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# Graph Coloring Applications and Defining Sets in Graph Theory


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# Contents

# 1 Introduction

We meet graphs in different ways in real life without notice. Graphs describe the connectedness of systems; typically, they model transport or communication systems, electrical networks, etc. It is not difficult to imagine that having a graph with very good connective properties could be useful in many communication problems, for instance the design of telephone networks [26], parallel computers, and neural networks. It is possible to construct a special type of highly connected and yet very sparse graphs using bipartite graphs as building blocks, see [2].

Another example is addressing schemes for computer networks. In any modern computer, data is perpetually moving throughout the system. To route this data efficiently is an important task. A natural course to pursue is to represent a computer network as a graph and provide each vertex of this graph with an address. Perhaps less obvious is that these addresses could be made to reflect the distances between the vertices in the graph. If this were the case, then a parcel of data could be optimally routed around a network simply by examining local information at each step. When a parcel reaches a vertex, it needs only to compare the addresses of its neighbors with the address of its destination and move to the neighbor closest to its target. There are graphs which naturally have this addressing property, see [2].

In this work, we focus the attention on graph coloring. Graph coloring has an extensive history and theory. A classical example is *map coloring problem*, where countries sharing a common frontier must be given different colors on the map, the question is how many colors does the cartographer need?
There are different ways to color a graph: namely vertex coloring, edge coloring, total coloring, list coloring, etc.

There are more combinatorial applications of graphs in real problems. For example interpretation of edge coloring in *timetabling*. From a practical point of view, bipartite graphs form a model of interaction between two different types of objects, be they sets and their elements, jobs and workers, or telephone changes and cities. The desire to model such interaction is extremely common and many recreational and much more serious problems can be phrased in terms of problems on bipartite graphs.

The concept of defining set has been firstly studied in block designs, see [31] and latin squares, under the name critical sets. Mahmoodian introduced the concept of a defining set for a $\chi(G)$-coloring of a graph $G$, [23]. Morril and Pritikin [27] extended this to $k$-coloring of a graph $G$, where $k \geq \chi(G)$.

A defining set of a given set, say $A$, gives information about $A$ which are enough to determine $A$ uniquely. The concepts of critical sets in latin squares, and defining sets of graph colorings, have applications in cryptography [11]. Another useful aspect is that few information rather than the graph completely has to be stored in memory.

As in the case of block designs, finding defining sets seems to be difficult problem, and there is not a general conclusion. Hence we confine us here to some special types of graphs like bipartite graphs, complete graphs, etc.

In this work, four new concepts of defining sets are introduced:

- Defining sets for perfect (maximum) matchings
- Defining sets for independent sets
- Defining sets for edge colorings
- Defining set for maximal (maximum) clique

Furthermore some algorithms to find and construct the defining sets are introduced. A review on some known kinds of defining sets in graph theory is also incorporated. In chapter 2 the basic definitions and some relevant notations used in this work are inroduced.

chapter 3 discusses the maximum and perfect matchings and a new concept for a defining set for perfect matching.

Different kinds of graph colorings and their applications are the subject of chapter 4.

Chapter 5 deals with defining sets in graph coloring. New results are discussed along with already existing research results. An algorithm is introduced, which enables to determine a defining set of a graph coloring.

In chapter 6, cliques are discussed. An algorithm for the determination of cliques using their defining sets. Several examples are included.

# 2 Fundamental concepts

In this chapter we introduce the basic definitions and some relevant notations which we shall use throughout this work.

**Definition 2.1** *A* ***graph*** *$G$ is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge, e. g. $uv \in E(G)$, two vertices, $u, v \in V(G)$, ( not necessarily distinct ) called its endpoints. A graph is finite if its vertex set and edge set are finite. The size of $V(G)$ is denoted by $n(G)$ and of $E(G)$ by $e(G)$. The* ***null*** *graph is the graph whose vertex set and edge set are empty.*

**Definition 2.2** *The* ***complement*** *$\overline{G}$ of a simple graph $G$ is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.*

**Definition 2.3** *A* ***loop*** *is an edge whose endpoints are equal.* ***Multiple*** *edges are having the same pair of endpoints. A* ***simple*** *graph is a graph having no loops or multiple edges.*

**Definition 2.4** *When $u$ and $v$ are endpoints of an edge, i.e. $e = uv, e \in E(G)$, they are* ***adjacent*** *and* ***neighbors****.*

**Definition 2.5** *If vertex $v$ is an endpoint of edge $e$, then $v$ and $e$ are* ***incident****. The* ***degree*** *of vertex $v$ (in a loopless graph) is the number of incident edges and is denoted by $d_G(v)$.*

**Definition 2.6** *The maximum degree of graph $G$ is denoted by $\Delta(G)$, the minimum degree by $\delta(G)$, and $G$ is* ***regular*** *if $\Delta(G) = \delta(G)$. It is* ***$k$-regular*** *if every vertex in $G$ has degree $k$.*

**Definition 2.7** *Let $G$ be a loopless graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. The* ***adjacency matrix*** *of $G$, written $A(G)$, is the $n$-by-$n$ matrix in which entry $a_{i,j}$ is the number of edges in $G$ with endpoints $\{v_i, v_j\}$. The* ***incidence matrix*** *$M(G)$ is the $n$-by-$m$ matrix in which entry $m_{i,j}$ is 1 if $v_i$ is an endpoint of $e_j$ otherwise is 0.*

**Definition 2.8** *A* ***clique*** *in a graph is a set of pairwise adjacent vertices. An* ***independent set*** *or* ***stable set*** *in a graph is a set of pairwise nonadjacent vertices.*

**Definition 2.9** *The* ***clique number*** *of a graph $G$, written $\omega(G)$ is the maximum size of a clique in $G$. The* ***independence number****, written $\alpha(G)$, of a graph is the maximum size of an independent set of vertices.*

**Definition 2.10** *The* ***clique cover number*** *$\theta(G)$ of a graph $G$ is the minimum number of cliques in $G$ needed to cover $G$.*

**Definition 2.11** *A* ***walk*** *of length $n$ in a graph $G$ is a finite non-null sequence $W = v_0e_1v_1e_2v_2 \ldots e_nv_n$, whose terms are alternately vertices and edges, such that, for*
*$1 \leq i \leq n$, the ends of $e_i$ are $v_{i-1}$ and $v_i$. $W$ is called a* ***path*** *if vertices and edges in $W$ are distinct and is denoted by $P_n$. A closed path of length $n$, i.e. $v_0 = v_n$, is called* ***cycle*** *and denoted by $C_n$. A cycle $C_n$ is* ***odd*** *or* ***even*** *as $n$ is odd or even. The length of shortest cycle in $G$ is called* ***girth****.*

**Definition 2.12** *A* ***subgraph*** *of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in $H$ is the same as in $G$. We then write $H \subseteq G$ and say that "$G$ contains $H$".*

**Definition 2.13** *An* ***induced subgraph****, $G[U]$, is a subgraph of $G$ whose vertex set is $U \subset V(G)$ and two vertices are adjacent in $G[U]$ if and only if these vertices are adjacent in $G$.*

**Definition 2.14** *A* ***spanning subgraph*** *of $G$ is subgraph with vertex set $V(G)$. A* ***spanning tree*** *is a spanning subgraph which is a tree. Analogously one can define a* ***spanning cycle****.*

**Definition 2.15** *A graph $G$ is* ***connected*** *if each pair of vertices in $G$ belongs to a path; otherwise $G$ is disconnected.*

**Definition 2.16** *A* ***separating set*** *or* ***vertex cut*** *of graph $G$ is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The* ***connectivity*** *of $G$, written $\kappa(G)$, is the minimum size of vertex set $S$ such that $G - S$ is $k$-****connected*** *if its connectivity is aat least $k$.*

**Definition 2.17** *In a connected graph, the length of shortest path between two vertices $u$ and $v$ is called* ***distance*** *between $u$ and $v$ and is denoted by $d(u, v)$.*

**Definition 2.18** *A* ***Hamiltonian graph*** *is a graph with a spanning cycle, also called a* ***Hamiltonian cycle****.*

**Definition 2.19** *A graph with no cycle is* ***acyclic****. A* ***forest*** *is an acyclic graph. A* ***tree*** *is a connected acyclic graph. A* ***leaf*** *(or* ***pendant vertex****) is a vertex of degree 1. A* ***star*** *is a tree consisting of one vertex adjacent to all the others.*

**Definition 2.20** *The graph obtained by taking the union of graphs $G$ and $H$ with disjoint vertex sets is the* ***join*** *or* ***sum****, written $G+H$, i.e. $V(G+H) = V(G) \cup V(H)$ and all vertices in $V(G)$ are joined to $V(H)$.*

**Definition 2.21** *A graph is said to be* ***planar*** *if it can be drawn in the plane so that its edges intersect only at their ends.*

**Definition 2.22** *A graph $G$ is called $k$-**partite** graph if its vertex set $V(G)$ can be partitioned into $k$ sets $V_1, \ldots, V_k$ independent sets. $G$ is called **complete** $k$-partite graph if every vertex in $V_i$ is adjacent to all the other vertices in $V_j, j \neq i$. If $|V_i| = n_i, i \in 1, \ldots, k$ then $G$ is denoted by $K_{n_1,\ldots,n_k}$.*
*If $|V_i| = |V_j|$ for all $i, j \in 1, \ldots, k$, then $G$ is called a **balanced** complete $r$-partite graph.*

**Definition 2.23** *A graph $G$ is **bipartite** if $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge has one end in $V_1$ and the other end in $V_2$. $G$ is complete bipartite if each vertex of $V_1$ is joined to each vertex of $V_2$. We call a graph $m$ by $n$ bipartite if $|V_1| = m$ and $|V_2| = n$, and a graph is a **balanced** bipartite graph when $|V_1| = |V_2|$.*

**Definition 2.24** *A **complete graph** is a simple graph whose vertices are pairwise adjacent; the complete graph with $n$ vertices is denoted by $K_n$. A **complete bipartite** graph is simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. It is denoted be $K_{m,n}$ when the partite sets have size $m$ and $n$.*

**Definition 2.25** *An **isomorphism** from a simple graph $G$ to a simple graph $H$ is a bijection (one-to-one function) $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "$G$ is **isomorphic to** $H$", written $G \simeq H$, if there is an isomorphism from $G$ to $H$.*

**Definition 2.26** *The vertex **chromatic number** of a graph $G$ is the minimum number of colors required to color the vertices of $G$ so that no two adjacent vertices have the same color. It is denoted by $\chi(G)$.*

**Definition 2.27** *Let $A = \{A_1, A_2, \ldots, A_m\}$ be a collection of subsets of a set $Y$. A **system of distinct representatives** (SDR) for $A$ is a set of distinct elements $a_1, a_2, \ldots, a_m$ in $Y$ such that $a_i \in A_i$.*

Here we consider (simple) graphs which are finite, undirected, with no loops or multiple edges. Let G be a graph, we denote its vertex set, edge set, maximum- and minimum degree of its vertices by $V(G), E(G), \Delta(G)$ and $\delta(G)$ respectively.

