and PI-PM in the incomplete market. Some of our main results are given in the Sections 4.2, 4.3 and 4.4. This chapter is concluded by numerical examples.

In Chapter 5 we start by pointing out some weaknesses of one-factor term structure models for the interest rates. Then our results in Chapter 4 are extended to a two-factor term structure model for interest rates. More precisely, we consider a two-factor Vasiček model for interest rates. We consider both complete and incomplete financial markets and examine the utility maximization problem for the cases of M-PM, ES-PM and PI-PM. We obtain the corresponding optimal terminal wealth and optimal policies in both complete and incomplete markets. Our results are stated in the Sections 5.1, 5.2, 5.3 and 5.4. This chapter is concluded by numerical examples as well.

2. Financial market and portfolio optimization problem

2.1. Financial markets and stochastic interest rates

We consider an arbitrage free financial market where $n+2$ assets are continuously traded over a fixed finite time-horizon $[0, T]$. The uncertainty in this market is modeled by a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We denote by $\mathcal{F} = (\mathcal{F}(t))_{t\in[0,T]}$ the augmented filtration generated by an $(m+1)$-dimensional standard Brownian motion

$$W^{\mathbb{P}}(t) = \left(W_r^{\mathbb{P}}(t), W_S^{\mathbb{P}}(t)\right)^{\top} = \left(W_1^{\mathbb{P}}(t), \cdots, W_m^{\mathbb{P}}(t), W_S^{\mathbb{P}}(t)\right)^{\top}$$

under the probability measure $\mathbb{P}$, where $\top$ indicates a transpose. We assume that $\mathbb{F} = \mathcal{F}(T)$. It is assumed further that all stochastic processes are adapted to $\mathcal{F}$ and all stated processes are well defined. Moreover, we assume throughout this thesis that all equalities as well as inequalities hold $\mathbb{P}$-almost surely.

One of the assets is a risk-free money market account (M) following the differential equation

$$dM(t) = M(t)\, r(t)\, dt, \quad \text{with} \quad M(0) = 1. \tag{2.1}$$

As a result,

$$M(t) = \exp\left(\int_0^t r(s) ds\right), \tag{2.2}$$

where r is the *instantaneous interest rate*, usually called *short-term interest rate* or *spot interest rate*, at which the money market account accrues. From Equations (2.1) and (2.2) it means that investing a unit amount in the money market account at time 0 yields the value $\exp\left(\int_0^t r(s)ds\right)$ at time t. The amount $\beta(t,T)$, at time t that is equivalent to one unit of currency payable at time T given by

$$\beta(t,T) = \frac{M(t)}{M(T)} = \exp\left(-\int_t^T r(s)ds\right) \tag{2.3}$$

is the *discount factor*. From a general point of view, the interest rates $r(t)$ are modeled by

$$\begin{aligned} r(t) &= g(t, Y(t)) \\ dY(t) &= \mu_r(t,Y)\,dt + \sigma_r(t,Y)\,dW_r^{\mathbb{P}}(t), \end{aligned} \tag{2.4}$$

where $W_r^{\mathbb{P}}$ is an m-dimensional Brownian motion under the probability measure $\mathbb{P}$, $g : \mathbb{R}^d \to \mathbb{R}$ is a deterministic function, Y with values in $\mathbb{R}^d$ is the vector of interest rate factors in the normal form, μ_r with values in $\mathbb{R}^d$ is the drift of interest rate factors and σ_r with values in $\mathbb{R}^{d\times m}$ is a root of the variance covariance matrix of the interest rate factors in the normal form. It is assumed that μ_r and σ_r are regular enough to guarantee existence and uniqueness of a strong solution to (2.4).

Throughout this work, we consider the case $d = m$. If $m = 0$, we have constant interest rates. For $m = 1$, we have a class of one-factor models of interest rates including:

- Ho-Lee model [35]

$$dr(t) = a(t)\,dt + \sigma_r dW_r(t)$$

- Vasiček model [63]

$$dr(t) = a(b - r(t))\,dt + \sigma_r dW_r(t)$$

- Cox-Ingersoll-Ross (CIR) [12]

$$dr(t) = a(b - r(t))\,dt + \sigma_r\sqrt{r(t)}dW_r(t) \tag{2.5}$$

- Hull-White (extended Vasiček model) [37]

$$dr(t) = a(t)(b(t) - r(t))\,dt + \sigma_r(t)\,dW_r(t)$$

- Hull-White (extended CIR model) [37]

$$dr(t) = a(t)(b(t) - r(t))\,dt + \sigma_r(t)\sqrt{r(t)}dW_r(t)$$

In the case $m = 2$, we obtain a class of two-factor interest rate models, for example

- Two-factor Vasiček model

$$\begin{aligned} r(t) &= g(t, Y(t)) := \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t) \quad \text{with} \\ dY_1(t) &= (\nu - b_{11}Y_1(t) - b_{12}Y_2(t))\,dt + \sigma_1 dW_1(t) \\ dY_2(t) &= (\nu - b_{21}Y_1(t) - b_{22}Y_2(t))\,dt + \sigma_2 dW_2(t), \end{aligned}$$

  where all the coefficients are constants.

- Two-factor CIR model

$$\begin{aligned} r(t) &= g(t, Y(t)) := \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t) \quad \text{with} \\ dY_1(t) &= (\nu - b_{11}Y_1(t) - b_{12}Y_2(t))\,dt + \sigma_1\sqrt{Y_1(t)}dW_1(t) \\ dY_2(t) &= (\nu - b_{21}Y_1(t) - b_{22}Y_2(t))\,dt + \sigma_2\sqrt{Y_2(t)}dW_2(t), \end{aligned}$$

  where all the coefficients are constants.

There are extended versions with non-constant parameters and even interest rate models of more than two factors but these are not discussed here. We refer the interested reader to [15], [65] and [60], among others, for details about the interest rate models.

We have $n + 1$ risky investment opportunities at the market as well, which can be n bonds and/or one stock, whose price processes $P(t) \in \mathbb{R}^{n+1}$ satisfy the stochastic differential equation (SDE)

$$dP(t) = P(t)\left(\mu(t)\,dt + \sigma(t)\,dW^{\mathbb{P}}(t)\right), \quad P(0) = p \in \mathbb{R}, \tag{2.6}$$

where $\mu(t) \in \mathbb{R}^{n+1}$ is the vector of instantaneous mean returns of the risky assets and $\sigma(t) \in \mathbb{R}^{(n+1)\times(m+1)}$ is the volatility matrix of the risky assets . We assume here also that μ and σ are sufficiently regular to ensure existence and uniqueness of a strong solution to (2.6). We can write μ and σ in matrix form as:

$$\mu = \begin{pmatrix} \mu_B \\ \mu_S \end{pmatrix} = \begin{pmatrix} \mu_{B_1} \\ \vdots \\ \mu_{B_n} \\ \mu_S \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} \sigma_B & 0 \\ \sigma_{Sr} & \sigma_S \end{pmatrix} = \begin{pmatrix} \sigma^1_{B_1} & \cdots & \sigma^m_{B_1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \sigma^1_{B_n} & \cdots & \sigma^m_{B_n} & 0 \\ \sigma^1_{Sr} & \cdots & \sigma^m_{Sr} & \sigma_S \end{pmatrix},$$

respectively and $\sigma\sigma^\top$ is assumed to be non-singular.

Definition 2.1 *A zero coupon bond (which will be called here only bond) with maturity T is a security which guarantees its holder a unity amount of money to be paid on the date T.*

Among the risky assets, we have n bonds in the market and the price of each bond at time $t \in [0,T]$ maturing at finite time $T_i > T$ is denoted by $B(t,T_i)$ or simply B_i, for $1 \leq i \leq n$. Throughout this thesis, we assume each bond price to dependent on the interest rate only, independent of liquidity risk, default risk or any other factors. The price process of each bond B_i is assumed to evolve according to the following SDE

$$dB_i = B_i \left(\mu_{B_i}(t,Y)\,dt + \sigma_{B_i}^\top(t,Y)\,dW_r^{\mathbb{P}}(t)\right),$$

where $\mu_{B_i}(t,\cdot) : \mathbb{R}^m \to \mathbb{R}$ is the drift rate of the bond B_i under the physical probability measure $\mathbb{P}$ and $\sigma_{B_i}(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is the volatility of the bond B_i due to the vector $Y = (Y(t))_{t\in[0,T]}$ of m interest rate factors.

The remaining asset (risky asset) is a stock and the price process at time $t \in [0,T]$ of a stock is denoted by $S(t)$. It's dynamics are considered to follow the SDE

$$dS(t) = S(t)\left(\mu_S(t)\,dt + \sigma_{Sr}^\top(t)\,dW_r^{\mathbb{P}}(t) + \sigma_S(t)\,dW_S^{\mathbb{P}}(t)\right),$$

where $\sigma_{Sr} \in \mathbb{R}^m$ is the volatility of the stock with respect to $W_r^{\mathbb{P}}$, and $\sigma_S \in \mathbb{R}$ is the volatility of the stock with respect to $W_S^{\mathbb{P}}$.

Assumption 2.2 *We assume that $\mu_B = (\mu_{B_1}, \cdots, \mu_{B_n})^\top$ and μ_S take the form*

$$\begin{aligned}\mu_B(t,\cdot) &= \bar{\mu}_B + r(t)\,\mathbb{1}_n,\\ \mu_S(t) &= \bar{\mu}_S + r(t),\end{aligned}$$

where $\bar{\mu}_B \in \mathbb{R}^n$ and $\bar{\mu}_S \in \mathbb{R}$ are constants, and $\mathbb{1}_n$ denotes the n-dimensional vector whose entries are all equal to 1.

Remark: We need this assumption to be able to compute $\int \mu(t)\,dt$ later. If the structure of $\mu(t)$ is known, for most results deterministic $\bar{\mu}(t)$ should be possible.

We consider an investor or portfolio manager endowed with an initial wealth x to be invested in the financial assets provided at the market. He/she is allowed to trade continuously during the investment period. We denote the fractions of wealth invested in the bonds and a stock at time $t \in [0, T]$ by $\varphi_B(t) \in \mathbb{R}^n$ and $\varphi_S(t) \in \mathbb{R}$, respectively. $\varphi(t) := (\varphi_B(t), \varphi_S(t))^\top$ is a stochastic process. It follows that the fraction of wealth invested in the money market account is given by $\left(1 - \varphi(t)^\top \mathbb{1}_{n+1}\right)$. Before we give the wealth process of an investor, let us first make some assumptions which will be considered throughout this thesis.

Assumption 2.3

- *The market is frictionless.*
- *The investor has no knowledge about the future prices of the securities.*
- *The portfolio process $\varphi(t)$ is self-financing.*
- *Money is invested only in bonds, stock or money market account.*
- *There is no consumption during the investment period.*
- *The assets are perfectly divisible.*

Putting the assumptions above into consideration, the wealth process of an investor at time $t \in [0, T]$, represented by $X(t)$, under the probability measure $\mathbb{P}$ is ruled by the following stochastic differential equation

$$\begin{aligned} dX(t) =& dX^M(t) + dX^B(t) + dX^S(t) \\ =& \left(1 - \varphi^\top(t) \mathbb{1}_{n+1}\right) X(t) r(t) dt \\ &+ X(t) \varphi_B^\top(t) \left(\mu_B(t) dt + \sigma_B^\top(t) dW_r^{\mathbb{P}}(t)\right) \\ &+ X(t) \varphi_S(t) \left(\mu_S(t) dt + \sigma_{S,Y}^\top(t) dW_r^{\mathbb{P}}(t) + \sigma_S(t) dW_S^{\mathbb{P}}(t)\right), \end{aligned}$$

where X^M, X^B and X^S denote the wealth in the money market account, bonds and the stock, respectively.

Definition 2.4 *The $(n+1)$-dimensional stochastic process $\varphi(t)$ is called portfolio process if it fulfills the following conditions: It is self-financing with*

- *$\varphi(t)$ is $\mathcal{F}_t$-measurable,*

- $\int_0^T \|\varphi(t)\|^2 \, dt < \infty$.

The wealth process $X(t)$ is a process controlled by the portfolio process $\varphi(t)$. This means that the wealth process of an investor is driven by his/her trading strategy.

Definition 2.5 *A self-financing portfolio process $\varphi(t)$ is called an arbitrage opportunity, if the corresponding wealth process satisfies*

- $X(0) = 0$,
- $X(T) \geq 0$,
- $P(X(T) > 0) > 0$.

Of course, the investor's hope is to find a portfolio process such that his/her terminal wealth is non-negative for sure and strictly positive with some positive probability without any investment (zero initial wealth). In other words, an arbitrage opportunity is the opportunity of making some money without risk.

The prices $(P(t))_{t\in[0,T]}$ of financial securities are said to fulfill the no-arbitrage condition if there is no arbitrage opportunity.

Definition 2.6 *A probability measure $\mathbb{Q}$ on $(\Omega, \mathbb{F})$ is called an equivalent martingale measure with respect to $\mathbb{P}$, if $\mathbb{Q} \sim \mathbb{P}$ and P is a martingale under $\mathbb{Q}$ with respect to a given numeraire.*

We denote the set of these equivalent martingale measures by $\mathcal{M}^e$. The following theorem, so-called Fundamental Theorem of Asset Pricing, relates the no-arbitrage condition to the existence of equivalent martingale measures as studied for example by Harrison and Pliska [32].

Theorem 2.7 (Fundamental Theorem of Asset Pricing) *:*
For the prices $(P(t))_{t\in[0,T]}$ of financial securities in a market modeled on a finite probability space $(\Omega, \mathbb{F}, \mathbb{P})$ the following are equivalent:

(i) $(P(t))_{t\in[0,T]}$ *satisfy no-arbitrage condition,*

(ii) $\mathcal{M}^e \neq \emptyset$.

If the numeraire is the money market account, as in our case, a probability measure $\mathbb{Q} \in \mathcal{M}^e$ is a so-called "risk-neutral" probability constructed such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \mid_{\mathcal{F}(t)} = Z^{\Theta}(t) := \exp\left\{ -\frac{1}{2}\int_0^t \|\Theta(s)\|^2 ds - \int_0^t \Theta^\top(s)\, dW^{\mathbb{P}}(s) \right\}, \tag{2.7}$$

where the vector

$$\Theta(t) = (\theta_r(t), \theta_S(t))^\top = (\theta_1(t), \cdots, \theta_m(t), \theta_S(t))^\top \in \mathbb{R}^{m+1}$$

solves

$$(\mu(t) - r(t)\mathbb{1}_{n+1}) = \sigma(t)\Theta(t)$$

and its components are interpreted as the market prices of risks due to the m interest rate factors and a stock, respectively.

Assumption 2.8 *We assume that* $\int_0^T \|\Theta(t)\|^2 dt < \infty$, $\mathbb{P}$*-almost surely.*

Definition 2.9 *The financial market above will be called complete if* $n = m$, *and incomplete if* $n < m$.

If we are in the case of a complete market, $\Theta(t)$ can be determined uniquely which is equivalent to the uniqueness of $\mathbb{Q} \in \mathcal{M}^e$. But for the case of incomplete market, $\Theta(t)$ can not be determined uniquely.

According to the Girsanov Theorem [26], the processes defined by

$$W^{\mathbb{Q}}(t) = \begin{pmatrix} W_r^{\mathbb{Q}}(t) \\ W_S^{\mathbb{Q}}(t) \end{pmatrix} := \begin{pmatrix} W_r^{\mathbb{P}}(t) + \int_0^t \theta_r(s)\, ds \\ W_S^{\mathbb{P}}(t) + \int_0^t \theta_S(s)\, ds \end{pmatrix}$$

are Brownian motions under the probability measure $\mathbb{Q} \in \mathcal{M}^e$. This will help us to change from a world of physical probability measure $\mathbb{P}$ to the equivalent martingale measure $\mathbb{Q} \in \mathcal{M}^e$. Now, the price processes of the n bonds and a stock under $\mathbb{Q} \in \mathcal{M}^e$ are given by

$$\begin{aligned} dB_i(t) &= B_i(t)\left(r(t)\,dt + \sigma_{B_i}^\top(t,Y)\,dW_r^{\mathbb{Q}}(t)\right) \quad \text{and} \\ dS(t) &= S(t)\left(r(t)\,dt + \sigma_{Sr}^\top(t)\,dW_r^{\mathbb{Q}}(t) + \sigma_S(t)\,dW_S^{\mathbb{Q}}(t)\right), \end{aligned}$$

respectively with $1 \le i \le n$.

$$\begin{aligned} dX(t) =& dX^M(t) + dX^B(t) + dX^S(t) \\ =& \left(1 - \varphi^\top(t)\,\mathbb{1}_{n+1}\right)X(t)\,r(t)\,dt \\ &+ X(t)\,\varphi_B^\top(t)\left(r(t)\,dt\mathbb{1}_n + \sigma_B(t)\,dW_r^{\mathbb{Q}}(t)\right) \\ &+ X(t)\,\varphi_S(t)\left(r(t)\,dt + \sigma_{Sr}^\top(t)\,dW_r^{\mathbb{Q}}(t) + \sigma_S(t)\,dW_S^{\mathbb{Q}}(t)\right) \\ =& X(t)\,r(t)\,dt \\ &+ X(t)\left(\varphi_B^\top(t)\,\sigma_B(t) + \varphi_S(t)\,\sigma_{Sr}^\top(t)\right)dW_r^{\mathbb{Q}}(t) \\ &+ X(t)\,\varphi_S(t)\,\sigma_S(t)\,dW_S^{\mathbb{Q}}(t). \end{aligned}$$

Applying Itô's rule on the product of the processes $\beta(t)$ and $X(t)$ yields:

$$\begin{aligned} d\left(\beta(t)\,X(t)\right) =& \beta(t)\,X(t)\left(\varphi_B^\top(t)\,\sigma_B(t) + \varphi_S(t)\,\sigma_{Sr}^\top(t)\right)dW_r^{\mathbb{Q}}(t) \\ &+ \beta(t)\,X(t)\,\varphi_S(t)\,\sigma_S(t)\,dW_S^{\mathbb{Q}}(t), \end{aligned} \tag{2.8}$$

which can be written in integral form as

$$\begin{aligned} \beta(T)\,X(T) =& X(0) + \int_0^T \beta(t)\,X(t)\left(\varphi_B^\top(t)\,\sigma_B(t) + \varphi_S(t)\,\sigma_{Sr}^\top(t)\right)dW_r^{\mathbb{Q}}(t) \\ &+ \int_0^T \beta(t)\,X(t)\,\varphi_S(t)\,\sigma_S(t)\,dW_S^{\mathbb{Q}}(t) \\ =& X(0) + \int_0^T \beta(t)\,X(t)\,\varphi^\top(t)\,\sigma(t)\,dW^{\mathbb{Q}}(t). \end{aligned} \tag{2.9}$$

The deflator $H(T)$ for the cash flow paid at T is defined by

$$\begin{aligned} H(T) &:= \beta(T)\,Z^{\Theta}(T) \\ &= \exp\left\{-\int_0^T r(t)\,dt - \frac{1}{2}\int_0^T \|\Theta(t)\|^2\,dt - \int_0^T \Theta^\top(t)\,dW^{\mathbb{P}}(t)\right\}. \end{aligned}$$

2.2. Risk constraints

Risk, in layman's terms, is the possibility of harm or loss. Financial risk refers to the possibility of a monetary loss associated with investments. Financial risk is divided into the following "general categories": Market risk, Exchange risk, Credit risk, Operational risk and Interest rate risk, among others. The unifying theme for each category is that risk requires both exposure and uncertainty. If a bank decides not to loan to a business that is likely to default, there is also no risk for that bank because the bank has no exposure to the possibility of loss. If a bank already knows that a loan will default, there is no uncertainty and therefore no risk.
Why do we need risk measures in finance?
For instance in the modern portfolio selection theory: maximizing return for a given level of risk or minimizing risk for a given level of return, the decision depends on the risk measure used. Now the goal is to have a universal definition of a risk measure. Theoreticians have given a number of properties that a risk measure might or might not fulfill. Before we come to that let us first give the risk binary relations.

Definition 2.10 *Risk ordering: The binary relations* $\succsim$, $\succ$ *and* $\sim$, *where*

- $X \succsim Y$ *means that* X *is at least as risky as* Y,
- $X \succ Y$ *means that* X *is strictly riskier than* Y *and*
- $X \sim Y$ *means that* X *and* Y *are equally risky.*

Definition 2.11 *Let* χ *be some vector space of random variables that contains the constants. The numerical representation of risk binary relations* $\succsim$, $\succ$ *and* $\sim$ *is a function* $\rho : \chi \to \mathbb{R}$ *such that* $\forall X, Y \in \chi$

- $X \succsim Y \Leftrightarrow \rho(X) \geq \rho(Y)$
- $X \succ Y \Leftrightarrow \rho(X) > \rho(Y)$
- $X \sim Y \Leftrightarrow \rho(X) = \rho(Y)$

Every such function ρ *will be called risk measure.*

Empirically, risk depends on the probability of occurrence of loss and the amount of potential loss. So, that is why we try to measure risk in terms of moments or quantiles. In the mean-variance principle of Markowitz [47], variance is considered to be a measure of risk. One of the drawbacks of variance as a measure of risk is that it considers the positive deviations as a risk, which are rather gains. To avoid this problem, the downside risk measures which are only based on negative deviations are considered, for instance the so called "lower partial moments" studied by Fishburn in 1977 [21]. The lower partial moments are measures of shortfall or downside risk which consider only negative deviations from a target. The lower partial moment of order k is computed as follows:

$$\rho_{k,q}(X) := \int_{-\infty}^{q} (q - X)^k \, dF(X),$$

where F is the distribution function of the random variable (wealth) X, $q \in \mathbb{R}$ is the target (of wealth) below which the deviations are measured as risk and $k \in \mathbb{N}_0$ is a measure of the relative impact of small and large deviations.

If $k = 0$, we have what is known as shortfall probability:

$$\rho_{0,q} := \int_{-\infty}^{q} (q - X)^0 \, dF(X) = \int_{-\infty}^{q} f_X(t) \, dt,$$

where F is the distribution function of random variable X and f_X is its density function. Shortfall probability corresponds to the widely known Value at Risk (VaR) as a risk measure which is defined as

$$VaR_\alpha(\Gamma) := F_\Gamma^{-1}(1 - \alpha),$$
$$where \quad F_\Gamma(\nu) = P(\Gamma \leq \nu) = \int_{-\infty}^{\nu} f_\Gamma(t) \, dt,$$

$\Gamma := (q - X) \in \chi$, $0 \leq \alpha \leq 1$, F_Γ is the distribution function of Γ and f_Γ is its density function. VaR_α can be interpreted as the maximum loss with some prespecified probability α over a given time horizon. VaR has been a very popular risk measure, one of the reasons being that it is easily understood. However, it has some limitations, e.g. it focuses mainly on the probability of loss and ignores completely the magnitude of loss. One alternative which puts the magnitude of losses into account is a so-called Expected Shortfall (ESF) (it corresponds to the case of $k = 1$) defined by

$$ESF(\Gamma) := \rho_{1,q}(X) = \int_{-\infty}^{q} \Gamma f_\Gamma(t) \, dt.$$

ESF is interpreted as the average magnitude of the losses over a given time horizon.

Different people have different attitudes towards risk, but the above mentioned risk measures do not put the investor's preferences into consideration. The preferences are characterized by the utility function which is defined below.

Definition 2.12 *A function $U : (0, \infty) \to \mathbb{R}$ which is strictly concave, twice continuously differentiable and satisfies the Inada conditions:*

$$U'(0) := \lim_{x \to 0} U'(x) = +\infty \quad \textit{and} \quad U'(\infty) := \lim_{x \to \infty} U'(x) = 0,$$

is called utility function.

An example of a risk measure which takes the preferences into account is the so called Expected Utility Loss (EUL) defined by

$$EUL(\Gamma) = E[U(\Gamma)].$$

EUL can be interpreted as the investor's average utility from the magnitude of losses over a given time horizon.

Artzner et al. [2] proposed desirable properties a risk measure should have, those given in the definition below. The risk measures which fulfill these properties were named coherent risk measures.

Definition 2.13 *A mapping $\rho : \chi \to \mathbb{R}$ is called a coherent risk measure if for all $X, Y \in \chi$ it satisfies*

- *Monotonicity (M): if $X \leq Y$, then $\rho(X) \geq \rho(Y)$.*
- *Cash invariance (CI): if $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.*
- *Convexity (C): $\rho(\gamma X + (1 - \gamma) Y) \leq \gamma \rho(X) + (1 - \gamma) \rho(Y)$ for $0 \leq \gamma \leq 1$.*
- *Subadditivity (SA): $\rho(X + Y) \leq \rho(X) + \rho(Y)$,*

where $\chi \subseteq \mathcal{L}^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$.

Note that, X, Y are considered to be discounted here. We refer to [2] for the discussion and interpretation of the above properties. A risk measure which fulfills properties (M) and (CI) is called monetary risk measure (for details you can see [22] by Föllmer and Schied). VaR, EL and EUL are monetary risk measures. A monetary risk measure that satisfies property (C) is called convex risk measure. VaR is not a convex risk measure (in general) because it doesn't fulfill property (C) for all distributions of losses [22]. As a consequence it is not a coherent risk measure. EL and EUL are convex risk measures but not coherent risk measures because they fail to satisfy the condition (S).

2.3. Portfolio optimization problem

Definition 2.14 *A self-financing portfolio process φ is called admissible for the initial wealth $x > 0$, if the corresponding wealth process satisfies $X(t) > 0$ for all $t \in [0, T]$. Let us denote the set of admissible portfolio processes by $\mathcal{A}(x)$.*

Optimization without risk constraints

The aim of an investor with initial wealth x is to choose a portfolio process φ, from the admissible set $\mathcal{A}(x)$, that maximizes his/her expected utility from the terminal wealth $X(T)$. Now, the portfolio optimization problem is formulated as

$$\begin{aligned} &\max_{\varphi \in \tilde{\mathcal{A}}(x)} \mathbb{E}\left[U\left(X\left(T\right)\right)\right], \\ \text{where} \quad &\tilde{\mathcal{A}}(x) = \left\{\varphi \in \mathcal{A}(x) \mid \mathbb{E}\left[U\left(X\left(T\right)\right)^{-}\right] < \infty\right\}. \end{aligned} \tag{2.10}$$

We restrict ourselves to the admissible set $\tilde{\mathcal{A}}(x)$ because the expectation exists. But it can be equal to infinity.

There are two main approaches that can be used to solve the portfolio optimization problem (2.10), namely the martingale and the stochastic control approach. In this work we use martingale methodology. The martingale approach is mainly based on the decomposition of the portfolio problem (2.10) into two problems:

(i) The static optimization problem consisting of determination of the optimal terminal wealth and

(ii) the representation problem that deals with finding a trading strategy that replicates the optimal terminal wealth.

Static optimization problem

The static optimization problem, as we have mentioned before, is concerned with the determination of the optimal terminal wealth. More precisely: An investor endowed with initial wealth $x > 0$ has to find the terminal wealth that maximizes his/her expected utility. Here the Lagrange approach is applied which is composed of a primal and a dual problem. Before we state the static optimization problem, let us first give a remark that helps to construct the budget equation.

Remark 2.15 *Any terminal wealth $X(T)$, for a given initial wealth $x > 0$, has to satisfy the budget constraint:*

$$\mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,X(T)\right] = \mathbb{E}\left[\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T X(T)\right] \le x, \forall \mathbb{Q} \in \mathcal{M}^e,$$

since by (2.8) $(\beta(t)\,X(t))_{t\in[0,T]}$ is a $\mathbb{Q}$-supermartingale.

Now, we can state the static optimization problem as follows:

$$\begin{aligned} &\max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[U(X(T))\right] \\ &\text{s.t. } \mathbb{E}\left[\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T X(T)\right] \le x \quad \text{(budget constraint)}, \end{aligned} \tag{2.11}$$

where

$$\mathcal{B} := \left\{X(T) > 0 \mid X(T) \;\; \mathcal{F}_T\text{-measurable}, \mathbb{E}\left[U(X(T))\right]^- < \infty\right\}. \tag{2.12}$$

We employ the Lagrange approach to solve the problem (2.11) with the Lagrangian function L defined by

$$\begin{aligned} L(y, \mathbb{Q}, X(T)) &:= \mathbb{E}\left[U(X(T)) + y\left(x - \beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T X(T)\right)\right] \\ &= \mathbb{E}\left[U(X(T)) - y\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T X(T) + yx\right], \end{aligned} \tag{2.13}$$

where y is the Lagrangian multiplier. The function L is constructed by penalizing the objective function of the problem (2.11) with its constraint. The problem

(2.11) is equivalent to the primal problem (2.14), where we have the optimization problem without a constraint. With the Lagrangian approach we first solve the primal optimization problem and then its dual optimization problem which are stated below.
Primal problem:

$$\Psi(y,\mathbb{Q}) := \max_{X(T)\in\mathcal{B}(x)} L(y,\mathbb{Q},X(T)) \quad \text{for} \quad y>0 \quad \text{and} \quad \mathbb{Q}\in\mathcal{M}^e. \tag{2.14}$$

Dual problem:

$$\Phi(X(T)) := \min_{y>0,\mathbb{Q}\in\mathcal{M}^e} L(y,\mathbb{Q},X(T)) \quad \text{for} \quad X(T)\in\mathcal{B}(x). \tag{2.15}$$

In the case of an incomplete market, the set $\mathcal{M}^e$ is composed of infinitely many elements, where in the dual problem we have to find $y^* > 0$ and $\mathbb{Q}^* \in \mathcal{M}^e$ that minimize the Lagrangian function $L(y,\mathbb{Q},X(T))$ for a given $X(T)\in\mathcal{B}(x)$. But for the case of complete market the set of equivalent martingale measures is a singleton, i.e, $\mathcal{M}^e = \{\mathbb{Q}\}$, and we only have to find $y^* > 0$ that minimizes the Lagrangian function $L(y,X(T))$ for a given $X(T)\in\mathcal{B}(x)$ to solve the dual problem. Let us denote by I the inverse function of the derivative of the utility function U (as defined in 2.12). $I:(0,\infty)\to(0,\infty)$ is continuously differentiable, strictly decreasing and satisfies

$$\lim_{x\to\infty} I(x) = 0 \quad \text{and} \quad \lim_{x\to 0} I(x) = \infty.$$

The function V defined by

$$V(y) := \max_{x>0}\left[U(x) - xy\right] = U(I(y)) - yI(y) \text{ for all } y\in(0,\infty) \tag{2.16}$$

is the Legendre transform of $-U(-x)$ (see for example Karatzas et al. [41]). The function V is strictly convex, strictly decreasing, continuously differentiable and satisfies

$$\begin{aligned} &V'(y) = -I(y) \text{ for all } y\in(0,\infty), \\ &V'(0) := \lim_{y\to 0} V'(y) = -\infty, \quad V'(\infty) := \lim_{y\to\infty} V'(y) = 0 \end{aligned} \tag{2.17}$$

and it has the following bidual relation:

$$U(x) = \min_{y>0}(V(y) + xy) = V\left(U'(x)\right) + xU'(x) \text{ for all } x\in(0,\infty).$$

From (2.16) and (2.17) the following inequalities hold (see Karatzas et al. [41] and Cvitanic et al. [13], among others)

$$U(I(y)) \geq U(x) + y\left[I(y) - x\right] \text{ for all } x,y\in(0,\infty) \tag{2.18}$$

$$V\left(U'(x)\right) \leq V(y) - x\left[U'(x) - y\right] \text{ for all } x, y \in (0,\infty). \tag{2.19}$$

In our case we consider

$$V(y) := \max_{X(T)\in\mathcal{B}(x)} \left[U(X(T)) - yX(T)\right].$$

We define the value function

$$u(x) := \max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[U(X(T))\right]. \tag{2.20}$$

To exclude the trivial case we assume that

$$u(x) = \max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[U(X(T))\right] < \infty \text{ for all } x \in (0,\infty). \tag{2.21}$$

We define for the dual optimization problem

$$v(y) := \mathbb{E}\left[V\left(y\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)\right] = \Psi(y) - yx, \tag{2.22}$$

where Ψ is defined in the Equation (2.14) for $\mathbb{Q} \in \mathcal{M}^e$ fixed or in the case of a complete market.

Following the idea of Kramkov & Schachermayer [44], we state the following theorem that gives the solution to the static optimization problem (in other words, the solution to the primal and dual problems) in the case of a complete market.

Theorem 2.16 (Complete market case) *:*
Let U be a utility function as in Definition 2.12, consider a complete market, i.e., $\mathcal{M}^e = \{\mathbb{Q}\}$ and assume that the condition (2.21) is fulfilled. Denote by $u(x)$ and $v(y)$ the value functions

$$u(x) = \max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[U(X(T))\right], \; x \in (0,\infty). \tag{2.23}$$

$$v(y) = \mathbb{E}\left[V\left(y\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)\right], \; y \in (0,\infty). \tag{2.24}$$

Then we have:

- *The functions* $u(x)$ *and* $v(y)$ *are conjugate and* u *inherits the properties of the utility function* U *stated in Definition 2.12.*

- *The optimizer* $X^*(T) \in \mathcal{B}(x)$ *in (2.23) exists, is unique and satisfies* $X^* = I\left(y\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)$,
where $x \in (0,\infty)$ *and* $y \in (0,\infty)$ *are related via* $v'(y) = -x$.

- *The following hold true*

$$
\begin{aligned}
u'(x) &= \mathbb{E}\left[U'(X^*(T))\right] \text{ for all } x \in (0,\infty)\\
v'(y) &= \mathbb{E}\left[\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T V'\left(y\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)\right] \text{ for all } y \in (0,\infty).
\end{aligned}
$$

Proof: see Kramkov & Schachermayer [44] □

Now we drop the case of $\mathcal{M}^e$ being necessarily a singleton $\{\mathbb{Q}\}$ and consider a general case of an incomplete market. Kramkov & Schachermayer [44] considered a certain class $\mathcal{Y}(y)$ of supermartingales defined by

$$
\mathcal{Y}(y) := \begin{cases} Y \geq 0 : Y_0 = y \text{ and } \beta XY = (\beta(t)X(t)Y(t))_{0\leq t\leq T} \\ \text{is a supermartingale, for all } X \in \mathcal{B}(1), \end{cases}
$$

with $\mathcal{Y}(1)$ being a class of supermartingales extending the class of density processes of equivalent martingale measures $\mathbb{Q} \in \mathcal{M}^e$. In their setting (Kramkov & Schachermayer in [44]) a further enlargement of the set $\mathcal{M}^e$ is necessary for the general case, but in our setting this enlargement is not necessary as shown by Kramkov & Schachermayer [44].

The value function of the dual problem in the case of an incomplete market is defined by

$$
v(y) = \min_{\mathbb{Q}\in\mathcal{M}^e} \mathbb{E}\left[V\left(y\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)\right],\ y \in (0,\infty). \tag{2.25}
$$

Before we state the theorem that proves the existence and uniqueness of the solution of the static optimization problem in the case of an incomplete market, let us first give the definition of the asymptotic elasticity AE.

Definition 2.17 *The asymptotic elasticity of a utility function U is defined by*

$$AE(U) := \lim_{x\to\infty} \sup \frac{xU'(x)}{U(x)}$$

and U is said to have "a reasonable asymptotic elasticity" if $AE(U) < 1$.

The notion of the asymptotic elasticity was introduced by Kramkov & Schachamayer [44].

Theorem 2.18 (Incomplete market case) *: Suppose that a utility function U has a reasonable asymptotic elasticity, i.e.*

$$AE(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1.$$

Then

- $u(x) < \infty$, *for all* $x \in (0,\infty)$ *and* $v(y) < \infty$, *for all* $y \in (0,\infty)$. *The value functions u and v defined below are conjugate:*

  $$v(y) := \sup_{x>0} [u(x) - xy], \ y \in (0,\infty)$$
  $$u(x) := \inf_{y>0} [v(y) + xy], \ x \in (0,\infty).$$

  The value function u is continuously differentiable and the value function v is strictly convex on $\{v(y) < \infty\}$ and they satisfy

  $$u'(0) := \lim_{x\to 0} u'(x) = \infty, \ v'(\infty) := \lim_{x\to\infty} v'(y) = 0.$$

- *If $v(y) < \infty$, then the optimal solution $\mathbb{Q}^* \in \mathcal{M}^e$ to the dual problem (2.25) exists and is unique. The optimal solution*

  $$X^*(T) = I\left(y^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right) \in \mathcal{B}(x), \ \textit{for all } x \in (0,\infty)$$

  to the primal problem (2.20) exists and is unique, where $v(y) + xy$ attains its minimum at y^* and $y^* = u'(x)$.

Proof: see Kramkov & Schachamayer [44] □

From Theorem 2.18 we know that there exists a probability measure $\mathbb{Q}^* \in \mathcal{M}^e$ which solves the dual problem (2.25) and which is unique.

Now the question is, how we can calculate $\mathbb{Q}^* \in \mathcal{M}^e$ in (2.25)?

This question will be answered in Chapters 4 and 5 after specifying the model for the stochastic interest rates. Once the optimal $\mathbb{Q}^*$ has been found, the incomplete markets case can be handled similarly to the complete markets case.

Representation problem

After finding the optimal terminal wealth in the static optimization problem, we now have to find the portfolio process that replicates that terminal wealth. We use Malliavin calculus to solve the representation problem as studied also in [53]. The definitions and some necessary basics of Malliavin calculus are given in Appendix A.

If the conditions of Theorem A.7 on a discounted optimal terminal wealth $\beta(T) X^*(T)$ are fulfilled, then the process $\beta(T) X^*(T)$ can be represented as in Equation (A.17). Let us call this representation Clark-Ocone representation. To obtain the trading strategy that replicates the optimal terminal wealth $X^*(T)$, we compare the representation of $\beta(T) X(T)$ in the Equation (2.9) with Clark-Ocone representation of $\beta(T) X^*(T)$ which is also a task that will be discussed in Chapters 4 and 5 after specifying the model for the stochastic interest rates.

Optimization with a risk constraint

The optimal portfolio strategy that replicates the maximum expected utility from terminal wealth without consideration of a risk constraint may lead to extreme positions and as a consequence the optimal terminal wealth may not exceed even the initial wealth with a high probability. That is undesirable e.g. for a pension fund manager. The idea to this situation is to embed the risk quantity into the portfolio optimization problem. We consider the risk quantity to be measured in general by the function $\rho(X(T) - q)$, which is a function of the terminal wealth $X(T)$ to fall below a preset level $q > 0$. Let us call it shortfall risk. Shortfall risk might be Value at Risk, Expected Shortfall risk or Expected

Utility Loss as given in Section 2.2. The most convenient way to embed risk into our optimization problem is to impose the shortfall risk constraint in addition to a budget constraint, requiring shortfall risk to be maintained under some pre-specified level $\delta > 0$, i.e

$$\rho\left(X\left(T\right)-q\right)\leq\delta.$$

Then the dynamic optimization problem with risk constraint can be stated as follows:

$$\begin{aligned} &\max_{\varphi\in\tilde{\mathcal{A}}(x)}\mathbb{E}\left[U\left(X\left(T\right)\right)\right]\\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)X\left(T\right)\right]\leq x \quad \text{(budget constraint)}\\ &\rho\left(X\left(T\right)-q\right)\leq\delta \quad \text{(risk constraint)}, \end{aligned} \tag{2.26}$$

where $\tilde{\mathcal{A}}\left(x\right)$ is as defined in (2.10) and $\mathbb{Q}$ is the equivalent martingale measure w.r.t. $\mathbb{P}$. The static optimization problem corresponding to the dynamic problem (2.26) is stated as follows:

$$\begin{aligned} &\max_{X(T)\in\mathcal{B}(x)}\mathbb{E}\left[U\left(X\left(T\right)\right)\right]\\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)X\left(T\right)\right]\leq x \quad \text{(budget constraint)}\\ &\rho\left(X\left(T\right)-q\right)\leq\delta \quad \text{(risk constraint)}, \end{aligned} \tag{2.27}$$

where $\mathcal{B}\left(x\right)$ is as defined in (2.12).
The solutions to the problems (2.26) and (2.27) will be discussed in Chapter 3, 4 and 5 after specifying the utility function, interest rate model and risk measure used.

3. Portfolio optimization with deterministic interest rates

In this chapter we consider an arbitrage free financial market where a risk-free money market account M and a risky asset S are continuously traded over a fixed finite time-horizon $[0,T]$. Their respective prices $(M(t))_{t\in[0,T]}$ and $(S(t))_{t\in[0,T]}$ evolve according to the following equations

$$dM(t) = M(t)\,rdt, \text{ with } M(0) = 1, \tag{3.1}$$

$$dS(t) = S(t)\left(\mu dt + \sigma_S dW^{\mathbb{P}}(t)\right), \text{ with } S(0) = s > 0. \tag{3.2}$$

Here r is the interest rate of money market account and it is assumed to be constant throughout this chapter, μ is the drift of the stock, σ_S is the volatility of the stock and $W^{\mathbb{P}}$ is a standard 1-dimensional Brownian motion on a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We denote by $\mathcal{F}$ the augmented filtration generated by $W^{\mathbb{P}}$. μ and $\sigma_S > 0$ are also assumed to be constants.
As a result from Equation (3.1) and the interest rate being constant, we can identify the price of the money market account at time $t \in [0,T]$ as

$$M(t) = \exp\{rt\},\ t \in [0,T].$$

$\beta(t) := \frac{1}{M(t)} = \exp\{-rt\}$ will denote the corresponding discounting factor. The price process of the stock in Equation (3.2) is under the physical probability measure $\mathbb{P}$.

From a variant of the Fundamental Theorem of Asset Pricing (compare Theorem 2.7) there exists a risk-neutral probability measure $\mathbb{Q} \in \mathcal{M}^e$, where $\mathcal{M}^e$ is the set of equivalent martingale measures as in Definition 2.6. $\mathbb{Q} = \{\mathcal{M}^e\}$ is constructed as in Equation (2.7) and it is characterized in this case by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}(t)} = Z^{\theta}(t) := \exp\left\{-\frac{1}{2}\theta^2 t - \theta dW^{\mathbb{P}}(t)\right\},$$

where $\theta := \frac{\mu - r}{\sigma_S}$ is the market price of risk which is also a constant.
The deflator $H(T)$ for the cash flow paid at T is defined by

$$\begin{aligned} H(T) &:= \beta(T) Z^{\theta}(T) \\ &= \exp\left\{-rT - \frac{1}{2}\theta^2 T - \theta dW^{\mathbb{P}}(T)\right\}. \end{aligned}$$

Remark 3.1 *We are in a complete market because the number of uncertainty is equal to the number of risky assets and as a result $\mathcal{M}^e$ is singleton, i.e. $\mathcal{M}^e = \{\mathbb{Q}\}$, see [9] (Meta-theorem 8.3.1).*

According to the Girsanov Theorem the process

$$W^{\mathbb{Q}}(t) = W^{\mathbb{P}}(t) + \theta t$$

is a Brownian motion under $\mathbb{Q}$.
The price process of the stock under $\mathbb{Q}$ is given by

$$dS(t) = S(t)\left(rdt + \sigma_S dW^{\mathbb{Q}}(t)\right). \tag{3.3}$$

We consider an investor endowed with an initial wealth x to be invested in a money market account and/or a stock. He/she is allowed to trade continuously during the investment interval $[0, T]$. We denote the fraction of wealth invested in the money market account and the stock by φ_M and φ_S, respectively. It follows that $\varphi_M = 1 - \varphi_S$. The wealth process of an investor at time $t \in [0, T]$, denoted by $X(t)$, under the probability measure $\mathbb{P}$ is ruled by the following stochastic differential equation

$$\begin{aligned} dX(t) =& dX^M(t) + dX^S(t) \\ =& (1 - \varphi_S(t)) X(t) rdt \\ &+ X(t)\varphi_S(t)\left(\mu dt + \sigma_S dW^{\mathbb{P}}(t)\right), \end{aligned}$$

where X^M and X^S denote the wealth in the money market account and the stock, respectively.
Using the Equations (3.1) and (3.3) the wealth process $X(t)$ can be transformed to

$$dX(t) = X(t)\left(rdt + \varphi_S(t)\sigma_S dW^{\mathbb{Q}}(t)\right). \tag{3.4}$$

Equation (3.4) describes the wealth process under the probability measure $\mathbb{Q}$. By applying Itô's rule on the process $\beta(t) X(t)$ and then integrate, it gives

$$d(\beta(t) X(t)) = \beta(t) X(t) \varphi_S(t) \sigma_S dW^{\mathbb{Q}}(t)$$

and thus

$$\beta(T)X(T) = X(0) + \int_0^T \beta(t)X(t)\varphi_S(t)\sigma_S dW^{\mathbb{Q}}(t). \tag{3.5}$$

3.1. Merton optimization problem

The objective of an investor, given initial wealth $x > 0$, is to select a portfolio process φ from the admissible set $\mathcal{A}(x)$ (as in Definition 2.14) that maximizes his/her expected utility from the terminal wealth $X(T)$. We will restrict ourselves to the power utility function with a risk aversion parameter γ:

$$U(x) = \frac{x^\gamma}{\gamma} \text{ for } \gamma \in (-\infty, 1) \setminus \{0\} \text{ and } x > 0. \tag{3.6}$$

Now, the dynamic optimization problem is formulated mathematically as

$$\begin{aligned} &\max_{\varphi \in \tilde{\mathcal{A}}(x)} \mathbb{E}\left[\frac{(X(T))^\gamma}{\gamma}\right], \\ \text{where} \quad &\tilde{\mathcal{A}}(x) = \left\{\varphi \in \mathcal{A}(x) \mid \mathbb{E}\left[\left(\frac{(X(T))^\gamma}{\gamma}\right)^-\right] < \infty\right\}. \end{aligned} \tag{3.7}$$

Without any additional restriction, for instance risk constraints, the portfolio optimization problem (3.7) in the continuous-time market model was first solved by Robert C. Merton in [48] using a stochastic control approach. That is why we call it here Merton optimization problem. The martingale approach for solving the continuous-time portfolio optimization problem was introduced by Pliska (see [54]), Karatzas, Lehoczky and Shreve (see [40]), and Cox and Huang (see [10]). We solve Problem (3.7) using the martingale approach which consists of a static optimization and a representation problem as explained in Chapter 2.