# 3 Maximum matching

**Definition 3.1** *A* ***matching*** *in a graph $G$ is a set of edges with no shared endpoints. The vertices incident to the edges of a matching $M$ are* ***saturated*** *by $M$; the others are* ***unsaturated*** *(we say $M$-saturated and $M$-unsaturated).*

**Definition 3.2** *A* ***maximal*** *matching in a graph is a matching that cannot be enlarged by adding an edge. A* ***maximum*** *matching is a matching of maximum size among all matchings in the graph.*

A matching $M$ is maximal if every edge not in $M$ is incident to an edge already in $M$. From definitions we observe that every maximum matching is a maximal matching, but the converse need not hold.

**Example 1** *The smallest graph having a maximal matching that is not a maximum matching is $P_4$. If we take the middle edge, then we can add no other, i. e. the middle edge is a maximal matching, but the two end edges form a larger matching which is maximum matching.*

Many discrete problems can be formulated as problems about maximum matchings. Consider, for example, probably the most famous:
A set of boys each knows several girls, is it possible for the boys each to marry a girl that he knows?

This situation has a natural representation as the bipartite graph with bipartition $(V_1, V_2)$, where $V_1$ is the set of boys, $V_2$ the set of girls and an edge between a boy and a girl represents that they know one another. The marriage problem is then the problem: does a maximum matching of $G$ have $|V_1|$ edges?

**Definition 3.3** *We call the set of adjacent vertices in graph $G$ to a vertex $v \in V(G)$, the* ***neighborhood*** *of $v$ and denote it by $N(v)$. Analogously for a subset of vertices $S \subset V(G)$ we denote the set of all vertices of $G$ which are adjacent to at least on vertex in $S$ by $N(S)$.*

**Theorem 3.1** *( Hall's Theorem) Let $G$ be a bipartite graph with bipartition $(V_1, V_2)$. Then $G$ has a matching of $V_1$ into $V_2$ if and only if $|N(A)| \geq |A|$ for every $A \subseteq V_1$.*

The above theorem is often called **Hall's condition**.

Putting this theorem into the language of boys and girls we obtain an answer to the marriage problem:

**Theorem 3.2** *It is possible to marry a group of $n$ boys each to a girl that he knows if and only if every subset of $k$ boys communally know at least $k$ girls, for each $k = 1, \ldots, n$.*

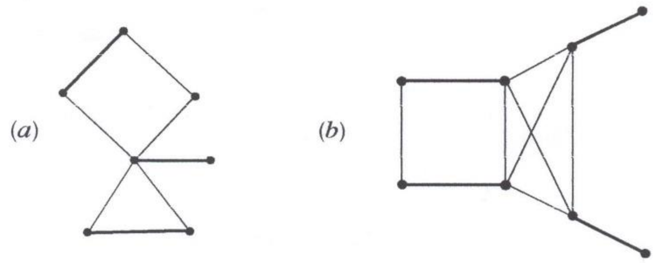
**Corollary 3.3** *Every bipartite graph $G$ has a maximum matching which covers all the vertices of maximum degree.*

One interesting type of maximum matchings is perfect matching.

## 3.1 Perfect matching

**Definition 3.4** *A **perfect** matching of a graph $G$ is a matching which covers every vertex of $G$.*

**Example 2** *In the following figure we can see the examples for maximum and perfect matchings:*




*(a) A maximum matching; (b) a perfect matching*

**Remark 3.1** *The necessary conditions for a connected graph $G$ to have a perfect matching are:*
*1)* $|V(G)| = 2k$
*2)* $|E(G)| \geq k$

The well known **Assignment Problem** is formulated as follows:
There are $n$ workers and $n$ jobs. The ability of worker $i$ to carry out job $j$ is $w(ij)$. What is the optimal assignment of workers to jobs, that is the permutation $\pi \in S_n$ which maximizes $\sum_{i=1}^{n}(w(i\pi(i))$?
Unfortunately, in practice, the assignment with maximum weight is a satisfactory solution. There are always hidden factors which cannot be entered into basic model. Rather than simply finding the assignment with the maximum weight, it might be better to find the $k$ best assignments to give some choice.

We can rephrase the problem as a problem on bipartite graph. Let $G = K_{n,n}$ have bipartition $(V_1, V_2)$, $V_1$ the set of workers and $V_2$ the set of jobs, and let each edge $e \in E(G)$ be given a weight $w(e)$. The weight of a matching $M$ is $w(M) = \sum_{e \in M} w(e)$. A matching with maximum weight is called an **optimal** matching. The problem now is to find k distinct perfect matchings $M_1, \ldots, M_k$ such that $w(M_1) \geq w(M_2) \geq \ldots \geq w(M_k) \geq w(M)$ for every perfect matchings $M \notin \{M_1, \ldots, M_k\}$. We call this problem the $k$ **Best Perfect Matchings Problem.**

Hall's theorem has many consequences, here we consider some of them for graphs with perfect matching in following section.

**Corollary 3.4** *A bipartite graph $G$ with bipartition $(V_1, V_2)$ has a perfect matching if and only if $|V_1| = |V_2|$ and $|N(A)| \geq |A|$ for each $A \subseteq V_1$.*

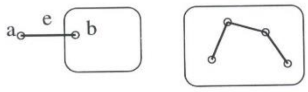
**Corollary 3.5** *(König) Every regular bipartite graph has a perfect matching.*

In this part we study the perfect matchings of some special graphs, the trees.

For trees we have the following results.

**Theorem 3.6** *Every forest has at most one perfect matching.*

**Proof :**
Induction by the number $k$ of edges of $G$.
$k = 0$. The forest consists of no edge and has no perfect matching.
Introduction step $k + 1$. Let the forest $G$ has $k + 1$ edges, $k \geq 0$.
Then $G$ has at least two leaves ( vertices of degree 1 ). Let $e$ be an edge which ends in a leaf $a$ and a vertex $b$.




Assume that $G$ has two different perfect matchings $M_1$ and $M_2$. Due to definition both matching have to contain $e$, i.e. they have to saturate all vertices of the forest $G' = G \setminus \{a, b\}$ otherwise they would not be perfect.
By induction hypothesis $G'$ has at most one perfect matching. Hence $M_1 \cup E(G') = M_2 \cup E(G')$.
The matchings $M_1$ and $M_2$ coincide on $G'$ and both contains $e$. Hence $M_1 = M_2$, a contradiction to our assumption.

**Corollary 3.7** *Every tree has at most one perfect matching.*

**Example 3** *$P_{2n+1}$ has 1 perfect matchings and $P_{2n}$ has no one.*


**Theorem 3.8** *Every cycle has at most two perfect matchings.*

**Proof :**
By removing one edge from a cycle we get a tree which according to Corollary 3.7 has at most one perfect matching. Now if this tree has a perfect matching, then by adding the removed edge we have also a perfect matching in the cycle and by considering the edges which were not in matching we get another perfect matching.

**Definition 3.5** *For a subset of vertices $S$ of a graph $G$ we denote by $c_G(S)$ the number of components of $G - S$ which have an odd number of vertices (odd components).*

**Theorem 3.9** *( Tutte's theorem) A nontrivial graph $G$ has a perfect matching iff for every proper subset $S$ of $V(G)$, $|S| \geq c_G(S)$.*

**Corollary 3.10** *$T$ has perfect matching iff for all $v \in V(T)$ the number of odd components of $G \setminus v$ is exactly 1.*

In the following there are some necessary conditions for trees having a perfect matching:

1. T should have $2n$ vertices.
2. T should contain no star, i.e. if T has two or more leaves with one endpoint in common then T has no perfect matching.

## 3.2 Defining Set of Perfect Matchings

Suppose that instead of knowing the complete set of a perfect matching $M$, just a subset $D$ of $M$ is specified, the question is; can we obtain $M$ from $D$? In the following we introduce a new concept which gives answer to this question.

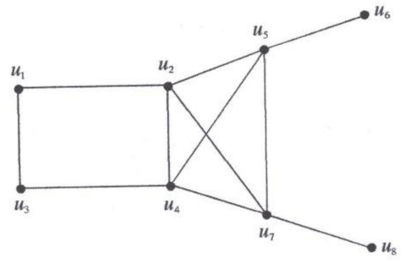
**Definition 3.6** *A defining set, $D$, of a perfect (respectively maximal) matching $M$ of a graph $G$ is a subset of $E(G)$ which can be extended to $M$ uniquely.*

It means with help of set $D$ we can determine a unique perfect (respectively maximal) matching $M$ which contains $D$ as a subset.

**Definition 3.7** *A defining set of a perfect (respectively maximal) matching $M$ of a graph $G$, with minimum cardinality is called minimum defining set and its cardinalty is denoted by $d_p(M)$ (respectively $d_m(M)$).*

To illustrate the previous definitions, we consider the following example;

**Example 4** *Consider the perfect matching $M$ shown in Example 2;*




*We claim that the set $D = \{u_1u_2\}$ is a defining set for this perfect matching. To construct the unique perfect matching which contains $\{u_1u_2\}$, we check all edges with one endpoints of degree one. It is clear that such edges, here $u_5u_6$ and $u_7u_8$ have to belong to the perfect matching. Now only choice for having a matching which covers all vertices of thid graph, is $u_3u_4$. This matching is perfect. $D$ has minimum cardinality, i.e. we need at least one edge to determine this perfect matching uniquely, hence $d_p(M) = 1$*

From previous Section 3.1 it is easy to see:

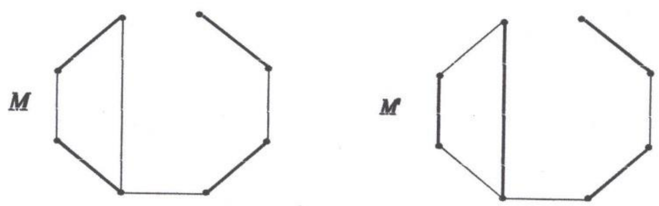
**Corollary 3.11**
$d_p(F) = d_p(T) = 0$ ,

$$d_p(C_n) = \begin{cases} 1 & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1 \end{cases}$$

*where $F$ is a forest, $T$ is a tree and $C_n$ is a cycle.*

**Remark 3.2** *It is easy to observe that for a graph $G$ with $d_G(v) \geq k$ for all vertices in $G$, then $d_p(G)$ (resp. $d_m(G)) \geq k$.*

**Example 5** *In the following there two perfect matchings $M$ and $M'$ of graph $G$ are shown.*



*Here we have* $|D(M)| = |D(M')| = 1$, *i.e. we can determine the perfect matching* $M$ *when only one of its vertex is given, although the perfect matchings* $M$ *and* $M'$ *have two edges in common. Hence It is not always true that* $|D| \geq |\bigcap_i M_i|$, *where* $M_i$ *is a perfect matching in* $G$.