Static optimization problem

The static optimization problem can be stated as follows:

$$\begin{aligned} &\max_{X(T) \in \mathcal{B}(x)} \mathbb{E}\left[\frac{(X(T))^\gamma}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}[\beta(T)X(T)] \le x \quad \text{(budget constraint)}, \end{aligned} \tag{3.8}$$

where

$$\mathcal{B}(x) := \left\{ X(T) > 0 \mid X(T)\ \mathcal{F}_T\text{-measurable}, \mathbb{E}\left[\left(\frac{(X(T))^{\gamma}}{\gamma}\right)^{-}\right] < \infty, X(0) = x \right\}.$$

Theorem 3.2 *Consider the utility function U as given in (3.6) and assume that*

$$\max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[U(X(T))\right] < \infty \text{ for all } x \in (0,\infty). \tag{3.9}$$

Then the solution for the Optimization Problem (3.8) is given by

$$X^*(T) = I(y^* H(T)), \tag{3.10}$$

where $y^* \in \mathbb{R}$ is obtained through $x = \mathbb{E}[H(T) I(yH(T))]$ with $I(x) = x^{\frac{1}{\gamma-1}}$.

Proof: We can use Theorem 2.16 since we have the assumption (3.9). The existence and uniqueness of the solution are guaranteed by Theorem 2.16. We prove here only the optimality of the solution. From Problem (3.8) and the definition of $X^*(T)$ we have

$$\begin{aligned}\mathbb{E}[H(T) X^*(T)] &= \mathbb{E}[H(T) I(yH(T))] \\ &= \mathbb{E}\left[H(T)(yH(T))^{\frac{1}{\gamma-1}}\right] = x.\end{aligned} \tag{3.11}$$

Let $X(T) \in \mathcal{B}(x)$ be arbitrary. Then

$$\begin{aligned}\mathbb{E}[U(X^*(T))] &\overset{1}{=} \mathbb{E}[U(I(y^* H(T)))] \\ &\overset{2}{\geq} \mathbb{E}\left[U(X(T)) + U'(I(yH(T)))(I(yH(T)) - X(T))\right] \\ &\overset{3}{=} \mathbb{E}[U(X(T)) + yH(T)(I(yH(T)) - X(T))] \\ &\overset{4}{=} \mathbb{E}[U(X(T))] + y\{\mathbb{E}[H(T) I(yH(T))] - \mathbb{E}[H(T) X(T)]\} \\ &\overset{5}{=} \mathbb{E}[U(X(T))] + \underbrace{y}_{>0}\underbrace{\{x - \mathbb{E}[H(T) X(T)]\}}_{\geq 0} \\ &\overset{6}{\geq} \mathbb{E}[U(X(T))].\end{aligned}$$

The equalities 1, 3, 4 and 5 follow from the form of X^* in (3.10), $U'(I(x)) = x$, linearity of expectation and Equation (3.11), respectively. The inequalities 2 and

6 are result of Equation (2.18) and the budget constraint, respectively. □

Representation problem

For the representation problem we have to find the trading strategy that generates the optimal payoff $X^*(T)$.

Theorem 3.3 *Suppose all the conditions of Theorem 3.2 are fulfilled. Then:*

(a) The optimal wealth $X^*(t)$ at any time $t \in [0,T]$ is given by

$$X^*(t) = (y^*)^{\frac{1}{\gamma-1}} (H(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)},$$

where y^* is given as in Theorem 3.2 and

$$M^H(t,T) = \left(r + \frac{1}{2}\theta^2\right)(T-t),$$
$$V^H(t,T) = \left(\frac{1}{2}\theta^2\right)(T-t).$$

(b) The portfolio process $\varphi^*(t) \in \tilde{\mathcal{A}}(x)$ at any time $t \in [0,T]$ that replicates the optimal wealth $X^*(t)$ is given by

$$\varphi^*(t) = (\varphi^*_S(t), \varphi^*_M(t)),$$
$$\textit{where} \quad \varphi^*_S(t) = \frac{1}{1-\gamma}\frac{\theta}{\sigma_S} \quad \textit{and} \quad \varphi^*_M(t) = 1 - \varphi^*_S(t).$$

Proof: See for example Ocone & Karatzas [53]. □

3.2. Optimization under bounded expected shortfall risk

In this section we consider an investor whose aim, given initial wealth x, is to find a portfolio process $\varphi \in \tilde{\mathcal{A}}(x)$ that maximizes his/her utility function (power utility function as defined in (3.6)) from terminal wealth in the presence of both a budget and a risk constraint. In particular, the risk quantity is measured here by present expected loss, i.e.

$$\rho\left(X\left(T\right)-q\right)=\mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)\left(X\left(T\right)-q\right)^{-}\right]\leq\delta, \tag{3.12}$$

where $q > 0$ is a pre-specified benchmark or a shortfall level and $\delta \geq 0$ is a bound for the present expected loss. Then, the dynamic optimization problem reads as

$$\begin{aligned}
&\max_{\varphi\in\tilde{\mathcal{A}}(x)} \mathbb{E}\left[\frac{\left(X\left(T\right)\right)^{\gamma}}{\gamma}\right]\\
&\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)X\left(T\right)\right]\leq x \quad \text{(budget constraint)}\\
&\qquad \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)\left(X\left(T\right)-q\right)^{-}\right]\leq\delta \quad \text{(risk constraint)}\,,
\end{aligned} \tag{3.13}$$

where $\tilde{\mathcal{A}}\left(x\right)$ is as defined in (2.10). The dynamic problem (3.13) can be as well splitted into the static and the representation problem.

Static optimization problem

The static optimization problem corresponding to the dynamic problem (3.13) is stated as

$$\begin{aligned}
&\max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{\left(X\left(T\right)\right)^{\gamma}}{\gamma}\right]\\
&\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)X\left(T\right)\right]\leq x\\
&\qquad \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)\left(X\left(T\right)-q\right)^{-}\right]\leq\delta,
\end{aligned} \tag{3.14}$$

where $\mathcal{B}\left(x\right)$ is as defined in (2.12).

Remark 3.4 *If $\delta=\infty$, it means the risk constraint has no bound, in that case we are faced with the Merton optimization problem, as studied in Section 3.1. If $\delta=0$, the risk constraint corresponds to $X\left(T\right)\geq q$. Then the optimization problem (3.14) is transformed to*

$$\begin{aligned}
&\max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{\left(X\left(T\right)\right)^{\gamma}}{\gamma}\right]\\
&s.t.\ \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)X\left(T\right)\right]\leq x\\
&\qquad X\left(T\right)\geq q.
\end{aligned} \tag{3.15}$$

This is a so-called portfolio insurer problem.

The minimum value of the risk bound $\underline{\delta}$ such that the risk constraint can be satisfied was studied by Gundel and Weber (see [29]) and the maximum value of the risk bound $\overline{\delta}$ such that the risk constraint holds with equality is given by the risk of the Merton portfolio.

Theorem 3.5 *(a) If* $\delta \in \left(\underline{\delta}, \overline{\delta}\right)$*, then the solution to the optimization problem (3.14) is given by*

$$X^*(T) = f\left(y_1^* H(T), y_2^* H(T)\right),$$

where

$$f(x_1, x_2) = \begin{cases} (x_1)^{\frac{1}{\gamma-1}} & \text{, for } x_1 \leq q^{\gamma-1} \\ q & \text{, for } q^{\gamma-1} < x_1 \leq q^{\gamma-1} + x_2 \\ (x_1 - x_2)^{\frac{1}{\gamma-1}} & \text{, for } x_1 > q^{\gamma-1} + x_2, \end{cases}$$

and $y_1^*, y_2^* > 0$ *solve the system of equations*

$$\begin{aligned} &\mathbb{E}\left[H(T) f\left(y_1^* H(T), y_2^* H(T)\right)\right] = x \\ &\mathbb{E}\left[H(T)\left(f\left(y_1^* H(T), y_2^* H(T)\right) - q\right)^-\right] = \delta. \end{aligned} \tag{3.16}$$

The optimal terminal wealth X^* *can be rewritten as*

$$\begin{aligned} X^*(T) = &\left(y_1^* H(T)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_{\left\{H(T) \leq \frac{q^{\gamma-1}}{y_1^*}\right\}} + q\mathbb{1}_{\left\{\frac{q^{\gamma-1}}{y_1^*} < H(T) \leq \frac{q^{\gamma-1}}{y_1^* - y_2^*}\right\}} \\ &+ \left(\left(y_1^* - y_2^*\right) H(T)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_{\left\{H(T) > \frac{q^{\gamma-1}}{y_1^* - y_2^*}\right\}}. \end{aligned}$$

(b) If $\delta = \underline{\delta}$ *and* $q\mathbb{E}[H(T)] \leq x$ *it holds* $\underline{\delta} = 0$*. Then the optimal terminal wealth to problem (3.14) is given by*

$$X^{PI}(T) = f_{PI}\left(y^{PI} H(T)\right)$$

where

$$f_{PI}(x) = \begin{cases} (x)^{\frac{1}{\gamma-1}} & \text{, for } x \leq q^{\gamma-1} \\ q & \text{, for } x > q^{\gamma-1} \end{cases}$$

and y^{PI} *solves the equation* $\mathbb{E}\left[H(T) f_{PI}(yH(T))\right] = x$.

Proof: See [25]. □

Representation problem

We have now to derive the trading strategies $\varphi^*(t) = (\varphi_S^*(t), \varphi_M^*(t))^\top$ and $\varphi^{PI}(t) = \left(\varphi_S^{PI}(t), \varphi_M^{PI}(t)\right)^\top$ that replicate the optimal terminal wealth $X^*(T)$ and $X^{PI}(T)$, respectively, from Theorem 3.5. This is done in the following theorem.

Theorem 3.6 *Consider all the conditions of Theorem 3.5. Then:*

(a) The optimal wealth $X^*(t)$ and $X^{PI}(t)$ at any time $t \in [0, T]$ are given by

$$\begin{aligned} &X^*(t) \\ &= (y_1^*)^{\frac{1}{\gamma-1}} (H(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} * \Phi\left(d_1^\delta(t)\right) \\ &+ e^{-M^H(t,T) + \frac{1}{2} V^H(t,T)} * \left(\Phi\left(d_2^\delta(t)\right) - \Phi\left(d_3^\delta(t)\right)\right) \\ &+ (y_1^* - y_2^*)^{\frac{1}{\gamma-1}} (H(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} * \Phi\left(d_4^\delta(t)\right) \end{aligned}$$

and

$$\begin{aligned} &X^{PI}(t) \\ &= \left(y_1^{PI}\right)^{\frac{1}{\gamma-1}} (H(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} * \Phi\left(d_1^{PI}(t)\right) \\ &+ e^{-M^H(t,T) + \frac{1}{2} V^H(t,T)} * \Phi\left(d_2^{PI}(t)\right), \end{aligned}$$

where

$$
\begin{aligned}
d_1^{\delta}(t) &= \frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^H(t,T)}, \\
d_2^{\delta}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \sqrt{V^H(t,T)}, \\
d_3^{\delta}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right) H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \sqrt{V^H(t,T)}, \\
d_4^{\delta}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right) H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} - \frac{\gamma}{1-\gamma}\sqrt{V^H(t,T)}, \\
d_1^{PI}(t) &= \frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI} H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^H(t,T)}, \\
d_2^{PI}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI} H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \sqrt{V^H(t,T)}.
\end{aligned}
\tag{3.17}
$$

$M^H(t,T)$ and $V^H(t,T)$ are as in Theorem 3.3, and $\Phi(\cdot)$ is the standard-normal probability distribution function.

(b) The trading strategies that replicate the optimal wealth $X^*(t)$ and $X^{PI}(t)$ are given by $\varphi^*(t) = \left(\varphi_S^*(t), \varphi_M^*(t)\right)^\top$ and $\varphi^{PI}(t) = \left(\varphi_S^{PI}(t), \varphi_M^{PI}(t)\right)^\top$, respectively. Hereby

$$
\begin{aligned}
\varphi_S^*(t) &= \frac{\theta}{(1-\gamma)\sigma_S} + \frac{\theta q e^{-M^H(t,T)+\frac{1}{2}V^H(t,T)} * \left(\Phi\left(d_2^{\delta}(t)\right)\right)}{(\gamma-1) X^*(t)\sigma_S} \\
&\quad - \frac{\theta q e^{-M^H(t,T)+\frac{1}{2}V^H(t,T)} * \left(\Phi\left(d_3^{\delta}(t)\right)\right)}{(\gamma-1) X^*(t)\sigma_S}, \\
\varphi_M^*(t) &= 1 - \varphi_S^*(t), \\
\varphi_S^{PI}(t) &= \frac{\theta}{(1-\gamma)\sigma_S} + \frac{\theta q e^{-M^H(t,T)+\frac{1}{2}V^H(t,T)} * \Phi\left(d_2^{PI}(t)\right)}{(\gamma-1) X^{PI}(t)\sigma_S}, \\
\varphi_M^{PI}(t) &= 1 - \varphi_S^{PI}(t).
\end{aligned}
$$

Proof: See [6] or [23]. □

4. Portfolio optimization with a one-factor Vasiček model

In the previous chapter we discussed the portfolio optimization problem under deterministic interest rates. This has the advantage that closed form solutions can be easily obtained. However, the assumption of deterministic interest rates is far from the real world, especially for long term investments. In this chapter we analyze portfolio optimization problems in which the interest rate dynamics of the economy is described by a one-factor Vasiček model. The presentation of the problem with one-factor Vasiček term structure without risk constraints and in a complete market goes back to Hainaut (see [30]) but we use Malliavin calculus to derive the optimal trading strategy of the investor. In this chapter we study also the portfolio optimization problem without risk constraints in an incomplete market. Furthermore, we investigate the problem of portfolio optimization in the presence of risk constraints in both a complete and an incomplete market.

Financial market models

We consider an arbitrage free financial market where a money market account M (a saving's account), a (zero) coupon bond B and a stock S are continuously traded over a fixed finite time-horizon $[0,T]$. The uncertainty in this market is modeled by a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$. The augmented filtration generated by a 2-dimensional standard Brownian motion: $W(t) = \left(W_r^{\mathbb{P}}(t), W_S^{\mathbb{P}}(t)\right)^\top$ under the probability measure $\mathbb{P}$ is denoted by $\mathcal{F}$.

The dynamics of these securities at time $t \in [0,T]$ are modeled by the following stochastic differential equations (SDEs):

$$dM(t) = M(t)\, r(t)\, dt \tag{4.1}$$

$$dB\left(t,\widetilde{T}\right) = B\left(t,\widetilde{T}\right)\left[\mu_B\left(t,\widetilde{T}\right) dt + \sigma_B\left(t,\widetilde{T}\right) dW_r^{\mathbb{P}}(t)\right] \tag{4.2}$$

$$dS(t) = S(t)\left[\mu_S(t)\, dt + \sigma_{rS} dW_r^{\mathbb{P}}(t) + \sigma_S dW_S^{\mathbb{P}}(t)\right], \tag{4.3}$$

where:

- $r(t)$ is the interest rate or the return of the money market account,
- $B\left(t,\widetilde{T}\right)$ is the bond price at time $t \in [0,T]$ with maturity $\widetilde{T} > T$,
- $\mu_B\left(t,\widetilde{T}\right)$ is the drift of the bond price at time $t \in [0,T]$ and from the Assumption 2.2 $\mu_B\left(t,\widetilde{T}\right) = \bar{\mu}_B + r(t)$, where $\bar{\mu}_B$ is a constant.
- $\widetilde{\sigma}_B(t) = \sigma_B\left(t,\widetilde{T}\right) > 0$ is the volatility of the bond price at time $t \in [0,T]$,
- $\mu_S(t)$ is the drift of the stock price at time $t \in [0,T]$ and again from the Assumption 2.2 $\mu_S(t) = \bar{\mu}_S + r(t)$, where $\bar{\mu}_S$ is a constant.
- $\sigma_S > 0$ is the volatility of the stock price and it is considered to be constant,
- σ_{rS} is the correlation of the stock price and the interest rate, and it is also considered to be constant.

The dynamics of the interest rates $r(t)$ are assumed to be stochastic and we consider in particular, throughout this chapter, the one-factor Vasiček model whose dynamics are characterized by the following SDE:

$$dr(t) = a\left(b^{\mathbb{P}} - r(t)\right)dt + \sigma_r dW_r^{\mathbb{P}}(t), \tag{4.4}$$

where the constants a, $b^{\mathbb{P}}$, σ_r are the speed of the mean reversion, the level of mean reversion with respect to the probability measure $\mathbb{P}$ and the volatility of the interest rate $r(t)$, respectively.

We can rewrite the dynamics of the bond and stock prices in the matrix form as

$$d\begin{pmatrix} B\left(t,\widetilde{T}\right) \\ S(t) \end{pmatrix} = \begin{pmatrix} \mu_B\left(t,\widetilde{T}\right) \\ \mu_S(t) \end{pmatrix} dt + \underbrace{\begin{pmatrix} \sigma_B\left(t,\widetilde{T}\right) & 0 \\ \sigma_{Sr} & \sigma_S \end{pmatrix}}_{=:A} \begin{pmatrix} dW_r^{\mathbb{P}}(t) \\ dW_S^{\mathbb{P}}(t) \end{pmatrix}. \tag{4.5}$$

Therefore

$$A^{-1} = \begin{pmatrix} \frac{1}{\widetilde{\sigma}_B} & 0 \\ -\frac{\sigma_{Sr}}{\widetilde{\sigma}_B \sigma_S} & \frac{1}{\sigma_S} \end{pmatrix} \text{ with } \widetilde{\sigma}_B = \widetilde{\sigma}_B(t). \tag{4.6}$$

Remark 4.1 *Note that the number of uncertainty is equal to the number of risky assets. Therefore, we are still in a complete market.*

Due to the absence of arbitrage and completeness of this financial market, there exists a unique equivalent martingale measure with respect to $\mathbb{P}$ denoted by $\mathbb{Q}$, i.e. under $\mathbb{Q}$ the discounted prices of securities are martingales, and characterized by

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = Z^{\Theta}(t) = \exp\left(-\frac{1}{2}\int_0^t \|\Theta(u)\|^2 du - \int_0^t \Theta^{\top}(u)\, dW^{\mathbb{P}}(u)\right), \quad (4.7)$$

where the vector $\Theta(t) = (\theta_r(t), \theta_S(t))^{\top}$ is composed of the market price of risk of the bond $(\theta_r(t))$ and the market price of risk of the stock $(\theta_S(t))$, which are given by

$$\begin{aligned} \Theta(t) = \begin{pmatrix} \theta_r(t) \\ \theta_S(t) \end{pmatrix} &= A^{-1}\begin{pmatrix} \mu_B\left(t,\widetilde{T}\right) - r(t) \\ \mu_S(t) - r(t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\mu_B\left(t,\widetilde{T}\right)-r(t)}{\widetilde{\sigma}_B} \\ \frac{\mu_S(t)-r(t)}{\sigma_S} - \frac{\sigma_{Sr}}{\sigma_S}\left(\frac{\mu_B\left(t,\widetilde{T}\right)-r(t)}{\widetilde{\sigma}_B}\right) \end{pmatrix} = \begin{pmatrix} \frac{\bar{\mu}_B}{\widetilde{\sigma}_B} \\ \frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}}{\sigma_S}\left(\frac{\bar{\mu}_B}{\widetilde{\sigma}_B}\right) \end{pmatrix}. \end{aligned} \quad (4.8)$$

Let us define

$$\lambda_S := \frac{\mu_S(t) - r(t)}{\sigma_S} = \frac{\bar{\mu}_S}{\sigma_S}. \quad (4.9)$$

The processes

$$W_r^{\mathbb{Q}}(t) := W_r^{\mathbb{P}}(t) + \int_0^t \theta_r(u)\, du, \text{ for } t \in [0,T] \quad (4.10)$$

$$W_S^{\mathbb{Q}}(t) := W_S^{\mathbb{P}}(t) + \int_0^t \theta_S(u)\, du, \text{ for } t \in [0,T], \quad (4.11)$$

from the Girsanov Theorem, are Brownian motions with respect to the probability measure $\mathbb{Q}$. Therefore, the dynamics of $r(t)$, $B\left(t,\widetilde{T}\right)$ and $S(t)$ in terms of Brownian motions $W_r^{\mathbb{Q}}$ and $W_S^{\mathbb{Q}}$ are then given by:

$$\begin{aligned} dr(t) &= a\left(b^{\mathbb{P}} - \sigma_r\frac{\theta_r(t)}{a} - r(t)\right)dt + \sigma_r\left(dW_r^{\mathbb{P}}(t) + \theta_r(t)\,dt\right) \\ &= a\left(b^{\mathbb{Q}}(t) - r(t)\right)dt + \sigma_r dW_r^{\mathbb{Q}}(t), \end{aligned} \quad (4.12)$$

where $b^{\mathbb{Q}}(t) = b^{\mathbb{P}} - \sigma_r \frac{\theta_r(t)}{a}$ is the mean reversion of $r(t)$ with respect to the probability measure $\mathbb{Q}$ and will be denoted by $b(t)$ in order to ease the notations. That means we have a Hull-White model under $\mathbb{Q}$ (extended Vasiček with a and σ_r being constants)

$$\begin{aligned}\frac{dB\left(t,\widetilde{T}\right)}{B\left(t,\widetilde{T}\right)} &= \mu_B\left(t,\widetilde{T}\right)dt + \sigma_B\left(t,\widetilde{T}\right)dW_r^{\mathbb{P}}(t)\\ &= \mu_B\left(t,\widetilde{T}\right)dt + \sigma_B\left(t,\widetilde{T}\right)d\left(W_r^{\mathbb{Q}}(t) - \int_0^t \theta_r(u)\,du\right)\\ &= r(t)\,dt + \sigma_B\left(t,\widetilde{T}\right)dW_r^{\mathbb{Q}}(t)\end{aligned} \tag{4.13}$$

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \mu_S(t)\,dt + \sigma_{Sr}dW_r^{\mathbb{P}}(t) + \sigma_S dW_S^{\mathbb{P}}(t)\\ &= \mu_S(t)\,dt + \sigma_{Sr}d\left(W_r^{\mathbb{Q}}(t) - \int_0^t \theta_r(u)\,du\right)\\ &+ \sigma_S d\left(W_S^{\mathbb{Q}}(t) - \int_0^t \lambda_S(u)\,du + \frac{\sigma_{Sr}}{\sigma_S}\int_0^t \theta_r(u)\,du\right)\\ &= r(t)\,dt + \sigma_{Sr}dW_r^{\mathbb{Q}}(t) + \sigma_S dW_S^{\mathbb{Q}}(t)\,.\end{aligned} \tag{4.14}$$

Remark 4.2 *(a) Following Musiela and Rutkowski [50] the bond price volatility in the Vasiček model is a deterministic, bounded and continuous function* $\widetilde{\sigma}_B = \widetilde{\sigma}_B\left(\cdot,\widetilde{T}\right) : [0,T] \to \mathbb{R}$ *which is defined by*

$$\sigma_B\left(t,\widetilde{T}\right) := -\sigma_r n\left(t,\widetilde{T}\right),$$

where

$$n\left(t,\widetilde{T}\right) = \frac{1}{a}\left(1 - e^{-a\left(\widetilde{T}-t\right)}\right).$$

Therefore, the dynamics of the bond price under the probability measure $\mathbb{Q}$ *in Equation (4.13) are given by*

$$dB\left(t,\widetilde{T}\right) = B\left(t,\widetilde{T}\right)\left(r(t)\,dt - \sigma_r n\left(t,\widetilde{T}\right)dW_r^{\mathbb{Q}}(t)\right).$$

(b) The market price of risk is also a deterministic, bounded and continuous function $\Theta : [0,T] \to \mathbb{R}^2$ *which is given in (4.8).*

Remark 4.3 *The stochastic differential equation (4.15) that characterizes the dynamics of the stochastic interest rate under the probability measure* $\mathbb{P}$ *in the Vasiček model:*

$$dr(t) = a\left(b^{\mathbb{P}} - r(t)\right)dt + \sigma_r dW_r^{\mathbb{P}}(t) \tag{4.15}$$

has the unique solution

$$r(t) = r(0)e^{-at} + b^{\mathbb{P}}\left(1 - e^{-at}\right) + e^{-at}\sigma_r \int_0^t e^{au} dW_r^{\mathbb{P}}(u). \tag{4.16}$$

We can define a unique deflator $H(t,s)$ at time t, for a payment at time $s \geq t$ as

$$\begin{aligned} H(t,s) :=& \frac{\beta(s)}{\beta(t)} \frac{Z^{\Theta}(s)}{Z^{\Theta}(t)} \\ =& \exp\left(-\int_t^s r(u)\,du - \frac{1}{2}\int_t^s \|\Theta(u)\|^2 du - \int_t^s \Theta^{\top}(u)\,dW^{\mathbb{P}}(u)\right), \end{aligned} \tag{4.17}$$

with β as defined in (2.3) and Z^{Θ} as in (4.7). For $t = 0$ we write

$$H(s) := \exp\left(-\int_0^s r(u)\,du - \frac{1}{2}\int_0^s \|\Theta(u)\|^2 du - \int_0^s \Theta^{\top}(u)\,dW^{\mathbb{P}}(u)\right). \tag{4.18}$$

Let us now consider an investor who initially has the wealth $x > 0$. We denote the fraction of wealth invested in the money market account, bond and stock at time $t \in [0,T]$ by the vector $\varphi(t) = (\varphi_M(t), \varphi_B(t), \varphi_S(t))^{\top}$, with $\varphi_M(t)$ for the money market account, φ_B for the bond and φ_S for the stock. According to Assumptions 2.3 it follows that the fraction of wealth invested in the money market account is given by $(1 - \varphi_B(t) - \varphi_S(t))$.

The wealth process of an investor at time $t \in [0,T]$, denoted here also by $X(t)$, under the probability measure $\mathbb{Q}$ (using the Equations (4.1), (4.2) and (4.14)) follows the SDE:

$$\begin{aligned}
dX(t) =& dX^M(t) + dX^B(t) + dX^S(t) \\
=& (1 - \varphi_B(t) - \varphi_S(t)) X(t) r(t) dt \\
& + X(t) \varphi_B(t) \left(r(t) dt - \sigma_r n\left(t, \widetilde{T}\right) dW_r^{\mathbb{Q}}(t) \right) \\
& + X(t) \varphi_S(t) \left(r(t) dt + \sigma_{Sr} dW_r^{\mathbb{Q}}(t) + \sigma_S dW_S^{\mathbb{Q}}(t) \right) \\
=& X(t) r(t) dt - \varphi_B(t) X(t) r(t) dt - \varphi_S(t) X(t) r(t) dt \\
& + X(t) \varphi_B(t) r(t) dt - X(t) \varphi_B(t) \sigma_r n\left(t, \widetilde{T}\right) dW_r^{\mathbb{Q}}(t) \\
& + X(t) \varphi_S(t) r(t) dt + X(t) \varphi_S(t) \sigma_{Sr} dW_r^{\mathbb{Q}}(t) \\
& + X(t) \varphi_S(t) \sigma_S dW_s^{\mathbb{Q}}(t) \\
=& X(t) \left[r(t) dt + \varphi_S(t) \sigma_S dW_S^{\mathbb{Q}}(t) \right. \\
& \left. + \left(\varphi_S(t) \sigma_{Sr} - \varphi_B(t) \sigma_r n\left(t, \widetilde{T}\right) \right) dW_r^{\mathbb{Q}}(t) \right],
\end{aligned} \tag{4.19}$$

where X^M, X^B and X^S refer to the wealth invested in the money market account, the bond and the stock, respectively. Applying the same procedures as used for (2.8) and (2.9) on the process $\beta(t) X(t)$ gives

$$\begin{aligned}
d(\beta(t) X(t)) =& \beta(t) X(t) \left(\varphi_S(t) \sigma_{Sr} - \varphi_B(t) \sigma_r n\left(t, \widetilde{T}\right) \right) dW_r^{\mathbb{Q}}(t) \\
& + \beta(t) X(t) \varphi_S(t) \sigma_S dW_S^{\mathbb{Q}}(t)
\end{aligned} \tag{4.20}$$

and thus

$$\begin{aligned}
\beta(T) X(T) =& X(0) + \int_0^T \beta(t) X(t) \left(\varphi_S(t) \sigma_{Sr} - \varphi_B(t) \sigma_r n\left(t, \widetilde{T}\right) \right) dW_r^{\mathbb{Q}}(t) \\
& + \int_0^T \beta(t) X(t) \varphi_S(t) \sigma_S dW_S^{\mathbb{Q}}(t).
\end{aligned} \tag{4.21}$$

4.1. Optimization without risk constraints in a complete market

In this section we focus on the portfolio optimization problem without risk constraints as studied in Section 3.1, but under the one-factor Vasiček model of

interest rates. The dynamic and the static optimization problems in these settings are stated as in Section 3.1.
The dynamic optimization problem is stated as in (3.7):

$$\begin{aligned} &\max_{\varphi \in \tilde{\mathcal{A}}(x)} \mathbb{E}\left[\frac{(X(T))^{\gamma}}{\gamma}\right], \\ &\text{where} \quad \tilde{\mathcal{A}}(x) = \left\{\varphi \in \mathcal{A}(x) \mid \mathbb{E}\left[\left(\frac{(X(T))^{\gamma}}{\gamma}\right)^{-}\right] < \infty\right\}. \end{aligned} \tag{4.22}$$

The corresponding static optimization problem is stated as

$$\begin{aligned} &\max_{X(T) \in \mathcal{B}(x)} \mathbb{E}\left[\frac{(X(T))^{\gamma}}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta(T) X(T)\right] \leq x \quad (\text{budget constraint}), \end{aligned} \tag{4.23}$$

where

$$\mathcal{B}(x) := \\ \left\{X(T) > 0 \mid X(T) \ \mathcal{F}_T\text{-measurable}, \mathbb{E}\left(\frac{(X(T))^{\gamma}}{\gamma}\right)^{-} < \infty, X(0) = x\right\}.$$

The following theorem characterizes the optimal terminal wealth, denoted by $X^*(T)$, and the trading strategy, denoted by the vector $\varphi^*(t) = (\varphi_M^*(t), \varphi_B^*(t), \varphi_S^*(t))^{\top}$, that generates $X^*(T)$.

Theorem 4.4 *(i) The solution to the static optimization problem (4.23) is given by*

$$X^*(T) = (y^* H(T))^{\frac{1}{\gamma-1}},$$

with y^* obtained through

$$\mathbb{E}\left[y^{\frac{1}{\gamma-1}} (H(T))^{\frac{\gamma}{\gamma-1}}\right] = x, \ \text{i.e.} \ (y^*)^{\frac{1}{\gamma-1}} = \frac{x}{\mathbb{E}\left[(H(T))^{\frac{\gamma}{\gamma-1}}\right]}.$$

(ii) The optimal wealth $X^*(t)$ at any time $t \in [0, T]$ is then

$$X^*(t) = (y^*)^{\frac{1}{\gamma-1}} (H(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)}.$$

Here,

$$
\begin{aligned}
M^{H}(t,T) &= \int_t^T \left(r(t)\, e^{-a(s-t)} + b^{\mathbb{P}} \left(1 - e^{-a(s-t)}\right) + \frac{1}{2} \left\| \Theta(s) \right\|^2 \right) ds, \\
V^{H}(t,T) &= \int_t^T \left[\left(\sigma_r n(s,T) + \theta_r(s)\right)^2 + \theta_S^2(s) \right] ds.
\end{aligned}
\tag{4.24}
$$

(iii) The trading strategy that replicates the optimal terminal wealth $X^*(t)$ is given by

$$
\begin{aligned}
\varphi^*(t) &= \left(\varphi_M^*(t), \varphi_B^*(t), \varphi_S^*(t)\right)^\top, \text{ where} \\
\varphi_B^*(t) &= \frac{1}{1-\gamma} \frac{1}{\sigma_r n\left(t, \widetilde{T}\right)} \left(\frac{\theta_S(t)\, \sigma_{Sr}}{\sigma_S} - \gamma \sigma_r n(t,T) - \theta_r(t) \right), \\
\varphi_S^*(t) &= \frac{1}{1-\gamma} \frac{\theta_S(t)}{\sigma_S} \quad \text{and} \\
\varphi_M^*(t) &= 1 - \varphi_B^*(t) - \varphi_S^*(t).
\end{aligned}
\tag{4.25}
$$

Proof:

(i) From Theorem 3.2 it suffices to verify that $\mathbb{E}\left[U\left(X^*(T)\right)\right] < \infty$, for all $x \in (0,\infty)$.

$$
\begin{aligned}
\mathbb{E}\left[U\left(X^*(T)\right)\right] &= \mathbb{E}\left[\frac{\left(X^*(T)\right)^\gamma}{\gamma}\right] \\
&= \mathbb{E}\left[\frac{\left(y^* H(T)\right)^{\frac{\gamma}{\gamma-1}}}{\gamma}\right] \\
&= \frac{y^*}{\gamma} \mathbb{E}^{\mathbb{Q}}\left[\beta(T) \left(y^* H(T)\right)^{\frac{1}{\gamma-1}}\right] \\
&= \frac{y^*}{\gamma} \mathbb{E}^{\mathbb{Q}}\left[\beta(T)\, X^*(T)\right] \\
&\overset{1}{\leq} \frac{y^*}{\gamma} x \overset{2}{<} \infty.
\end{aligned}
$$

The inequalities 1 and 2 hold because of the budget condition and $y^* \in (0,\infty)$, respectively.

(ii) From the Equations (4.7) and (4.21), Itô's lemma implies that the process $(H(t)\,X^*(t))_{t\in[0,T]}$ is an $\mathcal{F}(t)$-martingale, i.e.

$$H(t)\,X^*(t) = \mathbb{E}\left[H(T)\,X^*(T) \mid \mathcal{F}(t)\right]. \tag{4.26}$$

From (4.26) we have

$$\begin{aligned}
X^*(t) &= \mathbb{E}\left[\frac{H(T)}{H(t)} X^*(T) \mid \mathcal{F}(t)\right] \\
&= \mathbb{E}\left[\frac{H(T)}{H(t)}\left((y^* H(T))^{\frac{1}{\gamma-1}}\right) \mid \mathcal{F}(t)\right] \\
&= (y^*)^{\frac{1}{\gamma-1}}\,\mathbb{E}\left[\frac{(H(T))^{\frac{\gamma}{\gamma-1}}}{H(t)} \mid \mathcal{F}(t)\right] \\
&= (y^*)^{\frac{1}{\gamma-1}}\,\mathbb{E}\left[\frac{(H(t))^{\frac{\gamma}{\gamma-1}}}{H(t)}\left(\frac{H(T)}{H(t)}\right)^{\frac{\gamma}{\gamma-1}} \mid \mathcal{F}(t)\right] \\
&\overset{1}{=} (y^* H(t))^{\frac{1}{\gamma-1}}\,\mathbb{E}\left[\left(\frac{H(T)}{H(t)}\right)^{\frac{\gamma}{\gamma-1}}\right] \\
&\overset{2}{=} (y^*)^{\frac{1}{\gamma-1}}\,(H(t))^{\frac{1}{\gamma-1}}\,e^{\frac{\gamma}{1-\gamma}M^H(t,T)+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)}.
\end{aligned}$$

Equality 1 above holds because of the adaptedness and the Markov property of $(H(t))_{t\in[0,T]}$. From Equation (4.17) we know that

$$\frac{H(T)}{H(t)} = \exp\left(-\int_t^T r(u)\,du - \frac{1}{2}\int_t^T \|\Theta(u)\|^2\,du - \int_t^T \Theta^\top(u)\,dW^{\mathbb{P}}(u)\right).$$

By Remark 4.3 we get

$$\begin{aligned}
&\int_t^T r(s)\,ds \\
&= \int_t^T \left(r(t)\,e^{-a(s-t)} + b^{\mathbb{P}}\left(1-e^{-a(s-t)}\right)\right) ds + \int_t^T\int_t^s \sigma_r e^{-a(s-u)} dW_r^{\mathbb{P}}(u)\,ds \\
&= \int_t^T \left(r(t)\,e^{-a(s-t)} + b^{\mathbb{P}}\left(1-e^{-a(s-t)}\right)\right) ds + \sigma_r \int_t^T n(u,T)\,dW_r^{\mathbb{P}}(u),
\end{aligned} \tag{4.27}$$

where $n(u,T) = \int_u^T e^{-a(T-s)}ds = \frac{1-e^{-a(T-u)}}{a}$. The last equality of (4.27) follows from the Fubini theorem for stochastic integration (see Theorem 64 in [55]). Hence $\int_t^T r(s)\,ds$ is normally distributed and since $\Theta(t)$ is deterministic, it follows that

$$-\ln\left(\frac{H(T)}{H(t)}\right)\sim\mathcal{N}\left(M^{H}(t,T),\sqrt{V^{H}(t,T)}\right). \tag{4.28}$$

Hereby $M^{H}(t,T)$ and $V^{H}(t,T)$ are as given in (4.24).

(iii) We use Theorem A.7 and set $F=\beta(T)X^{*}(T)$ to compute the optimal trading strategy $\varphi^{*}(t)$ but we have to verify first that the process $\beta(T)Z^{\Theta}(T)X^{*}(T)\in\mathbb{D}_{1,1}$, where $\mathbb{D}_{1,1}$ is as in Definition A.2. Let us first show that $(r(t),\theta_r(t),\theta_S(t))^{\top}\in(\mathbb{D})^3$ ($\mathbb{D}$ is as defined in the Definition A.2) in order to use Proposition A.4 and the chain rule of Lemma A.5. From Remark 4.2 we know that $\theta_r(t),\theta_S(t)\in\mathbb{D}$. From Equation (4.4) we have

$$dr(t)=\underbrace{a\left(b^{\mathbb{Q}}(t)-r(t)\right)}_{=:\mu_r(t,r)}dt+\sigma_r dW_r^{\mathbb{Q}}(t). \tag{4.29}$$

Observe that μ_r and σ_r satisfy the conditions stated in Proposition A.4 and hence $r(t)\in\mathbb{D}$. From the Equation (4.16) we have

$$r(t)=r(0)e^{-at}+b^{\mathbb{Q}}(t)\left(1-e^{-at}\right)+e^{-at}\sigma_r\int_0^t e^{au}dW_r^{\mathbb{Q}}(u). \tag{4.30}$$

Using Theorem 3.18 and Corollary 3.19 in [18] yields

$$\begin{aligned}D_t r(s)&=D_t\left(r(0)e^{-as}+b^{\mathbb{Q}}(t)\left(1-e^{-as}\right)+\sigma_r\int_0^s e^{-a(s-u)}dW_r^{\mathbb{Q}}(u)\right)\\&=D_t\left(\sigma_r\int_0^s e^{-a(s-u)}dW_r^{\mathbb{Q}}(u)\right)\\&=\sigma_r\int_t^s D_t e^{-a(s-u)}dW_r^{\mathbb{Q}}(u)+\sigma_r e^{-a(s-t)}=\sigma_r e^{-a(s-t)},\end{aligned} \tag{4.31}$$

for $s\in(t,T]$. Using the same argument as above it follows that

$$\left(\int_0^T r(s)\,ds,\int_0^T\theta_r(s)\,dW_r^{\mathbb{Q}}(s),\int_0^T\theta_S(s)\,dW_S^{\mathbb{Q}}(s)\right)^{\top}\in(\mathbb{D})^3$$

with

$$\begin{aligned}D_t\int_0^T r(s)\,ds&=\int_t^T D_t r(s)\,ds=\int_t^T\sigma_r e^{-a(s-t)}ds=\sigma_r n(t,T),\\D_t\int_0^T\theta_r(s)\,dW_r^{\mathbb{Q}}(s)&=\int_t^T D_t\theta_r(s)\,dW_r^{\mathbb{Q}}(s)+\theta_r(t)=\theta_r(t)\ \text{ and}\\D_t\int_0^T\theta_S(s)\,dW_S^{\mathbb{Q}}(s)&=\int_t^T D_t\theta_S(s)\,dW_S^{\mathbb{Q}}(s)+\theta_S(t)=\theta_S(t).\end{aligned}$$

The chain rule in Lemma A.5 yields

$$\beta(T)\, Z^{\Theta}(T)\, X^*(T) = (y^*)^{\frac{1}{\gamma-1}} (H(T))^{\frac{\gamma}{\gamma-1}} \in \mathbb{D}_{1,1}.$$

Now we can use representation (A.17) of Theorem A.7 for the process $\beta(T)\, X^*(T)$.

$$\beta(T)\, X^*(T) = \mathbb{E}^{\mathbb{Q}}[\beta(T)\, X^*(T)] + \int_0^T \mathbb{E}^{\mathbb{Q}}[D_t(\beta(T)\, X^*(T)) \mid \mathcal{F}(t)]\, dW^{\mathbb{Q}}(t). \tag{4.32}$$

Since $D_t(\Theta(t)) = 0$ from the definition of the Malliavin derivative and $\Theta(t)$ being a deterministic function.

$$\begin{aligned} D_t(\beta(T)\, X^*(T)) =& \beta(T)\, D_t(X^*(T)) + X^*(T)\, D_t(\beta(T)) \\ =& \beta(T)\, D_t(X^*(T)) + X^*(T)\, \beta(T)\, D_t\left(-\int_0^T r(s)\, ds\right) \\ =& \beta(T)\, D_t(X^*(T)) + X^*(T)\, \beta(T)\, (-\sigma_r n(t,T)) \end{aligned} \tag{4.33}$$

from the product and chain rules. By using again Lemma A.5 gives

$$\begin{aligned} & D_t(X^*(T)) = D_t\left((y^* H(T))^{\frac{1}{\gamma-1}}\right) \\ &= \frac{1}{\gamma-1} (y^*)^{\frac{1}{\gamma-1}} D_t(H(T))\, (H(T))^{\frac{1}{\gamma-1}-1} \\ &= \frac{1}{\gamma-1} (y^* H(T))^{\frac{1}{\gamma-1}} \times \\ & \left(D_t\left(-\int_0^T r(s)\, ds e_1 + \frac{1}{2}\int_0^T \|\Theta(s)\|^2\, ds - \int_0^T \Theta^\top(s)\, dW^{\mathbb{Q}}(s)\right)\right) \\ &= \frac{1}{\gamma-1} (y^* H(T))^{\frac{1}{\gamma-1}} \left(-D_t\left(\int_0^T r(s)\, ds e_1\right) - \Theta^\top(t)\right) \\ &= \frac{1}{\gamma-1} (y^* H(T))^{\frac{1}{\gamma-1}} \left(-n(t,T)\, e_1 - \Theta^\top(t)\right) \end{aligned} \tag{4.34}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Combining all together we finally obtain

$$D_t(\beta(T)\, X^*(T)) = \frac{\gamma}{1-\gamma}\left(\sigma_r n(t,T)\, e_1 + \frac{1}{\gamma}\Theta^\top(t)\right) \beta(T)\, (y^* H(T))^{\frac{1}{\gamma-1}}. \tag{4.35}$$

Substituting the value of $D_t\left(\beta\left(T\right)X^*\left(T\right)\right)$ from the Equation (4.35) in the Equation (4.32) and comparing it with (4.21) gives the result.

□

4.2. Optimization with bounded expected shortfall risk in a complete market

In this section we focus on the portfolio optimization problem in the presence of risk constraints as discussed in Section 3.2, but we treat the problem under the one-factor Vasiček model of interest rates. Hainaut (see [30]) studied the same problem considering VaR as a risk measure. However, VaR measures the probability of losses and ignores the magnitude of losses. We consider here the expected loss, in particular present expected loss as presented in (3.12), which is an alternative of the risk measures that put the magnitude of losses into account. The dynamic optimization problem is stated as in (3.13):

$$\begin{aligned} &\max_{\varphi\in\tilde{\mathcal{A}}(x)} \mathbb{E}\left[\frac{\left(X\left(T\right)\right)^{\gamma}}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)X\left(T\right)\right] \leq x \quad \text{(budget constraint)} \\ &\qquad \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)\left(X\left(T\right)-q\right)^{-}\right] \leq \delta \quad \text{(risk constraint)}, \end{aligned} \tag{4.36}$$

where $\tilde{\mathcal{A}}\left(x\right)$ is as defined in (2.10).

The corresponding static problem reads as

$$\begin{aligned} &\max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{\left(X\left(T\right)\right)^{\gamma}}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)X\left(T\right)\right] \leq x \\ &\qquad \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)\left(X\left(T\right)-q\right)^{-}\right] \leq \delta, \end{aligned} \tag{4.37}$$

thereby $\mathcal{B}\left(x\right)$ is as defined in (2.12). The static portfolio insurer problem can be also stated as

$$\begin{aligned} &\max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{\left(X\left(T\right)\right)^{\gamma}}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)X\left(T\right)\right] \leq x \\ &\qquad X\left(T\right) \geq q, \end{aligned} \tag{4.38}$$

as given in Remark 3.4. Adding stochastic interest rates changes the deflator $H(T)$. But Theorem 3.5 remains valid since the optimal terminal wealth $X^*(T)$ is given as a function of H. The solutions to the problems (4.37) and (4.38) are restated in the next theorem.

Theorem 4.5 *Let q be a fixed benchmark and x be an initial wealth of an investor.*
(a) If $\delta \in (\underline{\delta}, \overline{\delta})$, then the solution to the Optimization problem (4.37) is given by

$$X^{\delta}(T) = f\left(y_1^* H(T), y_2^* H(T)\right), \tag{4.39}$$

where

$$f(x_1, x_2) = \begin{cases} (x_1)^{\frac{1}{\gamma-1}} & \text{, for } x_1 \leq q^{\gamma-1} \\ q & \text{, for } q^{\gamma-1} < x_1 \leq q^{\gamma-1} + x_2 \\ (x_1 - x_2)^{\frac{1}{\gamma-1}} & \text{, for } x_1 > q^{\gamma-1} + x_2, \end{cases}$$

and $y_1^*, y_2^* > 0$ solve the system of equations

$$\begin{aligned} &\mathbb{E}\left[H(T) f\left(y_1^* H(T), y_2^* H(T)\right)\right] = x \\ &\mathbb{E}\left[H(T)\left(f\left(y_1^* H(T), y_2^* H(T)\right) - q\right)^-\right] = \delta. \end{aligned} \tag{4.40}$$

(b) If $\delta = \underline{\delta}$ and $q\mathbb{E}[H(T)] \leq x$ it holds $\underline{\delta} = 0$. Then the optimal terminal wealth to the problem (4.38) is given by

$$X^{PI}(T) = f_{PI}\left(y^{PI} H(T)\right),$$

where

$$f_{PI}(x) = \begin{cases} (x)^{\frac{1}{\gamma-1}} & \text{, for } x \leq q^{\gamma-1} \\ q & \text{, for } x > q^{\gamma-1} \end{cases}$$

and y^{PI} solves the equation $\mathbb{E}\left[H(T) f_{PI}(yH(T))\right] = x$.

Proof: see [6]. □

Representation problem

We have now to find the trading strategies that generate the optimal terminal wealth $X^{\delta}(T)$ and $X^{PI}(T)$.

Theorem 4.6 *Suppose all the conditions of Theorem 4.5 are satisfied . Then:*

(a) The time-$t \in [0,T]$ optimal wealth $X^{\delta}(t)$ and $X^{PI}(t)$ are given by

$$
\begin{aligned}
&X^{\delta}(t) \\
&= (y_1^*)^{\frac{1}{\gamma-1}} (H(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} * \Phi\left(d_1^{\delta}(t)\right) \\
&+ q e^{-M^H(t,T) + \frac{1}{2} V^H(t,T)} * \left(\Phi\left(d_2^{\delta}(t)\right) - \Phi\left(d_3^{\delta}(t)\right)\right) \\
&+ (y_1^* - y_2^*)^{\frac{1}{\gamma-1}} (H(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} * \Phi\left(d_4^{\delta}(t)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&X^{PI}(t) \\
&= \left(y^{PI}\right)^{\frac{1}{\gamma-1}} (H(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} * \Phi\left(d_1^{PI}(t)\right) \\
&+ e^{-M^H(t,T) + \frac{1}{2} V^H(t,T)} * \Phi\left(d_2^{PI}(t)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
d_1^{\delta}(t) &= \frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^H(t,T)}, \\
d_2^{\delta}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \sqrt{V^H(t,T)}, \\
d_3^{\delta}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^* - y_2^*\right) H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \sqrt{V^H(t,T)}, \\
d_4^{\delta}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^* - y_2^*\right) H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} - \frac{\gamma}{1-\gamma}\sqrt{V^H(t,T)}, \\
d_1^{PI}(t) &= \frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI} H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^H(t,T)}, \\
d_2^{PI}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI} H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} + \sqrt{V^H(t,T)}.
\end{aligned}
$$

With $M^H(t,T)$ and $V^H(t,T)$ as in Theorem 4.4 and $\Phi(\cdot)$ is the standard-normal probability distribution function.

(b) The portfolio processes at any time $t \in [0,T]$ that replicate the optimal terminal wealth $X^{\delta}(t)$ and $X^{PI}(t)$ are given by

$$\varphi^{*}(t) = \left(\varphi^{*}_{M}(t), \varphi^{*}_{B}(t), \varphi^{*}_{S}(t)\right)^{\top} \quad \textit{and}$$
$$\varphi^{PI}(t) = \left(\varphi^{PI}_{M}(t), \varphi^{PI}_{B}(t), \varphi^{PI}_{S}(t)\right)^{\top},$$

respectively. Thereby

$$\begin{aligned}
\varphi^{*}_{B}(t) &= \frac{\gamma}{\gamma-1}\frac{1}{\sigma_r n\left(t,\widetilde{T}\right)}\left(\sigma_r n(t,T) + \frac{\theta_r}{\gamma}\right) + \frac{1}{1-\gamma}\frac{\left(\sigma_r n(t,T)+\theta_r\right)}{\sigma_r n\left(t,\widetilde{T}\right)}\times \\
&\quad \frac{q e^{-M^H(t,T)+\frac{1}{2}V^H(t,T)} * \left(\Phi\left(d_2^{\delta}(t)\right) - \Phi\left(d_3^{\delta}(t)\right)\right)}{X^{*}(t)}, \\
\varphi^{*}_{S}(t) &= \frac{1}{1-\gamma}\frac{\theta_S}{\sigma_S} + \frac{1}{\gamma-1}\frac{\theta_S q e^{-M^H(t,T)+\frac{1}{2}V^H(t,T)}}{\sigma_S}\times \\
&\quad \frac{\left(\Phi\left(d_2^{\delta}(t)\right) - \Phi\left(d_3^{\delta}(t)\right)\right)}{X^{*}(t)}, \\
\varphi^{*}_{M}(t) &= 1 - \varphi^{*}_{B}(t) - \varphi^{*}_{S}(t), \\
\varphi^{PI}_{B}(t) &= \frac{\gamma}{\gamma-1}\frac{1}{\sigma_r n\left(t,\widetilde{T}\right)}\left(\sigma_r n(t,T) + \frac{\theta_r}{\gamma}\right) + \frac{1}{1-\gamma}\frac{\left(\sigma_r n(t,T)+\theta_r\right)}{\sigma_r n\left(t,\widetilde{T}\right)}\times \\
&\quad \frac{q e^{-M^H(t,T)+\frac{1}{2}V^H(t,T)} * \Phi\left(d_2^{PI}(t)\right)}{X^{PI}(t)}, \\
\varphi^{PI}_{S}(t) &= \frac{1}{1-\gamma}\frac{\theta_S}{\sigma_S} + \frac{\theta_S}{\gamma-1} q e^{-M^H(t,T)+\frac{1}{2}V^H(t,T)}\times \\
&\quad \frac{\Phi\left(d_2^{PI}(t)\right)}{X^{PI}(t)} \qquad \textit{and} \\
\varphi^{PI}_{M}(t) &= 1 - \varphi^{PI}_{B}(t) - \varphi^{PI}_{S}(t).
\end{aligned} \tag{4.41}$$

Proof:

(a) We give the proof for $X^{\delta}(t)$, the proof for $X^{PI}(t)$ is similar and it is omitted here. Applying again Itô's lemma to the Equations (4.7) and (4.21) implies that the process $\left(H(t)X^{\delta}(t)\right)_{t\in[0,T]}$ is $\mathcal{F}(t)$-measurable, that means

$$H(t)X^{\delta}(t) = \mathbb{E}\left[H(T)X^{\delta}(T) \mid \mathcal{F}(t)\right]. \tag{4.42}$$

From (4.39) we can write $X^{\delta}(T)$ as

$$X^{\delta}(T) = \left(y_1^* H(T)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_A + q\mathbb{1}_B + \left(\left(y_1^* - y_2^*\right) H(T)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_C. \tag{4.43}$$

Hereby

$$\begin{aligned} &A := \left\{H(T) \leq \frac{q^{\gamma-1}}{y_1^*}\right\}, B := \left\{\frac{q^{\gamma-1}}{y_1^*} < H(T) \leq \frac{q^{\gamma-1}}{y_1^* - y_2^*}\right\} \text{ and} \\ &C := \left\{H(T) > \frac{q^{\gamma-1}}{y_1^* - y_2^*}\right\}. \end{aligned} \tag{4.44}$$

Hence

$$\begin{aligned} X^{\delta}(t) =& \mathbb{E}\left[\frac{H(T)}{H(t)} X^{\delta}(T) \mid \mathcal{F}(t)\right] \\ =& \mathbb{E}\left[\frac{H(T)}{H(t)} \left(y_1^* H(T)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_A \mid \mathcal{F}(t)\right] \\ &+ \mathbb{E}\left[\frac{H(T)}{H(t)} q\mathbb{1}_B \mid \mathcal{F}(t)\right] \\ &+ \mathbb{E}\left[\frac{H(T)}{H(t)} \left(\left(y_1^* - y_2^*\right) H(T)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_C \mid \mathcal{F}(t)\right]. \end{aligned} \tag{4.45}$$

We compute the conditional expectation of the first term of Equation (4.45) and the remaining two terms are calculated in Appendix B. We use the same idea as for the proof of Theorem 4.4 part (ii) of applying the Markov and martingale property of $(H(t))_{t\in[0,T]}$, and the fact that it is log-normally distributed.

$$\begin{aligned}
&\mathbb{E}\left[\frac{H(T)}{H(t)}\left(y_1^* H(T)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_A \mid \mathcal{F}(t)\right] \\
&= \left(y_1^*\right)^{\frac{1}{\gamma-1}} \mathbb{E}\left[\frac{(H(T))^{\frac{\gamma}{\gamma-1}}}{H(t)} \mathbb{1}_A \mid \mathcal{F}(t)\right] \\
&= \left(y_1^*\right)^{\frac{1}{\gamma-1}} \mathbb{E}\left[\frac{(H(t))^{\frac{\gamma}{\gamma-1}}}{H(t)}\left(\frac{H(T)}{H(t)}\right)^{\frac{\gamma}{\gamma-1}} \mathbb{1}_{\left\{\frac{H(T)}{H(t)} \leq \frac{q^{\gamma-1}}{y_1^* H(t)}\right\}} \mid \mathcal{F}(t)\right] \\
&= \left(y_1^* H(t)\right)^{\frac{1}{\gamma-1}} \mathbb{E}\left[\left(\frac{H(T)}{H(t)}\right)^{\frac{\gamma}{\gamma-1}} \mathbb{1}_{\left\{\frac{H(T)}{H(t)} \leq \frac{q^{\gamma-1}}{y_1^* H(t)}\right\}}\right] \\
&= \left(y_1^* H(t)\right)^{\frac{1}{\gamma-1}} \mathbb{E}\left[e^{\frac{\gamma}{1-\gamma} M^H(t,T)+\frac{\gamma}{1-\gamma} \sqrt{V^H(t,T)} x} \mathbb{1}_{\left\{e^{-M^H(t,T)-\sqrt{V^H(t,T)} x} \leq \frac{q^{\gamma-1}}{y_1^* H(t)}\right\}}\right] \\
&= \left(y_1^* H(t)\right)^{\frac{1}{\gamma-1}} \mathbb{E}\left[e^{\frac{\gamma}{1-\gamma} M^H(t,T)+\frac{\gamma}{1-\gamma} \sqrt{V^H(t,T)} x} \mathbb{1}_{\left\{x \geq -\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right)+M^H(t,T)}{\sqrt{V^H(t,T)}}\right\}}\right] \\
&= \left(y_1^* H(t)\right)^{\frac{1}{\gamma-1}} \frac{1}{\sqrt{2\pi}} \int_{-\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right)+M^H(t,T)}{\sqrt{V^H(t,T)}}}^{\infty} e^{\frac{\gamma}{1-\gamma} M^H(t,T)+\frac{\gamma}{1-\gamma} \sqrt{V^H(t,T)} x} e^{-\frac{x^2}{2}} dx \\
&= \left(y_1^* H(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T)+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} \times \\
&\qquad \frac{1}{\sqrt{2\pi}} \int_{-\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right)+M^H(t,T)}{\sqrt{V^H(t,T)}}}^{\infty} e^{-\frac{\left(x-\frac{\gamma}{1-\gamma}\sqrt{V^H(t,T)}\right)^2}{2}} dx \\
&= \left(y_1^*\right)^{\frac{1}{\gamma-1}} (H(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T)+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} * \Phi\left(d_1^\delta(t, H(t))\right).
\end{aligned}$$

(b) We prove the statement for $\varphi^*(t)$. The proof of $\varphi^{PI}(t)$ is quite similar and it is omitted here.
Let us define a function ψ as

$$\begin{aligned}
\psi(H(T)) :=& H(T) X^\delta(T) \\
&= \left(y_1^*\right)^{\frac{1}{\gamma-1}} (H(T))^{\frac{\gamma}{\gamma-1}} \mathbb{1}_A + H(T) q \mathbb{1}_B \\
&+ \left(y_1^* - y_2^*\right)^{\frac{1}{\gamma-1}} (H(T))^{\frac{\gamma}{\gamma-1}} \mathbb{1}_C,
\end{aligned}$$

with A, B and C as given in (4.44). From Definition A.8 we can directly observe (with $H(T)$ playing a role of F and ψ playing a role of φ) that

$$\psi\left(H\left(T\right)\right) \in PC^{1}\left(\left(0, \frac{q^{\gamma-1}}{y_1^*}\right) \cup \left(\frac{q^{\gamma-1}}{y_1^*}, \frac{q^{\gamma-1}}{y_1^*-y_2^*}\right) \cup \left(\frac{q^{\gamma-1}}{y_1^*-y_2^*}, \infty\right)\right).$$

The break points are $0, \frac{q^{\gamma-1}}{y_1^*}, \frac{q^{\gamma-1}}{y_1^*-y_2^*}$ and ∞. We have shown in the proof of Theorem 4.4 part (iii) that $\left(r\left(t\right), \theta_r\left(t\right), \theta_S\left(t\right)\right)^\top \in \left(\mathbb{D}\right)^3$. It follows from Lemma A.5 that $H\left(T\right) \in \mathbb{D}_{1,1}$. Using Proposition A.9 it holds $\psi\left(H\left(T\right)\right) \in \mathbb{D}_{1,1}$. Now we are in a position to apply representation (A.17), in Theorem A.7, for the process $\beta\left(T\right) X^\delta\left(T\right)$ since $Z^\Theta\left(T\right)\beta\left(T\right)X^\delta\left(T\right) = \psi\left(H\left(T\right)\right) \in \mathbb{D}_{1,1}$.

$$\begin{aligned}\beta\left(T\right) X^\delta\left(T\right) =& \mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right) X^\delta\left(T\right)\right] \\ &+ \int_0^T \mathbb{E}^{\mathbb{Q}}\left[D_t\left(\beta\left(T\right) X^\delta\left(T\right)\right) \mid \mathcal{F}\left(t\right)\right] dW^{\mathbb{Q}}\left(t\right)\end{aligned} \tag{4.46}$$

since $D_t\left(\Theta\left(u\right)\right) = 0, u > t$, which follows from $\Theta\left(u\right)$ being a deterministic function.

$$\begin{aligned}D_t\left(\beta\left(T\right) X^\delta\left(T\right)\right) &= \beta\left(T\right) D_t\left(X^\delta\left(T\right)\right) + X^\delta\left(T\right) D_t\left(\beta\left(T\right)\right) \\ =&\beta\left(T\right) D_t\left(X^\delta\left(T\right)\right) + X^\delta\left(T\right)\beta\left(T\right) D_t\left(-\int_0^T r\left(s\right) ds\right)\end{aligned} \tag{4.47}$$

from the product and chain rules of Malliavin calculus. Applying the arguments and the ideas used for the proof of Theorem 4.4 part (iii) we obtain

$$\begin{aligned}D_t\left(X^\delta\left(T\right)\right) =& \frac{1}{1-\gamma}\left(\sigma_r n\left(t,T\right) e_1 + \Theta^\top\left(t\right)\right)\left(y_1^* H\left(T\right)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_A \\ &+ \frac{1}{1-\gamma}\left(\sigma_r n\left(t,T\right) e_1 + \Theta^\top\left(t\right)\right)\left(\left(y_1^* - y_2^*\right) H\left(T\right)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_C,\end{aligned} \tag{4.48}$$

with $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Combining Equations (4.47) and (4.48) gives

$$
\begin{aligned}
&D_t\left(\beta(T)X^\delta(T)\right) \\
=&\frac{\gamma}{1-\gamma}\left(\sigma_r n(t,T)e_1 + \frac{1}{\gamma}\Theta^\top(t)\right)\beta(T)\left(y_1^* H(T)\right)^{\frac{1}{\gamma-1}}\mathbb{1}_A \\
&-\sigma_r n(t,T)e_1\beta(T)q\mathbb{1}_B \\
&+\frac{\gamma}{1-\gamma}\left(\sigma_r n(t,T)e_1 + \frac{1}{\gamma}\Theta^\top(t)\right)\beta(T)\left(\left(y_1^*-y_2^*\right)H(T)\right)^{\frac{1}{\gamma-1}}\mathbb{1}_C \\
=&\frac{\gamma}{1-\gamma}\left(\sigma_r n(t,T)e_1 + \frac{1}{\gamma}\Theta^\top(t)\right)\beta(T)\left(y_1^* H(T)\right)^{\frac{1}{\gamma-1}}\mathbb{1}_A \\
&-\sigma_r n(t,T)e_1\beta(T)q\mathbb{1}_B \\
&-\frac{\gamma}{1-\gamma}\left(\sigma_r n(t,T)e_1 + \frac{1}{\gamma}\Theta^\top(t)\right)e_1\beta(T)q\mathbb{1}_B \\
&+\frac{\gamma}{1-\gamma}\left(\sigma_r n(t,T)e_1 + \frac{1}{\gamma}\Theta^\top(t)\right)e_1\beta(T)q\mathbb{1}_B \\
&+\frac{\gamma}{1-\gamma}\left(\sigma_r n(t,T)e_1 + \frac{1}{\gamma}\Theta^\top(t)\right)\beta(T)\left(\left(y_1^*-y_2^*\right)H(T)\right)^{\frac{1}{\gamma-1}}\mathbb{1}_C.
\end{aligned}
$$

After some rearrangements we get

$$
\begin{aligned}
D_t\left(\beta(T)X^\delta(T)\right) =&\frac{\gamma}{1-\gamma}\left(\sigma_r n(t,T)e_1 + \frac{1}{\gamma}\Theta^\top(t)\right)\beta(T)X^\delta(T) \\
&+\frac{1}{\gamma-1}\left(\sigma_r n(t,T)e_1 - \Theta^\top(t)\right)e_1\beta(T)q\mathbb{1}_B
\end{aligned}
\tag{4.49}
$$

with X^δ as given in (4.43).

Plugging the value of $D_t\left(\beta(T)X^\delta(T)\right)$ from Equation (4.49) in Equation (4.46) and comparing it with (4.21), with $X(T)$ replaced by $X^\delta(T)$, we obtain

$$\begin{aligned}
&\varphi_S(t)\,\sigma_{Sr} - \varphi_B(t)\,\sigma_r n\left(t,\widetilde{T}\right) \\
&= \frac{\gamma}{1-\gamma}\left(\sigma_r n(t,T) + \frac{1}{\gamma}\theta_r(t)\right)\frac{\mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,X^{\delta}(T)\mid\mathcal{F}(t)\right]}{\beta(t)\,X^{\delta}(t)} \\
&\quad + \frac{1}{\gamma-1}\left(\sigma_r n(t,T) - \theta_r(t)\right)\frac{\mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,(T)\,q\mathbb{1}_B\mid\mathcal{F}(t)\right]}{\beta(t)\,X^{\delta}(t)} \\
&\varphi_S(t)\,\sigma_S = \frac{1}{1-\gamma}\theta_S(t)\,\frac{\mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,X^{\delta}(T)\mid\mathcal{F}(t)\right]}{\beta(t)\,X^{\delta}(t)} \\
&\qquad + \frac{1}{\gamma-1}\theta_S(t)\,\frac{\mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,(T)\,q\mathbb{1}_B\mid\mathcal{F}(t)\right]}{\beta(t)\,X^{\delta}(t)}.
\end{aligned} \tag{4.50}$$

We know from the martingale property of $\left(\beta(t)\,X^{\delta}(t)\right)_{t\in[0,T]}$ under $\mathbb{Q}$ that

$$\beta(t)\,X^{\delta}(t) = \mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,X^{\delta}(T)\mid\mathcal{F}(t)\right].$$

$\mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,q\mathbb{1}_B\mid\mathcal{F}(t)\right]$ is computed by applying the same arguments used for the proof of part (a). Solving the system of the Equation (4.50) simultaneously gives $\varphi_B^*(t)$ and $\varphi_S^*(t)$, and $\varphi_M^*(t)$ is obtained through $1-\varphi_B^*(t)-\varphi_S^*(t)$.

□

4.3. Optimization without risk constraints in an incomplete market

So far in this chapter we have been working in a complete financial market but the financial market might as well be incomplete. In this and the next section we turn to the world of incomplete financial markets. More precisely, we consider a financial market consisting of one money market account M and one stock S with prices evolving as in (4.1) and (4.3), respectively:

$$dM(t) = M(t)\,r(t)\,dt \tag{4.51}$$

$$dS(t) = S(t)\left[\mu_S(t)\,dt + \sigma_{Sr}dW_r^{\mathbb{P}}(t) + \sigma_S dW_S^{\mathbb{P}}(t)\right]. \tag{4.52}$$

The wealth process of an investor under the probability measure $\mathbb{P}$ is ruled by

$$\begin{aligned} dX(t) =& dX^M(t) + dX^S(t) \\ =& (1-\varphi_S(t)) X(t) r(t) dt \\ &+ X(t)\varphi_S(t)\left(\mu_S(t) dt + \sigma_{Sr} dW_r^{\mathbb{P}}(t) + \sigma_S dW_S^{\mathbb{P}}(t)\right), \end{aligned}$$

where X^M and X^S are as in (4.19).

Remark 4.7 *Since our market is assumed to be arbitrage free, from Theorem 2.7 we know that $\mathcal{M}^e \neq \emptyset$. We shall identify $\mathbb{Q} \in \mathcal{M}^e$ by its Radon-Nikodym derivative $Z^{\Theta}(t) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ defined by*

$$Z^{\Theta}(t) := \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = \exp\left\{-\frac{1}{2}\int_0^t \|\Theta(u)\|^2 du - \int_0^t \Theta^{\top}(u) dW^{\mathbb{P}}(u)\right\}.$$

for some $\Theta(t) = (\theta_r(t), \theta_S(t))^{\top}$ solving

$$\mu_S(t) - r(t) = \theta_r(t)\sigma_{Sr} + \theta_S(t)\sigma_S, \tag{4.53}$$

θ_r and θ_S are interpreted as the market prices of risk due to the interest rate r and the stock S, respectively.

Observe that $\Theta(t) := (\theta_r(t), \theta_S(t))^{\top}$ corresponds to ($\mathbb{Q} \in \mathcal{M}^e$). Now we have in the Equation (4.53) two unknowns (θ_r and θ_S) in one equation. So, the values of these unknowns cannot be determined uniquely, that means the set $\mathcal{M}^e$ is comprised of infinitely many elements. This is then an indication that we are in an incomplete market. Note also that the deflator $\frac{H(s)}{H(t)}$ at time t, for payment at time $s \geq t$ defined by

$$\begin{aligned} \frac{H(s)}{H(t)} :=& \frac{\beta(s) Z^{\Theta}(s)}{\beta(t) Z^{\Theta}(t)} = \frac{\beta(s)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_s}{\beta(t)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t} \\ =& \exp\left\{-\int_t^s r(u) du - \frac{1}{2}\int_t^s \|\Theta(u)\|^2 du - \int_t^s \Theta^{\top}(u) dW^{\mathbb{P}}(u)\right\}, \end{aligned} \tag{4.54}$$

where $\beta(t) = \frac{1}{M(t)}$, exists but it is not unique as it is the case in a complete market. According the Girsanov Theorem the processes

$$W^{\mathbb{Q}}(t) = \begin{pmatrix} W_r^{\mathbb{Q}}(t) \\ W_S^{\mathbb{Q}}(t) \end{pmatrix} := \begin{pmatrix} W_r^{\mathbb{P}}(t) + \int_0^t \theta_r(u) du \\ W_S^{\mathbb{P}}(t) + \int_0^t \theta_S(u) du \end{pmatrix}$$

are Brownian motions under the probability measure $\mathbb{Q} \in \mathcal{M}^e$. The dynamics of the interest rates $r(t)$ and the stock $S(t)$ under the measure $\mathbb{Q} \in \mathcal{M}^e$ are given by

$$\begin{aligned} dr(t) &= a\left(b^{\mathbb{Q}}(t) - r(t)\right)dt + \sigma_r dW_r^{\mathbb{Q}}(t) \\ dS(t) &= S(t)\left[r(t)\,dt + \sigma_{Sr} dW_r^{\mathbb{Q}}(t) + \sigma_S dW_S^{\mathbb{Q}}(t)\right]. \end{aligned}$$

Note that the dynamics of the money market account

$$dM(t) = M(t)\,r(t)\,dt$$

do not change because they do not contain a diffusion term.

The wealth process of an investor in terms of $W^{\mathbb{Q}}$, or under the probability measure $\mathbb{Q} \in \mathcal{M}^e$, is described by the following SDE

$$\begin{aligned} dX(t) =& (1 - \varphi_S(t))\,X(t)\,r(t)\,dt \\ &+ X(t)\,\varphi_S(t)\left(r(t)\,dt + \sigma_{Sr} dW_r^{\mathbb{Q}}(t) + \sigma_S dW_S^{\mathbb{Q}}(t)\right) \\ =& X(t)\left[r(t)\,dt + \varphi_S\left(\sigma_{Sr} dW_r^{\mathbb{Q}}(t) + \sigma_S dW_S^{\mathbb{Q}}(t)\right)\right]. \end{aligned}$$

By using Itô's rule on the process $\beta(t)\,X(t)$, as it is done in Chapter 3, it gives

$$d(\beta(t)\,X(t)) = \beta(t)\,X(t)\,\varphi_S(t)\left[\sigma_{Sr} dW_r^{\mathbb{Q}}(t) + \sigma_S dW_S^{\mathbb{Q}}(t)\right] \quad \text{and}$$

$$\begin{aligned} \beta(T)\,X(T) = X(0) &+ \int_0^T \beta(t)\,X(t)\,\varphi_S(t)\,\sigma_{Sr} dW_r^{\mathbb{Q}}(t) \\ &+ \int_0^T \beta(t)\,X(t)\,\varphi_S(t)\,\sigma_S dW_S^{\mathbb{Q}}(t). \end{aligned} \tag{4.55}$$

Static optimization problem

The static optimization problem in the case of incomplete market is stated as in the case of complete market (Problem (4.23)) and it is re-stated below

$$\begin{aligned} &\max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{(X(T))^{\gamma}}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta(T)\,X(T)\right] \le x \quad (\text{budget constraint}), \end{aligned} \tag{4.56}$$

where $\mathcal{B}(x)$ is as in (4.23).

We use a Lagrangian approach to solve Problem (4.56) with the Lagrangian function L defined by

$$\begin{aligned} L\left(y,\mathbb{Q},X\left(T\right)\right) &:= \mathbb{E}\left[\frac{\left(X\left(T\right)\right)^{\gamma}}{\gamma}+y\left(x-\beta\left(T\right)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_{T}X\left(T\right)\right)\right] \\ &= \mathbb{E}\left[\frac{\left(X\left(T\right)\right)^{\gamma}}{\gamma}-y\beta\left(T\right)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_{T}X\left(T\right)+yx\right], \end{aligned} \tag{4.57}$$

where y is the Lagrangian multiplier. The function L in (4.57) is constructed by penalizing the objective function of the Problem (4.56) by its constraint. Let us first solve the primal problem and then its dual problem, which are stated as follows:
Primal problem:

$$\Psi\left(y,\mathbb{Q}\right) := \max_{X(T)\in\mathcal{B}(x)} L\left(y,\mathbb{Q},X\left(T\right)\right) \quad \text{for} \quad y\in\left(0,\infty\right) \quad \text{and} \quad \mathbb{Q}\in\mathcal{M}^{e}. \tag{4.58}$$

Dual problem:

$$\Phi\left(X\left(T\right)\right) := \min_{y\in(0,\infty),\mathbb{Q}\in\mathcal{M}^{e}} L\left(y,\mathbb{Q},X\left(T\right)\right) \quad \text{for} \quad X\left(T\right)\in\mathcal{B}\left(x\right). \tag{4.59}$$

For the case of incomplete markets, the set $\mathcal{M}^{e}$ is composed of infinitely many elements, as mentioned before, where in the dual problem (4.59) we have to find $y^{*}\in(0,\infty)$ and $\mathbb{Q}^{*}\in\mathcal{M}^{e}$ that minimize the Lagrangian function $L\left(y,\mathbb{Q},X\left(T\right)\right)$ for a given $X\left(T\right)\in\mathcal{B}\left(x\right)$. But for the case of complete markets the set $\mathcal{M}^{e}$ is a singleton and we have only to find $y^{*}\in(0,\infty)$ that solves the problem (4.59) for a given $X\left(T\right)\in\mathcal{B}\left(x\right)$. The solution to the primal problem (4.58) for the incomplete market looks like for the complete market and is given in the following proposition without a proof because the proof is exactly as for complete markets for every $\mathbb{Q}$ fixed.

Proposition 4.8 *For given $\mathbb{Q}\in\mathcal{M}^{e}$, the solution to the primal problem (4.58) is given by $X^{*}\left(T\right)=f\left(y^{*},\mathbb{Q}\right)$, thereby*

$$f\left(y^{*},\mathbb{Q}\right)=\left(y^{*}\beta\left(T\right)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_{T}\right)^{\frac{1}{\gamma-1}}$$

and $y^{*}\in(0,\infty)$ solves $\mathbb{E}^{\mathbb{Q}}\left[\beta\left(T\right)f\left(y,\mathbb{Q}\right)\right]=x$.

By inserting $X^*(T)$ and y^* from Proposition 4.8 in the Equation (4.57) it gives

$$\begin{aligned} L(y^*, \mathbb{Q}, X^*(T)) &= \mathbb{E}\left[\frac{(X^*(T))^\gamma}{\gamma} - y^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T X^*(T) + y^* x\right] \\ &= \mathbb{E}\left[\frac{\left(y^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}}}{\gamma} - y^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \left(y^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} + y^* x\right] \\ &= \mathbb{E}\left[\frac{1-\gamma}{\gamma}\left(y^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}} + y^* x\right]. \end{aligned} \tag{4.60}$$

Recall from (4.54) and Remark 4.7 that

$$\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T = \beta(T) Z^\Theta(T) = \exp\left\{-\int_0^T r(t)\,dt - \frac{1}{2}\int_0^T \|\Theta(t)\|^2\,dt - \int_0^T \Theta^\top(t)\,dW^{\mathbb{P}}(t)\right\}$$

and

$$\mu_S(t) - r(t) = \theta_r(t)\sigma_{Sr} + \theta_S(t)\sigma_S. \tag{4.61}$$

We are now ready to state a theorem that gives the solution to the dual problem (4.59) for $X^*(T)$ and y^* from Proposition 4.8.

Theorem 4.9 *For y^* and $X^*(T)$ as in Proposition 4.8, the dual problem (4.59) is equivalent to*

$$\begin{aligned} &\min_{\Theta(t)=(\theta_r(t),\theta_S(t))} \mathbb{E}\left[\frac{1-\gamma}{\gamma}(y^*)^{\frac{\gamma}{\gamma-1}}\left(\beta(T)Z^\Theta(T)\right)^{\frac{\gamma}{\gamma-1}}\right] \\ &s.t.\quad \theta_r(t)\sigma_{Sr} + \theta_S(t)\sigma_S = \bar{\mu}_S. \end{aligned} \tag{4.62}$$

Then, the solution to (4.62) is given by $\Theta^*(t) = (\theta_r^*(t), \theta_S^*(t))^\top$, where

$$\theta_r^*(t) = \frac{\sigma_{Sr}\bar{\mu}_S - \gamma\sigma_S^2\sigma_r n(t,T)}{\sigma_S^2 + \sigma_{rS}^2} \quad \text{and}$$

$$\theta_S^*(t) = \frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}\theta_r^*(t)}{\sigma_S}.$$

Proof: The existence and uniqueness of the solution of problem (4.62) is guaranteed by the fact that the asymptotic elasticity of power utility function is less than one and Theorem 2.18. From (4.16)

$$\begin{aligned}
&\int_0^T r(t)\,dt \\
&= \int_0^T r(0)\,e^{-at}dt + \int_0^T b^{\mathbb{Q}}(t)\left(1-e^{-at}\right)dt + \int_0^T e^{-at}\sigma_r \int_0^t e^{au}dW_r^{\mathbb{Q}}(u)\,dt \\
&= \int_0^T r(0)\,e^{-at}dt + \int_0^T b^{\mathbb{P}}\left(1-e^{-at}\right)dt + \int_0^T \sigma_r n(t,T)\,dW_r^{\mathbb{P}}(t)\,.
\end{aligned}$$

Hence $\int_0^T r(t)\,dt$ is normally distributed and since $\Theta(t)$ is deterministic it follows that

$$\frac{\gamma}{1-\gamma}\left(\int_0^T r(t)\,dt + \frac{1}{2}\int_0^T \|\Theta(t)\|^2\,dt + \int_0^T \Theta^\top(t)\,dW^{\mathbb{P}}(t)\right) \sim \mathcal{N}\left(M,\sqrt{V}\right),$$

where

$$\begin{aligned}
M =& \frac{\gamma}{1-\gamma}\left[\int_0^T r(0)\,e^{-at}dt + \int_0^T b^{\mathbb{P}}\left(1-e^{-at}\right)dt + \frac{1}{2}\int_0^T \|\Theta(t)\|^2\,dt\right] \\
V =& Var\left[\frac{\gamma}{1-\gamma}\left(\int_0^T r(t)\,dt + \frac{1}{2}\int_0^T \|\Theta(t)\|^2\,dt + \int_0^T \Theta^\top(t)\,dW^{\mathbb{P}}(t)\right)\right] \\
=& \frac{\gamma^2}{(1-\gamma)^2}\left[\int_0^T \left[(\sigma_r n(t,T) + \theta_r(t))^2 + \theta_S^2(t)\right]dt\right].
\end{aligned}$$

Therefore,

$$L\left(y^*,\mathbb{Q},X^*(T)\right) = \frac{1-\gamma}{\gamma}\left(y^*\right)^{\frac{\gamma}{\gamma-1}}\exp\left\{M + \frac{1}{2}V\right\} + xy^*.$$

and

$$\min_{\mathbb{Q}\in\mathcal{M}^e} L\left(y^*,\mathbb{Q},X^*(T)\right) = \min_{\Theta(t)} \frac{1-\gamma}{\gamma}\left(y^*\right)^{\frac{\gamma}{\gamma-1}}\exp\left\{M(\theta_r,\theta_S) + \frac{1}{2}V(\theta_r,\theta_S)\right\},$$

since the market price of risk corresponds to the equivalent martingale measure and the term xy^* does not depend on $\mathbb{Q}$. As the relation (4.61) has to be fulfilled, then it acts as a constraint in the following optimization problem.

$$\begin{aligned}
&\min_{\Theta=(\theta_r,\theta_S)} \mathbb{E}\left[\frac{1-\gamma}{\gamma}\left(y^*\right)^{\frac{\gamma}{\gamma-1}}\exp\left\{M(\theta_r,\theta_S) + \frac{1}{2}V(\theta_r,\theta_S)\right\}\right] \\
&s.t.\ \ \theta_r(t)\,\sigma_{Sr} + \theta_S(t)\,\sigma_S = \mu_S(t) - r(t) = \bar{\mu}_S.
\end{aligned} \tag{4.63}$$

To solve the problem (4.63) let us set first

$$\theta_r(t) = \lambda(t) \tag{4.64}$$

and then express $\theta_S(t)$ in terms of $\lambda(t)$ as

$$\theta_S(t) = \frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}\lambda(t)}{\sigma_S}. \tag{4.65}$$

Let us define a function $g(\lambda)$ as

$$g(\lambda) := \frac{1-\gamma}{\gamma}(y^*)^{\frac{\gamma}{\gamma-1}} e^{M(\lambda)+\frac{1}{2}V(\lambda)}$$

where

$$\begin{aligned}
M(\lambda) =& \frac{\gamma}{1-\gamma}\left[\int_0^T r(0)e^{-at}dt + \int_0^T b^{\mathbb{P}}\left(1-e^{-at}\right)dt\right] \\
&+ \frac{\gamma}{1-\gamma}\frac{1}{2}\left[\int_0^T (\lambda(t))^2 dt + \int_0^T \left(\frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}\lambda(t)}{\sigma_S}\right)^2 dt\right], \\
V(\lambda) =& \frac{\gamma^2}{(1-\gamma)^2}\int_0^T \left[(\sigma_r n(t,T) + \lambda(t))^2 + \left(\frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}\lambda(t)}{\sigma_S}\right)^2\right] dt.
\end{aligned}$$

Now, the problem (4.63) is reduced to the following:

$$\min_{\lambda} g(\lambda). \tag{4.66}$$

(i) For $\gamma \in (0,1)$ g is minimum, if $M(\lambda) + \frac{1}{2}V(\lambda)$ is minimum.

$$\begin{aligned}
&M(\lambda) + \frac{1}{2}V(\lambda) = \\
&\int_0^T \frac{\gamma}{1-\gamma}\left[r(0)e^{-at} + b^{\mathbb{P}}\left(1-e^{-at}\right) + \frac{1}{2}\left((\lambda(t))^2 + \left(\frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}\lambda(t)}{\sigma_S}\right)^2\right)\right. \\
&\left. + \frac{1}{2}\frac{\gamma}{1-\gamma}\left((\sigma_r n(t,T) + \lambda(t))^2 + \left(\frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}\lambda(t)}{\sigma_S}\right)^2\right)\right] dt \\
&=: \int_0^T f(\lambda(t), t)\, dt.
\end{aligned}$$

So, g is minimum, if $\lambda(t)$ minimizes $f(\lambda(t), t)$ for all $t \in [0,T]$.

$$\begin{aligned}\frac{\partial f(\lambda,t)}{\partial \lambda} &= \frac{\gamma}{1-\gamma}\left[\lambda - \frac{\sigma_{Sr}}{\sigma_S}\left(\frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}\lambda}{\sigma_S}\right)\right. \\ &\quad \left. + \frac{\gamma}{1-\gamma}\left((\sigma_r n(t,T) + \lambda) - \frac{\sigma_{Sr}}{\sigma_S}\left(\frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}\lambda}{\sigma_S}\right)\right)\right] \\ &\overset{!}{=} 0.\end{aligned} \tag{4.67}$$

After some reshuffling and making λ in Equation (4.67) the subject, we obtain

$$\lambda^*(t) = \frac{\sigma_{Sr}\bar{\mu}_S - \gamma\sigma_S^2\sigma_r n(t,T)}{\sigma_S^2 + \sigma_{rS}^2}. \tag{4.68}$$

$\lambda^*(t)$ is a candidate minimizer of $f(\lambda(t), t)$. We have to prove that $\lambda^*(t)$ is a global minimizer of $f(\lambda(t), t)$. Let us use second order condition, i.e., we need to show that $\frac{\partial^2 f(\lambda,t)}{\partial \lambda^2} > 0$ at $\lambda^*(t)$ for all $t \in [0,T]$.

$$\begin{aligned}\frac{\partial^2 f(\lambda,t)}{\partial \lambda^2} &= \frac{\gamma}{1-\gamma}\left[1 + \frac{\sigma_{Sr}^2}{\sigma_S^2} + \frac{\gamma}{1-\gamma}\left(1 + \frac{\sigma_{Sr}^2}{\sigma_S^2}\right)\right] \\ &= \left(\frac{\gamma}{1-\gamma} + \frac{\gamma^2}{(1-\gamma)^2}\right)\left(1 + \frac{\sigma_{Sr}^2}{\sigma_S^2}\right) > 0.\end{aligned} \tag{4.69}$$

(ii) For $\gamma \in (-\infty, 0)$ g is minimum, if $M(\lambda) + \frac{1}{2}V(\lambda)$ is maximum. We take the same steps as for part (i) and we obtain $\lambda^*(t)$, as given in (4.68), as a candidate maximizer of $f(\lambda(t), t)$. What is left, is to show that $\frac{\partial^2 f(\lambda(t),t)}{\partial \lambda^2} < 0$ at $\lambda^*(t)$ for all $t \in [0,T]$, in order for $\lambda^*(t)$ to be a global maximizer of $f(\lambda(t), t)$. From (4.69) we have

$$\begin{aligned}\frac{\partial^2 f(\lambda,t)}{\partial \lambda^2} &= \left(\frac{\gamma}{1-\gamma} + \frac{\gamma^2}{(1-\gamma)^2}\right)\left(1 + \frac{\sigma_{Sr}^2}{\sigma_S^2}\right) \\ &= \underbrace{\frac{\gamma}{1-\gamma}}_{<0}\left(\underbrace{1 + \frac{\gamma}{1-\gamma}}_{>0}\right)\left(\underbrace{1 + \frac{\sigma_{Sr}^2}{\sigma_S^2}}_{>0}\right) < 0.\end{aligned}$$

□

After characterizing $\mathbb{Q}^* \in \mathcal{M}^e$, by

$$\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\right)_T = Z^{\Theta^*}(T) := \exp\left\{-\frac{1}{2}\int_0^T \|\Theta^*(u)\|^2 du - \int_0^T \Theta^{*,\top}(u)\, dW^{\mathbb{P}}(u)\right\},$$

with Θ^* as in Theorem 4.9, the same procedures as for the complete market are taken to derive the optimal trading strategy. The following theorem characterizes the trading strategy that generates the optimal terminal wealth $X^*(T)$.

Theorem 4.10 *Suppose that all the conditions of Proposition 4.8 and Theorem 4.9 are fulfilled. Then:*

(a) The optimal wealth at any time $t \in [0,T]$ is

$$\begin{aligned} & X^*(t) \\ & = (y^*)^{\frac{1}{\gamma-1}} (H^*(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^{H^*}(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{H^*}(t,T)}, \end{aligned}$$

where

$$\begin{aligned} H^*(t) &= \beta(t) Z^{\Theta^*}(t), \\ M^{H^*}(t,T) &= \int_t^T \left(r(t) e^{-a(s-t)} + b^{\mathbb{P}} \left(1 - e^{-a(s-t)}\right) + \frac{1}{2} \|\Theta^*(s)\|^2 \right) ds, \\ V^{H^*}(t,T) &= \int_t^T \left[(\sigma_r n(s,T) + \theta_r^*(s))^2 + (\theta_S^*(s))^2 \right] ds, \end{aligned}$$

and $\mathbb{Q}^* \in \mathcal{M}^e$ as in Theorem 4.9.