As we will see later there is a very close relationship between edge coloring of a graph and the class of its maximal matchings. In fact the edge color class is nothing else than class of maximal (and in some case perfect) matchings.

# 4 Graph coloring

We mentioned earlier that graphs model connectivity. Here, the viewpoint is that graphs model "incompatibility". For example, in a university we want to assign time slots for final examinations so that two courses with common students have different slots. The number of slots needed is the chromatic number of the graph in which two courses are adjacent if they have common student. There are many variations of graph coloring. In this chapter we introduce some of them.

## 4.1 Vertex coloring

**Definition 4.1** *A $k$-vertex* ***coloring*** *of a graph $G$ is an assignment of $k$ colors, $1, 2, \ldots, k$, to the vertices of $G$; the vertices of one color form a* ***color class****.*
*A $k$-coloring is* ***proper*** *if adjacent vertices have different colors. A graph is $k$-****colorable*** *if it has a proper $k$-coloring. The* ***chromatic number*** *$\chi(G)$ is the least $k$ such that $G$ is $k$-colorable; if $\chi(G) = k$, $G$ is said to be $k$-chromatic.*

**Remark 4.1** *In a proper coloring, each color class is an independent set, so $G$ is $k$-colorable if and only if $V(G)$ is the union of $k$ independent sets. Thus "$k$-colorable" and "$k$-partite" have the same meaning.*

**Definition 4.2** *A $k$-coloring $(V_1, V_2, \ldots, V_k)$ of $G$ is said to be* ***canonical*** *if $V_1$ is a maximal independent set of $G$, $V_2$ is a maximal independent set of $G - V_1$, $V_3$ is a maximal independent set of $G - (V_1 \cup V_2)$, and so on.*

**Remark 4.2** *If graph $G$ is $k$-colorable, then there exists a canonical $k$-coloring of $G$*

The chromatic number is the most famous graphical invariant; its fame being mainly due to the **Four Color Conjecture**, which asserts that all planar graphs are 4-colorable. This has been most challenging problem of combinatorics for over a century and has contributed more to the development of the field than any other single problem. Although today chromatic number and graph coloring attract attention for several other reason too, many of which arise from applied mathematical fields such as operations research, as well from problems of practical interest.

**A Storage Problem**
A company manufactures $n$ chemicals $C_1, C_2, \ldots, C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure the company wishes to partition its warehouse

into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned?

We obtain a graph $G$ on the vertex set $\{v_1, v_2, \ldots, v_n\}$ by joining two vertices $v_i$ and $v_j$ if and only if the chemicals $C_i$ and $C_j$ are incompatible. Now we must assign labels to the vertices so that the endpoints of each edge receive different labels. Since we are only interested in partitioning the vertices, and the labels have no numerical value, it is convenient to call them colors. It is now easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of $G$.

The chromatic number of a graph is the least number of independent sets into which its vertex set can be partitioned, Because every independent set is a subset of a maximal independent set, it suffices to determine all the maximal independent sets. A method for listing all the independent sets is given in [7]. By implementing this algorithm, one can obtain the list of all maximal independent sets of $G$. By repeatedly using this method for finding maximal independent sets, one can determine all canonical colorings of $G$. The least number of colors used in such a coloring is then the chromatic number of $G$.

**Definition 4.3** *A coloring of a graph is called* ***complete*** *if every pair of color classes contains an adjacent pair of vertices.*

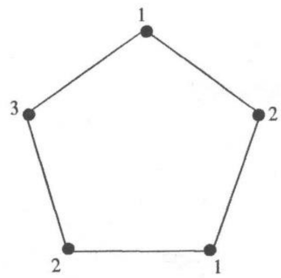
This definition corresponds to the complete graph $K_n$ which consists of $n$ vertices, every pair of distinct vertices being adjacent. In fact, if $G$ is a graph with a complete $n$-coloring we immediately obtain a corresponding isomorphism $\phi$ from $G$ *to* $K_n$. Conversely, given a isomorphism from a graph $G$ onto $K_n$ a complete $n$-coloring of $G$ is found by taking the color classes of vertices in $G$ to be the pre-images of the vertices of $K_n$.

**Proposition 4.1** *A graph $G$ has a complete $n$-coloring if and only if $K_n$ is an isomorphism image of $G$.*

Due to definitions of bipartite graphs it is easy to observe that:

**Remark 4.3** *A simple graph is 2-colorable if and only if it is bipartite.*

**Example 6** *According to Remark 4.3 $C_5$ and the Peterson graph have chromatic number at least 3, i.e. they are not bipartite, so we need at least 3 colors. Since they are 3-colorable, as shown below, they have chromatic number exactly 3.*



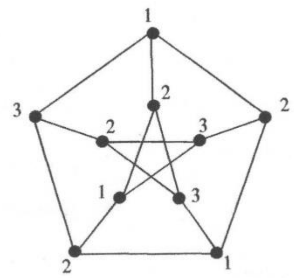




Figure 1: A vertex coloring for $C_5$ A vertex coloring for Peterson graph

**Proposition 4.2** *For every graph $G$, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$*

**Proof :**
The first bound holds because we need distinct colors for vertices of a clique. The second bound holds because each color class is an independent set and thus has at most $\alpha(G)$ vertices.

**Proposition 4.3** *Every $k$-chromatic graph has at least $k$ vertices of degree at least $k-1$.*

**Algorithm 4.1** *(Greedy Coloring [34])*
*The greedy coloring relative to a vertex ordering $v_1, \ldots, v_n$ of $V(G)$ is obtained by coloring vertices in the order $v_1, \ldots, v_n$, assigning to $v_i$ the smallest-indexed color not already used on its lower-indexed neighbors.*

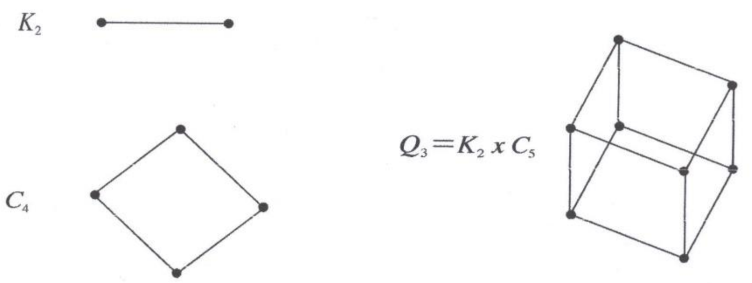
**Proposition 4.4** $\chi(G) \leq \Delta(G) + 1$.

**Theorem 4.5** *(Brooks' Theorem) If $G$ is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

**Definition 4.4** *The cartesian product of simple graphs $G$ and $H$ is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which $(u, v)$ is adjacent to $(u', v')$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.*

In other words, it is a graph which contains $|V(H)|$ "horizontal" copies of $G$ and $|V(H)|$ "vertical" copies of $H$. A horizontal copy and vertical copy have exactly one vertex in common. By the vertex $(i,j)$ in $G \times H$, we mean the vertex in intersection of the i-th copy of $G$ and the j-th copy of $H$.

**Example 7** *The m-by-n* ***grid*** *is the cartesian product of $P_m$ and $P_n$.*
*And for the n-* ***cube*** *$Q_n$, an n-regular simple graph with $2^n$ vertices and $n2^n$ edges, holds:*
$Q_n = Q_{n-1} \times K_2$




**Proposition 4.6** *(Vizing, Aberth)* $\chi(G \times H) = max\ \{\chi(G), \chi(H)\}$.

### 4.1.1 Color-critical graphs

**Definition 4.5** *A graph G is k-****color-critical*** *( or just critical ) if $\chi(H) < \chi(G) = k$ for every proper subgraph H of G.*

**$k$-critical graphs for small $k$**
Properly coloring a graph needs at least two colors if and only if the graph has an edge. Thus $K_2$ is the only 2-critical graph (similarly, $K_1$ is the only 1-critical graph). Since 2-colorable is the same as bipartite, the characterization of bipartite graphs, Theorem 4.21, implies that the 3-critical graphs are the odd cycles.

We can test 2-colorability of a graph $G$ by computing distances from a vertex $x$ (in each component).
Let $V_1 = \{u \in V(G) : d(u,x) \text{ is even}\}$, and let $V_2 = \{u \in V(G) : d(u,x) \text{ is odd}\}$.
The graph $G$ is bipartite if and only if $(V_1, V_2)$ is a bipartition, meaning that $G[V_1]$, and $G[V_2]$ are independent sets.

No good characterization of 4-critical graphs or test for 3-colorability is known.

Brook's theorem implies that the complete graphs and odd cycles are the only $k-1$-regular $k$-critical graphs. Gallai (1963) strengthened this by proving that in the subgraph of a $k$-critical graph induced by vertices of degree $k-1$, is a clique or an odd cycle.

The study of $\Delta$-critical graphs can be considered as one of approaches to improve Brook's theorem. Brook's theorem states that $\chi(G) \leq \Delta(G)$ whenever $3 \leq \omega(G) \leq \Delta(G)$. Bordin and Kostochka (1997) conjectured that $\omega(G) < \Delta(G)$ implies $\chi(G) \leq \Delta(G)$ if $\Delta \geq 9$. Reed (1999) proved that this is true when $\Delta(G) \geq 10^{14}$. Reed (1998) also conjectured that the chromatic number is bounded by average of the trivial upper and lower bounds; that is, $\chi(G) \leq \lceil \frac{\chi(G)+1+\omega(G)}{2} \rceil$.

Beutelspacher and Hering have shown that there are only 13 $\Delta$-critical graph on $2\Delta - 1$ vertices with $\Delta > 3$ and proved for these graphs $4 \leq \Delta \leq 8$.

In the following section we consider a type of graphs with the property that their chromatic number is equal with their maximum size of a clique.

## 4.2 Perfect graphs

A graph with high chromatic number is difficult to deal with. One may become interested in graphs whose trivial lower bound of $\omega(G)$ colors always suffices to color their vertices.