(b) The portfolio process that generates the wealth $X^*(t)$ is given by

$$\begin{aligned} \varphi_S^*(t) &= \frac{\bar{\mu}_S + \gamma \sigma_{Sr} \sigma_r n(t,T)}{(1-\gamma)(\sigma_S^2 + \sigma_{Sr}^2)} \\ \text{and} \quad \varphi_M^*(t) &= 1 - \varphi_S^*(t). \end{aligned}$$

Proof: The proof is as of Theorem 4.4 part (ii) and (iii) with $\Theta(t)$ replaced by $\Theta^*(t)$, since $\Theta(t)$ and $\Theta^*(t)$ share the same properties of being continuous, deterministic and bounded. Using the same argument discussed in Theorem 4.4 for part (a) the result is obtained directly, and for part (b) the representation (A.17) is used for the random variable $\beta(T) X^*(T)$.

$$\beta(T) X^*(T) = \mathbb{E}^{\mathbb{Q}^*}[\beta(T) X^*(T)] + \int_0^T \mathbb{E}^{\mathbb{Q}^*}[D_t(\beta(T) X^*(T)) \mid \mathcal{F}(t)]\, dW^{\mathbb{Q}^*}(t). \tag{4.70}$$

Using the idea of the proof of Theorem 4.4 we obtain

$$\begin{aligned} &D_t\left(\beta\left(T\right)X^*\left(T\right)\right)\\ &=\frac{\gamma}{1-\gamma}\left(\sigma_r n\left(t,T\right)e_1+\frac{1}{\gamma}\Theta^{*,\top}\left(t\right)\right)\beta\left(T\right)\left(y^*\beta\left(T\right)Z^{\Theta^*}\left(T\right)\right)^{\frac{1}{\gamma-1}} \end{aligned} \tag{4.71}$$

Substituting the value of $D_t\left(\beta\left(T\right)X^*\left(T\right)\right)$ from Equation (4.71) into Equation (4.70) and comparing it with (4.55) gives

$$\begin{aligned} \beta\left(t\right)X^*\left(t\right)\varphi_S\left(t\right)\sigma_{Sr}=&\frac{\gamma}{1-\gamma}\left(\sigma_r n\left(t,T\right)+\frac{1}{\gamma}\theta_r^*\left(t\right)\right)\times\\ &\mathbb{E}^{\mathbb{Q}^*}\left[\beta\left(T\right)\left(y^*H\left(T\right)\right)^{\frac{1}{\gamma-1}}\mid\mathcal{F}\left(t\right)\right] \end{aligned} \tag{4.72}$$

and

$$\beta\left(t\right)X^*\left(t\right)\varphi_S\left(t\right)\sigma_S=\frac{1}{1-\gamma}\theta_S^*\left(t\right)\mathbb{E}^{\mathbb{Q}^*}\left[\beta\left(T\right)\left(y^*H\left(T\right)\right)^{\frac{1}{\gamma-1}}\mid\mathcal{F}\left(t\right)\right] \tag{4.73}$$

Using Equation (4.72) and the fact that $\left(\beta\left(t\right)X^*\left(t\right)_{t\in[0,T]}\right)$ is a martingale under $\mathbb{Q}^*$ results in

$$\varphi_S^*\left(t\right)\sigma_{Sr}=\frac{\gamma}{1-\gamma}\left(\sigma_r n\left(t,T\right)+\frac{1}{\gamma}\theta_r^*\left(t\right)\right). \tag{4.74}$$

Substituting the value of $\theta_r^*\left(t\right)$ from Theorem 4.9 in (4.74) and after some rearrangements we obtain the results. □

Remark 4.11 *If we use the Equation (4.73) for the computation of $\varphi_S^*\left(t\right)$, it gives the same results.*

Proof: From Equation (4.73) and $\left(\beta\left(t\right)X^*\left(t\right)\right)_{t\in[0,T]}$ being a martingale process under $\mathbb{Q}^*$ we get

$$\varphi_S^*\left(t\right)\sigma_S=\frac{1}{1-\gamma}\theta_S^*\left(t\right).$$

Substituting the value of $\theta_S^*\left(t\right)$ from Theorem 4.9 in (4.3) gives

$$\begin{aligned} \varphi_S^*\left(t\right)=&\frac{1}{1-\gamma}\frac{\bar{\mu}_S}{\sigma_S^2}\\ &-\frac{1}{1-\gamma}\frac{\sigma_{rS}}{\sigma_S^2}\left(\frac{\sigma_{rS}\bar{\mu}_S-\gamma\sigma_S^2\sigma_r n\left(t,T\right)}{\sigma_S^2+\sigma_{rS}^2}\right)\\ =&\frac{\bar{\mu}_S+\gamma\sigma_{rS}\sigma_r n\left(t,T\right)}{\left(1-\gamma\right)\left(\sigma_S^2+\sigma_{rS}^2\right)}. \end{aligned}$$

□

Note that if S and r are uncorrelated, i.e. for $\sigma_{Sr} = 0$, we have $\varphi_S^*(t) = \frac{1}{1-\gamma}\frac{\bar{\mu}_S}{\sigma_S^2}$ which corresponds to the case of constant r. The same applies to $\gamma \to 0$ which corresponds to logarithmic utility function.

4.4. Optimization with bounded expected shortfall risk in an incomplete market

In this section we examine the portfolio optimization problem in an incomplete market, as studied in Section 4.3, in the presence of an additional risk constraint. We consider a portfolio manager who wishes to limit his/her expected loss, in particular present expected loss, as discussed in Section 4.2 and stated below for a revision.
The dynamic optimization problem:

$$\begin{aligned} &\max_{\varphi \in \tilde{\mathcal{A}}(x)} \mathbb{E}\left[\frac{(X(T))^\gamma}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta(T) X(T)\right] \le x \quad \text{(budget constraint)} \\ &\qquad \mathbb{E}^{\mathbb{Q}}\left[\beta(T)(X(T)-q)^-\right] \le \delta \quad \text{(risk constraint)}, \end{aligned} \tag{4.75}$$

where $\tilde{\mathcal{A}}(x)$ is as defined in (2.10) and $\varphi(t) = (\varphi_M(t), \varphi_S(t))^\top$ is a portfolio process composed of φ_M and φ_S refering to the fraction of wealth invested in the money market account and the stock, respectively. The differences between the problems (4.36) and (4.75) are:

- φ in (4.36) is composed of φ_M, φ_B and φ_S, while in (4.75) it is composed of φ_M and φ_S.
- in (4.36) $\mathcal{M}^e$ is singleton, i.e it can be determined uniquely, while in (4.75) $\mathcal{M}^e$ is made up of infinitely many elements, as discussed in Section 4.3.

The corresponding static optimization problem to (4.75) reads as

$$\begin{aligned} &\max_{X(T) \in \mathcal{B}(x)} \mathbb{E}\left[\frac{(X(T))^\gamma}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta(T) X(T)\right] \le x \\ &\qquad \mathbb{E}^{\mathbb{Q}}\left[\beta(T)(X(T)-q)^-\right] \le \delta, \end{aligned} \tag{4.76}$$

thereby $\mathcal{B}(x)$ is as defined in (2.12), and its static portfolio insurer problem as

$$\begin{aligned}\max_{X(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{(X(T))^{\gamma}}{\gamma}\right]&\\ \text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\beta(T)X(T)\right]\leq x&\\ X(T)\geq q.&\end{aligned} \tag{4.77}$$

We examine the problem (4.76). For the problem (4.77) it is quite similar and we will give only the results. We use the Lagrangian approach to solve the problem (4.76). The Lagrangian function $L(y_1,y_2,\mathbb{Q},X(T))$ is defined by

$$\begin{aligned}&L(y_1,y_2,\mathbb{Q},X(T))=\\ &\mathbb{E}\left[\frac{(X(T))^{\gamma}}{\gamma}-y_1\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T X(T)-y_2\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T (X(T)-q)^-\right]\\ &\qquad +y_1x+y_2\delta.\end{aligned} \tag{4.78}$$

Primal problem:

$$\begin{aligned}\Psi(y_1,y_2,\mathbb{Q}):=\max_{X(T)\in\mathcal{B}(x)} L(y_1,y_2,\mathbb{Q},X(T))&\\ \text{for}\quad y_1,y_2\in(0,\infty)\quad\text{and}\quad \mathbb{Q}\in\mathcal{M}^e.&\end{aligned} \tag{4.79}$$

Dual problem:

$$\Phi(X(T)):=\min_{y_1,y_2\in(0,\infty),\mathbb{Q}\in\mathcal{M}^e} L(y_1,y_2,\mathbb{Q},X(T))\quad\text{for}\quad X(T)\in\mathcal{B}(x). \tag{4.80}$$

The following proposition gives the solution to the primal problem (4.79) for $y_1,y_2\in(0,\infty)$ and $\mathbb{Q}\in\mathcal{M}^e$.

Proposition 4.12 *Let q be a fixed benchmark, x an initial wealth of an investor and $\mathbb{Q}\in\mathcal{M}^e$.*

(i) If $\delta\in\left(\underline{\delta},\overline{\delta}\right)$, then the solution of the primal problem (4.79) is given by

$$X^{\delta}_{inc}(T)=f(y_1^*,y_2^*,\mathbb{Q}), \tag{4.81}$$

where

$$\begin{aligned}f(y_1^*,y_2^*,\mathbb{Q})=&\left(y_1^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}\mathbb{1}_A+q\mathbb{1}_B\\ &+\left((y_1^*-y_2^*)\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}\mathbb{1}_C,\end{aligned} \tag{4.82}$$

$$
\begin{aligned}
A = A\left(y_1^*, y_2^*, \mathbb{Q}\right) &= \left\{ y_1^* \beta\left(T\right) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \leq q^{\gamma-1} \right\}, \\
B = B\left(y_1^*, y_2^*, \mathbb{Q}\right) &= \left\{ \frac{q^{\gamma-1}}{y_1^*} < \beta\left(T\right) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \leq \frac{q^{\gamma-1}}{y_1^* - y_2^*} \right\} \quad \textit{and} \\
C = C\left(y_1^*, y_2^*, \mathbb{Q}\right) &= \left\{ \beta\left(T\right) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T > \frac{q^{\gamma-1}}{y_1^* - y_2^*} \right\},
\end{aligned}
$$

such that $y_1^*, y_2^* \in (0, \infty)$ *solve the following system of equations*

$$
\begin{aligned}
\mathbb{E}\left[\beta\left(T\right) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T f\left(y_1, y_2, \mathbb{Q}\right)\right] &= x \\
\mathbb{E}\left[\beta\left(T\right) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \left(f\left(y_1, y_2, \mathbb{Q}\right) - q\right)^-\right] &= \delta.
\end{aligned}
$$

(ii) If $\delta = \underline{\delta}$*, i.e. in the case of portfolio insurer problem, the solution to the primal problem (4.79) is given by*

$$
X_{inc}^{PI}\left(T\right) = f\left(y^{PI}, \mathbb{Q}\right),
$$

where

$$
f\left(y^{PI}, \mathbb{Q}\right) = \left(y^{PI} \beta\left(T\right) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_{A^{PI}} + q \mathbb{1}_{B^{PI}}, \tag{4.83}
$$

$$
\begin{aligned}
A^{PI} = A^{PI}\left(y^{PI}, \mathbb{Q}\right) &= \left\{ \beta\left(T\right) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \leq \frac{q^{\gamma-1}}{y^{PI}} \right\} \quad \textit{and} \\
B^{PI} = B^{PI}\left(y^{PI}, \mathbb{Q}\right) &= \left\{ \frac{q^{\gamma-1}}{y^{PI}} < \beta\left(T\right) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \right\},
\end{aligned}
$$

such that y^{PI} *solves the equation*

$$
\mathbb{E}\left[\beta\left(T\right) \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T f\left(y^{PI}, \mathbb{Q}\right)\right] = x.
$$

Proof: The proof is exactly as for complete markets for every $\mathbb{Q} \in \mathcal{M}^e$ fixed. □

Let us now turn first to the dual problem (4.80) and use y_1^*, y_2^*, and $X_{inc}^{\delta}(T)$ from Proposition 4.12. What is left is to find $\mathbb{Q}^* \in \mathcal{M}^e$ that solves the dual problem (4.80). The Lagrangian function, for $\delta \in \left(\underline{\delta}, \overline{\delta}\right)$, is given by

$$\begin{aligned}
L\left(y_1^*, y_2^*, \mathbb{Q}, X_{inc}^{\delta}(T)\right) &= \mathbb{E}\left[\frac{\left(X_{inc}^{\delta}(T)\right)^{\gamma}}{\gamma} - y_1^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T X_{inc}^{\delta}(T)\right] \\
&- \mathbb{E}\left[y_2^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \left(X_{inc}^{\delta}(T) - q\right)^- - y_1^* x - y_2^*\delta\right] \\
=&\mathbb{E}\left[\frac{\left(f\left(y_1^*, y_2^*, \mathbb{Q}\right)\right)^{\gamma}}{\gamma} - y_1^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T f\left(y_1^*, y_2^*, \mathbb{Q}\right)\right] \\
&- \mathbb{E}\left[y_2^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T \left(f\left(y_1^*, y_2^*, \mathbb{Q}\right) - q\right)^- - y_1^* x - y_2^*\delta\right] \\
=&\mathbb{E}\left[\frac{(\gamma-1)}{\gamma}\left(y_1^*\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}} \mathbb{1}_A\right] \\
&+ \mathbb{E}\left[\left(\frac{q^{\gamma} - y_1^* q\gamma\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T}{\gamma}\right)\mathbb{1}_B\right] \\
&+ \mathbb{E}\left[\left(\frac{\left(y_1^* - y_2^*\right)}{\gamma} - y_1^* - y_2^*\right)\left(y_1^* - y_2^*\right)^{\frac{1}{\gamma-1}}\left(\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}} \mathbb{1}_C\right] \\
&+ y_1^* x + y_2^*\delta.
\end{aligned} \tag{4.84}$$

Remark 4.13 $\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T =: \beta(T) Z^{\Theta}(T)$ *is a log-normal random variable under* $\mathbb{P}$*:*

$$-\ln\beta(T) Z^{\Theta}(T) \sim \mathcal{N}\left(M^H(\theta_r, \theta_s), \sqrt{V^H(\theta_r, \theta_s)}\right),$$

where

$$\begin{aligned}
M^H(\theta_r, \theta_s) &= \int_0^T \left(r(0) e^{-at} + b^{\mathbb{P}}\left(1 - e^{-at}\right) + \frac{1}{2}\|\Theta(t)\|^2\right) dt \quad \text{and} \\
V^H(\theta_r, \theta_s) =& Var\left[\int_0^T \left(r(t) + \frac{1}{2}\int_0^T \|\Theta(t)\|^2\right) dt + \int_0^T \Theta^{\top}(t)\, dW^{\mathbb{P}}(t)\right] \\
=& \int_0^T \left[\left(\sigma_r n(t,T) + \theta_r(t)\right)^2 + \theta_S^2(t)\right] dt.
\end{aligned} \tag{4.85}$$

We deal with the first term of Equation (4.84) and the remaining other two terms containing random variables are computed analogously. From Remark

4.13 it follows

$$\begin{aligned}
&\mathbb{E}\left[\frac{(\gamma-1)}{\gamma}\left(y_1^*\beta(T)Z^\Theta(T)\right)^{\frac{\gamma}{\gamma-1}}\mathbb{1}_{\{A\}}\right]\\
&=\mathbb{E}\left[e^{\frac{\gamma}{1-\gamma}M^H+\frac{\gamma}{1-\gamma}\sqrt{V^H}x}\mathbb{1}_{\left\{x\leq\left(\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*}\right)+M^H}{\sqrt{V^H}}\right)\right\}}\right]\\
&=\left(\frac{1-\gamma}{\gamma}\right)\left(y_1^*\right)^{\frac{\gamma}{\gamma-1}}e^{\frac{\gamma}{1-\gamma}M^H+\left(\frac{\gamma}{1-\gamma}\right)^2\frac{1}{2}V^H}\Phi\left(d_1\right),
\end{aligned} \tag{4.86}$$

where

$$d_1=d_1\left(y_1^*,\theta_r,\theta_S\right)=\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*}\right)+M^H\left(\theta_r,\theta_S\right)}{\sqrt{V^H\left(\theta_r,\theta_S\right)}}+\frac{\gamma}{1-\gamma}\sqrt{V^H\left(\theta_r,\theta_S\right)}$$

and Φ is the cumulative distribution function of a standard normal random variable, and then we take the same procedure to calculate for other terms in (4.84). Therefore,

$$\begin{aligned}
&L\left(y_1^*,y_2^*,Z^\Theta(T),X_{inc}^\delta(T)\right)=\frac{1-\gamma}{\gamma}\left(y_1^*\right)^{\frac{\gamma}{\gamma-1}}e^{\frac{\gamma}{1-\gamma}M^H+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2V^H}\Phi\left(d_1\right)\\
&+\frac{q^\gamma}{\gamma}\left(\Phi\left(d_3\right)-\Phi\left(d_2\right)\right)-y_1^*qe^{-M^H+\frac{1}{2}V^H}\left(\Phi\left(d_5\right)-\Phi\left(d_4\right)\right)\\
&+\left(\frac{\left(y_1^*-y_2^*\right)}{\gamma}-y_1^*-y_2^*\right)\left(y_1^*-y_2^*\right)^{\frac{1}{\gamma-1}}e^{\frac{\gamma}{1-\gamma}M^H+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2V^H}\Phi\left(d_6\right)\\
&+y_1^*x+y_2^*\delta.
\end{aligned} \tag{4.87}$$

Thereby

$$\begin{aligned}
d_2=d_2\left(y_1^*,\theta_r,\theta_S\right)&:=-\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*}\right)+M^H}{\sqrt{V^H}}\\
d_3=d_3\left(y_1^*,y_2^*,\theta_r,\theta_S\right)&:=\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*-y_2^*}\right)+M^H}{\sqrt{V^H}}\\
d_4=d_4\left(y_1^*,\theta_r,\theta_S\right)&:=d_2+\sqrt{V^H}\\
d_5=d_5\left(y_1^*,y_2^*,\theta_r,\theta_S\right)&:=d_3-\sqrt{V^H}\\
d_6=d_6\left(y_1^*,y_2^*,\theta_r,\theta_S\right)&:=-\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*-y_2^*}\right)+M^H}{\sqrt{V^H}}-\frac{\gamma}{1-\gamma}\sqrt{V^H}.
\end{aligned} \tag{4.88}$$

Now, we can state the optimization problem as follows

$$\begin{aligned} &\min_{(\theta_r,\theta_S)} L\left(y_1^*, y_2^*, Z^{\Theta}(T), X_{inc}^{\delta}(T)\right) \\ &s.t.\ \ \theta_r(t)\sigma_{Sr} + \theta_S(t)\sigma_S = \bar{\mu}_S. \end{aligned} \tag{4.89}$$

The existence and uniqueness of a solution of problem (4.89) is guaranteed by the reasonable asymptotic elasticity of power utility function which is less than one and Theorem 5.9 part (iv) in [28]. The optimization problem (4.89) is solved analogously to Theorem 4.9, by setting first

$$\begin{aligned} &\theta_r(\lambda, t) = \lambda(t) \\ &\Rightarrow \theta_S(\lambda, t) = \frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{Sr}\lambda(t)}{\sigma_S}. \end{aligned} \tag{4.90}$$

We plug the values of $\theta_r(\lambda, t)$ and $\theta_S(\lambda, t)$ above in Equation (4.85) to express M and V in terms of λ and t.

Now we have to look for the λ^* that yields the minimum of

$$L(\lambda) = L(\theta_r(\lambda), \theta_S(\lambda)) = L\left(y_1^*, y_2^*, Z^{\Theta}(T), X_{inc}^{\delta}(T)\right).$$

As far as we know, λ^* can be only found numerically but cannot be calculated analytically. Then, we plug λ^* into Equation (4.90) and obtain the solution to problem (4.89).

For the case of $\delta = \underline{\delta}$ and from Proposition 4.12, the Lagrangian function is given by

$$\begin{aligned} L\left(y^{PI}, \mathbb{Q}, X_{inc}^{PI}(T)\right) &= \mathbb{E}\left[\frac{\left(X_{inc}^{PI}(T)\right)^{\gamma}}{\gamma} - y^{PI}\left(\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T X_{inc}^{PI}(T) - x\right)\right] \\ =&\mathbb{E}\left[\frac{(\gamma-1)}{\gamma}\left(y^{PI}\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}} \mathbb{1}_{A^{PI}} + \frac{q^{\gamma}}{\gamma}\mathbb{1}_{B^{PI}}\right] \\ &- y^{PI}\mathbb{E}\left[\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T q\mathbb{1}_{B^{PI}} - x\right]. \end{aligned} \tag{4.91}$$

Using Remark 4.13 we get

$$\begin{aligned} L\left(y^{PI}, \mathbb{Q}, X^{PI}(T)\right) =& \frac{1-\gamma}{\gamma}\left(y^{PI}\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{\gamma}{1-\gamma}M^H + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H} * \Phi\left(d_1^{PI}\right) \\ &+ \frac{q^{\gamma}}{\gamma}\Phi\left(d_2^{PI}\right) - y^{PI} q e^{-M^H + \frac{1}{2}V^H}\Phi\left(d_3^{PI}\right), \end{aligned} \tag{4.92}$$

where

$$
\begin{aligned}
d_1^{PI} &= d_1^{PI}\left(y^{PI},\theta_r,\theta_S\right) := \frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}}\right) + M^H\left(\theta_r,\theta_S\right)}{\sqrt{V^H\left(\theta_r,\theta_S\right)}} + \frac{\gamma}{1-\gamma}\sqrt{V^H\left(\theta_r,\theta_S\right)} \\
d_2^{PI} &= d_2^{PI}\left(y^{PI},\theta_r,\theta_S\right) := -\frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}}\right) + M^H}{\sqrt{V^H}} \\
d_3^{PI} &= d_3^{PI}\left(y^{PI},\theta_r,\theta_S\right) := -\frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}}\right) + M^H}{\sqrt{V^H}} - \sqrt{V^H}.
\end{aligned}
\tag{4.93}
$$

We state the optimization problem, as (4.89), for the portfolio insurer problem as follows

$$
\begin{aligned}
&\min_{(\theta_r,\theta_S)} L\left(y^{PI}, Z^{\Theta}(T), X_{inc}^{PI}(T)\right) \\
&s.t. \;\; \theta_r(t)\,\sigma_{Sr} + \theta_S(t)\,\sigma_S = \bar{\mu}_S.
\end{aligned}
\tag{4.94}
$$

To solve the Problem (4.94) we take the same procedures as for the case of $\delta \in \left(\underline{\delta},\overline{\delta}\right)$: set $\theta_r(\lambda,t) = \lambda$ and express θ_S in terms of λ and t using the constraint in (4.94) and then find the minimizer

$$\lambda^{PI}$$

of $L\left(y^{PI}, Z^{\Theta}, X_{inc}^{PI}(T)\right) = L\left(\theta_r(\lambda),\theta_S(\lambda)\right) =: L^{PI}(\lambda)$.Like for the case of $\delta \in \left(\underline{\delta},\overline{\delta}\right)$, λ^{PI} can be only found numerically but cannot be calculated analytically. We denote the market price of risk and the probability measure that correspond to λ^{PI} by Θ^{PI} and $\mathbb{Q}^{PI} \in \mathcal{M}^e$, respectively.

Theorem 4.14 *Suppose all the conditions of Proposition 4.12 are satisfied and we consider the equivalent martingale measures $\mathbb{Q}^* \in \mathcal{M}^e$ and $\mathbb{Q}^{PI} \in \mathcal{M}^e$ corresponding to $\Theta^*(t)$ and $\Theta^{PI}(t)$, respectively. Then:*

(a) The time-$t \in [0,T]$ optimal wealth $X_{inc}^{\delta}(t)$ and $X_{inc}^{PI}(t)$ are given by

$$
\begin{aligned}
&X_{inc}^{\delta}(t) \\
&= \left(y_1^*\right)^{\frac{1}{\gamma-1}}\left(H^*(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma}M^{H^*}(t,T)+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{H^*}(t,T)} * \Phi\left(d_1^{\delta}(t)\right) \\
&+ e^{-M^{H^*}(t,T)+\frac{1}{2}V^{H^*}(t,T)} * \left(\Phi\left(d_2^{\delta}(t)\right) - \Phi\left(d_3^{\delta}(t)\right)\right) \\
&+ \left(y_1^* - y_2^*\right)^{\frac{1}{\gamma-1}}\left(H^*(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma}M^{H^*}(t,T)+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{H^*}(t,T)} * \Phi\left(d_4^{\delta}(t)\right)
\end{aligned}
$$

and

$$\begin{aligned}
&X_{inc}^{PI}(t)\\
&= \left(y^{PI}\right)^{\frac{1}{\gamma-1}} \left(H^{PI}(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^{H^{PI}}(t,T)+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{H^{PI}}(t,T)} \times\\
&\Phi\left(d_1^{PI}(t)\right) + e^{-M^{H^{PI}}(t,T)+\frac{1}{2}V^{H^{PI}}(t,T)} * \Phi\left(d_2^{PI}(t)\right),
\end{aligned}$$

where

$$\begin{aligned}
H^*(t) &= \beta(t)\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\right)_t,\\
H^{PI}(t) &= \beta(t)\left(\frac{d\mathbb{Q}^{PI}}{d\mathbb{P}}\right)_t,\\
M^{H^*}(t,T) &= \int_t^T \left(r(t)e^{-as} + b^{\mathbb{P}}\left(1-e^{-as}\right) + \frac{1}{2}\left\|\Theta^*(s)\right\|^2\right) ds,\\
V^{H^*}(t,T) &= \int_t^T \left[\left(\sigma_r n(s,T) + \theta_r^*(s)\right)^2 + \left(\theta_S^*\right)^2(s)\right] ds,\\
M^{H^{PI}}(t,T) &= \int_t^T \left(r(t)e^{-as} + b^{\mathbb{P}}\left(1-e^{-as}\right) + \frac{1}{2}\left\|\Theta^{PI}(s)\right\|^2\right) ds,\\
V^{H^{PI}}(t,T) &= \int_t^T \left[\left(\sigma_r n(s,T) + \theta_r^{PI}(s)\right)^2 + \left(\theta_S^{PI}\right)^2(s)\right] ds,
\end{aligned}$$

$$\begin{aligned}
d_1^{\delta}(t) &= \frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H^*(t)}\right) + M^{H^*}(t,T)}{\sqrt{V^{H^*}(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^{H^*}(t,T)},\\
d_2^{\delta}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H^*(t)}\right) + M^{H^*}(t,T)}{\sqrt{V^{H^*}(t,T)}} + \sqrt{V^{H^*}(t,T)},\\
d_3^{\delta}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right)H^*(t)}\right) + M^{H^*}(t,T)}{\sqrt{V^{H^*}(t,T)}} + \sqrt{V^{H^*}(t,T)},\\
d_4^{\delta}(t) &= -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right)H^*(t)}\right) + M^{H^*}(t,T)}{\sqrt{V^{H^*}(t,T)}} - \frac{\gamma}{1-\gamma}\sqrt{V^{H^*}(t,T)},
\end{aligned}$$

$$d_1^{PI}(t) = \frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}H^{PI}(t)}\right) + M^{H^{PI}}(t,T)}{\sqrt{V^{H^{PI}}(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^{H^{PI}}(t,T)},$$
$$d_2^{PI}(t) = -\frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}H^{PI}(t)}\right) + M^{H^{PI}}(t,T)}{\sqrt{V^{H^{PI}}(t,T)}} + \sqrt{V^{H^{PI}}(t,T)}$$

and $\Phi(\cdot)$ is the standard-normal probability distribution function.

(b) *The portfolio processes at any time $t \in [0,T]$ that replicate the terminal wealth $X_{inc}^{\delta}(t)$ and $X_{inc}^{PI}(t)$ are given by $\varphi^*(t) = (\varphi_M^*(t), \varphi_S^*(t))^\top$ and $\varphi^{PI}(t) = \left(\varphi_M^{PI}(t), \varphi_S^{PI}(t)\right)^\top$, respectively. Thereby*

$$\begin{aligned}
\varphi_S^*(t) &= \frac{1}{1-\gamma}\frac{\theta_S^*(t)}{\sigma_S} + \frac{1}{\gamma-1}\frac{\theta_S^*(t)}{\sigma_S}\times \\
&\quad \frac{qe^{-M^{H^*}(t,T)+\frac{1}{2}V^{H^*}(t,T)} * \left(\Phi\left(d_2^{\delta}(t)\right) - \Phi\left(d_3^{\delta}(t)\right)\right)}{X_{inc}^{\delta}(t)}, \\
\varphi_M^*(t) &= 1 - \varphi_S^*(t), \\
\varphi_S^{PI}(t) &= \frac{1}{1-\gamma}\frac{\theta_S^{PI}(t)}{\sigma_S} + \frac{1}{\gamma-1}\frac{\theta_S^{PI}(t)}{\sigma_S}qe^{-M^{H^{PI}}(t,T)+\frac{1}{2}V^{H^{PI}}(t,T)}\times \\
&\quad \frac{\Phi\left(d_2^{PI}(t)\right)}{X_{inc}^{PI}(t)} \quad \text{and} \\
\varphi_M^{PI}(t) &= 1 - \varphi_S^{PI}(t).
\end{aligned} \tag{4.95}$$

Proof: After specifying $\Theta^*(t)$ and $\Theta^{PI}(t)$ the proof is as of Theorem 4.6, except with $\Theta(t)$ replaced by $\Theta^*(t)$ and $\Theta^{PI}(t)$. □

4.5. Numerical examples

Let us conclude this chapter with numerical examples. Table 4.1 shows the parameters considered. Figure 4.1 plots the profiles of the optimal terminal wealth of different portfolio managers depending on the deflator $H(T)$ (as given by (4.18)) in a complete market. The managers considered are the Expected Shortfall Portfolio Manager (ES-PM), Portfolio Insurer Portfolio Manager (PI-PM)

Table 4.1.: Table of parameters

a	0.4077	$\bar{\mu}_B$	0.02	γ	0.5
$b^{\mathbb{P}}$	0.0212	$\bar{\mu}_S$	0.03	x	1
σ_r	0.0457	T	10	q	0.95
σ_{Sr}	-0.015	$\widetilde{T}$	12		

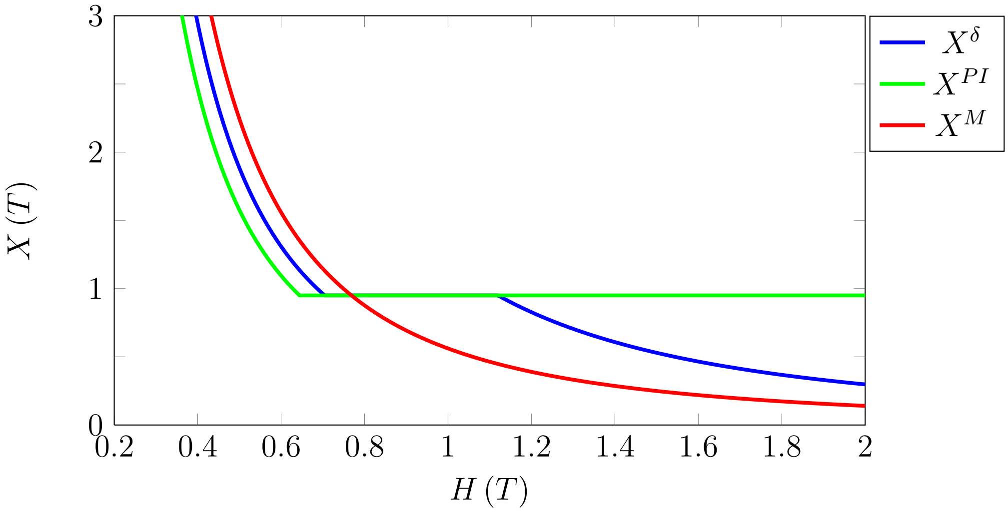

Figure 4.1.: Terminal wealth for $\delta = 0.1$.

and Merton Portfolio Manager (M-PM), with their optimal terminal wealth denoted by X^{δ}, X^{PI} and X^{M} respectively. The upper limit of ES-PM's risk δ is set to 0.1. Before we compare the performances of these managers let us first separate $H(T)$ into three regions: $H(T) \in \left(0, \frac{q^{\gamma-1}}{y_1^*}\right]$ (i.e. good state), $H(T) \in \left(\frac{q^{\gamma-1}}{y_1^*}, \frac{q^{\gamma-1}}{y_1^*-y_2^*}\right]$ (i.e. intermediate state) and $H(T) \in \left(\frac{q^{\gamma-1}}{y_1^*-y_2^*}, \infty\right)$ (i.e. bad state). In the good state M-PM outperforms both ES-PM and PI-PM. In the intermediate state ES-PM behaves like PI-PM. In the bad state PI-PM outperforms both ES-PM and M-PM, because PI-PM totally insures himself/herself against any losses. In the same region ES-PM partially insures himself/herself against losses and M-PM is totally exposed to all losses. If the value of δ is increased, then the behavior of ES-PM tends to M-PM as it is shown in the Figure 4.2 when δ is set to 0.2. If δ is decreased, then the behavior of ES-PM tends to PI-PM.

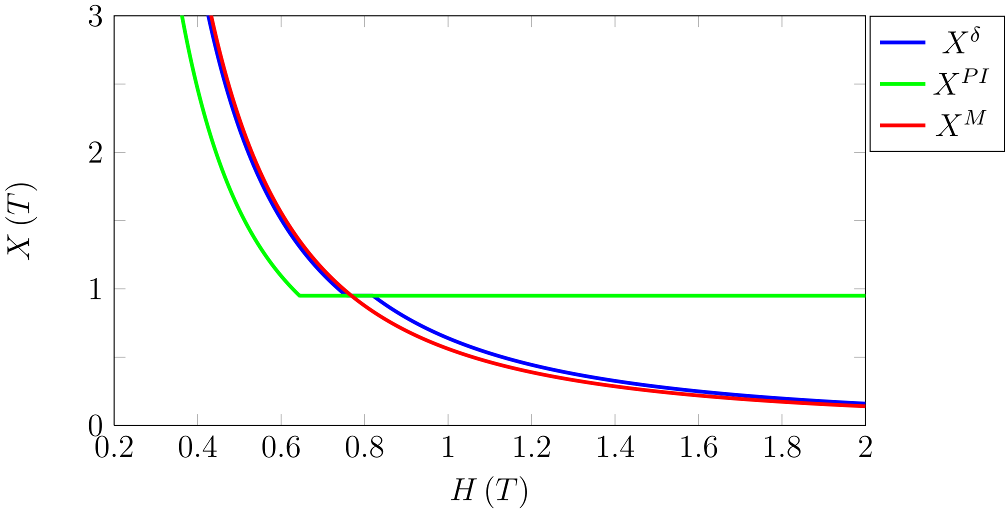

Figure 4.2.: Terminal wealth for $\delta = 0.2$.

The profiles of the optimal terminal wealth in an incomplete market case look similar to the complete market, since considering an incomplete market changes the value of $H(T)$ and the optimal terminal wealth are given as functions of $H(T)$.

Figures 4.3 - 4.8 depict the optimal trading strategies as functions of time t in both complete and incomplete markets without risk constraint. $\varphi^*_{M,co}$, $\varphi^*_{B,co}$, $\varphi^*_{S,co}$ and $\varphi^*_{R,co}$ denote the optimal fractions of wealth invested in money market account M, bond B, stock S and risky assets R (i.e., $\varphi^*_{R,co} = 1 - \varphi^*_{M,co}$) respectively in the complete market case. $\varphi^*_{M,inc}$, $\varphi^*_{S,inc}$ and $\varphi^*_{R,inc}$ refer to the optimal fractions of wealth invested in M, S and R (i.e., $\varphi^*_{R,co} = 1 - \varphi^*_{M,inc}$) respectively in an incomplete market case. In Figure 4.3 we set $\gamma = 0.7$ and $\sigma_{Sr} = -0.25$. We can see that $\varphi^*_{B,co}$ as well as $\varphi^*_{S,inc}$ increase and $\varphi^*_{M,co}$ as well as $\varphi^*_{M,inc}$ decrease as t increases. The increase and the decrease get sharper as t tends to T. This figure illustrates that in a complete market money is borrowed from M and S, and invested in B, whereas in an incomplete market the positions of an investor in M and S are positive as it is clearly shown in Figure 4.4. If $\gamma = 0.7$ and $\sigma_{Sr} = 0.25$, Figure 4.5 reveals that B and S are more attractive than when σ_{Sr} is negative, i.e., more money is taken from M and invested in B and S.

If $\gamma = -20$ and $\sigma_{Sr} = -0.25$, Figure 4.6 shows that $\varphi^*_{M,co}$ as well as $\varphi^*_{M,inc}$

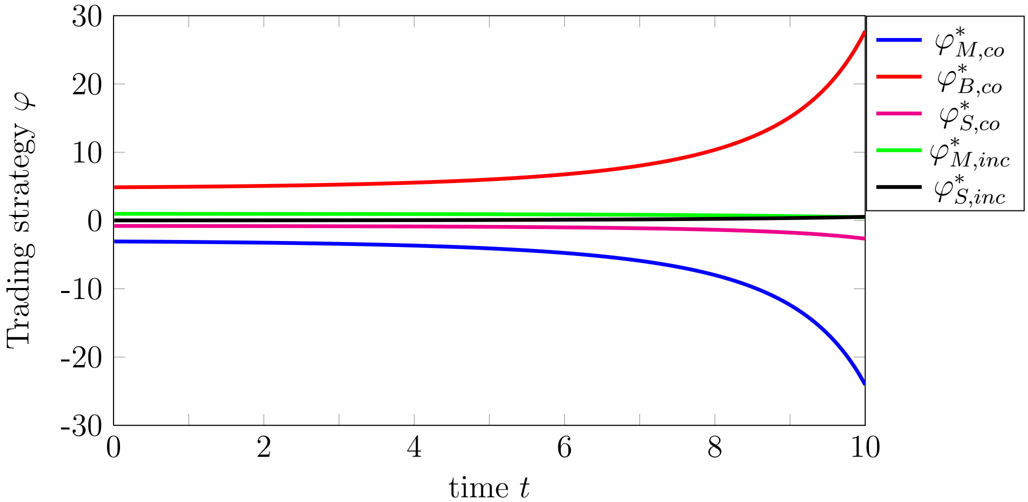

Figure 4.3.: Optimal trading strategies for $\gamma = 0.7$ and $\sigma_{Sr} = -0.25$.

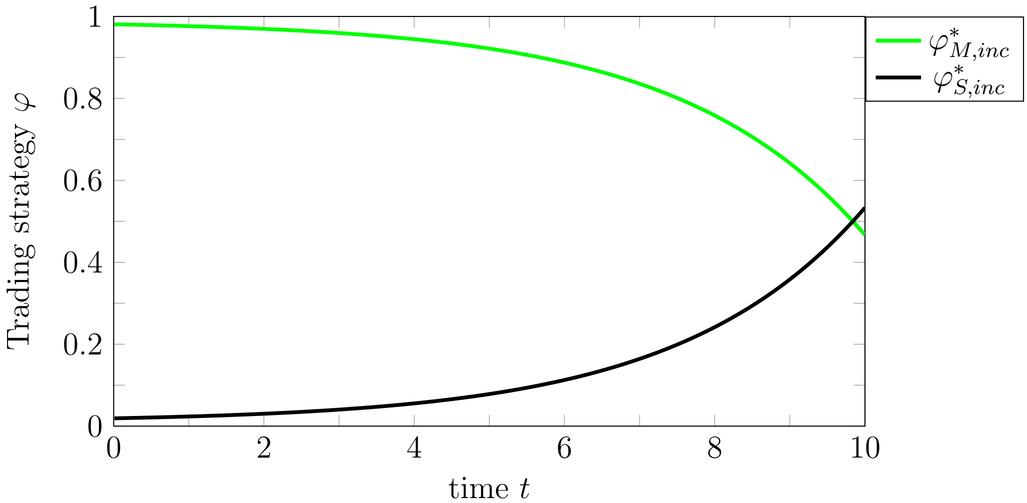

Figure 4.4.: Optimal trading strategies in an incomplete market for $\gamma = 0.7$ and $\sigma_{Sr} = -0.25$.

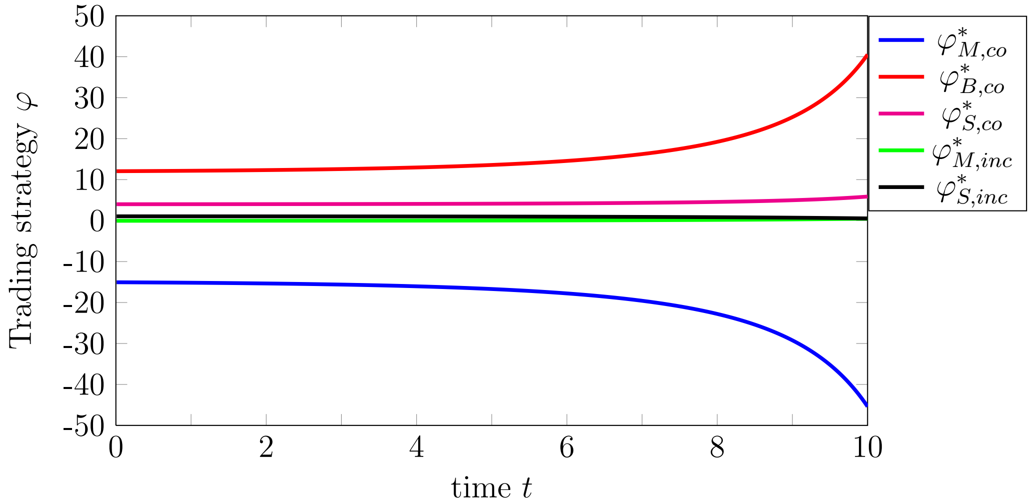

Figure 4.5.: Optimal trading strategies for $\gamma = 0.7$ and $\sigma_{Sr} = 0.25$.

increase and $\varphi^*_{B,co}$, $\varphi^*_{S,co}$ as well as $\varphi^*_{S,inc}$ decrease as t increases. The increase or decrease of fractions of wealth get sharper as t tends to T. We can see further that $\varphi^*_{M,inc}$ is always greater than $\varphi^*_{S,inc}$. Moreover, the differences in the fractions of wealth in both complete and incomplete market are smaller than for the case of $\gamma = 0.7$. For comparison of the fractions of wealth invested in risky assets in the case of complete and incomplete market Figures 4.7 and 4.8 show $\varphi^*_{R,co}$ and $\varphi^*_{R,inc}$ for $\gamma = 0.7$ and $\sigma_{Sr} = -0.25$, and $\gamma = -20$ and $\sigma_{Sr} = -0.25$.

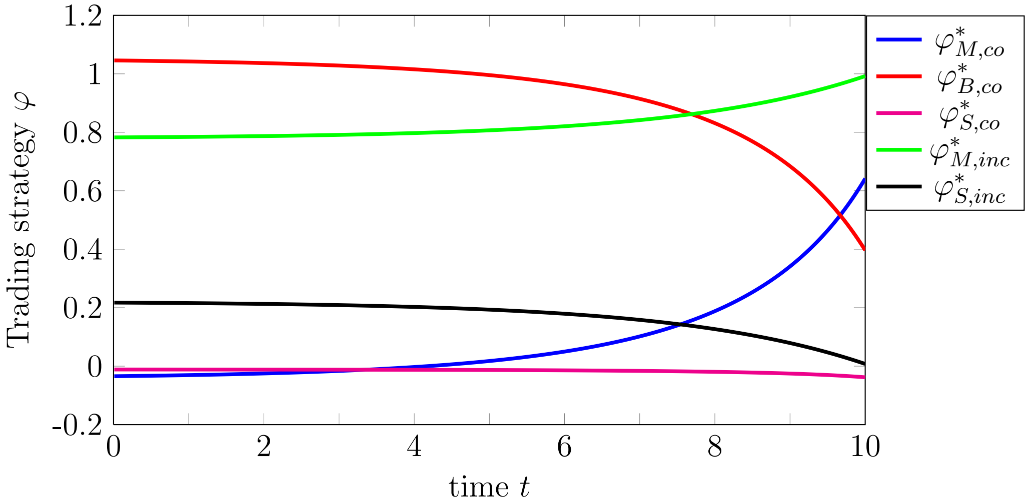

Figure 4.6.: Optimal trading strategies for $\gamma = -20$ and $\sigma_{Sr} = -0.25$.

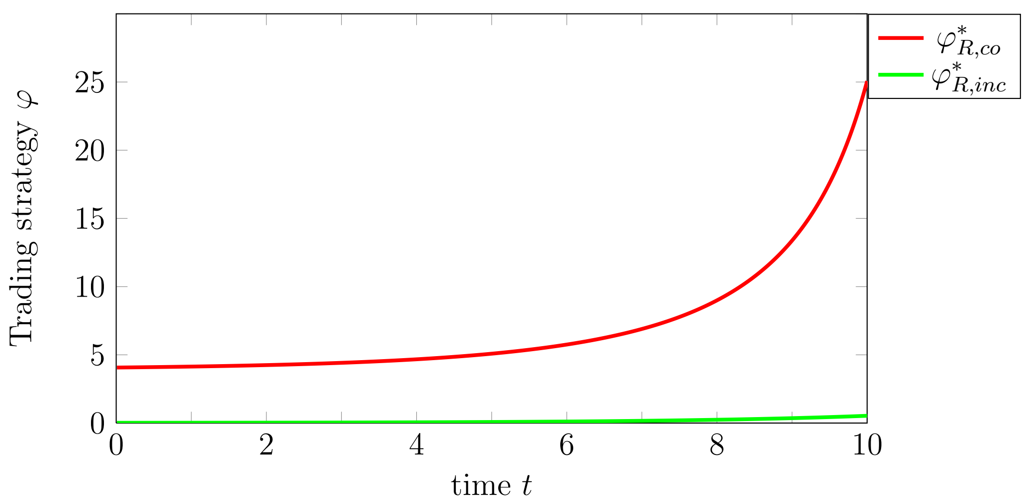

Figure 4.7.: Optimal trading strategies for $\gamma = 0.7$ and $\sigma_{Sr} = -0.25$.

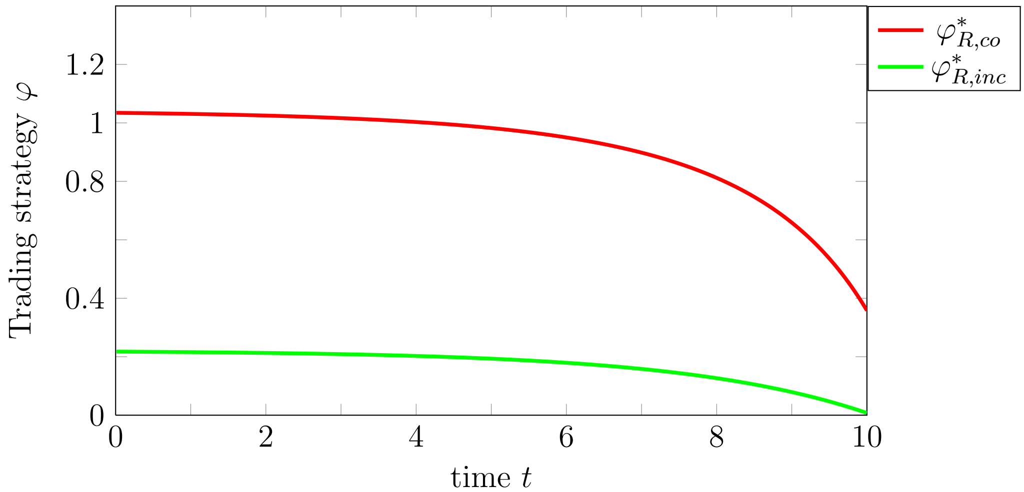

Figure 4.8.: Optimal trading strategies for $\gamma = -20$ and $\sigma_{Sr} = -0.25$.

5. Portfolio optimization with a two-factor Vasiček model

In this chapter we extend the results of the previous one-factor to the two-factor Vasiček model. Recall that in the previous chapter we considered the interest rates to be described by the one-factor Vasiček model. Before we go further, let us first motivate two-factor models of interest rates by pointing out some drawbacks of the one-factor models in general. In one-factor models, the interest rate $r(t)$ is the principal coordinate with which the yield curve can be characterized and which in turn characterizes bond prices. Recall from the previous chapter that the Vasiček model assumes the evolution of $r(t)$ to be given by the stochastic differential equation

$$dr(t) = a(b - r(t))\,dt + \sigma_r dW_r(t)\,.$$

Then, the bond price at time $t \in [0, T]$ maturing at time T is given by

$$B(t,T) = \exp\{m(t,T) - n(t,T)\,r(t)\}$$

and the continuously compounded spot rates are given by

$$R(t,T) = \frac{m(t,T) - n(t,T)\,r(t)}{T-t} =: a(t,T) + b(t,T)\,r(t)\,,$$

where $n(t,T)$ is as given in the Remark 4.2 and

$$m(t,T) = \frac{\sigma_r^2}{2}\int_t^T (n(u,T))^2\,du - ab\int_t^T n(u,T)\,du.$$

Now, if we consider a payoff depending on the joint distribution of two such rates, of different maturities (T_1 and T_2), at time $t < T_1 < T_2$, then the correlation between these two rates is computed as

$$\begin{aligned}&Corr\,(R(t,T_1)\,, R(t,T_1))\\ &= Corr\,(a(t,T_1) + b(t,T_1)\,r(t)\,, a(t,T_2) + b(t,T_2)\,r(t)) = 1.\end{aligned}$$

That means, in particular that even if $T_1 \ll T_2$ (say $T_1 = t+1, T_2 = t+30$) interest rates are perfectly correlated. Thus all bonds are perfectly correlated. Therefore, a shock to the interest rate curve is transmitted equally and almost rigidly through all the maturities. In the real world, interest rates are known to exhibit some decorrelation. If correlations play a more relevant role, or if higher precision is needed, we need to use multifactor short rate models.

Jamshidian and Zhu in [38] (table 1), in their empirical study using JPY, USD and DEM as data, showed that, under the objective measure, the one-factor model can explain from 67.7% to 75.8% of the total variations.The two-factor model can explain from 83.9% to 91.2% of the total variations, whereas the three-factor model can explain from 93% to 94.3%. Rebonato in [56] (table 3.2) carried out a related study on the UK market and came up with more optimistic results, that a one-factor model can explain 92.17% of the total variations and a two-factor model already explains 99.10% of the total variations, whereas a three-factor model explains 99.71% of the total variations.

What is done with one-factor or two-factor models can be extended to three or more-factor models. But as the factors of the model are increased, the analytic tractability of the model is usually much reduced. In this chapter we consider a two-factor model: The two-factor Vasiček.

We will investigate in this chapter the portfolio optimization problem with and without risk constraint in both a complete and an incomplete market in the view of a two-factor Vasiček model of interest rate.

The complete financial market model

We assume that the uncertainty in the market is modeled by a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$. $\mathcal{F}$ denotes the augmented filtration generated by a 3-dimensional standard Brownian motion

$$W^{\mathbb{P}}(t) = \left(W_r^{\mathbb{P}}(t), W_S^{\mathbb{P}}(t)\right)^\top = \left(W_1^{\mathbb{P}}(t), W_2^{\mathbb{P}}(t), W_S^{\mathbb{P}}(t)\right)^\top$$

under the probability measure $\mathbb{P}$, with $W_r^{\mathbb{P}}(t) = \left(W_1^{\mathbb{P}}(t), W_2^{\mathbb{P}}(t)\right)^\top$. The components of $W^{\mathbb{P}}(t)$ are assumed to be independent.

We consider an arbitrage free financial market where a money market account M (a saving's account), two bonds B_1 and B_2, with the maturities T_1 and T_2, respectively, and a stock S are continuously traded over a fixed finite time-horizon $[0, T]$. We assume that $T_1, T_2 > T$ and $T_1 \neq T_2$.

The price dynamics $M(t)$ of the money market account at time $t \in [0, T]$ are modeled by

$$dM(t) = M(t)\, r(t)\, dt, \text{ with } M(0) = 1, \tag{5.1}$$

as in the previous chapters, which results into

$$M(t) = \exp\left(\int_0^t r(s) ds\right), \tag{5.2}$$

and the discounting factor

$$\beta(t, T) := \frac{M(t)}{M(T)} = \exp\left(-\int_t^T r(s) ds\right). \tag{5.3}$$

As we have mentioned before, throughout this chapter we will consider the interest rate $r(t)$ in Equation (5.2) to be modeled by a two-factor Vasiček model, described by

$$r(t) = \epsilon_0 + \epsilon_1 Y_1(t) + \epsilon_2 Y_2(t)$$

where ϵ_0, ϵ_1 and ϵ_2 are constants, and $Y_1(t)$ and $Y_2(t)$ are factors given by the following system of SDEs

$$\begin{aligned} dY_1(t) &= \underbrace{\left(\nu_1^{\mathbb{P}}(t) - b_{11} Y_1(t) - b_{12} Y_2(t)\right)}_{=\mu_1^{\mathbb{P}}} dt + \sigma_1 dW_1^{\mathbb{P}}(t) \\ dY_2(t) &= \underbrace{\left(\nu_2^{\mathbb{P}}(t) - b_{21} Y_1(t) - b_{22} Y_2(t)\right)}_{=\mu_2^{\mathbb{P}}} dt + \sigma_2 dW_2^{\mathbb{P}}(t). \end{aligned}$$

We assume that the matrix

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

has strictly positive eigenvalues a_1 and a_2 to guarantee the mean-reversion of the factors $Y_1(t)$ and $Y_2(t)$.

By Itô's formula, the price processes $B(t, T_1)$ and $B(t, T_2)$ of the bonds B_1 and B_2, maturing at time T_1 and T_2 respectively, evolve according to the following

SDEs (see [65])

$$
\begin{aligned}
dB(t,T_1) &= B(t,T_1)\left[\mu_{1B}(t,T_1)\,dt + \sigma_{11}(t,T_1)\,dW_1^{\mathbb{P}}(t) + \sigma_{12}(t,T_1)\,dW_2^{\mathbb{P}}(t)\right] \\
dB(t,T_2) &= B(t,T_2)\left[\mu_{2B}(t,T_2)\,dt + \sigma_{21}(t,T_2)\,dW_1^{\mathbb{P}}(t) + \sigma_{22}(t,T_2)\,dW_2^{\mathbb{P}}(t)\right],
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
\mu_{1B}(t,T_1) =& \frac{1}{B(t,T_1)}\left[\frac{\partial B(t,T_1)}{\partial t} + \mu_1^{\mathbb{P}}\frac{\partial B(t,T_1)}{\partial Y_1} + \mu_2^{\mathbb{P}}\frac{\partial B(t,T_1)}{\partial Y_2}\right] \\
&+ \frac{1}{B(t,T_1)}\left[\frac{\sigma_1^2}{2}\frac{\partial^2 B(t,T_1)}{\partial Y_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 B(t,T_1)}{\partial Y_2^2}\right], \\
\mu_{2B}(t,T_2) =& \frac{1}{B(t,T_2)}\left[\frac{\partial B(t,T_2)}{\partial t} + \mu_1^{\mathbb{P}}\frac{\partial B(t,T_2)}{\partial Y_1} + \mu_2^{\mathbb{P}}\frac{\partial B(t,T_2)}{\partial Y_2}\right] \\
&+ \frac{1}{B(t,T_2)}\left[\frac{\sigma_1^2}{2}\frac{\partial^2 B(t,T_2)}{\partial Y_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 B(t,T_2)}{\partial Y_2^2}\right], \\
\sigma_{11}(t,T_1) =& \frac{\sigma_1}{B(t,T_1)}\frac{\partial B(t,T_1)}{\partial Y_1}, \quad \sigma_{12}(t,T_1) = \frac{\sigma_2}{B(t,T_1)}\frac{\partial B(t,T_1)}{\partial Y_2}, \\
\sigma_{21}(t,T_2) =& \frac{\sigma_1}{B(t,T_2)}\frac{\partial B(t,T_2)}{\partial Y_1} \text{ and } \sigma_{22}(t,T_2) = \frac{\sigma_2}{B(t,T_2)}\frac{\partial B(t,T_2)}{\partial Y_2}.
\end{aligned}
$$

Another asset we have in the market is a stock S. Its price process $S(t)$ at time $t \in [0,T]$ is assumed to be determined by

$$
dS(t) = S(t)\left[\mu_S^{\mathbb{P}}(t)\,dt + \sigma_{1S}dW_1^{\mathbb{P}}(t) + \sigma_{2S}dW_2^{\mathbb{P}}(t) + \sigma_S dW_S^{\mathbb{P}}(t)\right]. \tag{5.5}
$$

We refer to $\mu_S^{\mathbb{P}}(t)$ as the drift of the stock under the probability measure $\mathbb{P}$ and the constants σ_{1S}, σ_{2S} and σ_S as the volatility of the stock w.r.t. $W_1^{\mathbb{P}}$, $W_2^{\mathbb{P}}$ and $W_S^{\mathbb{P}}$, respectively. Recall from Assumption 2.2 that

$$
\begin{aligned}
\mu_{1B}(t,T_1) &= \bar{\mu}_{1B} + r(t) \\
\mu_{2B}(t,T_2) &= \bar{\mu}_{2B} + r(t) \\
\mu_S^{\mathbb{P}}(t) &= \bar{\mu}_S + r(t),
\end{aligned}
$$

with $\bar{\mu}_{1B}$, $\bar{\mu}_{2B}$ and $\bar{\mu}_S$ considered to be constants. Note that $\bar{\mu}_{1B}, \bar{\mu}_{2B}$ may also be allowed to depend on T_1, T_2.

Remark 5.1 *Due to the assumption of the arbitrage free financial market and Theorem 2.7 we know that $\mathcal{M}^e \neq \emptyset$. We shall characterize $\tilde{\mathbb{Q}} \in \mathcal{M}^e$ by its*

Radon-Nikodym derivative $Z^{\tilde{\Theta}}(t) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ *defined by*

$$Z^{\tilde{\Theta}}(t) := \left(\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\right)_t = \exp\left\{-\frac{1}{2}\int_0^t \left\|\tilde{\Theta}(s)\right\|^2 ds - \int_0^t \tilde{\Theta}^\top(s)\, dW^{\mathbb{P}}(s)\right\},$$

for some $\tilde{\Theta}(t) = \left(\tilde{\theta}_1(t), \tilde{\theta}_2(t), \tilde{\theta}_S(t)\right)^\top$ *solving*

$$\begin{aligned}
\bar{\mu}_{1B} &= \tilde{\theta}_1(t)\,\sigma_{11}(t,T_1) + \tilde{\theta}_2(t)\,\sigma_{12}(t,T_1)\\
\bar{\mu}_{2B} &= \tilde{\theta}_1(t)\,\sigma_{21}(t,T_2) + \tilde{\theta}_2(t)\,\sigma_{22}(t,T_2)\\
\bar{\mu}_S &= \tilde{\theta}_1(t)\,\tilde{\sigma}_{1S} + \tilde{\theta}_2(t)\,\tilde{\sigma}_{2S} + \tilde{\theta}_S(t)\,\tilde{\sigma}_S,
\end{aligned} \tag{5.6}$$

where $\tilde{\theta}_1(t)$, $\tilde{\theta}_2(t)$ *and* $\tilde{\theta}_S(t)$ *are interpreted as the market prices of risk due to the factors* Y_1, Y_2 *and the stock* S*, respectively.*

Note that $\tilde{\Theta}(t) = \left(\tilde{\theta}_1(t), \tilde{\theta}_2(t), \tilde{\theta}_S(t)\right)^\top$ corresponds to $\tilde{\mathbb{Q}} \in \mathcal{M}^e$. In the system the Equations (5.6) we have three unknowns ($\tilde{\theta}_1$, $\tilde{\theta}_2$ and $\tilde{\theta}_S$) and three equations. So the values of these unknowns exist and can be determined uniquely, if the volatility matrix

$$\sigma(t) = \begin{pmatrix} \sigma_{11}(t,T_1) & \sigma_{12}(t,T_1) & 0\\ \sigma_{21}(t,T_2) & \sigma_{22}(t,T_2) & 0\\ \tilde{\sigma}_{1S} & \tilde{\sigma}_{2S} & \tilde{\sigma}_S \end{pmatrix}$$

is non-singular, which implies we have one equivalent martingale measure $\tilde{\mathbb{Q}}$ with respect to the probability measure $\mathbb{P}$ and that means we are in a complete market.

According to the Girsanov Theorem,

$$\widetilde{W}^{\tilde{\mathbb{Q}}}(t) = \begin{pmatrix} \widetilde{W}_1^{\tilde{\mathbb{Q}}}(t)\\ \widetilde{W}_2^{\tilde{\mathbb{Q}}}(t)\\ \widetilde{W}_S^{\tilde{\mathbb{Q}}}(t)\end{pmatrix} := \begin{pmatrix} W_1^{\mathbb{P}}(t) + \int_0^t \tilde{\theta}_1(s)\,ds\\ W_2^{\mathbb{P}}(t) + \int_0^t \tilde{\theta}_2(s)\,ds\\ W_S^{\mathbb{P}}(t) + \int_0^t \tilde{\theta}_S(s)\,ds\end{pmatrix}$$

are Brownian motions under the probability measure $\tilde{\mathbb{Q}}$. The dynamics of the interest rate factors Y_1 and Y_2 under the measure $\tilde{\mathbb{Q}}$ are given by

$$\begin{aligned}
&dY_1(t) = \left(\tilde{\nu}_1 - b_{11}Y_1(t) - b_{12}Y_2(t)\right)dt + \sigma_1 d\widetilde{W}_1^{\tilde{\mathbb{Q}}}(t)\\
&dY_2(t) = \left(\tilde{\nu}_2 - b_{21}Y_1(t) - b_{22}Y_2(t)\right)dt + \sigma_2 d\widetilde{W}_2^{\tilde{\mathbb{Q}}}(t) \quad \text{and}\\
&r(t) = \tilde{\epsilon}_0 + \tilde{\epsilon}_1 Y_1(t) + \tilde{\epsilon}_2 Y_2(t),
\end{aligned} \tag{5.7}$$

where $\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\epsilon}_0, \tilde{\epsilon}_1, \tilde{\epsilon}_2$ and matrix $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ are considered to be constants. The strictly positive eigenvalues a_1 and a_2 of the matrix B assumed above guarantee the mean-reversion of the canonical factors $r_1(t)$ and $r_2(t)$ in Equation (5.8) as well. The two-factor Vasiček model in its normal form (Equation (5.7)) seems to be over-parametrized. To avoid this over-parametrization, the model (5.7) can be transformed to its canonical form (see for example [61]):

$$\begin{aligned} dr_1(t) &= -a_1 r_1(t)\,dt + dW_1^{\hat{\mathbb{Q}}}(t) \\ dr_2(t) &= \left(-a_{21} r_1(t) - a_2 r_2(t)\right)dt + dW_2^{\hat{\mathbb{Q}}}(t) \quad \text{and} \\ r(t) &= \delta_0 + \delta_1 r_1(t) + \delta_2 r_2(t), \end{aligned} \tag{5.8}$$

where $a_1 > 0, a_{21} > 0, a_2 > 0, \delta_0, \delta_1$ and δ_2 are constants and the process $W^{\hat{\mathbb{Q}}}(t)$ defined by

$$W^{\hat{\mathbb{Q}}}(t) = \begin{pmatrix} W_1^{\hat{\mathbb{Q}}}(t) \\ W_2^{\hat{\mathbb{Q}}}(t) \\ W_S^{\hat{\mathbb{Q}}}(t) \end{pmatrix} := \begin{pmatrix} W_1^{\mathbb{P}}(t) + \int_0^t \hat{\theta}_1(s)\,ds \\ W_2^{\mathbb{P}}(t) + \int_0^t \hat{\theta}_2(s)\,ds \\ W_S^{\mathbb{P}}(t) + \int_0^t \hat{\theta}_S(s)\,ds \end{pmatrix}.$$

is a Brownian motion under the equivalent martingale measure $\hat{\mathbb{Q}}$ w.r.t. the probability measure $\mathbb{P}$, characterized by

$$Z^{\hat{\Theta}}(t) := \left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_t = \exp\left\{-\frac{1}{2}\int_0^t \left\|\hat{\Theta}(s)\right\|^2 ds - \int_0^t \hat{\Theta}^\top(s)\,dW^{\mathbb{P}}(s)\right\}. \tag{5.9}$$

Thus, the price processes of the bonds B_1 and B_2 under the probability measure $\hat{\mathbb{Q}}$ are described by

$$\begin{aligned} \frac{dB(t,T_1)}{B(t,T_1)} &= r(t)\,dt + \hat{\sigma}_{11}(t,T_1)\,dW_1^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_{12}(t,T_1)\,dW_2^{\hat{\mathbb{Q}}}(t) \quad \text{and} \\ \frac{dB(t,T_2)}{B(t,T_2)} &= r(t)\,dt + \hat{\sigma}_{21}(t,T_2)\,dW_1^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_{22}(t,T_2)\,dW_2^{\hat{\mathbb{Q}}}(t), \\ &\text{where} \\ \hat{\sigma}_{i1}(t,T_i) &= -\frac{1}{a_1}\left(\delta_1 - \frac{a_{21}\delta_2}{a_2}\right)\left(1 - e^{-a_1(T_i-t)}\right) \\ &\quad - \frac{a_{21}\delta_2}{a_2(a_1-a_2)}\left(e^{-a_2(T_i-t)} - e^{-a_1(T_i-t)}\right) \quad \text{and} \\ \hat{\sigma}_{i2}(t,T_i) &= -\frac{\delta_2}{a_2}\left(1 - e^{-a_2(T_i-t)}\right) \quad \text{for} \quad i = \{1,2\}. \end{aligned} \tag{5.10}$$

The price process of the stock under $\hat{\mathbb{Q}}$ can be transformed to

$$dS(t) = S(t)\left[r(t)\,dt + \hat{\sigma}_{1S}dW_1^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_{2S}dW_2^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_S dW_S^{\hat{\mathbb{Q}}}(t)\right]. \tag{5.11}$$

The deflator $\hat{H}(T)$ for the cash flow paid at T is defined by

$$\begin{aligned}\hat{H}(T) &:= \beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \\ &= \exp\left\{-\int_0^T r(t)\,dt\right\}\exp\left\{-\frac{1}{2}\int_0^T \left\|\hat{\Theta}(t)\right\|^2 dt - \int_0^T \hat{\Theta}^\top(t)\,dW^{\mathbb{P}}(t)\right\} \\ &= \exp\left\{-\int_0^T r(t)\,dt - \frac{1}{2}\int_0^T \left\|\hat{\Theta}(t)\right\|^2 dt - \int_0^T \hat{\Theta}^\top(t)\,dW^{\mathbb{P}}(t)\right\},\end{aligned} \tag{5.12}$$

where $\hat{\Theta}(t) = \left(\hat{\theta}_1(t), \hat{\theta}_2(t), \hat{\theta}_S(t)\right)$.

As in Chapter 4 we will consider a complete and an incomplete market setting.

- In the complete market setting (Sections 5.1, 5.2) we allow investment in S, B_1, B_2, M and thus, $\hat{\Theta}$ is the unique solution of

$$\begin{aligned}\bar{\mu}_{1B} &= \hat{\theta}_1(t)\,\hat{\sigma}_{11}(t,T_1) + \hat{\theta}_2(t)\,\hat{\sigma}_{12}(t,T_1) \\ \bar{\mu}_{2B} &= \hat{\theta}_1(t)\,\hat{\sigma}_{21}(t,T_2) + \hat{\theta}_2(t)\,\hat{\sigma}_{22}(t,T_2) \\ \bar{\mu}_S &= \hat{\theta}_1(t)\,\hat{\sigma}_{1S} + \hat{\theta}_2(t)\,\hat{\sigma}_{2S} + \hat{\theta}_S(t)\,\hat{\sigma}_S.\end{aligned} \tag{5.13}$$

- In the incomplete market setting (Sections 5.3, 5.4), we do not allow investment in B_2 and thus, $\hat{\Theta}$ will be a solution of

$$\begin{aligned}\bar{\mu}_{1B} &= \hat{\theta}_1(t)\,\hat{\sigma}_{11}(t,T_1) + \hat{\theta}_2(t)\,\hat{\sigma}_{12}(t,T_1) \\ \bar{\mu}_S &= \hat{\theta}_1(t)\,\hat{\sigma}_{1S} + \hat{\theta}_2(t)\,\hat{\sigma}_{2S} + \hat{\theta}_S(t)\,\hat{\sigma}_S.\end{aligned} \tag{5.14}$$

We begin with the complete market case and consider an investor or a portfolio manager endowed with a positive initial wealth x at time $t = 0$ for the investment in the assets in the financial market. The fractions of wealth invested in B_1, B_2 and S at time $t \in [0,T]$ are denoted by the vector $\hat{\varphi}_c(t) = (\hat{\varphi}_1(t), \hat{\varphi}_2(t), \hat{\varphi}_S(t))^\top$ whose first component is for B_1, second for B_2 and the third for S. According to Assumption 2.3, the fraction of wealth $\hat{\varphi}_M(t)$ invested in M is given by

$\left(1-\hat{\varphi}_c(t)^\top \mathbb{1}_3\right)$. The wealth process of an investor at time $t \in [0,T]$, denoted by $X_c(t)$, under the probability measure $\hat{\mathbb{Q}}$, is governed by the following SDE:

$$\begin{aligned}
dX_c(t) &= dX^M(t) + dX^{B_1}(t) + dX^{B_2}(t) + dX^S(t) \\
&= \left(1-\varphi_c(t)^\top \mathbb{1}_3\right) X_c(t)\, r(t)\, dt \\
&\quad + X_c(t)\,\hat{\varphi}_1(t)\left(r(t)\,dt + \hat{\sigma}_{11} dW_1^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_{12} dW_2^{\hat{\mathbb{Q}}}(t)\right) \\
&\quad + X_c(t)\,\hat{\varphi}_2(t)\left(r(t)\,dt + \hat{\sigma}_{21} dW_1^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_{22} dW_2^{\hat{\mathbb{Q}}}(t)\right) \\
&\quad + X_c(t)\,\hat{\varphi}_S(t)\left(r(t)\,dt + \hat{\sigma}_{1S} dW_1^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_{2S} dW_2^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_S dW_S^{\hat{\mathbb{Q}}}(t)\right) \\
&= X_c(t)\left[r(t)\,dt + \hat{\varphi}_S(t)\,\hat{\sigma}_S dW_S^{\hat{\mathbb{Q}}}(t)\right] \\
&\quad + X_c(t)\left(\hat{\varphi}_S(t)\,\hat{\sigma}_{1S} + \hat{\varphi}_1(t)\,\hat{\sigma}_{11} + \hat{\varphi}_2(t)\,\hat{\sigma}_{21}\right) dW_1^{\hat{\mathbb{Q}}}(t) \\
&\quad + X_c(t)\left(\hat{\varphi}_S(t)\,\hat{\sigma}_{2S} + \hat{\varphi}_1(t)\,\hat{\sigma}_{12} + \hat{\varphi}_2(t)\,\hat{\sigma}_{22}\right) dW_2^{\hat{\mathbb{Q}}}(t),
\end{aligned} \tag{5.15}$$

where $X^M(t), X^{B_1}(t), X^{B_2}(t)$ and $X^S(t)$ refer to the amount of wealth in the money market account M, bonds B_1 and B_2, and the stock S, respectively. By applying Ito's rule to the process $\beta(t)\,X_c(t)$ we get

$$\begin{aligned}
d\left(\beta(t)\,X_c(t)\right) &= X_c(t)\,\beta(t)\left(\hat{\varphi}_S(t)\,\hat{\sigma}_{1S} + \hat{\varphi}_1(t)\,\hat{\sigma}_{11} + \hat{\varphi}_2(t)\,\hat{\sigma}_{21}\right) dW_1^{\hat{\mathbb{Q}}}(t) \\
&\quad + X_c(t)\,\beta(t)\left(\hat{\varphi}_S(t)\,\hat{\sigma}_{2S} + \hat{\varphi}_1(t)\,\hat{\sigma}_{12} + \hat{\varphi}_2(t)\,\hat{\sigma}_{22}\right) dW_2^{\hat{\mathbb{Q}}}(t) \\
&\quad + X_c(t)\,\beta(t)\,\hat{\varphi}_S(t)\,\hat{\sigma}_S dW_S^{\hat{\mathbb{Q}}}(t)
\end{aligned}$$

and thus

$$\begin{aligned}
\beta(T)\,X_c(T) &= X_c(0) \\
&\quad + \int_0^T X_c(t)\,\beta(t)\left(\hat{\varphi}_S(t)\,\hat{\sigma}_{1S} + \hat{\varphi}_1(t)\,\hat{\sigma}_{11} + \hat{\varphi}_2(t)\,\hat{\sigma}_{21}\right) dW_1^{\hat{\mathbb{Q}}}(t) \\
&\quad + \int_0^T X_c(t)\,\beta(t)\left(\hat{\varphi}_S(t)\,\hat{\sigma}_{2S} + \hat{\varphi}_1(t)\,\hat{\sigma}_{12} + \hat{\varphi}_2(t)\,\hat{\sigma}_{22}\right) dW_2^{\hat{\mathbb{Q}}}(t) \\
&\quad + \int_0^T X_c(t)\,\beta(t)\,\hat{\varphi}_S(t)\,\hat{\sigma}_S dW_S^{\hat{\mathbb{Q}}}(t).
\end{aligned} \tag{5.16}$$

5.1. Optimization without risk constraints in a complete market

In this section we investigate the portfolio optimization problem without risk constraints in a complete market, as studied in section 4.1, but under the two-factor Vasiček model. The dynamic and static optimization problems in these settings are stated as in Section 4.1 and we will consider the assumptions and definitions made therein.

The Dynamic portfolio optimization problem is restated as

$$\max_{\varphi\in\mathcal{A}(x)} \mathbb{E}\left(\frac{(X_c(T))^{\gamma}}{\gamma}\right)$$
$$s.t.\;\; \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)X_c(T)\right] = \mathbb{E}\left[\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T X_c(T)\right] \leq x \quad \text{(budget constraint)}, \tag{5.17}$$

with $\hat{\mathbb{Q}}$ given by $\hat{\Theta}$ as in (5.13).

Static optimization problem:

$$\begin{aligned} &\max_{X_c(T)\in\overline{\mathcal{B}}(x)} \mathbb{E}\left[\frac{(X_c(T))^{\gamma}}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}\left[\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T X_c(T)\right] = x, \end{aligned} \tag{5.18}$$

where

$$\overline{\mathcal{B}}(x) := \left\{X_c(T) > 0 \mid X_c(T)\;\; \mathcal{F}_T\text{-measurable}, \mathbb{E}\left[\frac{(X_c(T))^{\gamma}}{\gamma}\right]^{-} < \infty\right\}$$

and the budget constraint is binding.

Proposition 5.2 *The solution to the static optimization problem (5.18) is given by*

$$X_c^*(T) = \left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}, \tag{5.19}$$

with y^* obtained through

$$\mathbb{E}\left[y^{\frac{1}{\gamma-1}}\left(\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}}\right]=x.$$

Proof: The proof is as of Theorem 3.2 since the optimal terminal wealth is given as a function of deflator. We only have to replace $H(T)$ by $\hat{H}(T)=\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T$.
□

Before we state a theorem that gives a trading strategy which replicates the optimal terminal wealth $X_c^*(T)$, let us first find a more tractable representation of $\int_0^T r(t)\,dt$. Let

$$A=\begin{pmatrix}a_1 & 0\\ a_{21} & a_2\end{pmatrix},\quad A^{-1}=\begin{pmatrix}\tilde{a}_{11} & \tilde{a}_{12}\\ \tilde{a}_{21} & \tilde{a}_{22}\end{pmatrix},\quad W_r^{\hat{\mathbb{Q}}}(t)=\begin{pmatrix}W_1^{\hat{\mathbb{Q}}}(t)\\ W_2^{\hat{\mathbb{Q}}}(t)\end{pmatrix}$$
$$R(t)=\begin{pmatrix}r_1(t)\\ r_2(t)\end{pmatrix},\quad \delta=\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}\quad\text{and}\quad P(t)=R(t)\,e^{A(t-s)},\ \text{for } s>t \text{ fixed.}$$

Using Ito's lemma and (5.8) on $P(t)$ we obtain

$$\begin{aligned}dP(t)&=(dR(t))\,e^{A(t-s)}+Ae^{A(t-s)}R(t)\,dt\\&=e^{A(t-s)}\left(-AR(t)+dW_r^{\hat{\mathbb{Q}}}\right)+Ae^{A(t-s)}R(t)\,dt\\&=e^{A(t-s)}dW_r^{\hat{\mathbb{Q}}}\end{aligned}$$

$$\begin{aligned}&\Rightarrow R(t)\,e^{A(t-s)}-R(s)=\int_s^t e^{A(u-s)}dW_r^{\hat{\mathbb{Q}}}(u)\\&\Rightarrow R(t)=R(s)\,e^{-A(t-s)}+e^{-A(t-s)}\int_s^t e^{A(u-s)}dW_r^{\hat{\mathbb{Q}}}(u)\\&\qquad=R(s)\,e^{-A(t-s)}+\int_s^t e^{-A(t-u)}dW_r^{\hat{\mathbb{Q}}}(u).\end{aligned}$$

$$\int_0^T R(t)\,dt = R(0)\int_0^T e^{-At}dt + \int_0^T \int_0^t e^{-A(t-u)} dW_r^{\hat{\mathbb{Q}}}(u)\,dt.$$

$$\int_0^T e^{-At}dt = A^{-1}\left(I - e^{-AT}\right).$$

$$\begin{aligned}
\int_0^T \int_0^t e^{-A(t-u)} dW_r^{\hat{\mathbb{Q}}}(u)\,dt &= \int_0^T \int_0^t e^{-At}e^{-Au} dW_r^{\hat{\mathbb{Q}}}(u)\,dt \\
&= \int_0^T \int_u^T e^{-At}e^{-Au} dt dW_r^{\hat{\mathbb{Q}}}(u) \\
&= \int_0^T e^{-Au}\int_u^T e^{-At} dt dW_r^{\hat{\mathbb{Q}}}(u) \\
&= \int_0^T A^{-1}\left(I - e^{-A(T-u)}\right) dW_r^{\hat{\mathbb{Q}}}(u)
\end{aligned}$$

Thus,

$$\int_0^T R(t)\,dt = \int_0^T R(0)\,e^{-At}dt + \int_0^T A^{-1}\left(I - e^{-A(T-u)}\right) dW_r^{\hat{\mathbb{Q}}}(u).$$

We transform A to its Jordan canonical form by choosing a non-singular matrix $P := \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ with its inverse denoted by $P^{-1} := \begin{pmatrix} \tilde{p}_{11} & \tilde{p}_{12} \\ \tilde{p}_{21} & \tilde{p}_{22} \end{pmatrix}$ such that

$$J := P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ k & \lambda_2 \end{pmatrix}. \quad \text{It follows that} \quad A = PJP^{-1},$$

and consequently

$$e^A = Pe^J P^{-1}.$$

If $\lambda_1 \neq \lambda_2$, then the columns of P are eigenvectors of A and $k = 0$. If $\lambda_1 = \lambda_2$, then k might be zero, but it can also happen that $k \neq 0$, in which case P might be chosen so that $k = 1$. We consider $\lambda_1 \neq \lambda_2$ and define

$$\Lambda(t) := \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix}.$$

Then

$$\begin{aligned}
\int_0^T R(t)dt &= \int_0^T Pe^{-Jt}P^{-1}R(0)dt + \int_0^T A^{-1}\left(I - Pe^{-J(T-t)}P^{-1}\right)dW_r^{\hat{\mathbb{Q}}}(t) \\
&= \int_0^T P\Lambda(t)\,P^{-1}R(0)dt + \int_0^T A^{-1}\left(I - P\Lambda(T-t)\,P^{-1}\right)dW_r^{\hat{\mathbb{Q}}}(t).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\int_0^T r(t)dt =& \int_0^T \delta_0 dt + \int_0^T \delta^\top P\Lambda(T-t)P^{-1}R(0)dt \\ &+ \int_0^T \delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right)dW_r^{\hat{\mathbb{Q}}}(t) \\ =& \int_0^T \delta_0 dt + \int_0^T \delta^\top P\Lambda(T-t)P^{-1}R(0)dt \\ &+ \int_0^T \delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right)d\left(W_r^{\mathbb{P}}(t) + \int_0^t \hat{\theta}_r(s)ds\right) \\ =& \int_0^T \delta_0 dt + \int_0^T \mu^{\mathbb{P}}(t)dt \\ &+ \int_0^T \delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right)dW_r^{\mathbb{P}}(t)\end{aligned} \tag{5.20}$$

where $\hat{\theta}_r(t) = \begin{pmatrix}\hat{\theta}_1(t)\\ \hat{\theta}_2(t)\end{pmatrix}$ and $W_r^{\mathbb{P}} = \begin{pmatrix}W_1^{\mathbb{P}}\\ W_2^{\mathbb{P}}\end{pmatrix}$,

$$A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right) = \begin{pmatrix} f_{11}(t,T) & f_{12}(t,T)\\ f_{21}(t,T) & f_{22}(t,T)\end{pmatrix}, \tag{5.21}$$

where

$$\begin{aligned} f_{11}(t,T) =& \tilde{a}_{11} - \left(\tilde{a}_{11}p_{11}\tilde{p}_{21} + \tilde{a}_{12}p_{21}\tilde{p}_{11}\right)e^{-\lambda_1(T-t)} \\ &- \left(\tilde{a}_{11}p_{11}\tilde{p}_{21} + \tilde{a}_{12}p_{22}\tilde{p}_{21}\right)e^{-\lambda_2(T-t)}, \\ f_{12}(t,T) =& \tilde{a}_{12} - \left(\tilde{a}_{11}p_{11}\tilde{p}_{12} + \tilde{a}_{12}p_{21}\tilde{p}_{12}\right)e^{-\lambda_1(T-t)} \\ &- \left(\tilde{a}_{11}p_{12}\tilde{p}_{22} + \tilde{a}_{12}p_{21}\tilde{p}_{22}\right)e^{-\lambda_2(T-t)}, \\ f_{21}(t,T) =& \tilde{a}_{21} - \left(\tilde{a}_{21}p_{11}\tilde{p}_{11} + \tilde{a}_{22}p_{21}\tilde{p}_{11}\right)e^{-\lambda_1(T-t)} \\ &- \left(\tilde{a}_{21}p_{12}\tilde{p}_{21} + \tilde{a}_{22}p_{12}\tilde{p}_{21}\right)e^{-\lambda_2(T-t)}, \\ f_{22}(t,T) =& \tilde{a}_{22} - \left(\tilde{a}_{21}p_{11}\tilde{p}_{12} + \tilde{a}_{22}p_{21}\tilde{p}_{12}\right)e^{-\lambda_1(T-t)} \\ &- \left(\tilde{a}_{21}p_{12}\tilde{p}_{22} + \tilde{a}_{22}p_{21}\tilde{p}_{22}\right)e^{-\lambda_2(T-t)}. \end{aligned}$$

Note that

$$\mu^{\mathbb{P}}(t) = \delta^\top P\Lambda(T-t)P^{-1}R(0) + \delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right)\hat{\theta}_r(t)$$

does not depend on $\hat{\theta}_r$ since $\int_0^T r(t)dt$ is considered under $\mathbb{P}$.

Representation problem

Theorem 5.3 *Suppose all the conditions of Proposition 5.2 are satisfied and we consider the equivalent martingale measure $\hat{\mathbb{Q}}$. Then:*

(a) The time-$t \in [0,T]$ optimal wealth is given by

$$X_c^*(t) = (y^*)^{\frac{1}{\gamma-1}} \left(\hat{H}(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma}M^{\hat{H}}(t,T)+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{\hat{H}}(t,T)}.$$

Here,

$$\begin{aligned}
\hat{H}(t) &= \beta(t)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_t, \\
M^{\hat{H}}(t,T) &= \int_t^T \left(\delta_0 + \mu^{\mathbb{P}}(s) + \frac{1}{2}\left\|\hat{\Theta}(s)\right\|^2\right) ds, \\
V^{\hat{H}}(t,T) &= \int_s^T \left[\left(\delta^\top A^{-1}\left(I - P\Lambda(T-s)P^{-1}\right) + \hat{\theta}_r^\top(s)\right)^2 + \hat{\theta}_S^2(s)\right] ds.
\end{aligned}$$

(b) The portfolio process at any time $t \in [0,T]$ that replicates the terminal wealth $X_c^*(t)$ is given by $\varphi_c^*(t) = \left(\varphi_1^*(t), \varphi_2^*(t), \varphi_S^*(t)\right)^\top$ where

$$\begin{aligned}
\varphi_1^*(t) &= \frac{\hat{\sigma}_{21}(t,T_2)}{(1-\gamma)G(t,T_1,T_2)}\left(\gamma F_2(t,T) + \hat{\theta}_2(t) - \frac{\hat{\theta}_S(t)\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right) \\
&\quad - \frac{\hat{\sigma}_{22}(t,T_2)}{(1-\gamma)G(t,T_1,T_2)}\left(\gamma F_1(t,T) + \hat{\theta}_1(t) - \frac{\hat{\sigma}_{1S}\hat{\theta}_S(t)}{\hat{\sigma}_S}\right), \\
\varphi_2^*(t) &= \frac{1}{(1-\gamma)\hat{\sigma}_{21}(t,T_2)}\left(\gamma F_1(t,T) + \hat{\theta}_1(t) - \frac{\hat{\sigma}_{1S}\hat{\theta}_S(t)}{\hat{\sigma}_S}\right) \\
&\quad - \frac{\hat{\sigma}_{11}(t,T_1)}{(1-\gamma)G(t,T_1,T_2)}\left(\gamma F_2(t,T) + \hat{\theta}_2(t) - \frac{\hat{\theta}_S(t)\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right) \\
&\quad + \frac{\hat{\sigma}_{11}(t,T_1)\hat{\sigma}_{22}(t,T_2)}{(1-\gamma)\hat{\sigma}_{21}(t,T_2)G(t,T_1,T_2)}\left(\gamma F_1(t,T) + \hat{\theta}_1(t) - \frac{\hat{\sigma}_{1S}\hat{\theta}_S(t)}{\hat{\sigma}_S}\right), \\
\varphi_S^*(t) &= \frac{\hat{\theta}_S(t)}{(1-\gamma)\hat{\sigma}_S} \quad \textit{and} \\
\varphi_M^*(t) &= 1 - \varphi_c^*(t)^\top \mathbb{1}_3,
\end{aligned} \tag{5.22}$$

where

$$\begin{aligned} F_1(t,T) &:= \delta_1 f_{11}(t,T) + \delta_2 f_{21}(t,T) \\ F_2(t,T) &:= \delta_1 f_{12}(t,T) + \delta_2 f_{22}(t,T) \\ G(t,T_1,T_2) &:= \hat{\sigma}_{12}(t,T_1)\,\hat{\sigma}_{21}(t,T_2) - \hat{\sigma}_{22}(t,T_2)\,\hat{\sigma}_{11}(t,T_1). \end{aligned}$$

Proof:

(a) As proved in Theorem 4.25 part (ii), with $(H(t))_{t\in[0,T]}$ replaced by $\left(\hat{H}(t)\right)_{t\in[0,T]}$, since $\left(\hat{H}(t)\right)_{t\in[0,T]}$ is also a lognormally distributed, martingale and Markov process.

(b) Using the same argument as for the proof of Theorem 4.25 part (iii) it can be observed that $(r(t), \theta_1(t), \theta_2(t), \theta_S(t),)^\top \in (\mathbb{D})^4$. This implies from Lemma A.5 that $\hat{H}(T)\,X_c^*(T) \in \mathbb{D}_{1,1}$. Now we can apply the Clark-Ocone formula under change of measure (Theorem A.7) to the process $\beta(T)\,X_c^*(T)$:

$$\begin{aligned} \beta(T)X_c^*(T) &= \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\,X_c^*(T)\right] \\ &+ \int_0^T \mathbb{E}^{\hat{\mathbb{Q}}}\left[D_t\left(\beta(T)\,X_c^*(T)\right) \mid \mathcal{F}(t)\right] dW^{\hat{\mathbb{Q}}}(t) \\ &- \int_0^T \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\,X_c^*(T)\int_t^T D_t\hat{\Theta}^\top(u)\,dW^{\hat{\mathbb{Q}}}(u) \mid \mathcal{F}(t)\right] dW^{\hat{\mathbb{Q}}}(t). \end{aligned} \tag{5.23}$$

Moreover,

$$\int_0^T \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\,X_c^*(T)\int_t^T D_t\hat{\Theta}^\top(u)\,dW^{\hat{\mathbb{Q}}}(u) \mid \mathcal{F}(t)\right] dW^{\hat{\mathbb{Q}}}(t) = 0 \tag{5.24}$$

because $\Theta(u)$ is deterministic, and the Malliavin derivative of a deterministic function is zero.