Clearly, $\alpha(G) \leq \theta(G)$ since a stable set $S$ can have at most one vertex in each clique of the partition. Similarly, $\omega(G) \leq \chi(G)$.

**Definition 4.6** *Graph $G$ is called $\alpha$-**perfect** if*
$\alpha(G(U)) = \theta(G(U)) \quad forall \quad U \subseteq V(G)$

**Definition 4.7** *Graph $G$ is called $\chi$-**perfect**, or simply perfect, if*
$\chi(G(U)) = \omega(G(U)) \quad$ *for all* $\quad H \subseteq V(G)$

**Example 8** *If $G$ is a bipartite graph, then $\alpha(G) = \theta(G)$. Thus, a bipartite graph is $\alpha$-perfect. A bipartite graph is also $\chi$-perfect because if it has an edge, then*
$\chi(G) = 2 = \omega(G)$,
*and if $G$ has no edges, then*
$\chi(G) = 1 = \omega(G)$.

**Example 9** *If $G$ consists of an odd cycle of length $2k+1 > 3$ without chords* [1]*, then $G$ is not $\alpha$-perfect because $\alpha(G) = k$ and $\theta(G) = k+1$. (A minimum partition*

[1] An edge joining two non-consecutive vertices of a cycle.

*of G consists of k 2-cliques and one 1-clique.) Moreover, G is not $\chi$-perfect because $\chi(G) = 3$ and $\omega(G) = 2$.*

**Remark 4.4** *Let $G(V_1), \ldots, G(v_n), n \in \mathbf{N}$ be the connected componenets of a graph G. Then :*
*(i) $\alpha(G) = \alpha(G(V_1)) + \ldots + \alpha(G(V_n))$*
*(ii) $\theta(G) = \theta(G(V_1)) + \ldots + \theta(G(V_n))$*
*(iii) $\chi(G) = max\chi(G(V_i))$*
*(iv) $\omega(G) = max\omega(G(V_i))$*

**Theorem 4.7** *A graph G is $\alpha$-perfect if, and only if, its complementary graph $\bar{G}$ is $\chi$-perfect.*

**Proof :**
clearly,
$\alpha(G(U)) = \omega(\bar{G}(U))$,
$\theta(G(U)) = \chi(\bar{G}(U))$, for all $U \subset V(G)$
Thus, $\alpha(G(U)) = \theta(G(U))$ is equivallent to $\omega(\bar{G}(U)) = \chi(\bar{G}(U))$

**Corollary 4.8** *If either G or its complementary graph $\bar{G}$ contains an odd cycle of length $> 3$ without chords, then G is neither $\alpha$-perfect nor $\chi$-perfect.*

**Proof :**
Let $A$ be the vertex set of such a cycle of $G$. Then, from Example 9 $\alpha(G(A)) \neq \theta(G(A))$, $\omega(G(A)) \neq \chi(G(A))$. Thus, $G$ is neither $\alpha$-perfect nor $\chi$-perfect. If the coplementary graph $\bar{G}$ contains such a cycle, then it is neither $\alpha$-perfect nor $\chi$-perfect, and, from Theorem 4.7, $G$ is neither $\alpha$-perfect nor $\chi$-perfect.

This result and study of various classes of graphs ( Berge (1963),(1967),(1969)) suggest the following conjecture:

**The strong perfect graph conjecture (SPGC)**
For a graph $G$, the following conditions are equivalent:
(1) $G$ is $\alpha$-perfect,
(2) $G$ is $\chi$-perfect,
(3) Every odd cycle of length $> 3$ of G or $\bar{G}$ contains a chord.

It was showen above that (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (3). If (3) $\Rightarrow$ (1), $then$(3) $\Rightarrow$ (2). If (3) $\Rightarrow$ (2), $then$(3) $\Rightarrow$ (1). This conjecture is still unproved.

Lovasz proved (1971) the equivalence of (1) and (2), which is remarked as "Perfect Graph Theorem". Thus, from this point on we only need to speak of "perfect graphs".

Meyniel proved (1975) the **SPGC** for some certain classes of graphs:

**Theorem 4.9** *(Meyniel)*
*Every graph, whose odd cycles are of length* $\geq 5$ *contains at least two chords, is* $\chi$*-perfect.*

**Theorem 4.10** *Let* $G$ *be a connented graph with cut set* $A$ *that is a clique, and* $G(V_i \cup A)$ *is* $\chi$*-perfect for all connected component* $G(V_i), 1 \leq i \leq n, n \in \mathbf{N}$ *of* $G(V(G) - A)$*. Then* $G$ *is* $\chi$*-perfect.*

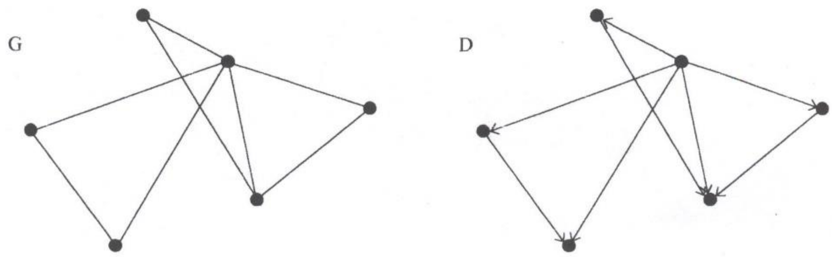
**Proof :**
It suffices to show that $\omega(G) = \chi(G)$.
If $\omega(G) = k$, then there exists a $k$-clique in at least one graph, $G(V_i \cup A), i \in 1, \ldots, n$.
For all other $V_j, j \neq i$, we have $\chi(G(V_j \cup A)) = \omega(G(V_j \cup A)) \leq k$.
Thus, $G$ is $k$-colorable and
$k = \omega(G) \leq \chi(G) \leq k$.
Hence, $\omega(G) = \chi(G)$.

## 4.3 Comparability graphs

**Definition 4.8** *A graph* $G = (V, E)$ *is called a* ***comparability*** *graph if it is possible to direct its edges so that the resulting graph* $D(G) = (V(G), D(E))$ *satisfies:*
*(i)* $(u, v) \in D(E), (v, w) \in D(E) \Rightarrow (u, w) \in D(E)$ *(transitivity)*
*(ii)* $(u, v) \in D(E) \Rightarrow (v, u) \in D(E)$ *(anti-symmetry).*

Each subgraph of a comparability graph is a comparability graph.

**Example 10** *The following graph G is a comparability graph, because there is direction D which satisfies the condition of above definition.*




**Example 11** *Let $(M, \leq)$ be an ordered set. Consider graph G whose vertices are the elements of M and two vertices $x, y$ are adjacent iff the corresponding elements of M are related, i.e. $x \leq y$ or $y \leq x$. Then graph G is a comparabiliyt graph.*

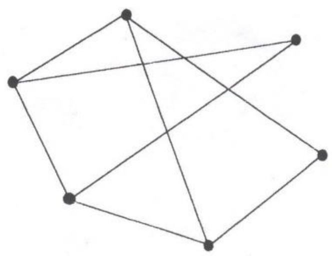
**Theorem 4.11** *Every comparability graph is perfect.*

**Proof :**
It suffices to show that if $G = (V(G), D(G))$ is the graph of an order relation, then $\chi(G) = \omega(G)$.
Let $t(v)$ denote the length of the longest path from $v$ plus one, $v \in V(G)$. Since $G$ has no cycle, $t(v) < \infty$ for all $v \in V(G)$. If $k = max\{t(v)|v \in V(G)\}$, then there exists a $k$-clique. There exist no $(k+1)$-cliques (because this clique would contain a path passing through all its vertices, and the longest path contains only $k$ vertices). Thus $\omega(G) = k$.
Consider $k$ colors denoted by $1, 2, \ldots, k$ and color each vertex $v$ with color $t(v)$. Two adjacent vertices cannot have the same color, because if there is an arc directed from $v$ to $w$, then $t(v) > t(w)$. Thus $\chi(G) \leq k$.
Since $\chi(G) \geq \omega(G) = k$, we have $\chi(G) = k = \omega(G)$.

## 4.4 Triangulated graphs

**Definition 4.9** *A graph G is called **triangulated** (or chordal) if each of its cycle of at least length 4 has a chord.*

**Example 12** *The graph in following pictuer is triangulated.*



**Remark 4.5** *A subgraph of a triangulated graph is also a triangulated graph.*

**Example 13** *A tree is a triangulated graph.*

**Example 14** *A complete graph is a triangulated graph.*

**Theorem 4.12** *If $G$ is a triangulated connected graph, then $G$ is complete or each minimal vertex cut set of $G$ is a clique.*

**Proof :**
Let $G$ be triangulated and not complete. Let $A$ be a minimal cut set of $G$. Removing $A$ creats several connected componenets $G(V_1), \ldots, G(V_n)$. Each vertex $a \in A$ is joined to each of these componenets. ( otherwise, $A - \{a\}$ would be a cut set of $G$, which contradicts the minimality of $A$.)
Let $u_1, u_2$ be two vertices in $A$. There exists a $(u_1, u_2)$-path $W_1 = u_1, v_{1,1}, \ldots, v_{1,p}, u_2$, where $v_{1,1}, \ldots, v_{1,p} \in V_1$.
Assume that $W_1$ is a path of this type with minimum length. There also exists a $u_1, u_2$-path $W_2 = u_2, v_{2,1}, \ldots, v_{2,q}, u_1$, where $v_{2,1}, \ldots, v_{2,q} \in V_2$.
Assume $W_2$ is a path of this type with minimum length.
The cycle $W_1 + W_2 = u_1, v_{1,1}, \ldots, v_{1,p}, u_2, v_{2,1}, \ldots, v_{2,q}, u_1$ has no chords of the following type:

- $(u_1, v_{1,i}), \quad i \neq 1$
- $(v_{1,i}, v_{1,j}), \quad i \neq j - 1$ } because of minimality of $W_1$
- $(u_2, v_{1,i}), \quad i \neq p$
- $(v_{1,i}, v_{2,j})$ because $W_1$ and $W_2$ are two distinct connected componnents of $G_{V-A}$

- $(u_2, v_{2,i}), \quad i \neq 1$
- $(v_{2,i}, v_{2,j}), \quad i \neq j-1$ } because of minimality of $W_2$
- $(u_1, v_{2,i}), \quad i \neq q$

Since the graph $G$ is triangulated, cycle $W_1 + W_2$, which has a length of at least 4, possesses a chord. This chord must necessarily be $(u_1, u_2)$.
Thus, any two vertices of $A$ are adjacent, and $A$ is a clique.