The product rule of Malliavin calculus and Proposition 5.2 yield

$$\begin{aligned} & D_t\left(\beta(T)\,X_c^*(T)\right) \\ &= X_c^*(T)\,D_t\beta(T) + \beta(T)\,D_tX_c^*(T) \\ &= \left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} D_t\beta(T) + \beta(T)\,D_t\left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}. \end{aligned} \tag{5.25}$$

Using the chain rule of Malliavin calculus on $\beta(T)$ and the last equality of Equation (5.20) gives

$$\begin{aligned} D_t(\beta(T)) = &- \beta(T) D_t \int_0^T r(u)du = -\beta(T) \int_t^T D_t r(u)du \\ &= -\beta(T)\delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right), \end{aligned} \tag{5.26}$$

since $\delta^\top A^{-1}(I - P\Lambda(T-t)P^{-1})$ is deterministic.
We apply again the chain and product rule to
$\left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}$ and obtain

$$\begin{aligned} &D_t\left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} = -\frac{1}{\gamma-1}\left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \times \\ &D_t\left(\int_0^T r(u)\,du - \frac{1}{2}\int_0^T \left\|\hat{\Theta}(u)\right\|^2 du + \int_0^T \hat{\Theta}^\top(u)\,dW^{\hat{\mathbb{Q}}}(u)\right) \\ &= \frac{1}{1-\gamma}\left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \times \\ &\qquad \left(\delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right)e_3 + \hat{\Theta}^\top(t)\right), \end{aligned} \tag{5.27}$$

where $e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Inserting Equations (5.27) and (5.26) into Equation (5.25) gives

$$\begin{aligned} &D_t(\beta(T)X_c^*(T)) \\ &\quad = \left(\frac{\gamma}{1-\gamma}\delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right)e_3 + \frac{1}{1-\gamma}\hat{\Theta}^\top(t)\right) \times \\ &\qquad\qquad \beta(T)\left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}. \end{aligned} \tag{5.28}$$

Plugging Equations (5.28) and (5.24) into Equation (5.23) results in

$$\begin{aligned} &\beta(T)X_c^*(T) = \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)X_c^*(T)\right] + \\ &\int_0^T \left(\frac{\gamma}{1-\gamma}\delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right)e_3 + \frac{1}{1-\gamma}\hat{\Theta}^\top(t)\right) \times \\ &\mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mid \mathcal{F}(t)\right] dW^{\hat{\mathbb{Q}}}(t). \end{aligned} \tag{5.29}$$

From Corollary 2.9 in [40] we have that

$$\beta(t) X_c^*(t) = \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mid \mathcal{F}(t)\right]. \tag{5.30}$$

Using Equations (5.30) and (5.29), the process $\beta(T) X_c^*(T)$ can be expressed as

$$\beta(T) X_c^*(T) = \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T) X_c^*(T)\right] + \int_0^T \left(\frac{\gamma}{1-\gamma}\delta^\top A^{-1}\left(I - P\Lambda(T-t) P^{-1}\right) e_3 + \frac{1}{1-\gamma}\hat{\Theta}^\top(t)\right)\beta(t) X_c^*(t)\, dW^{\hat{\mathbb{Q}}}(t). \tag{5.31}$$

By using the value of $A^{-1}\left(I - P\Lambda(T-t) P^{-1}\right)$ in (5.21) and comparing the expressions of $\beta(T) X_c^*(T)$ in (5.31) and (5.16),follows the results.

□

5.2. Optimization with bounded expected shortfall risk in a complete market

We will study in this section the portfolio optimization problem in the presence of a bounded risk constraint in a complete market setting as discussed in Section 4.2, but we will consider the two-factor Vasiček model rather than one-factor Vasiček used in Chapter 4.

The dynamic and static optimization problems in this setting look like those of Section 4.2 or Section 3.2, since these optimization problems depend partially on the deflator $H(T)$, but not on its structure and changing the model of interest rates, modifies the structure of the deflator $H(T)$. Let us consider the assumptions made in Section 4.2 and recall below the dynamic and static optimization problems.

The dynamic optimization problem:

$$\begin{aligned}
&\max_{\varphi\in\tilde{\mathcal{A}}(x)} \mathbb{E}\left[\frac{(X_c(T))^\gamma}{\gamma}\right] \\
&\text{s.t. } \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)X_c(T)\right]\leq x \quad \text{(budget constraint)} \\
&\qquad \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\left(X_c(T)-q\right)^-\right]\leq\delta \quad \text{(risk constraint)},
\end{aligned} \tag{5.32}$$

with $\tilde{\mathcal{A}}(x)$ as defined in (2.10).

The static optimization problem:

$$\begin{aligned}
&\max_{X_c(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{(X_c(T))^\gamma}{\gamma}\right] \\
&\text{s.t. } \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)X_c(T)\right]\leq x \\
&\qquad \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\left(X_c(T)-q\right)^-\right]\leq\delta,
\end{aligned} \tag{5.33}$$

$\mathcal{B}(x)$ is as defined in (2.12).

The static portfolio insurer problem:

$$\begin{aligned}
&\max_{X_c(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{(X_c(T))^\gamma}{\gamma}\right] \\
&\text{s.t. } \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)X_c(T)\right]\leq x \\
&\qquad X_c(T)\geq q
\end{aligned} \tag{5.34}$$

The optimal terminal wealth for the problems (5.33) and (5.34) are of the same form as in Theorem 4.5 and Theorem 3.5 for the same reason mentioned above, that the change of interest rate model, modifies the structure of the deflator $H(T)$ and the optimal terminal wealth X_c^* depends on the outcome of $H(T)$, but not on its structure. We restate below a proposition that characterizes the solutions of the problems (5.33) and (5.34).

Proposition 5.4 *Let q be a fixed benchmark and x an initial wealth of an investor. (i) If $\delta\in\left(\underline{\delta},\overline{\delta}\right)$, then the solution of the problem (5.33) is given by*

$$X_c^\delta(T)=f\left(y_1^*,y_2^*,\hat{\mathbb{Q}}\right), \tag{5.35}$$

where

$$f\left(y_1^*, y_2^*, \hat{\mathbb{Q}}\right) = \left(y_1^* \beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_A + q\mathbb{1}_B + \left(\left(y_1^* - y_2^*\right)\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_C, \tag{5.36}$$

$$A = A\left(y_1^*, y_2^*, \hat{\mathbb{Q}}\right) = \left\{\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \leq \frac{q^{\gamma-1}}{y_1^*}\right\}$$

$$B = B\left(y_1^*, y_2^*, \hat{\mathbb{Q}}\right) = \left\{\frac{q^{\gamma-1}}{y_1^*} < \beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \leq \frac{q^{\gamma-1}}{y_1^* - y_2^*}\right\} \quad \textit{and}$$

$$C = C\left(y_1^*, y_2^*, \hat{\mathbb{Q}}\right) = \left\{\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T > \frac{q^{\gamma-1}}{y_1^* - y_2^*}\right\},$$

such that $y_1^*, y_2^* \in (0, \infty)$ *solve the following system of equations*

$$\mathbb{E}\left[\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T f\left(y_1, y_2, \hat{\mathbb{Q}}\right)\right] = x$$

$$\mathbb{E}\left[\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \left(f\left(y_1, y_2, \hat{\mathbb{Q}}\right) - q\right)^-\right] = \delta.$$

(ii) If $\delta = \underline{\delta}$*, i.e in the case of portfolio insurer problem, then the solution to the problem (5.34) is given by*

$$X_c^{PI}(T) = f\left(y^{PI}, \hat{\mathbb{Q}}\right),$$

where

$$f\left(y^{PI}, \hat{\mathbb{Q}}\right) = \left(y^{PI} \beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_{A^{PI}} + q\mathbb{1}_{B^{PI}}, \tag{5.37}$$

$$A^{PI} = A^{PI}\left(y^{PI}, \hat{\mathbb{Q}}\right) = \left\{\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \leq \frac{q^{\gamma-1}}{y^{PI}}\right\} \quad \textit{and}$$

$$B^{PI} = B^{PI}\left(y^{PI}, \hat{\mathbb{Q}}\right) = \left\{\frac{q^{\gamma-1}}{y^{PI}} < \beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right\},$$

such that y^{PI} *solves the equation*

$$\mathbb{E}\left[\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T f\left(y^{PI}, \hat{\mathbb{Q}}\right)\right] = x$$

Representation problem:

The following theorem gives the trading strategies that generate the optimal terminal wealth $X_c^{\delta}(T)$ and $X_c^{PI}(T)$ in Proposition 5.4.

Theorem 5.5 *Suppose all conditions of Proposition 5.4 are satisfied and we consider the equivalent martingale measure $\hat{\mathbb{Q}}$. Then:*

(a) The time-$t \in [0,T]$ optimal wealth $X_c^{\delta}(t)$ and $X_c^{PI}(t)$ are given by

$$
\begin{aligned}
&X_c^{\delta}(t) \\
&= (y_1^*)^{\frac{1}{\gamma-1}} \left(\hat{H}(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^{\hat{H}}(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{\hat{H}}(t,T)} * \Phi\left(d_1^{\delta}(t)\right) \\
&+ e^{-M^{\hat{H}}(t,T) + \frac{1}{2} V^{\hat{H}}(t,T)} * \left(\Phi\left(d_2^{\delta}(t)\right) - \Phi\left(d_3^{\delta}(t)\right)\right) \\
&+ (y_1^* - y_2^*)^{\frac{1}{\gamma-1}} \left(\hat{H}(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^{\hat{H}}(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{\hat{H}}(t,T)} * \Phi\left(d_4^{\delta}(t)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&X^{PI}(t) \\
&= \left(y_1^{PI}\right)^{\frac{1}{\gamma-1}} \left(\hat{H}(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^{\hat{H}}(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{\hat{H}}(t,T)} * \Phi\left(d_1^{PI}(t)\right) \\
&+ e^{-M^{\hat{H}}(t,T) + \frac{1}{2} V^{\hat{H}}(t,T)} * \Phi\left(d_2^{PI}(t)\right),
\end{aligned}
$$

where

$$d_1^\delta(t) = \frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*\hat{H}(t)}\right) + M^{\hat{H}}(t,T)}{\sqrt{V^{\hat{H}}(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^{\hat{H}}(t,T)},$$

$$d_2^\delta(t) = -\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*\hat{H}(t)}\right) + M^{\hat{H}}(t,T)}{\sqrt{V^{\hat{H}}(t,T)}} + \sqrt{V^{\hat{H}}(t,T)},$$

$$d_3^\delta(t) = -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right)\hat{H}(t)}\right) + M^{\hat{H}}(t,T)}{\sqrt{V^{\hat{H}}(t,T)}} + \sqrt{V^{\hat{H}}(t,T)},$$

$$d_4^\delta(t) = -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right)\hat{H}(t)}\right) + M^{\hat{H}}(t,T)}{\sqrt{V^{\hat{H}}(t,T)}} - \frac{\gamma}{1-\gamma}\sqrt{V^{\hat{H}}(t,T)},$$

$$d_1^{PI}(t) = \frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}\hat{H}(t)}\right) + M^{\hat{H}}(t,T)}{\sqrt{V^{\hat{H}}(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^{\hat{H}}(t,T)},$$

$$d_2^{PI}(t) = -\frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}\hat{H}(t)}\right) + M^{\hat{H}}(t,T)}{\sqrt{V^{\hat{H}}(t,T)}} + \sqrt{V^{\hat{H}}(t,T)}.$$

Thereby $\hat{H}(t)$, $M^{\hat{H}}(t,T)$ *and* $V^{\hat{H}}(t,T)$ *are as given in Theorem 5.3 and* $\Phi(\cdot)$ *is the standard normal probability distribution function.*

(b) *The portfolio processes at any time* $t \in [0,T]$ *which replicate the optimal terminal wealth* $X_c^\delta(t)$ *and* $X_c^{PI}(t)$ *are given by*

$$\varphi_c^\delta(t) = \left(\varphi_1^\delta(t), \varphi_2^\delta(t), \varphi_S^\delta(t)\right)^\top \quad \textit{and}$$
$$\varphi_c^{PI}(t) = \left(\varphi_1^{PI}(t), \varphi_2^{PI}(t), \varphi_S^{PI}(t)\right)^\top \quad \textit{respectively,}$$

where

$$\begin{aligned}\varphi_1^\delta(t) = &\left(-\hat{\sigma}_{22}\left(\gamma F_1(t,T) + \hat{\theta}_1(t)\right) + \hat{\sigma}_{21}\left(\gamma F_2(t,T) + \hat{\theta}_2(t)\right)\right)\aleph_1^\delta(t,T,T_1,T_2)\\ &+ \hat{\theta}_S(t)\left(\hat{\sigma}_{22}\hat{\sigma}_{1S} - \hat{\sigma}_{21}\hat{\sigma}_{2S}\right)\left(\aleph_1^\delta(t,T,T_1,T_2) + \hbar_1^\delta(t,T,T_1,T_2)\right)\\ &+ \left(\hat{\sigma}_{22}\left(\hat{\theta}_1(t) - F_1(t,T)\right) - \hat{\sigma}_{21}\left(\hat{\theta}_2(t) - F_2(t,T)\right)\right)\hbar_1^\delta(t,T,T_1,T_2),\end{aligned}$$

$$
\begin{aligned}
&\varphi_2^{\delta}(t) = \\
&-\left(\gamma F_1(t,T) + \hat{\theta}_2(t)\right) \aleph_2^{\delta}(t,T,T_1,T_2) - \left(\hat{\theta}_2(t) + F_2(t,T)\right) \hbar_2^{\delta}(t,T,T_1,T_2) \\
&+ \left(\hat{\sigma}_{22}\left(\gamma F_1(t,T) + \hat{\theta}_1(t)\right) - \hat{\sigma}_{21}\left(\gamma F_2(t,T) + \hat{\theta}_2(t)\right)\right) \frac{\hat{\sigma}_{12}}{\hat{\sigma}_{22}} \aleph_1^{\delta}(t,T,T_1,T_2) \\
&- \frac{\hat{\theta}_S(t)\,\hat{\sigma}_{12}}{\hat{\sigma}_{22}} \left(\hat{\sigma}_{22}\hat{\sigma}_{1S} - \hat{\sigma}_{21}\hat{\sigma}_{2S}\right) \left(\aleph_1^{\delta}(t,T,T_1,T_2) + \hbar_1^{\delta}(t,T,T_1,T_2)\right) \\
&- \frac{\hat{\sigma}_{12}}{\hat{\sigma}_{22}} \left(\hat{\sigma}_{22}\left(\hat{\theta}_1(t) + F_1(t,T)\right) + \hat{\sigma}_{21}\left(\hat{\theta}_2(t) - F_2(t,T)\right)\right) \hbar_1^{\delta}(t,T,T_1,T_2) \\
&+ \frac{\hat{\sigma}_{2S}\hat{\theta}_S(t)}{\hat{\sigma}_{22}} \left(\aleph_2^{\delta}(t,T,T_1,T_2) + \hbar_2^{\delta}(t,T,T_1,T_2)\right),
\end{aligned}
$$

$$
\begin{aligned}
\varphi_S^{\delta}(t) &= \frac{\hat{\theta}_S(t)\,\hat{\sigma}_{22}}{\hat{\sigma}_S} \left(\aleph_2^{\delta}(t,T,T_1,T_2) + \hbar_2^{\delta}(t,T,T_1,T_2)\right), \\
\varphi_M^{\delta}(t) &= 1 - \varphi_c^{\delta}(t)^{\top} \mathbb{1}_3,
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_1^{PI}(t) = \\
&\quad \left(-\hat{\sigma}_{22}\left(\gamma F_1(t,T) + \hat{\theta}_1(t)\right) + \hat{\sigma}_{21}\left(\gamma F_2(t,T) + \hat{\theta}_2(t)\right)\right) \aleph_1^{PI}(t,T,T_1,T_2) \\
&\quad + \hat{\theta}_S(t)\left(\hat{\sigma}_{22}\hat{\sigma}_{1S} - \hat{\sigma}_{21}\hat{\sigma}_{2S}\right) \left(\aleph_1^{PI}(t,T,T_1,T_2) + \hbar_1^{\delta}(t,T,T_1,T_2)\right) \\
&\quad + \left(\hat{\sigma}_{22}\left(\hat{\theta}_1(t) - F_1(t,T)\right) - \hat{\sigma}_{21}\left(\hat{\theta}_2(t) - F_2(t,T)\right)\right) \hbar_1^{PI}(t,T,T_1,T_2),
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_2^{PI}(t) = \\
&-\left(\gamma F_1(t,T) + \hat{\theta}_2(t)\right) \aleph_2^{PI}(t,T,T_1,T_2) - \left(\hat{\theta}_2(t) + F_2(t,T)\right) \hbar_2^{PI}(t,T,T_1,T_2) \\
&+ \left(\hat{\sigma}_{22}\left(\gamma F_1(t,T) + \hat{\theta}_1(t)\right) - \hat{\sigma}_{21}\left(\gamma F_2(t,T) + \hat{\theta}_2(t)\right)\right) \frac{\hat{\sigma}_{12}}{\hat{\sigma}_{22}} \aleph_1^{PI}(t,T,T_1,T_2) \\
&- \frac{\hat{\theta}_S(t)\,\hat{\sigma}_{12}}{\hat{\sigma}_{22}} \left(\hat{\sigma}_{22}\hat{\sigma}_{1S} - \hat{\sigma}_{21}\hat{\sigma}_{2S}\right) \left(\aleph_1^{PI}(t,T,T_1,T_2) + \hbar_1^{PI}(t,T,T_1,T_2)\right) \\
&- \frac{\hat{\sigma}_{12}}{\hat{\sigma}_{22}} \left(\hat{\sigma}_{22}\left(\hat{\theta}_1(t) + F_1(t,T)\right) + \hat{\sigma}_{21}\left(\hat{\theta}_2(t) - F_2(t,T)\right)\right) \hbar_1^{PI}(t,T,T_1,T_2) \\
&+ \frac{\hat{\sigma}_{2S}\hat{\theta}_S(t)}{\hat{\sigma}_{22}} \left(\aleph_2^{PI}(t,T,T_1,T_2) + \hbar_2^{PI}(t,T,T_1,T_2)\right),
\end{aligned}
$$

$$
\begin{aligned}
\varphi_S^{PI}(t) &= \frac{\hat{\theta}_S(t)\,\hat{\sigma}_{22}}{\hat{\sigma}_S} \left(\aleph_2^{PI}(t,T,T_1,T_2) + \hbar_2^{PI}(t,T,T_1,T_2)\right), \\
\varphi_M^{PI}(t) &= 1 - \varphi_c^{PI}(t)^{\top} \mathbb{1}_3.
\end{aligned}
$$

Where,

$$\aleph_1^{\delta}(t,T,T_1,T_2) := \frac{1}{(1-\gamma)(\hat{\sigma}_{22}\hat{\sigma}_{11} - \hat{\sigma}_{12}\hat{\sigma}_{21})},$$
$$\hbar_1^{\delta}(t,T,T_1,T_2) :=$$
$$\frac{qe^{-M^{\hat{H}}(t,T)+\frac{1}{2}V^{\hat{H}}(t,T)} * \left(\Phi\left(d_2^{\delta}(t)\right) - \Phi\left(d_3^{\delta}(t)\right)\right)}{(1-\gamma) X_c^{\delta}(t)(\hat{\sigma}_{22}\hat{\sigma}_{11} - \hat{\sigma}_{12}\hat{\sigma}_{21})},$$
$$\aleph_2^{\delta}(t,T,T_1,T_2) := \frac{1}{(1-\gamma)\hat{\sigma}_{22}},$$
$$\hbar_2^{\delta}(t,T,T_1,T_2) :=$$
$$\frac{qe^{-M^{\hat{H}}(t,T)+\frac{1}{2}V^{\hat{H}}(t,T)} * \left(\Phi\left(d_2^{\delta}(t)\right) - \Phi\left(d_3^{\delta}(t)\right)\right)}{(1-\gamma) X_c^{\delta}(t)\hat{\sigma}_{22}},$$

$$\begin{aligned}
\aleph_1^{PI}(t,T,T_1,T_2) &:= \frac{1}{(1-\gamma)(\hat{\sigma}_{22}\hat{\sigma}_{11} - \hat{\sigma}_{12}\hat{\sigma}_{21})},\\
\hbar_1^{PI}(t,T,T_1,T_2) &:= \frac{qe^{-M^{\hat{H}}(t,T)+\frac{1}{2}V^{\hat{H}}(t,T)}\Phi\left(d_2^{PI}(t)\right)}{(1-\gamma) X_c^{PI}(t)(\hat{\sigma}_{22}\hat{\sigma}_{11} - \hat{\sigma}_{12}\hat{\sigma}_{21})},\\
\aleph_2^{PI}(t,T,T_1,T_2) &:= \frac{1}{(1-\gamma)\hat{\sigma}_{22}\hat{\sigma}_{11}},\\
\hbar_2^{PI}(t,T,T_1,T_2) &:= \frac{qe^{-M^{\hat{H}}(t,T)+\frac{1}{2}V^{\hat{H}}(t,T)}\Phi\left(d_2^{PI}(t)\right)}{(1-\gamma) X_c^{PI}(t)\hat{\sigma}_{22}},\\
&\textit{and} \quad F_1(t,T) \ \textit{and} \ F_2(t,T) \ \textit{as in (5.22).}
\end{aligned}$$

Proof:

(a) See the proof of Theorem 4.6 part (a) and Theorem 5.3 part (a).

(b) We will prove only the case of $\varphi_c^{\delta}(t)$, i.e., $\delta \in \left(\underline{\delta}, \overline{\delta}\right)$. For the optimal portfolio insurer trading strategy $\varphi_c^{PI}(t)$, i.e. in case $\delta = \underline{\delta}$, it is proved analogously. We use the same idea used to prove Theorem 4.6 part (b) and Theorem 5.3 part (b). Note that $\hat{H}(T) X_c^{\delta}(T) \in \mathbb{D}_{1,1}$ from the proof of Theorem 5.3 part (b) and Proposition A.9. Therefore, we can apply Theorem A.7 to $\beta(T) X_c^{\delta}(T)$. The Clark-Ocone representation under change of measure of $\beta(T)\, X_c^{\delta}(T)$ reads as

$$\begin{aligned}\beta(T)X_c^{\delta}(T) &= \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\, X_c^{\delta}(T)\right] \\ &+ \int_0^T \mathbb{E}^{\hat{\mathbb{Q}}}\left[D_t\left(\beta(T)\, X_c^{\delta}(T)\right) \mid \mathcal{F}(t)\right] dW^{\hat{\mathbb{Q}}}(t) \\ &- \int_0^T \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\, X_c^{\delta}(T) \int_t^T D_t \hat{\Theta}^{\top}(u)\, dW^{\hat{\mathbb{Q}}}(u) \mid \mathcal{F}(t)\right] dW^{\hat{\mathbb{Q}}}(t).\end{aligned} \tag{5.38}$$

The third term in (5.38) is equal to zero since $\Theta(u)$ is a deterministic function, as we have seen in the proof of Theorem 5.3. In Proposition 5.4, for the case of $\delta \in \left(\underline{\delta}, \overline{\delta}\right)$, the optimal terminal wealth is given by

$$\begin{aligned}X_c^{\delta}(T) = &\left(y_1^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_A + q\mathbb{1}_B \\ &+ \left((y_1^* - y_2^*)\,\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_C.\end{aligned}$$

Thus,

$$\begin{aligned}\beta(T)\, X_c^{\delta}(T) = &\beta(T)\left(y_1^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_A + \beta(T)\, q\mathbb{1}_B \\ &+ \beta(T)\left((y_1^* - y_2^*)\,\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_C.\end{aligned}$$

In the following we use Proposition A.9 to compute the Malliavin derivative of $\beta(T)\, X_c^{\delta}(T)$:

$$\begin{aligned}D_t\left(\beta(T)\, X_c^{\delta}(T)\right) &= D_t\left(\beta(T)\left(y_1^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_A\right) \\ &+ D_t\left(\beta(T)\, q\mathbb{1}_B\right) + D_t\left(\beta(T)\left((y_1^* - y_2^*)\,\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_C\right).\end{aligned} \tag{5.39}$$

We compute first the Malliavin derivative of the first term of (5.39) by using the same idea as in the proof of Theorem 5.3. In particular, from

Equation (5.28) we get

$$D_t\left(\beta(T)\left(y_1^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}\mathbb{1}_A\right)=$$
$$\frac{1}{1-\gamma}\left(\gamma\delta^\top A^{-1}\left(I-P\Lambda(T-t)P^{-1}\right)e_3+\hat{\Theta}^\top(t)\right)\times \tag{5.40}$$
$$\beta(T)\left(y^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}\mathbb{1}_A,$$

e_3 as given in (5.27). The second and the third term of Equation (5.39) are calculated analogously.
Thus,

$$\begin{aligned}
&D_t\left(\beta(T)X_c^\delta(T)\right)\\
=&\frac{1}{1-\gamma}\left(\gamma\delta^\top A^{-1}\left(I-P\Lambda(T-t)P^{-1}\right)e_3+\hat{\Theta}^\top(t)\right)\times\\
&\quad\beta(T)\left(y_1^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}\mathbb{1}_A\\
&-\delta^\top A^{-1}\left(I-P\Lambda(T-t)P^{-1}\right)\beta(T)q\mathbb{1}_B\\
&+\frac{1}{1-\gamma}\left(\gamma\delta^\top A^{-1}\left(I-P\Lambda(T-t)P^{-1}\right)e_3+\hat{\Theta}^\top(t)\right)\times\\
&\quad\beta(T)\left(\left(y_1^*-y_2^*\right)\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}\mathbb{1}_C.
\end{aligned} \tag{5.41}$$

After some re-arrangements of Equation (5.41) we obtain

$$\begin{aligned}
&D_t\left(\beta(T)X_c^\delta(T)\right)=\\
&\frac{1}{1-\gamma}\left(\gamma\delta^\top A^{-1}\left(I-P\Lambda(T-t)P^{-1}\right)e_3+\hat{\Theta}^\top(t)\right)\beta(T)X_c^\delta(T)\\
&-\frac{1}{1-\gamma}\left(\delta^\top A^{-1}\left(I-P\Lambda(T-t)P^{-1}\right)e_3+\hat{\Theta}^\top(t)\right)\beta(T)q\mathbb{1}_B.
\end{aligned} \tag{5.42}$$

Therefore,

$$\begin{aligned}
\beta(T)X_c^{\delta}(T) =& \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)X_c^{\delta}(T)\right] \\
&+\int_0^T \frac{1}{1-\gamma}\left(\gamma\delta^{\top}A^{-1}\left(I-P\Lambda(T-t)P^{-1}\right)e_3+\hat{\Theta}^{\top}(t)\right)\times \\
&\mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)X_c^{\delta}(T)\mid\mathcal{F}(t)\right]dW^{\hat{\mathbb{Q}}}(t) \\
&-\int_0^T \frac{1}{1-\gamma}\left(\delta^{\top}A^{-1}\left(I-P\Lambda(T-t)P^{-1}\right)e_3+\hat{\Theta}^{\top}(t)\right)\times \\
&\mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)q\mathbb{1}_B\mid\mathcal{F}(t)\right]dW^{\hat{\mathbb{Q}}}(t)\,.
\end{aligned} \tag{5.43}$$

We compare (5.43) and (5.16), and we obtain Equations (5.44), (5.45) and (5.46) below, which we have to solve simultaneously for $\varphi_1^{\delta}(t),\varphi_2^{\delta}(t)$ and $\varphi_S^{\delta}(t)$.

$$\begin{aligned}
&\beta(t)X_c^{\delta}(t)\left(\varphi_S^{\delta}(t)\hat{\sigma}_{1S}+\varphi_1^{\delta}(t)\hat{\sigma}_{11}+\varphi_2^{\delta}(t)\hat{\sigma}_{21}\right) \\
&=\frac{1}{1-\gamma}\left(\gamma F_1(t,T)+\hat{\theta}_1(t)\right)\mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)X_c^{\delta}(T)\mid\mathcal{F}(t)\right] \\
&\quad-\frac{1}{1-\gamma}\left(F_1(t,T)+\hat{\theta}_1(t)\right)\mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)q\mathbb{1}_B\mid\mathcal{F}(t)\right],
\end{aligned} \tag{5.44}$$

$$\begin{aligned}
&\beta(t)X_c^{\delta}(t)\left(\varphi_S^{\delta}(t)\hat{\sigma}_{2S}+\varphi_1^{\delta}(t)\hat{\sigma}_{12}+\varphi_2^{\delta}(t)\hat{\sigma}_{22}\right) \\
&=\frac{1}{1-\gamma}\left(\gamma F_2(t,T)+\hat{\theta}_2(t)\right)\mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)X_c^{\delta}(T)\mid\mathcal{F}(t)\right] \\
&\quad-\frac{1}{1-\gamma}\left(F_2(t,T)+\hat{\theta}_2(t)\right)\mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)q\mathbb{1}_B\mid\mathcal{F}(t)\right],
\end{aligned} \tag{5.45}$$

$$\beta(t)X_c^{\delta}(t)\varphi_S^{\delta}(t)\hat{\sigma}_S=\hat{\theta}_S(t)\mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\left(X_c^{\delta}(T)+q\mathbb{1}_B\right)\mid\mathcal{F}(t)\right]. \tag{5.46}$$

Solving Equation (5.46) for $\varphi_S^{\delta}(t)$ we get

$$\varphi_S^{\delta}(t)=\frac{\hat{\theta}_S(t)\hat{\sigma}_{22}}{\hat{\sigma}_S}\left(\aleph_2^{\delta}(t,T,T_1,T_2)+\hbar_2^{\delta}(t,T,T_1,T_2)\right). \tag{5.47}$$

Substituting (5.47) in (5.45) and (5.44), and solving them simultaneously for $\varphi_1^{\delta}(t)$ and $\varphi_2^{\delta}(t)$ gives the result.

□

5.3. Optimization without risk constraint in an incomplete market

In the previous sections of this chapter we have studied the portfolio optimization problem in a complete market, but as we have mentioned in Chapter 4 a market might as well be incomplete. Now, we turn to the world of an incomplete market in this and the next section of this chapter. The aim of this section is to analyze portfolio optimization problem without risk constraints in an incomplete market when the interest rates are described by the two-factor Vasiček model.

We consider a financial market composed of a money market account M, a bond B_1 and a stock S, whose dynamics under the probability measure $\mathbb{P}$ are described by (5.1), (5.4) and (5.5), respectively:

$$\begin{aligned} dM(t) &= M(t)\,r(t)\,dt, \\ dB(t,T_1) &= B(t,T_1)\left[\mu_{1B}(t,T_1)\,dt + \sigma_{11}(t,T_1)\,dW_1^{\mathbb{P}}(t) + \sigma_{12}(t,T_1)\,dW_2^{\mathbb{P}}(t)\right], \\ dS(t) &= S(t)\left[\mu_S^{\mathbb{P}}(t)\,dt + \sigma_{1S}dW_1^{\mathbb{P}}(t) + \sigma_{2S}dW_2^{\mathbb{P}}(t) + \sigma_S dW_S^{\mathbb{P}}(t)\right]. \end{aligned} \tag{5.48}$$

Note that from Definition 2.9 we are in an incomplete market.

Remark 5.6 *At this financial market (composed of M, B_1 and S), as in (5.14), we have*

$$\begin{aligned} \bar{\mu}_{1B} &= \hat{\theta}_1(t)\,\hat{\sigma}_{11}(t,T_1) + \hat{\theta}_2(t)\,\hat{\sigma}_{12}(t,T_1) \\ \bar{\mu}_S &= \hat{\theta}_1(t)\,\hat{\sigma}_{1S} + \hat{\theta}_2(t)\,\hat{\sigma}_{2S} + \hat{\theta}_S(t)\,\hat{\sigma}_S. \end{aligned} \tag{5.49}$$

A vector $\hat{\Theta}(t) = \left(\hat{\theta}_1(t), \hat{\theta}_2(t), \hat{\theta}_S(t)\right)^\top$ correspond to $\hat{\mathbb{Q}} \in \mathcal{M}^e$. In the system of Equations (5.49) we have three unknowns ($\hat{\theta}_1(t), \hat{\theta}_2(t)$ and $\hat{\theta}_S(t)$) and two equations. So, the values of these unknowns cannot be determined uniquely, which means the set $\mathcal{M}^e$ is composed of infinitely many elements. $\hat{\mathbb{Q}} \in \mathcal{M}^e$ will be characterized by its Radon-Nikodym derivative $Z^{\hat{\Theta}}(t) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$Z^{\hat{\Theta}}(t) := \left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_t = \exp\left\{-\frac{1}{2}\int_0^t \left\|\hat{\Theta}(u)\right\|^2 du - \int_0^t \hat{\Theta}^\top(u)\,dW^{\mathbb{P}}(u)\right\}.$$

Note that the differences between Section 5.1 and this section are:

- In Section 5.1 we considered a financial market composed of the assets M, B_1, B_2 and S while in this section we consider a market with assets M, B_1 and S.
- In Section 5.1 $\mathcal{M}^e$ is singleton, whereas in this section $\mathcal{M}^e$ is made up of infinitely many elements.

These differences are severe, since we now have to deal with an incomplete market and to find an optimal $\hat{\Theta}^*$ by the dual problem. This should serve as an example how incomplete market may be modeled and solved. Now $\hat{\Theta}$ is a solution of (5.49).

We consider again an investor endowed with initial wealth $x > 0$ for the investment. We denote the fractions of wealth invested in the assets at time $t \in [0,T]$ in the market by the vector $\varphi_{inc}(t) = (\varphi_M(t), \varphi_1(t), \varphi_S(t))^\top$, with $\varphi_M(t)$, $\varphi_1(t)$ and $\varphi_S(t)$ corresponding to M, B_1 and S, respectively.

The wealth process of an investor at time $t \in [0,T]$, denoted by $X_{inc}(t)$, under the probability measure $\hat{\mathbb{Q}} \in \mathcal{M}^e$ (using Equations (5.1), (5.10) and (5.11)) follows the SDE:

$$\begin{aligned}
dX_{inc}(t) &= dX^M(t) + dX^{B_1}(t) + dX^S(t) \\
&= (1 - \varphi_1(t) - \varphi_S(t)) X_{inc}(t) r(t) dt \\
&\quad + X_{inc}(t) \varphi_1(t) \left(r(t) dt + \hat{\sigma}_{11} dW_1^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_{12} dW_2^{\hat{\mathbb{Q}}}(t) \right) \\
&\quad + X_{inc}(t) \varphi_S(t) \left(r(t) dt + \hat{\sigma}_{1S} dW_1^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_{2S} dW_2^{\hat{\mathbb{Q}}}(t) + \hat{\sigma}_S dW_S^{\hat{\mathbb{Q}}}(t) \right) \\
&= + X_{inc}(t) (\varphi_S(t) \hat{\sigma}_{1S} + \varphi_1(t) \hat{\sigma}_{11}) dW_1^{\hat{\mathbb{Q}}}(t) \\
&\quad + X_{inc}(t) (\varphi_S(t) \hat{\sigma}_{2S} + \varphi_1(t) \hat{\sigma}_{12}) dW_2^{\hat{\mathbb{Q}}}(t) \\
&\quad + X_{inc}(t) \left[r(t) dt + \varphi_S(t) \hat{\sigma}_S dW_S^{\hat{\mathbb{Q}}}(t) \right],
\end{aligned} \tag{5.50}$$

where $X^M(t), X^{B_1}(t)$ and $X^S(t)$ are as given in (5.15). By applying Ito's rule on the process $\beta(t) X_{inc}(t)$ we get

$$\begin{aligned}
d(\beta(t) X_{inc}(t)) &= X_{inc}(t) \beta(t) (\varphi_S(t) \hat{\sigma}_{1S} + \varphi_1(t) \hat{\sigma}_{11}) dW_1^{\hat{\mathbb{Q}}}(t) \\
&\quad + X_{inc}(t) \beta(t) (\varphi_S(t) \hat{\sigma}_{2S} + \varphi_1(t) \hat{\sigma}_{12}) dW_2^{\hat{\mathbb{Q}}}(t) \\
&\quad + X_{inc}(t) \beta(t) \varphi_S(t) \hat{\sigma}_S dW_S^{\hat{\mathbb{Q}}}(t)
\end{aligned}$$

and thus

$$\begin{aligned}\beta(T)X_{inc}(T) = X_{inc}(0) &+ \int_0^T X_{inc}(t)\beta(t)\left(\varphi_S(t)\hat{\sigma}_{1S} + \varphi_1(t)\hat{\sigma}_{11}\right)dW_1^{\hat{\mathbb{Q}}}(t) \\ &+ \int_0^T X_{inc}(t)\beta(t)\left(\varphi_S(t)\hat{\sigma}_{2S} + \varphi_1(t)\hat{\sigma}_{12}\right)dW_2^{\hat{\mathbb{Q}}}(t) \\ &+ \int_0^T X_{inc}(t)\beta(t)\varphi_S(t)\hat{\sigma}_S dW_S^{\hat{\mathbb{Q}}}(t).\end{aligned} \tag{5.51}$$

The aim of an investor is to choose a portfolio process from $\mathcal{A}(x)$ which maximizes the expected utility of his/her terminal wealth $X_{inc}(T)$. The dynamic and the static optimization problems are constructed as in the case of a complete market (see (5.17) and (5.18)), with $X_c(T)$ in (5.17) and (5.18) replaced by $X_{inc}(T)$.

Before we state a proposition that gives the optimal terminal wealth, let us first recall the primal and dual problem for the terminal wealth $X_{inc}(T)$.

Primal problem:

$$\Psi\left(y,\hat{\mathbb{Q}}\right) := \max_{X_{inc}(T)\in\overline{\mathcal{B}}(x)} L\left(y,\hat{\mathbb{Q}},X_{inc}(T)\right) \quad \text{for} \quad y>0 \quad \text{and} \quad \hat{\mathbb{Q}}\in\mathcal{M}^e,$$

where L is the Lagrangian function defined by

$$L\left(y,\hat{\mathbb{Q}},X_{inc}(T)\right) := \mathbb{E}\left[\frac{\left(X_{inc}(T)\right)^{\gamma}}{\gamma} - y\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T X_{inc}(T) + yx\right], \tag{5.52}$$

and constructed as in (4.57).

Dual problem:

$$\Phi\left(X_{inc}(T)\right) := \min_{y>0,\hat{\mathbb{Q}}\in\mathcal{M}^e} L\left(y,\hat{\mathbb{Q}},X_{inc}(T)\right) \quad \text{for} \quad X_{inc}(T)\in\overline{\mathcal{B}}(x). \tag{5.53}$$

For a given $\hat{\mathbb{Q}}\in\mathcal{M}^e$, the solution to the primal problem (5.3) is similar to the case of complete market and it is restated in the following proposition.

Proposition 5.7 *For $\hat{\mathbb{Q}} \in \mathcal{M}^e$, the solution to the primal problem (5.3) is given by $X^*_{inc}(T) = f_{inc}\left(y^*, \hat{\mathbb{Q}}\right)$, where*

$$f_{inc}\left(y^*, \hat{\mathbb{Q}}\right) = \left(y^* \beta(T) \left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}}, \tag{5.54}$$

thereby $y^* > 0$ is obtained through

$$(y^*)^{\frac{1}{\gamma-1}} = \frac{x}{\mathbb{E}\left[\left(\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}}\right]}. \tag{5.55}$$

If we plug the value of $X^*_{inc}(T)$ in (5.52), after some rearrangements, we get

$$L\left(y^*, \hat{\mathbb{Q}}\right) = \mathbb{E}\left[\frac{1-\gamma}{\gamma}\left(f_{inc}\left(y^*, \hat{\mathbb{Q}}\right)\right)^{\gamma} + yx\right]. \tag{5.56}$$

We have now all the ingredients to state a theorem that gives a solution to the dual problem (5.53) for $X^*_{inc}(T)$ and y^* from Proposition 5.7.

Theorem 5.8 *For y^* and $X^*_{inc}(T)$ as given in Proposition 5.7, the dual problem (5.53) is equivalent to*

$$\begin{aligned} &\min_{\hat{\Theta}(t)=\left(\hat{\theta}_1(t),\hat{\theta}_2(t),\hat{\theta}_S(t)\right)} \mathbb{E}\left[\frac{1-\gamma}{\gamma}\left(f_{inc}\left(y^*, \hat{\mathbb{Q}}\right)\right)^{\gamma} + y^* x\right] \\ &\text{s.t.}\quad \hat{\theta}_1(t)\hat{\sigma}_{11}(t,T_1) + \hat{\theta}_2(t)\hat{\sigma}_{12}(t,T_1) = \bar{\mu}_{1B} \\ &\qquad\quad \hat{\theta}_1(t)\hat{\sigma}_{1S} + \hat{\theta}_2(t)\hat{\sigma}_{2S} + \hat{\theta}_S(t)\hat{\sigma}_S = \bar{\mu}_S, \end{aligned} \tag{5.57}$$

since there is one to one correspondence between $\hat{\Theta}$ and $\hat{\mathbb{Q}} \in \mathcal{M}^e$ together with Remark 5.6. Then, the solution to the problem (5.57) is given by $\Theta^*(t) =$

$\left(\theta_1^*(t), \theta_2^*(t), \theta_S^*(t)\right)^\top$ *corresponding to* $\mathbb{Q}^* \in \mathcal{M}^e$*, where*

$$
\begin{aligned}
\theta_1^*(t) &= \frac{\bar{\mu}_{1B}}{\hat{\sigma}_{11}} + \frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}} \frac{h_1(t)}{h_2(t)}, \\
\theta_2^*(t) &= \frac{h_1(t)}{h_2(t)} \quad \text{and} \\
\theta_S^*(t) &= \frac{\bar{\mu}_S}{\hat{\sigma}_S} - \frac{\hat{\sigma}_{1S}\bar{\mu}_{1B}}{\hat{\sigma}_S\hat{\sigma}_{11}} + \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)\frac{h_1(t)}{h_2(t)} \quad \text{with} \\
h_1(t) &= \frac{\hat{\sigma}_{12}\bar{\mu}_{1B}}{\hat{\sigma}_{11}^2} - \frac{\bar{\mu}_S}{\hat{\sigma}_S}\left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right) + \frac{\hat{\sigma}_{1S}\bar{\mu}_{1B}}{\hat{\sigma}_S\hat{\sigma}_{11}}\left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right) \\
&\quad + \gamma\left(\delta_1 f_{11}(t,T)\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}} - \delta_1 f_{12}(t,T) + \delta_2 f_{21}(t,T)\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}} - \delta_2 f_{22}(t,T)\right) \quad \text{and} \\
h_2(t) &= \left(\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}}\right)^2 + 1 + \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)^2.
\end{aligned}
\tag{5.58}
$$

Hereby $\delta_1, \delta_2, f_{11}(t,T), f_{12}(t,T), f_{21}(t,T)$ *and* $f_{22}(t,T)$ *are as given in (5.21).*

Proof: We have

$$
\frac{\gamma}{1-\gamma}\left(\int_0^T r(t)\,dt + \frac{1}{2}\int_0^T \left\|\hat{\Theta}(t)\right\|^2 dt + \int_0^T \hat{\Theta}^\top dW^{\mathbb{P}}(t)\right) \sim \mathcal{N}\left(M, \sqrt{V}\right),
$$

where

$$
\begin{aligned}
M &= \frac{\gamma}{1-\gamma}\left[\int_0^T \delta_0 dt + \int_0^T \mu^{\mathbb{P}}(t) dt + \frac{1}{2}\int_0^T \left\|\hat{\Theta}(t)\right\|^2 dt\right] \quad \text{and} \\
V &= Var\left[\frac{\gamma}{1-\gamma}\left(\int_0^T r(t)\,dt + \frac{1}{2}\int_0^T \left\|\hat{\Theta}(t)\right\|^2 dt + \int_0^T \hat{\Theta}^\top(t)\,dW^{\mathbb{P}}(t)\right)\right] \\
&= \frac{\gamma^2}{(1-\gamma)^2}\left\{\int_0^T \left[\left(\delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right) + \hat{\theta}_r^\top(t)\right)^2 + \hat{\theta}_S^2(t)\right] dt\right\}.
\end{aligned}
\tag{5.59}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{1-\gamma}{\gamma}\left(f_{inc}\left(y^*, \hat{\mathbb{Q}}\right)\right)^\gamma + y^* x\right] \\
&= \frac{1-\gamma}{\gamma}(y^*)^{\frac{\gamma}{\gamma-1}} e^{M+\frac{1}{2}V} + y^* x =: g\left(\hat{\theta}_1(t), \hat{\theta}_2(t), \hat{\theta}_S(t)\right).
\end{aligned}
\tag{5.60}
$$

Thus, we minimize the function g under two constraints since the relations in Equation (5.49) have to be fulfilled

$$\begin{aligned} &\min_{\left(\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_S\right)} g\left(\hat{\theta}_1\left(t\right),\hat{\theta}_2\left(t\right),\hat{\theta}_S\left(t\right)\right) \\ &s.t.\ \hat{\theta}_1\left(t\right)\hat{\sigma}_{11}\left(t,T_1\right)+\hat{\theta}_2\left(t\right)\left(t\right)\hat{\sigma}_{12}\left(t,T_1\right)=\bar{\mu}_{1B} \\ &\qquad \hat{\theta}_1\left(t\right)\hat{\sigma}_{1S}+\hat{\theta}_2\left(t\right)\hat{\sigma}_{2S}+\hat{\theta}_S\left(t\right)\hat{\sigma}_S=\bar{\mu}_S. \end{aligned} \tag{5.61}$$

We reduce the problem (5.61) from minimizing in three dimensions to one dimension by expressing the two variables in terms of the remaining variable: we set

$$\begin{aligned} &\hat{\theta}_2\left(t\right)=\lambda\left(t\right) \\ &\Rightarrow \hat{\theta}_1\left(\lambda\left(t\right),t\right)=\frac{\bar{\mu}_{1B}}{\hat{\sigma}_{11}}-\frac{\hat{\sigma}_{12}\lambda\left(t\right)}{\hat{\sigma}_{11}}, \\ &\quad \hat{\theta}_S\left(\lambda\left(t\right),t\right)=\frac{\bar{\mu}_S}{\hat{\sigma}_S}-\frac{\hat{\sigma}_{1S}\bar{\mu}_{1B}}{\hat{\sigma}_S\hat{\sigma}_{11}}+\left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}}-\frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)\lambda\left(t\right), \end{aligned} \tag{5.62}$$

and the function g in (5.60) can be described in terms of λ as

$$g\left(\lambda\right):=\frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}}e^{M(\lambda)+\frac{1}{2}V(\lambda)}+y^*x,$$

where

$$\begin{aligned} &M\left(\lambda\right)=\int_0^T\frac{\gamma}{1-\gamma}\left[\delta_0+\mu^{\mathbb{P}}\left(t\right)+\frac{1}{2}\left\|\hat{\Theta}\left(\lambda\left(t\right),t\right)\right\|^2\right]dt, \\ &V\left(\lambda\right)= \\ &\int_0^T\frac{\gamma^2}{\left(1-\gamma\right)^2}\left[\left(\delta^\top A^-\left(I-P\Lambda\left(T-t\right)P^-\right)+\hat{\theta}_r^\top\left(\lambda\left(t\right),t\right)\right)^2+\hat{\theta}_S^2\left(\lambda\left(t\right),t\right)\right]dt. \end{aligned} \tag{5.63}$$

Now the problem (5.61) is reduced to:

$$\min_\lambda g\left(\lambda\right). \tag{5.64}$$

(i) For $\gamma \in (0,1)$ g is minimum, if $M(\lambda) + \frac{1}{2}V(\lambda)$ is minimum.

$$M(\lambda) + \frac{1}{2}V(\lambda) =$$
$$\int_0^T \frac{\gamma}{1-\gamma}\left[\delta_0 + \mu^{\mathbb{P}}(t) + \frac{1}{2}\left\|\hat{\Theta}(\lambda(t),t)\right\|^2 + \right.$$
$$\left.\frac{1}{2}\frac{\gamma}{(1-\gamma)}\left(\left(\delta^\top A^-\left(I - P\Lambda(T-t)P^-\right) + \hat{\theta}_r^\top(\lambda(t),t)\right)^2 + \hat{\theta}_S^2(\lambda(t),t)\right)\right]dt$$
$$=: \int_0^T f(\lambda(t),t)\,dt.$$

So, g is minimum, if $\lambda(t)$ minimizes $f(\lambda(t),t)$ for all $t \in [0,T]$.

$$\begin{aligned}\frac{\partial f(\lambda,t)}{\partial\lambda} =& \frac{\gamma}{1-\gamma}\left[\hat{\Theta}^\top(\lambda,t)\,\hat{\Theta}_\lambda(\lambda,t)\right]\\ &+ \frac{\gamma^2}{(1-\gamma)^2}\left[\left(\delta^\top A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right) + \hat{\theta}_r^\top(\lambda,t)\right)\hat{\theta}_{\lambda,r}(\lambda,t)\right.\\ &\qquad\left. + \hat{\theta}_S(\lambda,t)\,\hat{\theta}_{\lambda,S}(\lambda,t)\right]\\ &\stackrel{!}{=} 0.\end{aligned} \tag{5.65}$$

where

$$\hat{\theta}_{\lambda,r}(\lambda,t) := \frac{\partial\hat{\theta}_r(\lambda,t)}{\partial\lambda} = \begin{pmatrix} -\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}} \\ 1 \end{pmatrix},$$
$$\hat{\Theta}_\lambda(\lambda,t) := \frac{\partial\hat{\Theta}(\lambda,t)}{\partial\lambda} = \begin{pmatrix} -\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}} \\ 1 \\ \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right) \end{pmatrix}.$$

After some rearrangements we find that

$$\lambda^*(t) = \frac{h_1(t)}{h_2(t)}. \tag{5.66}$$

Hereby

$$\begin{aligned}h_1(t) =& \frac{\hat{\sigma}_{12}\bar{\mu}_{1B}}{\hat{\sigma}_{11}^2} - \frac{\bar{\mu}_S}{\hat{\sigma}_S}\left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right) + \frac{\hat{\sigma}_{1S}\bar{\mu}_{1B}}{\hat{\sigma}_S\hat{\sigma}_{11}}\left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)\\ &+ \gamma\left(\delta_1 f_{11}(t,T)\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}} - \delta_1 f_{12}(t,T) + \delta_2 f_{21}(t,T)\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}} - \delta_2 f_{22}(t,T)\right)\end{aligned} \tag{5.67}$$

and

$$h_2(t) = \left(\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}}\right)^2 + 1 + \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)^2. \tag{5.68}$$

$\lambda^*(t)$ up to now is a candidate minimizer of $f(\lambda(t), t)$ for all $t \in [0, T]$. We now have to show that $\lambda^*(t)$ is a global minimizer of $f(\lambda(t), t)$ for all $t \in [0, T]$. We use the second order condition, as we have done in Chapter 4, i.e., we need to show that $\frac{\partial^2 f(\lambda,t)}{\partial \lambda^2} > 0$ at $\lambda^*(t)$ for all $t \in [0, T]$.

$$\begin{aligned} \frac{\partial^2 f(\lambda, t)}{\partial \lambda^2} &= \frac{\gamma}{1-\gamma}\left[\left(\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}}\right)^2 + 1 + \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)^2\right] \\ &\quad + \frac{\gamma^2}{(1-\gamma)^2}\left[\left(\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}}\right)^2 + 1 + \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)^2\right] \\ &\quad > 0. \end{aligned} \tag{5.69}$$

(ii) For $\gamma \in (-\infty, 0)$ g is minimum, if $M(\lambda) + \frac{1}{2}V(\lambda)$ is maximum. We take the same steps as for part (i) and we obtain $\lambda^*(t)$, as given in (5.66), as a candidate maximizer of $f(\lambda(t), t)$. What is left, is to show that $\frac{\partial^2 f(\lambda(t),t)}{\partial \lambda^2} < 0$ at $\lambda^*(t)$ for all $t \in [0, T]$, in order for $\lambda^*(t)$ to be a global maximizer of $f(\lambda(t), t)$. From (5.69) we have

$$\begin{aligned} \frac{\partial^2 f(\lambda, t)}{\partial \lambda^2} &= \left(\frac{\gamma}{1-\gamma} + \frac{\gamma^2}{(1-\gamma)^2}\right)\left(\left(\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}}\right)^2 + 1 + \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)^2\right) \\ &= \underbrace{\frac{\gamma}{1-\gamma}}_{<0}\left(\underbrace{1 + \frac{\gamma}{1-\gamma}}_{>0}\right)\left(\underbrace{\left(\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}}\right)^2 + 1 + \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)^2}_{>0}\right) \\ &< 0. \end{aligned}$$

Substituting the value of λ^* back into Equations (5.62), we obtain the result.

□

Representation problem:

Theorem 5.9 *Suppose all the conditions of Theorem 5.8 are satisfied. Then:*

(a) The time-$t \in [0,T]$ optimal wealth is given by

$$X_{inc}^{*}(t) = (y^{*})^{\frac{1}{\gamma-1}} \left(H^{*}(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^{H^{*}}(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^{2} V^{H^{*}}(t,T)}.$$

Thereby

$$\begin{aligned}
H^{*}(t) &= \beta(t) \left(\frac{d\mathbb{Q}^{*}}{d\mathbb{P}}\right)_{t}, \\
M^{H^{*}}(t,T) &= \int_{t}^{T} \delta_{0} ds + \int_{t}^{T} \mu^{\mathbb{P}}(s) ds + \frac{1}{2}\int_{0}^{T} \|\Theta^{*}(s)\|^{2} ds, \\
V^{H^{*}}(t,T) &= \int_{s}^{T} \left[\left(\delta^{\top} A^{-1}\left(I - P\Lambda(T-s)P^{-1}\right) + \theta_{r}^{*,\top}(s)\right)^{2} + (\theta^{*})_{S}^{2}(s)\right] ds.
\end{aligned}$$

(b) The portfolio process at any time $t \in [0,T]$ that replicates $X_{inc}^{*}(t)$ is given by $\varphi_{inc}^{*}(t) = \left(\varphi_{M,inc}^{*}(t), \varphi_{1,inc}^{*}(t), \varphi_{S,inc}^{*}(t)\right)^{\top}$, where

$$\begin{aligned}
\varphi_{1,inc}^{*}(t) &= \frac{1}{1-\gamma} \frac{\gamma F_{2}(t,T) + \theta_{2}^{*}(t)}{\hat{\sigma}_{12}} - \frac{\varphi_{S,inc}^{*}(t)\hat{\sigma}_{2S}}{\hat{\sigma}_{12}}, \\
\varphi_{S,inc}^{*}(t) &= \frac{1}{1-\gamma} \frac{\theta_{S}^{*}(t)}{\hat{\sigma}_{S}}, \\
\varphi_{M,inc}^{*}(t) &= \left(1 - \varphi_{1,inc}^{*}(t) - \varphi_{S,inc}^{*}(t)\right).
\end{aligned} \tag{5.70}$$

Hereby

$$\begin{aligned}
F_{1}(t,T) &= \delta_{1} f_{11}(t,T) + \delta_{2} f_{21}(t,T), \\
F_{2}(t,T) &= \delta_{1} f_{12}(t,T) + \delta_{2} f_{22}(t,T).
\end{aligned}$$

Proof: After specifying $\mathbb{Q}^{*} \in \mathcal{M}$ in Theorem 5.8, the same procedures as for the case of a complete market are taken to prove this theorem (see the proof of Theorem 5.3). Using Equation (5.31) and replacing $X_{c}^{*}(T)$ and $\hat{\mathbb{Q}}$ by $X_{inc}^{*}(T)$ and $\mathbb{Q}^{*}$, respectively, then $\beta(T) X_{inc}^{*}(T)$ can be expressed as

$$\begin{aligned}
\beta(T) X_{inc}^{*}(T) &= \mathbb{E}^{\mathbb{Q}^{*}}\left[\beta(T) X_{inc}^{*}(T)\right] \\
&+ \int_{0}^{T} \left(\frac{\gamma}{1-\gamma} \delta^{\top} A^{-1}\left(I - P\Lambda(T-t)P^{-1}\right) e_{3} + \frac{1}{1-\gamma}\Theta^{*,\top}(t)\right) \times \\
&\quad \beta(t) X_{inc}^{*}(t) dW^{\mathbb{Q}^{*}}(t).
\end{aligned} \tag{5.71}$$

Comparing (5.71) with (5.51) we obtain

$$\frac{1}{1-\gamma}\left(\gamma\begin{pmatrix}F_1(t,T)\\F_2(t,T)\\0\end{pmatrix}+\begin{pmatrix}\theta_1^*(t)\\\theta_2^*(t)\\\theta_S^*(t)\end{pmatrix}\right)=\begin{pmatrix}\varphi_S(t)\hat{\sigma}_{1S}+\varphi_1(t)\hat{\sigma}_{11}\\\varphi_S(t)\hat{\sigma}_{2S}+\varphi_1(t)\hat{\sigma}_{12}\\\varphi_S(t)\hat{\sigma}_S\end{pmatrix}, \tag{5.72}$$

and thus

$$\varphi_{S,inc}^*(t)=\frac{1}{1-\gamma}\frac{\theta_S^*(t)}{\hat{\sigma}_S}$$

and

$$\varphi_{S,inc}^*(t)\hat{\sigma}_{1S}+\varphi_1(t)\hat{\sigma}_{11}=\frac{1}{1-\gamma}\left(\gamma F_1(t,T)+\theta_1^*(t)\right) \tag{5.73}$$

$$\varphi_{S,inc}^*(t)\hat{\sigma}_{2S}+\varphi_1(t)\hat{\sigma}_{12}=\frac{1}{1-\gamma}\left(\gamma F_2(t,T)+\theta_2^*(t)\right). \tag{5.74}$$

We can use either (5.73) or (5.74) to find $\varphi_{1,inc}^*(t)$. Using (5.74) results in

$$\varphi_{1,inc}^*(t)=\frac{1}{\gamma-1}\frac{\gamma F_2(t,T)+\theta_2^*(t)}{\hat{\sigma}_{12}}-\frac{\varphi_{S,inc}^*(t)\hat{\sigma}_{2S}}{\hat{\sigma}_{12}}.$$

□

Remark 5.10 *If we use Equation (5.73) for the computation of $\varphi_{1,inc}^*(t)$, it gives the same results.*

Proof: From the Equation (5.73) we have

$$\varphi_{1,inc}^*(t)=\frac{1}{1-\gamma}\frac{\left(\gamma F_1(t,T)+\theta_1^*(t)\right)}{\hat{\sigma}_{11}}-\frac{\varphi_{S,inc}^*(t)\hat{\sigma}_{1S}}{\hat{\sigma}_{11}} \tag{5.75}$$

Using the values of $\varphi_{1,inc}^*$ and $\varphi_{S,inc}^*$ from (5.70) we have to show that

$$\begin{aligned}&\frac{1}{1-\gamma}\frac{\gamma F_2(t,T)+\theta_2^*(t)}{\hat{\sigma}_{12}}-\frac{1}{1-\gamma}\frac{\theta_S^*(t)\hat{\sigma}_{2S}}{\hat{\sigma}_S\hat{\sigma}_{12}}\\&-\left(\frac{1}{1-\gamma}\frac{\left(\gamma F_1(t,T)+\theta_1^*(t)\right)}{\hat{\sigma}_{11}}-\frac{1}{1-\gamma}\frac{\theta_S^*(t)\hat{\sigma}_{1S}}{\hat{\sigma}_S\hat{\sigma}_{11}}\right)\overset{!}{=}0\end{aligned} \tag{5.76}$$

Plugging the values of $\theta_2^*(t), \theta_2^*(t)$ and $\theta_S^*(t)$ from (5.58) in (5.76) we get

$$
\begin{aligned}
&\frac{\gamma F_2(t,T)}{\hat{\sigma}_{12}} + \frac{1}{\hat{\sigma}_{12}}\frac{h_1(t)}{h_2(t)} - \frac{(\gamma F_1(t,T))}{\hat{\sigma}_{11}} - \frac{\bar{\mu}_{1B}}{\hat{\sigma}_{11}^2} + \frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}^2}\frac{h_1(t)}{h_2(t)} \\
&+ \left(\frac{\hat{\sigma}_{1S}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S\hat{\sigma}_{12}}\right)\left(\frac{\bar{\mu}_S}{\hat{\sigma}_S} - \frac{\hat{\sigma}_{1S}\bar{\mu}_{1B}}{\hat{\sigma}_S\hat{\sigma}_{11}}\right) \\
&+ \left(\frac{\hat{\sigma}_{1S}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S\hat{\sigma}_{12}}\right)\left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)\frac{h_1(t)}{h_2(t)} \\
&= \frac{\gamma F_2(t,T)}{\hat{\sigma}_{12}} - \frac{(\gamma F_1(t,T))}{\hat{\sigma}_{11}} - \frac{\bar{\mu}_{1B}}{\hat{\sigma}_{11}^2} + \left(\frac{\hat{\sigma}_{1S}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S\hat{\sigma}_{12}}\right)\left(\frac{\bar{\mu}_S}{\hat{\sigma}_S} - \frac{\hat{\sigma}_{1S}\bar{\mu}_{1B}}{\hat{\sigma}_S\hat{\sigma}_{11}}\right) \\
&+ \frac{1}{\hat{\sigma}_{12}}\left(\left(\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}}\right)^2 + 1 + \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)^2\right)\frac{h_1(t)}{h_2(t)}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{\hat{\sigma}_{12}}\left[\gamma\left(F_2(t,T) - (F_1(t,T))\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}}\right) - \frac{\hat{\sigma}_{12}\bar{\mu}_{1B}}{\hat{\sigma}_{11}^2}\right] \\
&+ \frac{1}{\hat{\sigma}_{12}}\left(\frac{\hat{\sigma}_{12}\hat{\sigma}_{1S}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)\left(\frac{\bar{\mu}_S}{\hat{\sigma}_S}\right) - \frac{1}{\hat{\sigma}_{12}}\left(\frac{\hat{\sigma}_{12}\hat{\sigma}_{1S}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)\frac{\hat{\sigma}_{1S}\bar{\mu}_{1B}}{\hat{\sigma}_S\hat{\sigma}_{11}} \\
&+ \frac{1}{\hat{\sigma}_{12}}\left(\left(\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{11}}\right)^2 + 1 + \left(\frac{\hat{\sigma}_{1S}\hat{\sigma}_{12}}{\hat{\sigma}_S\hat{\sigma}_{11}} - \frac{\hat{\sigma}_{2S}}{\hat{\sigma}_S}\right)^2\right)\frac{h_1(t)}{h_2(t)} \\
&= 0.
\end{aligned}
$$

The last equality holds from the definitions of F_1 and F_2 in Theorem 5.3, h_1 and h_2 in Theorem 5.8. □

5.4. Optimization with bounded expected shortfall risk in an incomplete market

In this section we focus on the portfolio optimization problem in an incomplete market with risk constraints in the two-factor Vasiček model of interest rates. We will consider the market model studied in the previous section. In what follows, we review the dynamic and the static optimization problems as given in Section 5.2, with the difference that $\mathcal{M}^e$ is not a singleton as in Section 5.2. It is rather made up of infinitely many elements.

The dynamic optimization problem:

$$\begin{aligned} &\max_{\varphi\in\tilde{\mathcal{A}}(x)} \mathbb{E}\left[\frac{(X_{inc}(T))^{\gamma}}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\, X_{inc}(T)\right] \le x \quad \text{(budget constraint)} \\ &\qquad \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\left(X_{inc}(T)-q\right)^{-}\right] \le \delta \quad \text{(risk constraint)}, \end{aligned} \tag{5.77}$$

with $\tilde{\mathcal{A}}(x)$ as defined in (2.10) and $\hat{\mathbb{Q}} \in \mathcal{M}^e$.

The static optimization problem:

$$\begin{aligned} &\max_{X_{inc}(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{(X_{inc}(T))^{\gamma}}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\, X_{inc}(T)\right] \le x \\ &\qquad \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\left(X_{inc}(T)-q\right)^{-}\right] \le \delta, \end{aligned} \tag{5.78}$$

$\mathcal{B}(x)$ is as defined in (2.12).

The static portfolio insurer problem:

$$\begin{aligned} &\max_{X_{inc}(T)\in\mathcal{B}(x)} \mathbb{E}\left[\frac{(X_{inc}(T))^{\gamma}}{\gamma}\right] \\ &\text{s.t. } \mathbb{E}^{\hat{\mathbb{Q}}}\left[\beta(T)\, X_{inc}(T)\right] \le x \\ &\qquad X_{inc}(T) \ge q. \end{aligned} \tag{5.79}$$

The optimal wealth for problems (5.78) and (5.79) are as in Theorem 4.5, Theorem 3.5, Proposition 4.12. We restate below a proposition that characterizes the solutions of the problems (5.78) and (5.79) for a sake of revision.

Proposition 5.11 *Let q be a fixed benchmark, x an initial wealth of an investor and $\hat{\mathbb{Q}} \in \mathcal{M}^e$. (i) If $\delta \in \left(\underline{\delta}, \overline{\delta}\right)$, then the solution of the problem (5.78) is given by*

$$X^{\delta}_{inc}(T) = f_{inc}\left(y_1^*, y_2^*, \hat{\mathbb{Q}}\right), \tag{5.80}$$

where

$$f_{inc}\left(y_1^*, y_2^*, \hat{\mathbb{Q}}\right) = \left(y_1^* \beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_A + q\mathbb{1}_B + \left(\left(y_1^* - y_2^*\right)\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_C, \tag{5.81}$$

$$A = A\left(y_1^*, \hat{\mathbb{Q}}\right) := \left\{\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \leq \frac{q^{\gamma-1}}{y_1^*}\right\},$$

$$B = B\left(y_1^*, y_2^*, \hat{\mathbb{Q}}\right) := \left\{\frac{q^{\gamma-1}}{y_1^*} < \beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \leq \frac{q^{\gamma-1}}{y_1^* - y_2^*}\right\}, \text{ and}$$

$$C = C\left(y_1^*, y_2^*, \hat{\mathbb{Q}}\right) := \left\{\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T > \frac{q^{\gamma-1}}{y_1^* - y_2^*}\right\},$$

such that $y_1^*, y_2^* \in (0, \infty)$ *solve the following system of equations*

$$\mathbb{E}\left[\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T f_{inc}\left(y_1, y_2, \hat{\mathbb{Q}}\right)\right] = x$$

$$\mathbb{E}\left[\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \left(f_{inc}\left(y_1, y_2, \hat{\mathbb{Q}}\right) - q\right)^-\right] = \delta.$$

(ii) If $\delta = \underline{\delta}$*, i.e in the case of the portfolio insurer problem, the solution to (5.79) is given by*

$$X_{inc}^{PI}(T) = f_{inc}\left(y^{PI}, \hat{\mathbb{Q}}\right),$$

where

$$f_{inc}\left(y^{PI}, \hat{\mathbb{Q}}\right) = \left(y^{PI}\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{1}{\gamma-1}} \mathbb{1}_{A^{PI}} + q\mathbb{1}_{B^{PI}}, \tag{5.82}$$

$$A^{PI} = A^{PI}\left(y^{PI}, \hat{\mathbb{Q}}\right) := \left\{\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \leq \frac{q^{\gamma-1}}{y^{PI}}\right\}, \quad \text{and}$$

$$B^{PI} = B^{PI}\left(y^{PI}, \hat{\mathbb{Q}}\right) := \left\{\frac{q^{\gamma-1}}{y^{PI}} < \beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right\},$$

such that y^{PI} *solves the equation*

$$\mathbb{E}\left[\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T f_{inc}\left(y^{PI}, \hat{\mathbb{Q}}\right)\right] = x.$$

We use the same idea as in Section 4.4 to solve the dual problem below for the optimal terminal wealth $X_{inc}^{\delta}(T)$ as given in the Proposition 5.11, for the case of $X_{inc}^{PI}(T)$ it is done analogously.

Dual problem:

$$\Phi\left(X_{inc}^{\delta}(T)\right) := \min_{\hat{\mathbb{Q}}\in\mathcal{M}^e} L\left(\hat{\mathbb{Q}}, X_{inc}^{\delta}(T)\right), \tag{5.83}$$

where

$$\begin{aligned}
L\left(\hat{\mathbb{Q}}, X_{inc}^{\delta}(T)\right) &= \mathbb{E}\left[\frac{\left(X_{inc}^{\delta}(T)\right)^{\gamma}}{\gamma} - y_1^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T X_{inc}^{\delta}(T)\right] \\
&- \mathbb{E}\left[y_2^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \left(X_{inc}^{\delta}(T)-q\right)^{-} - y_1^*x - y_2^*\delta\right] \\
=&\mathbb{E}\left[\frac{\left(f_{inc}\left(y_1^*,y_2^*,\hat{\mathbb{Q}}\right)\right)^{\gamma}}{\gamma} - y_1^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T f\left(y_1^*,y_2^*,\hat{\mathbb{Q}}\right)\right] \\
&- \mathbb{E}\left[y_2^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T \left(f_{inc}\left(y_1^*,y_2^*,\hat{\mathbb{Q}}\right)-q\right)^{-} - y_1^*x - y_2^*\delta\right] \\
=&\mathbb{E}\left[\frac{(\gamma-1)}{\gamma}\left(y_1^*\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}} \mathbb{1}_A\right] \\
&+ \mathbb{E}\left[\left(\frac{q^{\gamma} - y_1^* q\gamma\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T}{\gamma}\right)\mathbb{1}_B\right] \\
&+ \mathbb{E}\left[\left(\frac{\left(y_1^*(1-\gamma) - y_2^*(1+\gamma)\right)}{\gamma}\right)\left(y_1^*-y_2^*\right)^{\frac{1}{\gamma-1}}\left(\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}}\mathbb{1}_C\right] \\
&+ y_1^*x + y_2^*\delta.
\end{aligned} \tag{5.84}$$

Remark 5.12 *$\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T$ is a log-normal random variable under $\mathbb{P}$:*

$$-\ln\left\{\beta(T)\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right)_T\right\} \sim \mathcal{N}\left(M^H(\theta_1,\theta_2,\theta_s), \sqrt{V^H(\theta_1,\theta_2,\theta_s)}\right)$$

where $M^H(\theta_1,\theta_2,\theta_s) = \frac{1-\gamma}{\gamma}M(\theta_1,\theta_2,\theta_s)$ *and* $V^H(\theta_1,\theta_2,\theta_s) = \frac{1-\gamma}{\gamma}V(\theta_1,\theta_2,\theta_s)$ *with* $M(\theta_1,\theta_2,\theta_s)$ *and* $V(\theta_1,\theta_2,\theta_s)$ *as given in (5.59).*

Therefore,

$$
\begin{aligned}
L\left(\hat{\mathbb{Q}}, X^{\delta}_{inc}(T)\right) &= \frac{1-\gamma}{\gamma}\left(y_1^*\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{\gamma}{1-\gamma}M^H+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2V^H} * \Phi(d_1) \\
&+ \frac{q^\gamma}{\gamma}\left(\Phi(d_3)-\Phi(d_2)\right) - y_1^* q e^{-M^H+\frac{1}{2}V^H}\left(\Phi(d_5)-\Phi(d_4)\right) \\
&+ \left(\frac{\left(y_1^*(1-\gamma)-y_2^*(1+\gamma)\right)}{\gamma}\right)\left(y_1^*-y_2^*\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma}M^H+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2V^H}\Phi(d_6) \\
&+ y_1^* x + y_2^*\delta =: L(\theta_1,\theta_2,\theta_S),
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= d_1(y_1^*,\theta_1,\theta_2,\theta_S) := \frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*}\right)+M^H}{\sqrt{V^H}} + \frac{\gamma}{1-\gamma}\sqrt{V^H}, \\
d_2 &= d_2(y_1^*,\theta_1,\theta_2,\theta_S) := -\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*}\right)+M^H}{\sqrt{V^H}}, \\
d_3 &= d_3(y_1^*,y_2^*,\theta_1,\theta_2,\theta_S) := \frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*-y_2^*}\right)+M^H}{\sqrt{V^H}}, \\
d_4 &= d_4(y_1^*,\theta_1,\theta_2,\theta_S) := d_2+\sqrt{V^H}, \\
d_5 &= d_5(y_1^*,y_2^*,\theta_1,\theta_2,\theta_S) := d_3-\sqrt{V^H}, \\
d_6 &= d_6(y_1^*,y_2^*,\theta_1,\theta_2,\theta_S) := -\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^*-y_2^*}\right)+M^H}{\sqrt{V^H}} - \frac{\gamma}{1-\gamma}\sqrt{V^H}.
\end{aligned}
\tag{5.85}
$$

We state the following optimization problem with the relations (5.49) considered as constraints.

$$
\begin{aligned}
&\min_{(\theta_1,\theta_2,\theta_S)} L\left(\theta_1,\theta_2,\theta_S,X^{\delta}_{inc}(T)\right) \\
&s.t.\quad \sigma_{11}(t,T_1)\theta_1(t)+\sigma_{12}(t,T_1)\theta_2(t) = \bar{\mu}_{1B} \\
&\qquad\quad \sigma_{1S}\theta_1(t)+\sigma_{2S}\theta_2(t)+\sigma_S\theta_S(t) = \bar{\mu}_S.
\end{aligned}
\tag{5.86}
$$

To solve the problem (5.86) we use the same procedure as used to prove Theorem

5.8 by setting first

$$\begin{aligned}
&\theta_2(\lambda,t) = \lambda(t)\\
&\Rightarrow \theta_1(\lambda,t) = \frac{\bar{\mu}_{1B}}{\sigma_{11}(t,T_1)} - \frac{\sigma_{12}(t,T_1)\lambda(t)}{\sigma_{11}(t,T_1)} \quad \text{and}\\
&\theta_S(\lambda,t) = \frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{1S}\bar{\mu}_{1B}}{\sigma_S\sigma_{11}(t,T_1)} + \left(\frac{\sigma_{1S}\sigma_{12}(t,T_1)}{\sigma_S\sigma_{11}(t,T_1)} - \frac{\sigma_{2S}}{\sigma_S}\right)\lambda(t).
\end{aligned} \tag{5.87}$$

Then, problem (5.86) is reduced to

$$\min_{\lambda} L(\lambda). \tag{5.88}$$

As far as we know, the solution λ^* to the problem (5.88) can be only found numerically but can't be calculated analytically. Then, we plug λ^* in Equation (5.87) and obtain the solution of the problem (5.86). We summarize all together in the following theorem.

Theorem 5.13 *Consider the problem*

$$\begin{aligned}
\min_{\Theta=(\theta_1,\theta_2,\theta_S)} \quad & L\left(\theta_1,\theta_2,\theta_S,X^{\delta}_{inc}(T)\right)\\
s.t. \quad & \sigma_{11}(t,T_1)\theta_1(t) + \sigma_{12}(t,T_1)\theta_2(t) = \bar{\mu}_{1B}\\
& \sigma_{1S}\theta_1(t) + \sigma_{2S}\theta_2(t) + \sigma_S\theta_S(t) = \bar{\mu}_S.
\end{aligned}$$

Then, the optimal solution is given by $\Theta^*(t) = (\theta_1^*(t), \theta_2^*(t), \theta_S^*(t))^\top$, *where*

$$\begin{aligned}
&\theta_1^*(t) = \frac{\bar{\mu}_{1B}}{\sigma_{11}(t,T_1)} - \frac{\sigma_{12}(t,T_1)}{\sigma_{11}(t,T_1)}\lambda^*,\\
&\theta_2^*(t) = \lambda^* \quad \textit{and}\\
&\theta_S^*(t) = \frac{\bar{\mu}_S}{\sigma_S} - \frac{\sigma_{1S}\bar{\mu}_{1B}}{\sigma_S\sigma_{11}(t,T_1)} - \left(\frac{\sigma_{1S}\sigma_{12}(t,T_1)}{\sigma_S\sigma_{11}(t,T_1)} + \frac{\sigma_{2S}}{\sigma_S(t)}\right)\lambda^*.
\end{aligned}$$

For the case of $\delta = \underline{\delta}$ and from Proposition 5.11, the Lagrangian function is given by

$$\begin{aligned}
L\left(\mathbb{Q},X^{PI}_{inc}(T)\right) &= \mathbb{E}\left[\frac{\left(X^{PI}(T)\right)^{\gamma}}{\gamma} - y^{PI}\left(\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T X^{PI}(T) - x\right)\right]\\
=&\mathbb{E}\left[\frac{(\gamma-1)}{\gamma}\left(y^{PI}\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T\right)^{\frac{\gamma}{\gamma-1}}\mathbb{1}_{A^{PI}} + \frac{q^{\gamma}}{\gamma}\mathbb{1}_{B^{PI}}\right]\\
&- y^{PI}\mathbb{E}\left[\beta(T)\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T q\mathbb{1}_{B^{PI}} - x\right].
\end{aligned} \tag{5.89}$$

Using Remark 5.12 we get

$$
\begin{aligned}
L\left(\mathbb{Q}, X_{inc}^{PI}(T)\right) =& \frac{1-\gamma}{\gamma}\left(y^{PI}\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{\gamma}{1-\gamma}M^H+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H} * \Phi\left(d_1^{PI}\right) \\
&+\frac{q^\gamma}{\gamma}\Phi\left(d_2^{PI}\right) - y^{PI} q e^{-M^H+\frac{1}{2}V^H}\Phi\left(d_3^{PI}\right),
\end{aligned}
\tag{5.90}
$$

where

$$
\begin{aligned}
d_1^{PI} &= d_1^{PI}\left(y^{PI}, \theta_1, \theta_2, \theta_S\right) := \frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}}\right) + M^H\left(\theta_1, \theta_2, \theta_S\right)}{\sqrt{V^H\left(\theta_1, \theta_2, \theta_S\right)}} + \frac{\gamma}{1-\gamma}\sqrt{V^H\left(\theta_1, \theta_2, \theta_S\right)}, \\
d_2^{PI} &= d_2^{PI}\left(y^{PI}, \theta_1, \theta_2, \theta_S\right) := -\frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}}\right) + M^H}{\sqrt{V^H}}, \\
d_3^{PI} &= d_3^{PI}\left(y^{PI}, \theta_1, \theta_2, \theta_S\right) := -\frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI}}\right) + M^H}{\sqrt{V^H}} - \sqrt{V^H}.
\end{aligned}
\tag{5.91}
$$

We state the optimization problem, as (5.86), for the portfolio insurer problem as follows

$$
\begin{aligned}
&\min_{(\theta_1,\theta_2,\theta_S)} L\left(\theta_1, \theta_2, \theta_S, X_{inc}^{PI}(T)\right) \\
&s.t. \;\; \sigma_{11}(t,T_1)\theta_1(t) + \sigma_{12}(t,T_1)\theta_2(t) = \bar{\mu}_{1B} \\
&\qquad \sigma_{1S}\theta_1(t) + \sigma_{2S}\theta_2(t) + \sigma_S\theta_S(t) = \bar{\mu}_S.
\end{aligned}
\tag{5.92}
$$

To solve (5.92) we take the same steps as for the case of $\delta \in \left(\underline{\delta}, \overline{\delta}\right)$: set $\theta_2(\lambda, t) = \lambda(t)$ and express θ_1 and θ_S in terms of λ and t using the constraints in (5.92) and then find the minimizer λ^{PI} of

$$
L^{PI}(\lambda) = L\left(\theta_1(\lambda), \theta_2(\lambda), \theta_S(\lambda)\right) = L\left(\theta_1, \theta_2, \theta_S, X_{inc}^{PI}(T)\right).
$$

Like for the case of $\delta \in \left(\underline{\delta}, \overline{\delta}\right)$, λ^{PI} can be only found numerically but cannot be calculated analytically. We denote the market price of risk and the probability measure that correspond to λ^{PI} by Θ^{PI} and $\mathbb{Q}^{PI} \in \mathcal{M}^e$, respectively.

Representation problem

Theorem 5.14 *Suppose all conditions of the Proposition 5.11 are fulfilled and we consider the probability measures $\mathbb{Q}^* \in \mathcal{M}^e$ and $\mathbb{Q}^{PI} \in \mathcal{M}^e$. Then:*

(a) *The time-$t \in [0,T]$ optimal wealth $X^{\delta}_{inc}(t)$ and $X^{PI}_{inc}(t)$ are given by*

$$
\begin{aligned}
&X^{\delta}_{inc}(t) \\
&= (y_1^*)^{\frac{1}{\gamma-1}} (H^*(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^{H^*}(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{H^*}(t,T)} * \Phi\left(d_1^{\delta}(t)\right) \\
&+ e^{-M^{H^*}(t,T) + \frac{1}{2} V^{H^*}(t,T)} * \left(\Phi\left(d_2^{\delta}(t)\right) - \Phi\left(d_3^{\delta}(t)\right)\right) \\
&+ (y_1^* - y_2^*)^{\frac{1}{\gamma-1}} (H^*(t))^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^{H^*}(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{H^*}(t,T)} * \Phi\left(d_4^{\delta}(t)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&X^{PI}(t) = \left(y_1^{PI}\right)^{\frac{1}{\gamma-1}} \left(H^{PI}(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^{H^{PI}}(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^{H^{PI}}(t,T)} \times \\
&\Phi\left(d_1^{PI}(t)\right) + e^{-M^{H^{PI}}(t,T) + \frac{1}{2} V^{H^{PI}}(t,T)} * \Phi\left(d_2^{PI}(t)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&H^*(t) = \beta(t) \left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\right)_t, \\
&H^{PI}(t) = \beta(t) \left(\frac{d\mathbb{Q}^{PI}}{d\mathbb{P}}\right)_t, \\
&M^{H^*}(t,T) = \int_t^T \delta_0 ds + \int_t^T \mu^{\mathbb{P}}(s) ds + \frac{1}{2} \int_0^T \|\Theta^*(s)\|^2 ds, \\
&V^{H^*}(t,T) \\
&= \int_t^T \left[\left(\delta^\top A^{-1}\left(I - P\Lambda(T-s)P^{-1}\right) + \theta_r^{*,\top}(s)\right)^2 + (\theta^*)_S^2(s)\right] ds, \\
&M^{H^{PI}}(t,T) = \int_t^T \delta_0 ds + \int_t^T \mu^{\mathbb{P}}(s) ds + \frac{1}{2} \int_0^T \left\|\Theta^{PI}(s)\right\|^2 ds, \\
&V^{H^{PI}}(t,T) \\
&= \int_t^T \left[\left(\delta^\top A^{-1}\left(I - P\Lambda(T-s)P^{-1}\right) + \theta_r^{PI,\top}(s)\right)^2 + \left(\theta^{PI}\right)_S^2(s)\right] ds,
\end{aligned}
$$

$$d_1^{\delta}(t) = \frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H^*(t)}\right) + M^{H^*}(t,T)}{\sqrt{V^{H^*}(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^{H^*}(t,T)},$$
$$d_2^{\delta}(t) = -\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H^*(t)}\right) + M^{H^*}(t,T)}{\sqrt{V^{H^*}(t,T)}} + \sqrt{V^{H^*}(t,T)},$$
$$d_3^{\delta}(t) = -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right) H^*(t)}\right) + M^{H^*}(t,T)}{\sqrt{V^{H^*}(t,T)}} + \sqrt{V^{H^*}(t,T)},$$
$$d_4^{\delta}(t) = -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right) H^*(t)}\right) + M^{H^*}(t,T)}{\sqrt{V^{H^*}(t,T)}} - \frac{\gamma}{1-\gamma}\sqrt{V^{H^*}(t,T)},$$

$$d_1^{PI}(t) = \frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI} H^{PI}(t)}\right) + M^{H^{PI}}(t,T)}{\sqrt{V^{H^{PI}}(t,T)}} + \frac{\gamma}{1-\gamma}\sqrt{V^{H^{PI}}(t,T)},$$
$$d_2^{PI}(t) = -\frac{\ln\left(\frac{q^{\gamma-1}}{y^{PI} H^{PI}(t)}\right) + M^{H^{PI}}(t,T)}{\sqrt{V^{H^{PI}}(t,T)}} + \sqrt{V^{H^{PI}}(t,T)},$$

and $\Phi(\cdot)$ is the standard normal probability distribution function.

(b) *The portfolio processes at any time $t \in [0,T]$ which replicate the optimal terminal wealth $X_{inc}^{\delta}(t)$ and $X_{inc}^{PI}(t)$ are given by*

$$\varphi_{inc}^{\delta}(t) = \left(\varphi_{M,inc}^{\delta}(t), \varphi_{1,inc}^{\delta}(t), \varphi_{S,inc}^{\delta}(t)\right)^{\top} \quad \text{and}$$
$$\varphi_{inc}^{PI}(t) = \left(\varphi_{M,inc}^{PI}(t), \varphi_{1,inc}^{PI}(t), \varphi_{S,inc}^{PI}(t)\right)^{\top},$$

respectively, where

$$
\begin{aligned}
\varphi^{\delta}_{1,inc}(t) &= \frac{1}{\gamma-1}\left(\frac{\gamma F_1(t,T)+\theta^*_1(t)}{\hat{\sigma}_{11}}\right) + \frac{1}{1-\gamma}\left(\frac{\gamma F_1(t,T)+\theta^*_1(t)}{\hat{\sigma}_{11}}\right) \times \\
&\quad \frac{q e^{-M^{H^*}(t,T)+\frac{1}{2}V^{H^*}(t,T)}\left(\Phi\left(d^{\delta}_2(t)\right)-\Phi\left(d^{\delta}_3(t)\right)\right)}{X^{\delta}_{inc}(t)} \\
&\quad - \frac{\varphi^{\delta}_{S,inc}(t)\,\hat{\sigma}_{1S}}{\hat{\sigma}_{11}}, \\
\varphi^{\delta}_{S,inc}(t) &= \frac{1}{1-\gamma}\left(\frac{\theta^*_S(t)}{\hat{\sigma}_S}\right) + \frac{1}{\gamma-1}\left(\frac{\theta^*_S(t)}{\hat{\sigma}_S}\right) \times \\
&\quad \frac{q e^{-M^{H^*}(t,T)+\frac{1}{2}V^{H^*}(t,T)}\left(\Phi\left(d^{\delta}_2(t)\right)-\Phi\left(d^{\delta}_3(t)\right)\right)}{X^{\delta}_{inc}(t)}, \\
\varphi^{\delta}_{M,inc}(t) &= \left(1-\varphi^{\delta}_{1,inc}(t)-\varphi^{\delta}_{S,inc}(t)\right),
\end{aligned}
\tag{5.93}
$$

$$
\begin{aligned}
\varphi^{PI}_{1,inc}(t) &= \frac{1}{\gamma-1}\left(\frac{\gamma F_1(t,T)+\theta^{PI}_1(t)}{\hat{\sigma}_{11}}\right) + \frac{1}{1-\gamma}\left(\frac{\gamma F_1(t,T)+\theta^{PI}_1(t)}{\hat{\sigma}_{11}}\right) \times \\
&\quad \frac{q e^{-M^{H^{PI}}(t,T)+\frac{1}{2}V^{H^{PI}}(t,T)}\Phi\left(d^{PI}_2(t)\right)}{X^{PI}(t)} \\
&\quad - \frac{\varphi^{PI}_{S,inc}(t)\,\hat{\sigma}_{1S}}{\hat{\sigma}_{11}}, \\
\varphi^{PI}_{S,inc}(t) &= \frac{1}{1-\gamma}\left(\frac{\theta^{PI}_S(t)}{\hat{\sigma}_S}\right) + \frac{1}{\gamma-1}\left(\frac{\theta^{PI}_S(t)}{\hat{\sigma}_S}\right) \times \\
&\quad \frac{q e^{-M^{H^{PI}}(t,T)+\frac{1}{2}V^{H^{PI}}(t,T)}\Phi\left(d^{PI}_2(t)\right)}{X^{PI}(t)}, \\
\varphi^{PI}_{M,inc}(t) &= \left(1-\varphi^{PI}_{1,inc}(t)-\varphi^{PI}_{S,inc}(t)\right).
\end{aligned}
\tag{5.94}
$$

Proof: After identifying $\mathbb{Q}^* \in \mathcal{M}^e$ and $\mathbb{Q}^{PI} \in \mathcal{M}^e$ we take the same steps as for the proof of Theorem 5.5 and obtain the results. □

Table 5.1.: Table of parameters

$\bar{\mu}_{1B}$	0.02	σ_{1S}	-0.25	δ_1	0.002	a_{12}	0	T	10
$\bar{\mu}_{2B}$	0.03	σ_{2S}	0.25	δ_2	0.0109	a_{21}	-0.0134	T_1	12
$\bar{\mu}_S$	0.05	σ_S	0.20	a_1	0.1086	a_2	0.0942	T_2	15

5.5. Numerical examples

We conclude this chapter with numerical examples. Table 5.1 shows the parameters considered. Figures 5.1 - 5.3 depict the optimal trading strategies as functions of time t without risk constraints in a complete market. $\varphi^*_{1,co}$, $\varphi^*_{2,co}$, $\varphi^*_{S,co}$, $\varphi^*_{M,co}$ and $\varphi^*_{R,co}$ represent the optimal fractions of wealth invested in bond B_1 with maturity T_1, bond B_2 with maturity T_2, stock S, money market account M and risky assets R (i.e., $\varphi^*_{R,co} = 1 - \varphi^*_{M,co}$), respectively. In Figure 5.1, the risk aversion parameter γ is set to 0.7 and it can be seen that as the residual time horizon $(T - t)$ reduces, $\varphi^*_{1,co}$ and $\varphi^*_{R,co}$ increase, while $\varphi^*_{M,co}$ and $\varphi^*_{2,co}$ decrease. Figure 5.2 reveals that for γ equal to -20 the differences in the optimal trading strategies reduce (in other words, the aggressiveness of an investor reduces) compared to Figure 5.1 but their profiles remain similar. Figure 5.3 shows that $\varphi^*_{S,co}$ is not equal to zero.