**Theorem 4.13** *Every triangulated graph is perfect.*

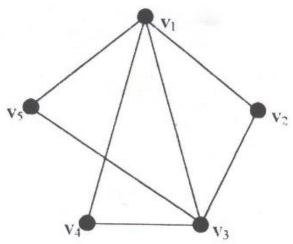
**Proof :**

Introduction by the number $k$ of vertices of $G$.
If $k \leq 3$. Then $\chi(G) = \omega(G)$.
Introduction step $k+1$. Let without lose of generality $G$ be connected and not complete.
By Theorem 4.12 $G$ has a minimal cut set $A \subset V(G)$, which is a clique.
Let $G(V_1), \ldots, G(V_n)$ be the connected components of $G(V(G) - A)$, then by induction hypothesis the subgraphs $G(V_1 \cup A), \ldots, G(V_n \cup A)$ are $\chi$-perfect. Hence by Theorem 4.10 $G$ is $\chi$-perfect too.

## 4.5 Interval graphs

**Definition 4.10** *Consider a family* $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ *of intervals on a line. The representative graph of* $\mathbf{A}$ *is defined to be a graph* $G$*, in which each vertex* $a_i$ *corresponds to an interval* $A_i$*, and with two vertices joined together if, and only if, the two corresponding intervals intersect. Such a graph is also called an* ***interval*** *graph.*

**Example 15** *Consider the intervals:*
$B_1 = [-1; 1]$, $B_2 = [0.5; 0.8]$, $B_3 = [-0.33; 0.6]$,
$B_4 = [0.25; 0.4]$ *and* $B_5 = [-0.2; 0.2]$, *the corresponding interval graph is given in the following picture.*

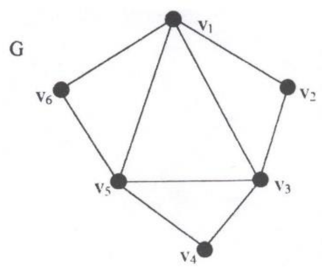


**Remark 4.6** *A subgraph of an interval graph is also an interval graph.*

**Theorem 4.14** *Every interval graph is triangulated.*

**Proof :**
Suppose that there is a cycle $C = v_1, v_2, \ldots, v_p, v_1$, $p > 3$ without chords. Let $A_i$ be the interval corresponding to vertex $v_i$ and $A_i = [x_i, y_i]$, $x_i, y_i \in \mathbf{R}$ , where $x_i \leq y_i$. Since $(v_{i-1}, v_i) \in E(G)$ for $i = 2, \ldots, p$ and $(v_{i-2}, v_i) \notin E(G)$ for $i = 3, \ldots, p$ (because $C$ has no chord), we have $A_{i-1} \cap A_i \neq \oslash$ for $i = 2, \ldots, p$ and $A_{i-2} \cap A_i = \oslash$ for $i = 3, \ldots, p$.
It follows: $x_1 \leq x_2 \leq y_1 < x_3 \leq y_2 < x_4 \leq y_3 < \ldots < x_p \leq y_{p-1} \leq y_p$. Therefore $A_p$ cannot overlap with $A_1$ ($y_1 < x_p$), which contradicts that $(v_p, v_1)$ is an edge of $G$.

**Remark 4.7** *The converse of this theorem is not true. Graph $G$ in following figure is triangulated, but we shall show that $G$ is not an interval graph.*
*Clearly, the intervals $A_1, A_2, A_3$ are pairwise disjoint and maybe placed in this order on the line. Then interval $A_5$ that intersects intervals $A_1$ and $A_3$ must also intersect interval $A_2$, which contradicts that vertices $v_2$ and $v_5$ are non-adjacent.*




**Corollary 4.15** *Every interval graph is perfect.*

**Proof :**
The proof follows immediately from Theorems 4.13 and 4.14

**Lemma 4.16** *If $G$ is an interval graph, then its complementary graph $\bar{G}$ is a comparability graph.*

**Proof :**
If graph $G = (V, E)$ represents a family of intervals $\mathbf{A}$, two vertices $x$ and $y$ are adjacent in $\bar{G}$ if and only if their corresponding intervals of $\mathbf{A}$ are disjoint.
Direct edge $(x, y)$ from $x$ to $y$ if the corresponding interval to $y$ is to the right of the corresponding interval to $x$ on the line. This produces a graph $(V, D(E))$ such that: $(x, y) \in D(E) \to (y, x) \notin D(E)$,

$(x, y) \in D(E), (y, z) \in D(E) \to (x, z) \in D(E)$.
Thus, $\bar{G}$ is a comparability graph.

## 4.6 Edge coloring

In previous section we introduced the application of vertex coloring in scheduling of final examinations of a university. Edge coloring problems arise when the object being scheduled are pairs of underlying elements.

**Example 16** *Edge coloring of $K_{2n}$.*
*In a league with $2n$ teams, we want to schedule games so that each pair of teams plays a game, but each team plays at most once a week. We can represent this problem by complete graph $K_{2n}$ in which the teams are the vertices Since each team must play $2n - 1$ others, the season lasts at least $2n - 1$ weeks. The games of each week must form a matching. We can schedule the season in $2n - 1$ weeks if and only if we can partition $E(K_{2n})$ into $2n - 1$ matchings. Since $K_{2n-1}$ is $2n - 1$-regular, these must be perfect matchings.*

**Definition 4.11** *A $k$-**edge-coloring** of $G$ is an assignment of $k$ colors, $1, 2, \ldots, k$, to the edges of $G$; the edges of one color form a **color class**. A $k$-edge-coloring is **proper** if adjacent edges have different colors; that is, if each color class is a matching. A graph is $k$-**edge-colorable** if it has a proper $k$-edge-coloring. The **edge-chromatic number** $\chi'(G)$ is the least $k$ such that $G$ is $k$-edge-colorable. Another name for $\chi'(G)$ is **chromatic index**.*

The chromatic index of a graph is the least number of matchings , in fact maximal matching, into which its edge set can be partitioned. Because every matching is a subset of a maximal matching, it suffices to determine all the maximal matching in a graph in order to find an edge coloring for it.

It is clear that the edges sharing a vertex in a graph need different colors, hence we have the following remark.

**Remark 4.8** $\chi'(G) \geq \Delta(G)$.

**Theorem 4.17** *(i)The chromatic index of cycle $C_n$ is given by*

$$\chi'(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$$

*(ii) The chromatic index of the complete graph $K_n$ is given by*

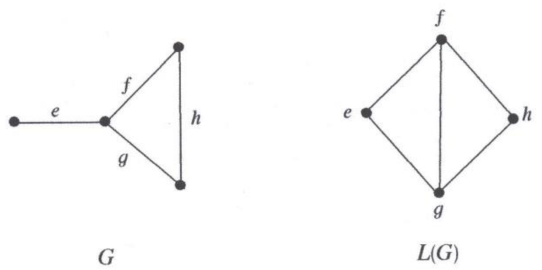
$$\chi'(K_n) = \begin{cases} n-1, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd} \end{cases}$$

### 4.6.1 Line graphs and edge coloring

Many questions about vertices have natural analogous for edges. Independent sets have no adjacent vertices; matchings have no "adjacent" edges . Vertex coloring partition vertices into independent sets; we can instead partition edges into matchings. These pairs of problems are related via line graphs. Coloring edges so that each color class is a matching amounts to proper vertex coloring of the line graph.

**Definition 4.12** *The **line graph** of $G$, written $L(G)$, is the simple graph whose vertices are the edges of $G$, with $ef \in E(L(G))$ when $e$ and $f$ have common endpoint in $G$.*

**Example 17** *In the following, a graph $G$ and its line graph $L(G)$ are shown:*



**Remark 4.9** *We have observed that always* $\chi'(G) \geq \Delta(G)$. *The upper bound* $\chi'(G) \leq 2\Delta(G) - 1$ *also follows easily. Color the edges in some order, always assigning the current edge the least-indexed color different from those already appearing on edges incident to it. Since no edge is incident to more than* $2(\Delta(G) - 1)$ *other edges, this never uses more than* $2\Delta(G) - 1$ *colors. The procedure is precisely greedy coloring for vertices of* $L(G)$.
$\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$

Vizing and Gupta independently proved that $\Delta(G) + 1$ colors suffice when $G$ is simple.

**Theorem 4.18** *(Vizing, Gupta) If* $G$ *is a simple graph, then* $\chi'(G) \leq \Delta(G) + 1$.

**Definition 4.13** *A simple graph* $G$ *is* ***Class 1*** *if* $\chi'(G) = \Delta(G)$. *It is* ***Class 2*** *if* $\chi'(G) = \Delta(G) + 1$.

**Theorem 4.19** *(Shanon) If* $G$ *is a graph, then* $\chi'(G) \leq \frac{3}{2}\Delta(G)$.

It is easy to observe that:

**Remark 4.10**
$L(C_n) = C_n$
$L(K_{1,n}) = K_n$

**Theorem 4.20** *Let* $G$ *be a graph and* $L(G)$ *the line graph of* $G$. *Then for each edge coloring of* $G$ *with* $k \in \mathbf{N}$ *colors, there is a corresponding* $k$*-vertex coloring of* $L(G)$ *and vice versa.*

**Proof :**
This theorem follows directly from definition of line graph.

### 4.6.2 Bipartite graphs and edge coloring

We have seen that the use of graphs to model relations between two disjoint sets has many important applications. These are the graphs whose vertex sets can be partitioned into two independent sets; bipartite graphs.

Another application of such graphs is in the timetabling in a school. In a school, there are m teachers $X_1, X_2, \ldots, X_m$ and $n$ classes $Y_1, Y_2, \ldots, Y_n$. Given that teacher

$X_i$ is required to teach class $Y_j$ for $p_{ij}$ hours, the problem is to schedule a complete timetable in the minimum possible number of hours. This problem can be again transferred into a graph. Consider a bipartite graph $G$, with bipartition $(V_1, V_2)$ where $V_1 = \{x_1, x_2, \ldots, x_m\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$ and vertices $x_i$ and $y_j$ are joined by $p_{ij}$ edges. Now, in any one hour, each teacher can teach at most one class, and each class can be taught by at most one teacher, this is our assumption. Thus a teaching schedule for one hour corresponds to a matching in the graph and, conversely, each matching corresponds to a possible assignment of teachers to classes for one hour. The problem, therefore, is to partition the edges of $G$ into as few matching as possible or, equivalently, to properly color the edges of $G$ with as few colors as possible. Since $G$ is bipartite, by Theorem 4.26 we know that $\chi'(G) = \Delta(G)$. Hence, if no teacher teaches for more than $p$ hours, and if no class is taught for more than $p$ hours, the teaching requirements can be scheduled in a $p$-hour timetable. Furthermore, there is a good algorithm for constructing such a timetable, see [7].