Figures 5.4 and 5.5 display the optimal trading strategies as functions of time t without risk constraints in an incomplete market for $\gamma = 0.7$ and $\gamma = -20$ respectively. $\varphi^*_{1,inc}$, $\varphi^*_{S,inc}$, $\varphi^*_{M,inc}$ and $\varphi^*_{R,inc}$ denote the optimal fraction of wealth invested in bond B_1 with maturity T_1, stock S, money market account M and risky assets R (i.e., $\varphi^*_{R,inc} = 1 - \varphi^*_{M,inc}$) respectively. Figure 5.4 as well as 5.5 illustrate that an investor in an incomplete market reduces his/her positions in the risky assets and increases his/her position in M as the residual time horizon reduces. If we compare an investor in a complete market with an investor in an incomplete market by the amount of money invested in R, we find that the one in a complete market is more aggressive than the one in an incomplete market.

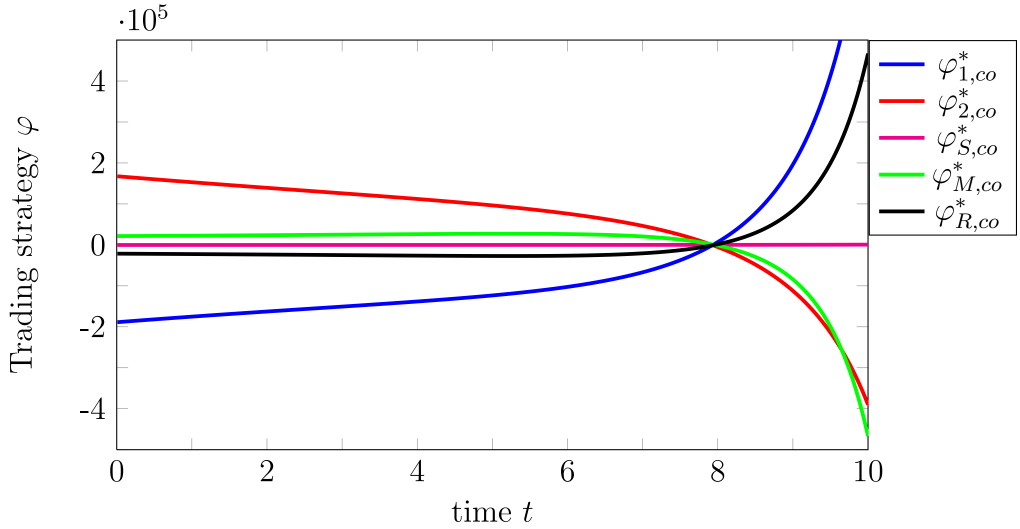

Figure 5.1.: Optimal trading strategies for $\gamma = 0.7$ in a complete market.

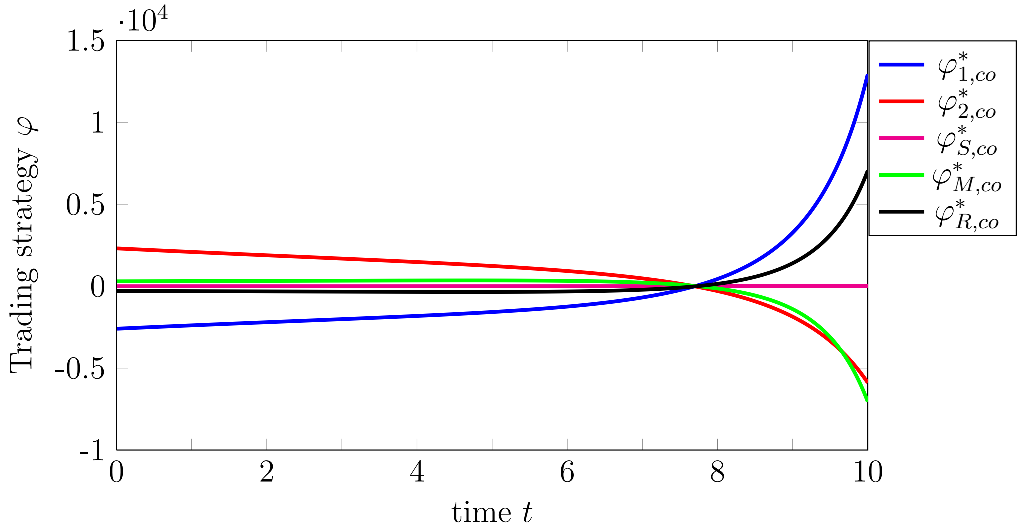

Figure 5.2.: Optimal trading strategies for $\gamma = -20$ in a complete market.

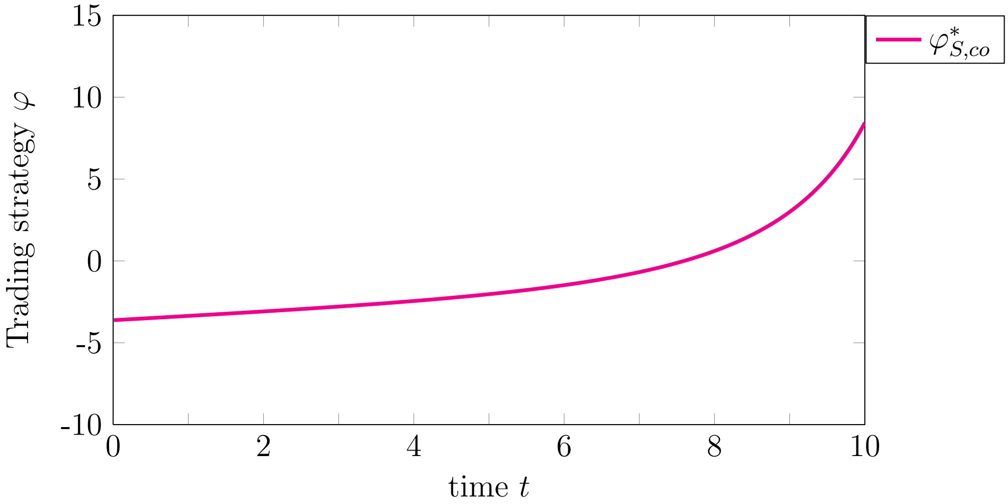

Figure 5.3.: Optimal trading strategies for $\gamma = -20$ in a complete market.

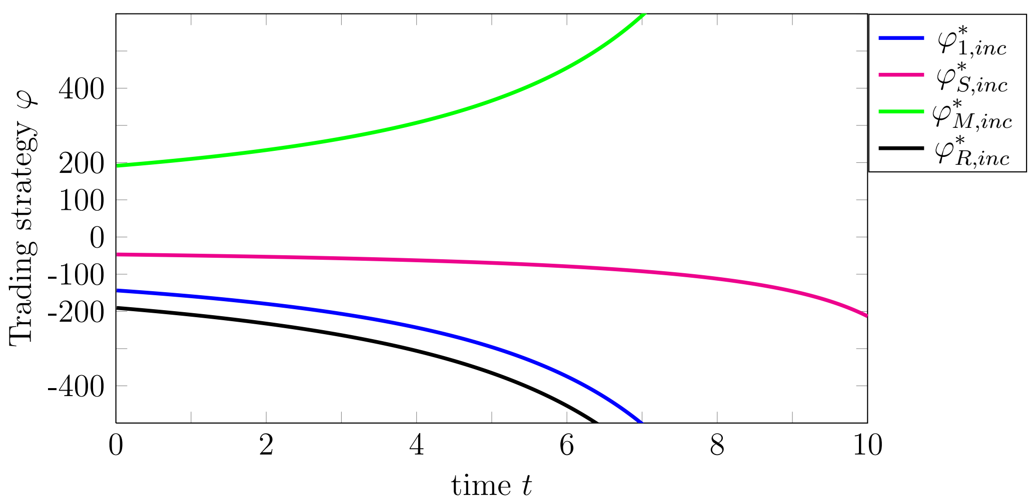

Figure 5.4.: Optimal trading strategies for $\gamma = 0.7$ in an incomplete market.

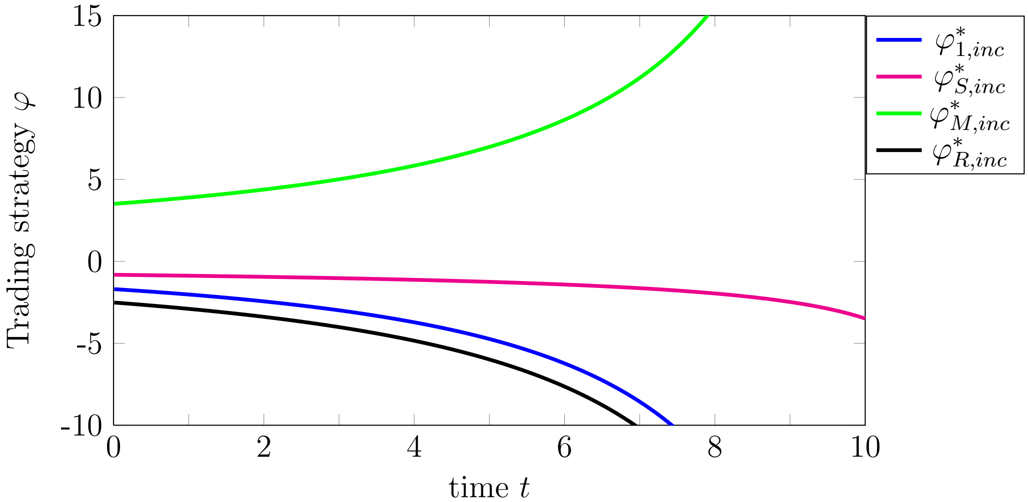

Figure 5.5.: Optimal trading strategies for $\gamma = -20$ in an incomplete market.

6. Conclusion

We conclude this thesis by giving a summary of what we have done in this work and provide an outlook.

6.1. Summary

We have investigated the expected utility maximization problem with limited present expected short-fall (PESF) risk when the interest rates are considered to follow a one-factor Vasiček term structure model in a complete market. We used the martingale approach to solve this problem. We computed the optimal trading strategies using Malliavin calculus. We studied further the same problem with limited present expected short-fall risk constraint and without risk constraints in the general case of an incomplete market. In particular we solved the dual problem explicitly for the optimization problem without risk constraints, i.e, Merton portfolio optimization problem. We derived the trading strategies using Malliavin calculus as well. We provided numerical examples to compare the behavior of the portfolio managers in both complete and incomplete financial markets.

We examined further the expected utility maximization problem without risk constraints and with limited PESF risk when the interest rates are allowed to be modeled by a two-factor Vasiček model. We solved the dual problem explicitly for the Merton optimization problem. We obtained the optimal policies in all cases using Malliavin calculus. We illustrated the behavior of portfolio managers in a complete and an incomplete market using numerical examples.

6.2. Outlook

In this work we considered PESF as a risk measure, which is in general not a coherent risk measure. This work might be extended to considering coherent risk measures, for example, conditional value at risk. We assumed the Lagrangian multipliers to exist which solve the budget and risk constraints with equality and we supposed deterministic market prices of risk in this thesis. Further work might be to relax this restriction and allow stochastic market prices of risk.

The Vasiček model for interest rates has been criticized for permitting negative interest rates, but it has turned out that in the presence of economic crisis interest rates may be negative. When there is no economic crisis interest rates are always positive and therefore the interest rates should be described by a model which does not allow negative values, for example, the Cox-Ingersoll-Ross (CIR) model whose dynamics are as given in (2.5):

$$dr\left(t\right) = a^{\mathbb{P}}\left(b^{\mathbb{P}} - r\left(t\right)\right)dt + \sigma_r\sqrt{r\left(t\right)}dW_r^{\mathbb{P}}\left(t\right),$$

for $t \in [0, T]$, $a^{\mathbb{P}}, b^{\mathbb{P}}, \sigma_r > 0$ and $2a^{\mathbb{P}}b^{\mathbb{P}} \geq \sigma_r$. Let us consider a complete financial market described in Chapter 4, i.e., with the money market account M, bond B (maturing at $\widetilde{T} > T$) and stock S. We provide some details in Appendix C. Considering two-factor CIR model for interest rates in a complete market case the portfolio optimization problem can be solved for a suitable choice of the form of market prices of risk. In an incomplete market case, in order to solve the dual problem (2.15) we need to know the distribution of $H\left(t\right)$ for $t \in [0, T]$ and it is not known when the interest rates are modeled by CIR. So, the further work might be to provide good numerical solution to find $H_{CIR}\left(t\right)$ for $t \in [0, T]$.

A. Malliavin calculus

In this section we give some definitions and necessary basics of Malliavin calculus that are needed for the solving of dynamic optimization problems. We refer the interested reader for more details about Malliavin calculus to Nualart & Pardoux [51] and Ocone [52], among others. We recall first the definitions of the gradient operator D and the class $\mathbb{D}_{1,1}$. These are applied on the probability space $(\Omega, \mathbb{F}, \mathbb{Q})$ and the Brownian motion $\left\{W^{\mathbb{Q}}(t) : t \leq T\right\}$. We denote by $\mathcal{F}(t)$ the augmented filtration generated by $W^{\mathbb{Q}}(t)$. Ω is represented here as the Wiener space, denoted by $C_0([0,T])$, of continuous functions
$\omega : [0,T] \to \mathbb{R}$ such that $\omega(0) = 0$, equipped with the uniform topology.

Let $\mathcal{S}$ denote the set of all random variables $F : C_0([0,T]) \to \mathbb{R}$ of the form $F = \varphi(\theta_1, \ldots, \theta_n)$, where $\varphi(x_1, \ldots, x_n) = \sum\limits_{\alpha=1}^{n} a_\alpha x^\alpha$ is a polynomial of degree n and $\theta_i = \int_0^T f_i(t) dW^{\mathbb{Q}}(t)$ for some deterministic function $f_i \in L^2([0,T])$.
We define the Cameron-Martin space $\mathcal{H}$ as follows

$$\mathcal{H} = \left\{\gamma : [0,T] \to \mathbb{R} : \gamma(t) = \int_0^t \dot{\gamma}(s)\, ds, |\gamma|^2_{\mathcal{H}} = \int_0^T (\dot{\gamma}(s))^2\, ds < \infty\right\}.$$

The derivative of a random variable

$$F(\omega) = \int_0^T f_\gamma(t) dW^{\mathbb{Q}}(t) \tag{A.1}$$

in the direction $\gamma \in \mathcal{H}$ (note that $\mathcal{H} \subset C_0([0,T])$) with

$$\gamma(t) = \int_0^t g(s)\, ds, \tag{A.2}$$

$g \in L^2([0,T])$ is defined as follows.

Definition A.1 *Let $F \in \mathcal{S}$ of the form (A.1) and $\gamma \in \mathcal{H}$ of the form (A.2). Then the directional derivative $D_\gamma F$ of a random variable F at the point $\omega \in C_0([0,T])$ in the direction γ is defined by*

$$\begin{aligned} D_\gamma F(\omega) : &= \frac{d}{d_\epsilon}\left[F(\omega+\epsilon\gamma)\right]_{\epsilon=0} \\ &= \lim_{\epsilon\to 0}\frac{F(\omega+\epsilon\gamma)-F(\omega)}{\epsilon} \end{aligned} \tag{A.3}$$

provided the limit exists i.e.

$$\begin{aligned} D_\gamma F(\omega) = (\nabla F(\omega), \gamma)_{L^2[0,T]} : &= \int_0^T \frac{d(\nabla F)}{dt}(t)\,\dot{\gamma}(t)\,dt \\ &= \int_0^T D_t F(\omega)\,\dot{\gamma}(t)\,dt. \end{aligned} \tag{A.4}$$

Let us assume that there exists $\psi(t,\omega) \in L^2([0,T]\times C_0([0,T]))$ such that $D_\gamma F(\omega) = \int_0^T \psi(t,\omega)\,g(t)\,dt$. Then F is differentiable and we set $D_t F(\omega) := \psi(t,\omega)$. Now $D_t F$ is called the Malliavin derivative of F.

Let $\|\cdot\|_{L^2}$ denote the $(L^2[0,T])^n$-norm ($n \geq 1$) i.e., for $\psi = (\psi_1,\ldots,\psi_n) \in (L^2[0,T])^n$

$$\|\psi\|_{L^2}^2 = \sum_{i=1}^n \int_0^T \psi_i^2(t)\,dt \tag{A.5}$$

and $|\cdot|$ is reserved for the Euclidean norm on $\mathbb{R}^n$.
We introduce now a norm $\|\cdot\|_{1,p}$ on the set $\mathcal{S}$ as

$$\|F\|_{1,p} := \left(\mathbb{E}\left[|F|^p + \left(\|DF\|_{L^2}^2\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \tag{A.6}$$

for each $p \geq 1$.

Definition A.2 *We define $\mathbb{D}_{1,p}$ as the Banach space which is the closure of $\mathcal{S}$ under the norm $\|\cdot\|_{1,p}$ and $\mathbb{D} := \bigcap_{p>1}\mathbb{D}_{1,p}$.*

Remark A.3 *$D_t F$ is well-defined on $\mathbb{D}_{1,1}$ (see Lemma 2.1 in [59]).*

The following proposition is proved in [57] Proposition 8.3.

Proposition A.4 *For $n \in \mathbb{N}$ we consider the n-dimensional SDE*

$$dY(t) = \mu(t, Y)\, dt + \sigma(t, Y)\, dW(t), \ t \in [0, T], \ Y(0) = y_0 \in \mathbb{R}^n, \ W(t) \in \mathbb{R}^d. \tag{A.7}$$

μ and σ are assumed to be measurable $\mathbb{R}^n$ and $\mathbb{R}^{n \times d}$-valued functions which are continuously differentiable and satisfy

$$\sup_{t \in [0,T], y \in \mathbb{R}^n} \left(\left| \frac{\partial}{\partial y_k} \mu_i(t, y) + \frac{\partial}{\partial y_k} \sigma_{ij}(t, y) \right| \right) < \infty, \tag{A.8}$$

$$\sup_{t \in [0,T]} \left(|\mu_i(t, 0) + \sigma_{ij}(t, 0)| \right) < \infty \tag{A.9}$$

for $i, k = 1, \cdots, n$, $j = 1, \cdots, d$. Then (A.7) has a unique continuous solution $(Y(t))_{t \in [0,T]}$ which satisfies $Y^k(s) \in \mathbb{D}$,

$$\begin{aligned} D_t Y(s) = (\sigma(t, Y))^\top &+ \int_t^s D_t Y(u) (\partial_y \mu(u, Y))^\top du \\ &+ \int_t^s D_t Y \sum_{j=1}^d (\partial_y \sigma_{\cdot j}(u, Y))^\top dW^j(u) \end{aligned} \tag{A.10}$$

for $t \in [0, s]$, and $D_t Y(s) = 0$ for $t \in (s, T]$. ∂_y denotes here the Jacobi matrix, i.e., $(\partial_y \mu)_{ij} = \frac{\partial}{\partial y_j} \mu_i$, and $\sigma_{\cdot j}$ is the j^{th} column of σ.

Lemma A.5 *Let $F = (F_1, \cdots, F_n)^\top \in (\mathbb{D})^n$ with values in $\mathbb{R}^n$. Let also $\Phi \in C^1(\mathbb{R}^n)$ be a real-valued function and assume that $\Phi(F) \in L^1$. If $\Phi'(F) \in L^q$ for some $q > 1$, then $\Phi(F) \in \mathbb{D}_{1,1}$ and*

$$D_t \Phi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) D_t F_i(t). \tag{A.11}$$

Proof: See the proof of the Lemma 2.1 in [53]. □

Theorem A.6 (Clark-Ocone formula) *Let $F \in \mathbb{D}_{1,1}$ be $\mathcal{F}_T$ measurable. Then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[(D_t F) \mid \mathcal{F}(t)]\, dW(t) \tag{A.12}$$

and it follows that

$$\mathbb{E}\left[F \mid \mathcal{F}(t)\right] = \mathbb{E}\left[F\right] + \int_0^t \mathbb{E}\left[(D_s F) \mid \mathcal{F}(s)\right] dW(s), \tag{A.13}$$

for $t \in [0, T]$.

Theorem A.7 (Clark-Ocone formula under change of measure) *Let* $F \in \mathbb{D}_{1,1}$ *be* $\mathcal{F}(T)$ *measurable,* $\Theta(t) = (\theta_1(t), \cdots, \theta_n(t))^\top \in (\mathbb{D}_{1,1})^n$ *be progressively measurable bounded process and*

$$\mathbb{E}^{\mathbb{Q}}\left[|F|\right] < \infty \tag{A.14}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T |D_t F|^2 \, dt\right)^{\frac{1}{2}}\right] < \infty \tag{A.15}$$

$$\mathbb{E}^{\mathbb{Q}}|F|\left(\int_0^T \left(\int_0^T D_t\Theta^\top(u)\, dW(u) + \int_0^T \Theta^\top(u)\, D_t\Theta(u)\, du\right)^2 dt\right)^{\frac{1}{2}} < \infty, \tag{A.16}$$

with

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T = Z(T) := \exp\left\{-\frac{1}{2}\int_0^T \|\Theta(s)\|^2 ds - \int_0^T \Theta^\top(s)\, dW^{\mathbb{P}}(s)\right\}.$$

Then $ZF \in \mathbb{D}_{1,1}$ *and*

$$F = \mathbb{E}^{\mathbb{Q}}[F] + \int_0^T \mathbb{E}^{\mathbb{Q}}\left[\left(D_t F - F\int_t^T D_t\Theta^\top(u)\, dW^{\mathbb{Q}}(u)\right) \mid \mathcal{F}(t)\right] dW^{\mathbb{Q}}(t) \tag{A.17}$$

for $t \in [0, T]$.

Proof: See the proof of Theorem 2.5 in [53]. □

Lakner & Nygren in [45] proved that the formula (A.17) is also applicable to the piecewise continuously differentiable function and below we give its definition.

Definition A.8 *A function* $\varphi : (a, b) \to \mathbb{R}$ *is called piecewise continuously differentiable if the following conditions are fulfilled:*

(i) *φ is continuous in (a,b);*

(ii) *There exist finitely many points $a = c_0 < c_1 < \cdots < c_{m+1} = b\,(m \geq 0)$ such that φ is continuously differentiable on (c_i, c_{i+1}) for $i \in \{0, \cdots, m\}$, we call these points beakpoints of φ;*

(iii) *The function $\varphi^{'}$ is bounded on every compact sub-interval of (a,b), where*

$$\varphi^{'}(x) := \begin{cases} \text{the derivative of } \varphi \text{ in } x & , \text{ if } x \in (a,b) \setminus \{c_1, \cdots, c_m\} \\ 0 & , \text{ if } x \in \{c_1, \cdots, c_m\} \end{cases}$$

(iv) *The limits*

$$\lim_{x \to a+} \varphi(x) \quad \lim_{x \to b-} \varphi(x)$$
$$\lim_{x \to a+} \varphi^{'}(x) \quad \lim_{x \to b-} \varphi^{'}(x)$$

exist;

(v)

$$\text{If } \lim_{x \to a+} \varphi^{'}(x) = \infty, \text{ then } \lim_{x \to a+} \varphi(x) = \infty$$
$$\text{and if } \lim_{x \to b-} \varphi^{'}(x) = \infty, \text{ then } \lim_{x \to b-} \varphi(x) = \infty.$$

The class of piecewise continuously differentiable functions is denoted by $PC^1(a,b)$.

Proposition A.9 *Suppose we have a function φ and a random variable F, s.t. $-\infty \leq a < b \leq \infty$, $\varphi \in PC^1(a,b)$ with breakpoints $c_1, \ldots, c_n$, $F \in \mathbb{D}_{1,1}$, and $\mathbb{Q}(F \in (a,b)) = 1$.*
If

$$\mathbb{E}^{\mathbb{Q}}\left[|\varphi(F)| + \left\|\varphi^{'}(F)\,DF\right\|_{L^2}\right] < \infty$$

and

$$\mathbb{Q}(F \in \{c_1, \ldots, c_n\}) = 0$$

Then $\varphi(F) \in \mathbb{D}_{1,1}$ and

$$D_t\varphi(F) = \varphi^{'}(F)\,D_tF.$$

Proof: See the Appendix in [45]. □

B. Proof of Theorem 4.6 part (a)

We compute here for the second and third terms of the Equation (4.45), i.e.,

$$\mathbb{E}\left[\frac{H(T)}{H(t)}q\mathbb{1}_B \mid \mathcal{F}(t)\right] \text{ and } \mathbb{E}\left[\frac{H(T)}{H(t)}\left(\left(y_1^* - y_2^*\right)H(T)\right)^{\frac{1}{\gamma-1}}\mathbb{1}_C \mid \mathcal{F}(t)\right]$$

with B and C as given in the Equation (4.44). We apply the idea used for the computation of the first term.

1 . For the second term, it holds

$$\begin{aligned}
&\mathbb{E}\left[\frac{H(T)}{H(t)}q\mathbb{1}_B \mid \mathcal{F}(t)\right] \\
&= \mathbb{E}\left[\frac{H(T)}{H(t)}q\mathbb{1}_{\left\{\frac{q^{\gamma-1}}{y_1^* H(t)} < \frac{H(T)}{H(t)} \leq \frac{q^{\gamma-1}}{\left(y_1^* - y_2^*\right)H(t)}\right\}} \mid \mathcal{F}(t)\right] \\
&= \mathbb{E}\left[qe^{-M^H(t,T)-\sqrt{V^H(t,T)}x}\mathbb{1}_{\left\{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right) < \ln e^{-M^H(t,T)-\sqrt{V^H(t,T)}x} \leq \ln\left(\frac{q^{\gamma-1}}{\left(y_1^* - y_2^*\right)H(t)}\right)\right\}}\right]
\end{aligned}$$

$$= \mathbb{E}\left[q e^{-M^H(t,T)-\sqrt{V^H(t,T)}x} \mathbb{1}_{\left\{ -\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right)+M^H(t,T)}{\sqrt{V^H(t,T)}} > x \geq -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right)H(t)}\right)+M^H(t,T)}{\sqrt{V^H(t,T)}} \right\}} \right]$$

$$= q\frac{1}{\sqrt{2\pi}} \int_{-\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right)H(t)}\right)+M^H(t,T)}{\sqrt{V^H(t,T)}}}^{-\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right)+M^H(t,T)}{\sqrt{V^H(t,T)}}} e^{-M^H(t,T)-\sqrt{V^H(t,T)}x} e^{-\frac{x^2}{2}}\, dx$$

$$= q e^{-M^H(t,T)+\frac{1}{2}V^H(t,T)} \frac{1}{\sqrt{2\pi}} \int_{-\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^*-y_2^*\right)H(t)}\right)+M^H(t,T)}{\sqrt{V^H(t,T)}}}^{-\frac{\ln\left(\frac{q^{\gamma-1}}{y_1^* H(t)}\right)+M^H(t,T)}{\sqrt{V^H(t,T)}}} e^{-\frac{\left(x+\sqrt{V^H(t,T)}\right)^2}{2}}\, dx$$

$$= q e^{-M^H(t,T)+\frac{1}{2}V^H(t,T)} * \left(\Phi\left(d_2^\delta\left(t, H\left(t\right)\right)\right) - \Phi\left(d_3^\delta\left(t, H\left(t\right)\right)\right)\right).$$

2. For the third term, we obtain

$$\mathbb{E}\left[\frac{H\left(T\right)}{H\left(t\right)}\left(\left(y_1^*-y_2^*\right)H\left(T\right)\right)^{\frac{1}{\gamma-1}} \mathbb{1}_C \mid \mathcal{F}\left(t\right)\right]$$
$$= \left(\left(y_1^*-y_2^*\right)\right)^{\frac{1}{\gamma-1}} \mathbb{E}\left[\frac{\left(H\left(T\right)\right)^{\frac{\gamma}{\gamma-1}}}{H\left(t\right)} \mathbb{1}_C \mid \mathcal{F}\left(t\right)\right]$$
$$= \left(\left(y_1^*-y_2^*\right)H\left(t\right)\right)^{\frac{1}{\gamma-1}} \mathbb{E}\left[\left(\frac{H\left(T\right)}{H\left(t\right)}\right)^{\frac{\gamma}{\gamma-1}} \mathbb{1}_C\right]$$
$$= \left(\left(y_1^*-y_2^*\right)H\left(t\right)\right)^{\frac{1}{\gamma-1}} \mathbb{E}\left[e^{\frac{\gamma}{1-\gamma}M^H(t,T)+\frac{\gamma}{1-\gamma}\sqrt{V^H(t,T)}x} \mathbb{1}_C\right].$$

$$
\begin{aligned}
C &= \left\{ \frac{H(T)}{H(t)} > \frac{q^{\gamma-1}}{(y_1^* - y_2^*)\, H(t)} \right\} \\
&= \left\{ e^{-M^H(t,T) - \sqrt{V^H(t,T)}x} > \frac{q^{\gamma-1}}{(y_1^* - y_2^*)\, H(t)} \right\} \\
&= \left\{ x < -\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^* - y_2^*\right) H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}} \right\}.
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
&= \left(\left(y_1^* - y_2^*\right) H(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} \times \\
&\qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{\ln\left(\frac{q^{\gamma-1}}{\left(y_1^* - y_2^*\right) H(t)}\right) + M^H(t,T)}{\sqrt{V^H(t,T)}}} e^{-\frac{\left(x - \frac{\gamma}{1-\gamma}\sqrt{V^H(t,T)}\right)^2}{2}} dx \\
&= \left(\left(y_1^* - y_2^*\right)\right)^{\frac{1}{\gamma-1}} \left(H(t)\right)^{\frac{1}{\gamma-1}} e^{\frac{\gamma}{1-\gamma} M^H(t,T) + \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2 V^H(t,T)} * \Phi\left(d_4^\delta\left(t, H(t)\right)\right).
\end{aligned}
$$

C. Optimization with CIR

In this appendix we study portfolio optimization when the interest rates are modeled by a one factor Cox-Ingersoll-Ross (CIR) model whose dynamics are as given in (2.5):

$$dr(t) = a^{\mathbb{P}}\left(b^{\mathbb{P}} - r(t)\right)dt + \sigma_r\sqrt{r(t)}dW_r^{\mathbb{P}}(t),$$

for $t \in [0, T]$, $a^{\mathbb{P}}, b^{\mathbb{P}}, \sigma_r > 0$ and $2a^{\mathbb{P}}b^{\mathbb{P}} \geq \sigma_r$. Let us consider a complete financial market described in Chapter 4, i.e., with the money market account M, bond B (maturing at $\widetilde{T} > T$) and stock S The dynamics of the securities under $\mathbb{P}$ are modeled by:

$$\begin{aligned}
dM(t) &= M(t)\,r(t)\,dt,\\
dB\left(t,\widetilde{T}\right) &= B\left(t,\widetilde{T}\right)\left[\mu_B\left(t,\widetilde{T}\right)dt + \sigma_B^{\mathbb{P}}\left(t,\widetilde{T}\right)dW_r^{\mathbb{P}}(t)\right],\\
dS(t) &= S(t)\left[\mu_S(t)\,dt + \sigma_{rS}dW_r^{\mathbb{P}}(t) + \sigma_S dW_S^{\mathbb{P}}(t)\right],
\end{aligned}$$

where

$$\begin{aligned}
\sigma_B^{\mathbb{P}}\left(t,\widetilde{T}\right) &= -\sigma_r n^{\mathbb{P}}\left(t,\widetilde{T}\right)\sqrt{r(t)},\\
n^{\mathbb{P}}\left(t,\widetilde{T}\right) &= \frac{\sinh\tau^{\mathbb{P}}(T-t)}{\tau^{\mathbb{P}}\cosh\tau^{\mathbb{P}}(T-t) + \frac{1}{2}b^{\mathbb{P}}\sinh\tau^{\mathbb{P}}(T-t)}\\
\text{and } \tau^{\mathbb{P}} &= \frac{\left(a^{\mathbb{P}} + 2\sigma_r^2\right)^{\frac{1}{2}}}{2}.
\end{aligned}$$

We assume that the market prices of risk associated to the interest rates to be given by $\theta_r(t) := \theta_r\sqrt{r(t)}$ and the one associated to the stock to be given by $\theta_S(t) := \theta_S - \frac{\sigma_{Sr}}{\sigma_S}\theta_r\sqrt{r(t)}$, where θ_r and θ_S are constants. We denote the vector of market price of risk by

$$\Theta_{CIR}(t) = \begin{pmatrix}\theta_r(t)\\ \theta_S(t)\end{pmatrix}.$$

The state price density process $Z^{\Theta_{CIR}}(t)$ at time $t \in [0,T]$ corresponding to $\Theta_{CIR}(t)$ is characterized by

$$Z^{\Theta_{CIR}}(t) = \left(\frac{d\mathbb{Q}_{CIR}}{d\mathbb{P}}\right)_t = \exp\left(-\frac{1}{2}\int_0^t \|\Theta_{CIR}(u)\|^2 du - \int_0^t \Theta_{CIR}^\top(u)\, dW^{\mathbb{P}}(u)\right),$$

where $\mathbb{Q}_{CIR} \in \mathcal{M}^e$. We define the deflator H_{CIR} by

$$\begin{aligned} H_{CIR}(t) :&= \beta(t)\, Z^{\Theta_{CIR}}(t) \\ &= \exp\left(-\int_0^t r(u)\, du - \frac{1}{2}\int_0^t \|\Theta_{CIR}(u)\|^2 du - \int_0^t \Theta_{CIR}^\top(u)\, dW^{\mathbb{P}}(u)\right), \end{aligned}$$

with $\beta(t) := exp\left(-\int_0^t r(u)\, du\right)$.
Due to the Girsanov Theorem the process

$$W^{\mathbb{Q}_{CIR}}(t) = \begin{pmatrix} W_r^{\mathbb{Q}_{CIR}}(t) \\ W_S^{\mathbb{Q}_{CIR}}(t) \end{pmatrix} := \begin{pmatrix} W_r^{\mathbb{P}}(t) + \int_0^t \theta_r(u)\, du \\ W_S^{\mathbb{P}}(t) + \int_0^t \theta_S(u)\, du \end{pmatrix}, \text{ for } t \in [0,T]$$

is a $\mathbb{Q}_{CIR}-$Brownian motion.
The dynamics of $r(t)$, $B\left(t,\widetilde{T}\right)$ and $S(t)$ under $\mathbb{Q}_{CIR}$ are ruled by

$$\begin{aligned} dr(t) &= a^{\mathbb{Q}_{CIR}}\left(b^{\mathbb{Q}_{CIR}} - r(t)\right) dt + \sigma_r \sqrt{r(t)} dW_r^{\mathbb{Q}_{CIR}}(t), \\ dB\left(t,\widetilde{T}\right) &= B\left(t,\widetilde{T}\right)\left[r(t)\, dt + \sigma_B^{\mathbb{Q}_{CIR}}\left(t,\widetilde{T}\right) dW_r^{\mathbb{Q}_{CIR}}(t)\right], \\ dS(t) &= S(t)\left[r(t)\, dt + \sigma_{rS} dW_r^{\mathbb{Q}_{CIR}}(t) + \sigma_S dW_S^{\mathbb{Q}_{CIR}}(t)\right], \end{aligned}$$

where

$$\begin{aligned} \sigma_B^{\mathbb{Q}_{CIR}}\left(t,\widetilde{T}\right) &= -\sigma_r n^{\mathbb{Q}_{CIR}}\left(t,\widetilde{T}\right)\sqrt{r(t)}, \\ n^{\mathbb{Q}_{CIR}}\left(t,\widetilde{T}\right) &= \frac{\sinh \tau^{\mathbb{Q}_{CIR}}(T-t)}{\tau^{\mathbb{Q}_{CIR}} \cosh \tau^{\mathbb{Q}_{CIR}}(T-t) + \frac{1}{2} b^{\mathbb{Q}_{CIR}} \sinh \tau^{\mathbb{Q}_{CIR}}(T-t)}, \\ \tau^{\mathbb{Q}_{CIR}} &= \frac{\left(a^{\mathbb{Q}_{CIR}} + 2\sigma_r^2\right)^{\frac{1}{2}}}{2}, \\ a^{\mathbb{Q}_{CIR}} &= a^{\mathbb{P}} + \sigma_r, \quad b^{\mathbb{Q}_{CIR}} = \frac{a^{\mathbb{P}} b^{\mathbb{P}}}{a^{\mathbb{P}} + \sigma_r}. \end{aligned}$$

The wealth process $X(t)$ of an investor under $\mathbb{Q}_{CIR}$ is ruled by

$$dX(t) = X(t)\left[r(t)\, dt + \varphi^\top(t)\, \sigma^{\mathbb{Q}_{CIR}}(t)\, dW^{\mathbb{Q}_{CIR}}(t)\right], \tag{C.1}$$

where

$$\sigma(t) = \begin{pmatrix} \sigma_B^{\mathbb{Q}_{CIR}}\left(t, \widetilde{T}\right) & 0 \\ \sigma_{Sr} & \sigma_S \end{pmatrix}$$

and $\varphi(t) = \begin{pmatrix} \varphi_B(t) \\ \varphi_S(t) \end{pmatrix}$, with $\varphi_B(t)$ and $\varphi_S(t)$ denoting the fraction of wealth invested in B and S respectively.
From Itô's lemma

$$\beta(T) X(T) = X(0) + \int_0^T X(t) \beta(t) \varphi^\top(t) \sigma^{\mathbb{Q}_{CIR}}(t) dW^{\mathbb{Q}_{CIR}}(t). \qquad \text{(C.2)}$$

Considering the interest rates to be modeled by CIR, the Problem (4.22) (without risk constraints) was studied by Deelstra & Koehl [17] and Kraft [43]. For the Problem (4.37), its solution (denoted here by $X_{CIR}^{\delta}(T)$) looks similar to the one given in Theorem 4.5 with $H(T)$ replaced by $H_{CIR}(T)$, since the optimal terminal wealth depends on the deflator and changing the model for the interest rates, modifies the deflator.

Proposition C.1 *If all conditions of Theorem 4.5 part (a) are satisfied with $H(T)$ replaced by $H_{CIR}(T)$, then the optimal trading strategy at any time $t \in [0,T]$ is given by $\varphi^{CIR}(t) = \left(\varphi_B^{CIR}(t), \varphi_S^{CIR}(t)\right)^\top$, where*

$$\varphi_B^{CIR}(t) = \frac{\mathbb{E}^{\mathbb{Q}_{CIR}}\left[R_B(t,T) \mid \mathcal{F}\right]}{\beta(t) X_{CIR}^{\delta}(t) \sigma_B^{\mathbb{Q}_{CIR}}(t)} - \frac{\sigma_{Sr}\varphi_S^{CIR}(t)}{\sigma_B^{\mathbb{Q}_{CIR}}(t)},$$
$$\varphi_S^{CIR}(t) = \frac{\mathbb{E}^{\mathbb{Q}_{CIR}}\left[R_S(t,T) \mid \mathcal{F}\right]}{\beta(t) X_{CIR}^{\delta}(t) \sigma_S}.$$

Thereby

$$\begin{aligned} R_B(t,T) = & \frac{\beta(T) X_{CIR}^{\delta}(T)}{1-\gamma} \times \\ & \left[\left(\gamma - \theta_r^2\right) \int_t^T \sqrt{r(u)} \pi(t,u,r) du + \gamma \int_t^T \theta_r \pi(t,u,r) dW_r^{\mathbb{Q}_{CIR}}(u) + \theta_r(t)\right] \\ & - \frac{1}{1-\gamma} \beta(T) q \mathbb{1}_{B_{CIR}} \times \\ & \left[\left(2 - \theta_r^2\right) \int_t^T \sqrt{r(u)} \pi(t,u,r) du + \theta_r \int_t^T \pi(t,u,r) dW_r^{\mathbb{Q}_{CIR}}(u) + \theta_r(t)\right], \end{aligned}$$

$$
\begin{aligned}
R_S(t,T) &= \frac{\beta(T)\, X^{\delta}_{CIR}(T)}{1-\gamma} \int_t^T \pi(t,u,r) \left(\frac{\theta_S \theta_r}{\sigma_S} - \frac{\sigma_{Sr}^2 \theta_r^2}{\sigma_S^2} \sqrt{r(u)} \right) du \\
&+ \frac{\beta(T)\, X^{\delta}_{CIR}(T)}{\gamma-1} \left[\gamma \int_t^T \frac{\sigma_{Sr}\theta_r}{\sigma_S} \pi(t,u,r)\, dW_S^{\mathbb{Q}_{CIR}}(u) - \theta_S(t) \right] \\
&+ \frac{1}{\gamma-1} \beta(T)\, q \left[\int_t^T \pi(t,u,r) \left(\frac{\theta_S \theta_r}{\sigma_S} - \frac{\sigma_{Sr}^2 \theta_r^2}{\sigma_S^2} \sqrt{r(u)} \right) du \right] \mathbb{1}_{B_{CIR}} \\
&+ \frac{1}{1-\gamma} \beta(T)\, q \left[\int_t^T \frac{\sigma_{Sr}\theta_r}{\sigma_S} \pi(t,u,r)\, dW_S^{\mathbb{Q}_{CIR}}(u) - \theta_S(t) \right] \mathbb{1}_{B_{CIR}}, \\
\pi(t,u,r) &= \frac{\sigma_r}{2} \exp\left\{ \int_t^u \left(-\frac{a^{\mathbb{Q}_{CIR}}}{2} - \left(\frac{a^{\mathbb{Q}_{CIR}} b^{\mathbb{Q}_{CIR}}}{2} - \frac{\sigma_r^2}{8} \right) \frac{1}{r(s)} ds \right) \right\}.
\end{aligned}
$$

Proof: We use the same idea as for the proof of Theorem 4.6 part (b) and the result by Alos & Ewald [1] for the computation of the Malliavin derivative of $\sqrt{r(t)}$ and $r(t)$ for $t \in [0,T]$:

$$
\begin{aligned}
D_t \sqrt{r(u)} = \pi(t,u,r) &:= \frac{\sigma_r}{2} \exp\left\{ \int_t^u \left(-\frac{a^{\mathbb{Q}_{CIR}}}{2} - \left(\frac{a^{\mathbb{Q}_{CIR}} b^{\mathbb{Q}_{CIR}}}{2} - \frac{\sigma_r^2}{8} \right) \frac{1}{r(s)} ds \right) \right\}, \\
D_t r(u) &= 2\sqrt{r(u)}\, \pi(t,u,r), \text{ for } u > t.
\end{aligned} \tag{C.3}
$$

Using representation (A.17) to $\beta(T)\, X^{\delta}_{CIR}(T)$ gives

$$
\beta(T)\, X^{\delta}_{CIR}(T) = \mathbb{E}^{\mathbb{Q}_{CIR}}\left[\beta(T)\, X^{\delta}_{CIR}(T)\right] + \int_0^T \mathbb{E}^{\mathbb{Q}_{CIR}}\left[\xi(t,T) \mid \mathcal{F}(t)\right] dW^{\mathbb{Q}_{CIR}}(t), \tag{C.4}
$$

where

$$
\xi(t,T) = \left(D_t\left(\beta(T)\, X^{\delta}_{CIR}(T)\right) - \beta(T)\, X^{\delta}_{CIR}(T) \int_t^T D_t \Theta^{\top}_{CIR}(u)\, dW^{\mathbb{Q}_{CIR}}(u) \right). \tag{C.5}
$$

Applying chain and product rules of Malliavin derivative, and using (C.3) to (C.5) and then comparing (C.4) with (C.2) gives the result. □

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