In the following there are some characterizations of bipartite graphs. We begin with one of the most widely used, which was obtained by König (1916).

**Theorem 4.21** *(König) A graph $G$ is bipartite if and only if $G$ has no cycle of odd length.*

**Property 4.22** *A connected bipartite graph has a unique bipartition.*

**Property 4.23** *A bipartite graph, without isolated vertices, which has $t$ connected components has $2^{t-1}$ bipartitions.*

**Corollary 4.24** *A connected graph $G$ is bipartite if and only if for every vertex $v$ there is no edge $xy$ with $d(v,x) = d(v,y)$.*

**Corollary 4.25** *For $k > 0$, every $k$-regular bipartite graph has a perfect matching.*

As we have mentioned $\chi'(G) \geq \Delta(G)$ for every graph $G$, but for bipartite graphs the above corollary improve the upper bound of Remark 4.9, achieving the trivial lower bound. Furthermore, there is a good algorithm to produce a proper $\Delta(G)$-edge-coloring in a bipartite graph $G$, which is presented in proof of König Coloring Theorem.

**Theorem 4.26** *(König's Coloring Theorem) For any bipartite graph $G$, $\chi'(G) = \Delta(G)$.*

**Proof :**
Corollary 4.25 states that every regular bipartite graph $H$ has a perfect matching. By induction on $\Delta(H)$, this yields a proper $\Delta(H)$-edge-coloring. It therefor suffices to show that for every bipartite graph $G$ with maximum degree $k$, there is a $k$-regular bipartite $H$ containing $G$.
To construct such a graph, first add vertices to smaller partite set of $G$, if necessary, to equalize the sizes. If the resulting $G'$ is not regular, then each partite set has a vertex with degree less than $k$. Add an edge with these two vertices as endpoints. Continue adding edges until the graph becomes $k$-regular; the resulting graph is $H$.

**Remark 4.11** *For a regular bipartite graph $G$, proper edge-coloring with $\Delta(G)$ colors is equivalent to decomposition into perfect matchings.*

Trees and forests, union of disjoint trees, are two special classes of bipartite graphs. They are certainly bipartite, since they contain no cycles of either parity. There are many characterizations of trees, in the following theorem, we give only two of them.

**Proposition 4.27** *The following statements are equivalent for a graph $G$:*
*(1) $G$ is a tree,*
*(2) each pair of vertices is joined by a unique path,*
*(3) $G$ is connected and $|V(G)| = |E(G)| + 1$*

**Theorem 4.28** *The product $G_1 \times G_2$ of two disjoint simple graphs is bipartite if and only if both $G_1$ and $G_2$ are bipartite.*

**Theorem 4.29** $K_{m,n} \simeq K_m \times K_n$

## 4.7 Total Coloring

Another kind of graph coloring is total coloring, before introducing this type of coloring we study the total graphs and their properties.

### 4.7.1 Total graphs

**Definition 4.14** *The total graph $G_T$ of a graph $G$ is the graph with $V(G_T) = V(G) \cup V(L(G))$, $L(G)$ is the line graph of $G$, and two vertices $u, v \in V(G_T)$ are adjacent if and only if:*

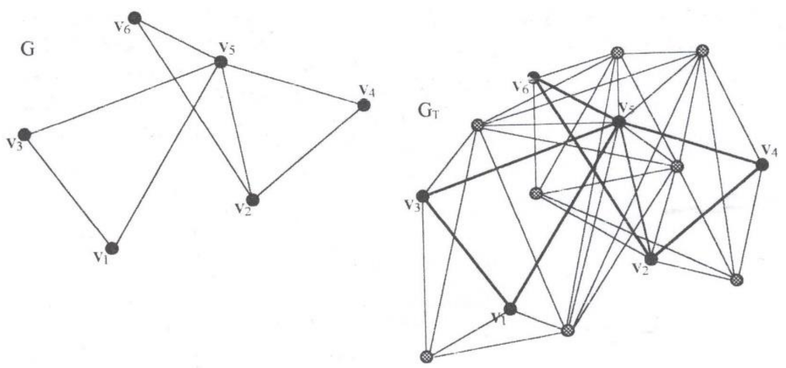
*(i) u and v are two adjacent vertices in G or,*
*(ii) u and v are two adjacent edges in G or,*
*(iii) u a vertex ,and v an edge in G are incident.*

In the following an algorithm is given to find the total graph of an arbitrary graph $G$.

**Algorithm 4.2**
Input: A graph $G$ with $V(G) = \{v_1, \ldots, v_{n(G)}\}$ and $E(G) = \{e_1, \ldots, e_{e(G)}\}$.
Output: The total graph of $G$.
(1) Put $V(G_T) := V(G)$,
(2) For $i = 1, \ldots, e(G)$ do:
Is $e_i = v_{i1}v_{i2}$, then expand $V(G_T)$ to a vertex $v_{n(G)+i}$
and $E(G_T)$ to the edges $v_{n(G)+i}v_{i1}$ and $v_{n(G)+i}v_{i2}$.
(3) For $i = 1, \ldots, e(G) - 1$ do:
For $j = i + 1, \ldots, e(G)$ do:
If $e_i$ and $e_j$ are adjacent then expand $E(G_T)$ to an edge $v_{n(G)+i}v_{n(G)+j}$.

**Definition 4.15** *The vertex $v_{n(G)+1}$ in above algorithm step (2), which is added to $V(G_T)$, is called with edge $e_i$ corresponding vertex.*

**Example 18** *In the following figure, a graph $G$ and its total graph $G_T$ are shown.*



### The Properties of Total Graphs

In this part we consider the parameters $\alpha(G_T), \theta(G_T), \chi(G_T)$ and $\omega(G_T)$ of a graph $G$.

**Definition 4.16** *Let $G_T$ be the total graph of $G$. Then for every $v \in V(G)$ we define*

$$N_T(v) := \{ \ u \in V(G_T) - V(G) | \ \text{the corresponding edge with } u \text{ of } G \text{ is incident with } v \ \}.$$

**Theorem 4.30** *Let $G_T$ be the total graph of graph $G$. Then,*
*a) If $\Delta(G) = 1$, then $\omega(G_T) = 3$.*
*b) If $\Delta(G) \neq 1$, then $\omega(G_T) = \Delta(G) + 1$.*

**Proof :**
a) Without lose of generality we can assume that $G$ is connected. Since $\Delta(G) = 1$, $G$ is the complete graph $K_2$. Therefore $G_T = K_3$ and $\omega(G) = 3$.
b) At first we show: (*) There is a clique $C \subset V(G_T)$ with $|C| = \Delta(G) + 1$.
Let $v \in V(G)$, consider $C(v) = v \cup N_T(v)$. Then $G(C(v))$ is complete, therefore $C(v)$ is a clique.
Clearly,

$$\max_{v \in V(G)} |C(v)| = \Delta(G) + 1.$$

If $v$ be a vertex in $G$ with $d_G(v) = \Delta(G)$, then the set $C(v)$ satisfies (*).
Suppose there is a maximal clique $C \subset V(G_T)$ with $|C| > \Delta(G) + 1$.
$|C \cap V(G)| = 0$, i.e. $C \subset V(G_T) - V(G)$.
It means the vertices in $C$ should be corresponding with edges of $G$, which pairwise have common end point.
Let $E_c \subset E(G)$ be the set of these edges and $V_c \subset V(G)$ be the set of vertices, which are the end points of edges of $E_c$. $|V_c|$ has more than one element $v \in V(G)$, otherwise $C \cup \{v\}$ is a clique, which is a contradiction to maximality of $C$.

**Theorem 4.31** *For total graph $G_T$ of graph $G$, $\theta(G_T) \leq n(G)$.*

**Proof :**
Let $V(G) = \{v_1, \ldots, v_{n(G)}\}$.
For all $i \in \{1, \ldots, n(G)\}$, $C(v_i) = \{v_i\} \cup N_T(v_i)$ is a clique of $G_T$.

Every vertex $u \in V(G_T)$ is in at least one set, $C(v_i), i \in \{1, \dots, n(G)\}$.
The following sets construct a clique decomposition of $G_T$
$C(v_1), C(v_2) - C(v_1), C(v_3) - (C(v_1) \cup C(v_2)), \dots, C(v_{n(G)}) - (C(v_1) \cup \dots \cup C(v_{n(G)-1}))$. Hence, $\theta(G_T) \leq n(G)$.

**Corollary 4.32** *For graph $G$, $\alpha(G_T) \leq n(G)$.*

**Proof :**
Since $\alpha(G_T) \leq \theta(G_T)$ , it follows from Theorem 4.31 $\alpha(G_T) \leq n(G)$.

**Theorem 4.33** *Let $G_T$ be the total graph of $G$, then $\chi(G_T) \leq 2\Delta(G) + 1$.*

**Proof :**
(1) for all $v \in V(G)$, $d_{G_T}(v) = 2d_G(v)$, because $E(G_T) - E(G)$ possesses $d_G(v)$ edges which are incident to $v$.
(2) Let $e = uw \in E(G)$. For vertex $v \in V(G_T) - V(G)$ corresponding with $e$ we have
$d_{G_T}(v) = (d_G(u) - 1) + (d_G(w) - 1) + 2 = d_G(u) + d_G(w)$
From (1) and (2) implies $d_{G_T}(v) \leq 2\Delta(G)$ for all $v \in V(G_T)$. Since $\chi(G_T) \leq \Delta(G) + 1$, it follows that $\chi(G_T) \leq 2\Delta(G) + 1$.

**Corollary 4.34** *For every graph $G$,*
*a) If $\Delta(G) = 1$, then $\chi(G_T) = 3$.*
*b) If $\Delta(G) \neq 1$, then $\Delta(G) + 1 \leq \chi(G_T) \leq 2\Delta(G) + 1$.*

**Proof :**
Since $\omega(G_T) \leq \chi(G_T)$, it follows from Theorems 4.30 and 4.33.

**Remark 4.12** *Let $G$ be a graph and $U_1 \subset V(G)$ an arbitrary subset. Then there is a set $U_2 \subset V(G_T)$ with $G[U_1]_T = G[U_2]$.*

**Theorem 4.35** *Let $G$ be a graph with $\omega(G) \geq 4$. Then the total graph of $G$ is not perfect.*

**Proof :**
Let $U$ be a clique of $G$ with 4 vertices. According to Theorem 4.30 we have $\omega(G[U])_T = 4$ for the subgraph $G[U]$. We will see later that $\chi''(G[U]) = \chi(G[U])_T$, hence by Theorem 4.38 $\chi(G[U])_T = 5$. Therefore $G[U]_T$ is as subgraph of $G_T$ not perfect and so $G_T$ is not perfect.

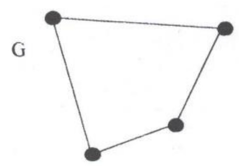
**Corollary 4.36** *The total graph of a perfect graph is not in general perfect.*

**Proof :**
It follows directly from Theorem 4.35, because for example every complete graph is triangulated and therefore perfect.

For a graph $G$ with $\omega(G) < 4$ there is no general statement for its perfectness. This we show with help of examples:

(i) Let $\omega(G) = 2$.
Without lose of generality suppose $G$ is connected. There are two cases:

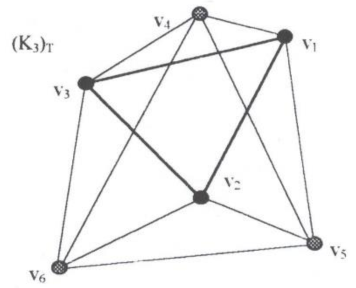
1) $G$ is a tree.
We will see later that total graphs of trees are perfect.
2)(i) $G$ has a cycle of length $\geq 4$ like following example:



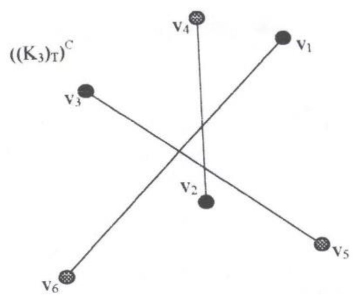

By theorems we have $\chi(G_T) = 4$, and by Theorem 4.30 $\omega(G_T) = 3$. Therefore $G_T$ is not perfect.

(ii) Let $\omega(G) = 3$

Consider the complete graph $G = K_3$ and its total graph $(K_3)_T$ in the following figure ($V(G) = \{v_1, v_2, v_3\}$).

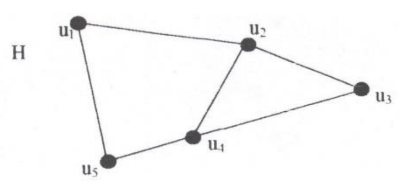



The complement of graph $(K_3)_T$ as shown in following picture is $\chi$-perfect.




Hence Theorem 4.7 implies that $(K_3)_T$ is perfect.

consider the following graph $H$ with $\omega(H) = 3$.



The subgraph $G[\{u_1, u_2, u_4, u_5\}]_T$ from $H_T$ by (i) case 2) is not perfect. Therefore $H_T$ is also not perfect.

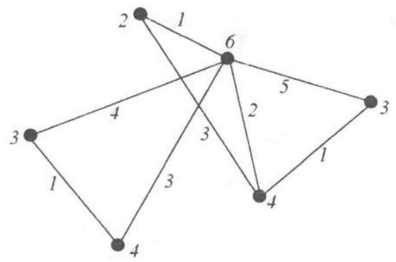
### 4.7.2 The total coloring conjecture

**Definition 4.17** *A $k$-**total-coloring** of $G$ is an assignment of $k$ colors, $1, 2, \ldots, k$, to the elements (vertices and edges) of $G$; the elements of one color form a **color class**. A $k$-total-coloring is **proper** if adjacent or incident elements have different colors. A graph is $k$-**total-colorable** if it has a proper $k$-total-coloring. The **total-chromatic number** $\chi''(G)$ is the least $k$ such that $G$ is $k$-total-colorable.*

**Theorem 4.37** *For every graph $G$,* $\chi''(G) \geq 1 + \Delta(G)$.

**Proof :**
let $v \in V(G)$ and $d_G(v) = \Delta(G)$. Since $v$ and all incident adges should have different colors, the theorem is true.

**Example 19** *In the following there is total coloring of graph $G$ with 6 colors. We will see in next theorems that 6 is the total chromatic number of $G$.*




Behzad and indepentdly from him Vizing had following conjecture, which is not yet completely proved.

**Total Coloring Conjecture (TCC)**
For every graph $G$ we have $\chi''(G) \leq \Delta(G) + 2$

In next sections different classes of graphs are represented, for which this conjecture is true or their total chromatic number is determined.

### 4.7.3 total coloring of some graphs

**Theorem 4.38** *(Behzad, Chartrand, Cooper)*

*(i)* $\chi''(K_n) = \begin{cases} n & n \text{ odd} \\ n+1 & n \text{ even} \end{cases}$

*(ii)* $\chi''(K_{m,n}) = max\{m, n\} + 1 + \delta_{m,n}$,
*where* $\delta_{m,n} = 1$ *if* $m = n$ *and* $\delta_{m,n} = 0$ *otherwise.*

**Corollary 4.39** *For every graph G with n vertices,*

$$\chi''(G) \leq \begin{cases} n & n \text{ odd} \\ n+1 & n \text{ even} \end{cases}$$

**Proof :**
Every graph $G$ with $n$ vertices is a subgraph of $K_n$. Therefore $\chi''(G) \leq \chi''(K_n)$.

**Theorem 4.40** *(Rosenfeld)*
*If G is a balanced complete r-partite graph, then*
$\chi''(G) \leq \Delta(G) + 2$.

**Theorem 4.41** *(Bermond)*
*If G is balanced complete r-partite graph with* $n(G)/r = n$, *then*

$$\chi''(G) = \begin{cases} \Delta(G) + 1, & \text{if } (r = 2) \text{ or } (r \text{ (even and } n \text{ odd)} \\ \Delta(G) + 1, & \text{if } (r \text{ odd or } (r \text{ even}, \neq 2 \text{ and } n \text{ even}) \end{cases}$$

**Theorem 4.42** *(Yap)*
*For every complete r-partite graph G,*
$\chi''(G) \leq \Delta(G) + 2$.

**Theorem 4.43** *Let G be a r-partite graph and* $V_1, \ldots, V_r$ *the correspondig partition sets. If*
$\Delta(G) = n(G) - min|V_i|$, *then*
$\chi''(G) \leq \Delta(G) + 2$.

**Proof :** $H$ be the complete $r$-partite graph with $V(H) = V(G)$ and the same partitions $V_1, \ldots, V_r$. $G$ is a subgraph of $H$ and we have $\Delta(G) = \Delta(G)$. Hence from the above theorem:
$\chi''(G) \leq \chi''(H) \leq \Delta(H) + 2 = \Delta(G) + 2$

**Theorem 4.44** *(Chew, Yap)*
*For every complete $r$-partite graph $G$ of odd order,*
$\chi''(G) = \Delta(G) + 1$

**Theorem 4.45** *(Kostachka)*
*For every graph $G$ with $\Delta(G) \leq 5$,*
$\chi''(G) \leq \Delta(G) + 2$.

**Theorem 4.46** *(Cleves,Jacobson)*
*Let $G$ be a $n$-cycle. Then*

$$\chi''(G) = \begin{cases} 3 & n \equiv 0 \quad mod \quad 3 \\ 4 & otherwise \end{cases}$$

**Theorem 4.47** *(Bollobas, Harris)*
*Let $G$ be a graph with high $\Delta(G)$, then $\chi''(G) < c\Delta(G)$, where $c$ is a constant with $11/6 < c < 2$.*

**Theorem 4.48** *(Borodin)*
*Let $G$ be planar graph, then:*
*(i)* $\chi''G) \leq \Delta(G) + 2$ *for* $\Delta(G) \notin \{6, 7, 8\}$,
*(ii)* $\chi''G) \leq \Delta(G) + 3$ *for* $\Delta(G) \in \{6, 7, 8\}$,
*(iii)* $\chi''G) = \Delta(G) + 1$ *for* $\Delta(G) \geq 14$.

**Theorem 4.49** *(Zhang, Zhang, Wang)*
*If $G$ is an exterior-planar graph with $\Delta(G) \geq 3$, then*
$\chi''(G) = \Delta(G) + 1$.

**Theorem 4.50** *(Zhang, Zhang, Wang)*
*If $G$ is a $(n(G) - 3)$-regular graph of odd order. Then*
$\chi''(G) \leq \Delta(G) + 2$.

**Theorem 4.51** *(Zhang, Zhang, Wang)*
*If G is a graph with exactly one vertex v , $d_G(v) = n(G) - 1$, then*
$\chi''(G) = \Delta(G) + 1$.

**Theorem 4.52** *(Yap, Wang, Zhang)*
*For every graph G with $\Delta(G) \geq n(G) - 4$,*
$\chi''(G) \leq \Delta(G) + 2$.

There are also some upper bound for for total chromatic number of arbitrary graph:

**Theorem 4.53** *(Hind)*
*For every graph G,*
*a)* $\chi''(G) \leq \Delta(G) + 2\lceil\sqrt{\Delta(G)}\rceil$
*b)* $\chi''(G) \leq \Delta(G) + 2\lceil\sqrt{\chi(G)}\rceil$

**Theorem 4.54** *(Hind)*
*For every graph G,*
$\chi''(G) \leq \Delta(G) + 2\lceil\frac{n(G)}{\Delta(G)}\rceil + 1$

McDiarmid and Reed shows that for $n \to \infty$ the number of graphs with $n$ vertices and total chromatic number $\chi''(G) > \Delta(G) + 1$ is "very small" and the number of graphs with $\chi''(G) > \Delta(G) + 2$ is "very very small".

**Total Coloring of Trees**

As we know trees are triangulated. Here at first we determine the total chromatic number of trees.

**Theorem 4.55** *The total graph $G_T$ of tree G is triangulated.*

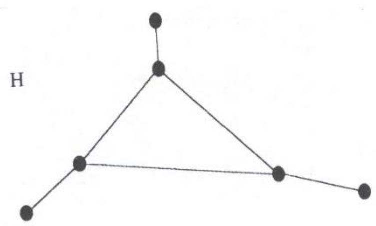
**Proof :**
Suppose $G_T$ has a cycle $W$ of length $l \geq 4$ without chords.
Let $V(G) = \{v_1, \ldots, v_n\}$. Considel the sets
$C(v_i) = \{v_i\} \cup N_T(v_i), i = 1, \cdots, n$.
$\{0, 1\}$ and also $C(v_i) \cap C(v_j) \cap C(v_k) = \emptyset$ for pairwise dinstinct $i, j, k \in \{1, \ldots, n\}$.
The sets $C(v_i), i = 1, \ldots, n$ are cliques. Therefore $W$ can contain at most two vertices of each of them.
Construct a cycle $W^*$ from $W$:
For all $i \in \{1, \ldots, n\}$ with $|N_T(v_i) \cap V(W)| = 2$,i.e. $N_T(v_i) \cap V(W) = \{u_{i1}, u_{i2}\}$, replace the edge $u_{i1}u_{i2}$ in $W$ with the sequence $u_{i1}v_i, v_i, v_iu_{i2}$.
For all vertices $u \in V(W^*) \cap (V(G_T) - V(G))$, the adjacent vertices with $u$ in $W^*$

are in $V(G)$.
Construct a cycle $W^{**}$ from $W^*$:
For all vertices $u \in V(W^*) \cap (V(G_T) - V(G))$, where $v_i u$ and $uv_j$, $i, j \in \{1, \ldots, n\}, i \neq j$ are incident edges with $u$ in $W^*$, replace the sequence $v_i u, u, uv_j$ in $W^*$ with edge $v_i v_j$, which is with $u$ corresponding edge in $G$.
Now we have $V(G) \cap V(W^{**}) = V(W^{**})$, it means $W^{**}$ is a cycle of $G$ which is a contradiction to the hypothesis that $G$ is a tree.

**Corollary 4.56** *The total graph of a tree is perfect.*

**Proof :**
Since every triangulate graph is perfect and according to theorem 4.55 the total graph of a tree is triangulate.

**Remark 4.13** *The converse of theorem 4.55 is not in general true, i.e. not for all triangulated graph $H$, there exists a tree $G$ with $G_T = H$. The total graph $G_T$ of any arbitrary graph $G$ must have at least $n(G)$ vertices with even degree, because for all $v \in V(G), d_{G_T}(v) = 2d_G(v)$. The triangulated graph $H$ in following figure dose not satisfies this condition.*




**Theorem 4.57** *Let $G$ be a tree, then:*
*a) If $\Delta(G) = 1$, then $\chi''(G) = 3$.*
*b) If $\Delta(G) \neq 1$, then $\chi''(G) = \Delta(G) + 1$*

**Proof :**
Since $\chi''(G) = \chi(G)$ and $G_T$ is perfect, by theorem 4.30 we have a) and b).

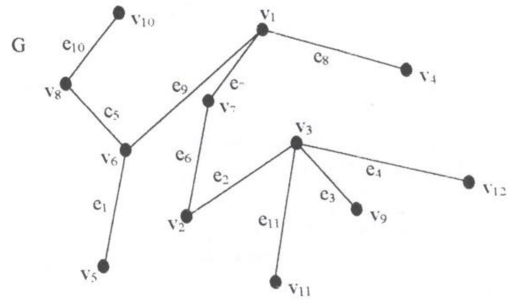
It is almost easy to find a total coloring of a tree, because it has no cycle.

**Algorithm 4.3**
Input: A tree with $V(G) = \{v_1, \ldots, v_{n(G)}\}$ and $E(G) = \{e_1, \ldots, e_{e(G)}\}$.
Output:A map $f : V(G) \cup E(G) \to \mathbf{N}$.
(1) define $f(v_1) := 1$.
Put $S := \{v_1\}$,
$T := \emptyset$.
(2) For $j = 1, \ldots, e(G)$ do:
If $e_j \notin T$ and incident with a vertex $v \in S$,
then define
$f(e_j) :=$ min $\{k \in \mathbf{N} | k \neq f(v)$ and $k \neq f(e)$ for all with $v$ incident $e \in T\}$
and put $T := T \cup \{e_j\}$.
(3) For $i = 2, \ldots, n(G)$ do:
If $v_i \notin S$ and incident with an edge $e \in T$,
then define
$f(v_j) :=$ min $\{k \in \mathbf{N} | k \neq f(e)$ and $k \neq f(v)$ for all with $v$ incident $v \in T\}$
and put $S := S \cup \{v_j\}$.
(4) If $S = V(G)$, then stop.
Otherwise go to (2).

**Theorem 4.58** *The map $f$ defined by algorithm 4.3 is a total coloring of tree $G$ with $\chi''(G)$ colors.*

**Proof :**
By definition the map $f$ in algorithm 4.3 step (1) and (2) is a total coloring.
For the coloring of an edge in step (2), there are maximum $\Delta(G)$ restrictions. Therefore $f(e) \leq \Delta(G) + 1$ for all $e \in E(G)$.
For the coloring of a vertex in step (3), there are exactly two restrictions. Therefore $f(v) \leq 3$ for all $v \in V(G)$. Hence according to theorem 4.58 the theorem is true.

**Example 20** *Consider the tree $G$ with $V(G) = \{v_1, \ldots, v_{12}\}$ , $E(G) = \{e_1, \ldots, e_{11}\}$. Here we find a total coloring $f : V(G) \cup E(G) \to \{1, \ldots, \chi''(G)\}$.*




*We apply the algorithm 4.3*
*(1)* $f(v_1) := 1$
*(2)* $f(v_2) := 2$
$f(e_8) := 3$
$f(e_9) := 4$
*(3)* $f(v_4) := 2$
$f(v_6) := 2$
$f(v_7) := 3$
*(2)* $f(e_1) := 1$
$f(e_5) := 3$
$f(e_6) := 1$
*(3)* $f(v_2) := 2$
$f(v_5) := 3$
$f(v_8) := 1$
*(2)* $f(e_2) := 3$
$f(e_{10}) := 2$
*(3)* $f(v_3) := 1$
$f(v_{10}) := 3$
*(2)* $f(e_3) := 2$
$f(e_4) := 4$
$f(e_{11}) := 5$
*(3)* $f(v_9) := 3$
$f(v_{11}) := 2$
$f(v_{12}) := 2$

*And by theorem 4.58 we have:* $\chi''(G) = 5$

## 4.8 List coloring

List coloring is a more general version of vertex coloring problem. We still pick a single color for each vertex, but the set of colors available at each vertex may be restricted. This type of restrictions have many uses in timetabling.

**Definition 4.18** *For each vertex $v$ in a graph $G$, let $L(v)$ denote a list of colors available at $v$. A* ***list coloring*** *is a proper coloring $f$ such that $f(v) \in L(v)$ for all $v$. A graph is* ***list $k$-colorable*** *if every assignment of $k$-element lists to the vertices permits a proper list coloring. The* ***list chromatic number*** $\chi - l(G)$ *is the minimum $k$ such that $G$ is list $k$-colorable.*

Since the lists could be identical, $\chi_l(G) \geq \chi(G)$. If the lists have size at least $1 + \Delta(G)$, then coloring the vertices in succession leaves available color at each vertex. This argument is analogous to the greedy coloring algorithm and proves that $\chi_l \leq 1 + \Delta(G)$.

Analogously we can define list $k$-edge-coloring for a graph.

**Definition 4.19** *For each edge $e$ in a graph $G$, let $L_e(e)$ denote a list of colors available at $e$. A* ***list edge-coloring*** *is a proper coloring $f$ such that $f(e) \in L_e(e)$ for all $e$. A graph is* ***list $k$-edge-colorable*** *if every assignment of $k$-element lists to the edges permits a proper list edge-coloring. The* ***list chromatic index*** $\chi' - l(G)$ *is the minimum $k$ such that $G$ is list $k$-edge-colorable.*

**Theorem 4.59** *(Galvin) Let $G$ be a bipartite graph. Then provided each edge receives a list of $\Delta(G)$ colors there is a proper edge coloring which respects the lists.*

**Theorem 4.60** *(Bordin, Kostochka, Woodall) Let $G$ be a bipartite graph, and for each edge $e \in E(G)$ let $|L_e(e)| = max\{d(x), d(y)\}$ where $x$ and $y$ are the endpoints of $e$. Then $G$ has an $L_e$-list edge coloring.*

Rather than placing lists on the edges, what if we place lists of allowed colors on the vertices? Consider the following problem.

Let $G$ be bipartite graph with bipartition $(V_1, V_2)$ and for each vertex $u \in V(G)$ let $L(U)$ be a set of colors assigned to $u$. Can $G$ have a proper edge-coloring in which each edge $e$ , with endpoints $x$ and $y$, receives a color which lies in $L(x) \cap L(y)$?

If we interpret such a coloring as a timetable in our usual way, with $V_1$ as the set of classes and $V_2$ as the set of teachers, then such a coloring corresponds to a timetable in which the list $L(u)$ corresponds to the set of hours in which the class or teacher is available.

The Theorem 4.60 gives the following result which is an answer to the above question in some special case.

**Theorem 4.61** *If* $|L(x) \cap L(y)| \geq max\{d(x), d(y)\}$ *for every pair of adjacent vertices* $x$ *and* $y$ *then the required coloring exists.*

## 4.9 Unique coloring

### 4.9.1 Uniquely vertex colorable graphs

**Definition 4.20** *A graph* $G$ *is called* ***uniquely*** *$k$-**vertex**-colorable, or simply uniquely $k$-colorable, if any two proper $k$-coloring of $G$ induce the same partition of $V(G)$.*

**Definition 4.21** *A graph $G$ with at least $k+1$ vertices is called* ***critically*** *uniquely colorable if it is uniquely $k$-colorable but no proper subgraph of it is so.*

**Theorem 4.62** *For all $k \geq 2$ and $g \geq 3$ there is a uniquely $k$-colorable graph whose girth* [2] *is at least $g$.*

If $G$ is uniquely $k$-colorable and its girth is at least $g > k$ then its minimal uniquely $k$-colorable subgraph ( which must be critically uniquely $k$-colorable ) must have at least $g$ vertices.

**Corollary 4.63** *For every $k \geq 3$ and $n$ there is a critically uniquely $k$-colorable graph with at least $n$ vertices.*

**Proposition 4.64** *If a graph $G$ is uniquely $k$-colorable, then in any $k$-coloring of $G$, every vertex $v$ of $G$ is adjacent with at least one vertex of every color different from that assigned to $v$.*

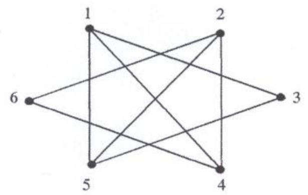
**Corollary 4.65** *If $G$ is uniquely $k$-colorable, then $\delta(G) \geq k-1$*

In the following there is a necessary condition for a graph to be uniquely $k$-colorable:

[2] The length of shortest cycle of $G$.

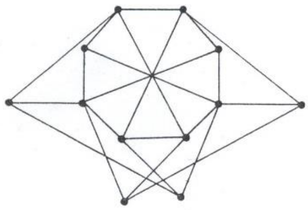
**Theorem 4.66** *For any $k$-coloring of a uniquely $k$-colorable graph $G$, the subgraph induced the union of any two color classes is connected.*

The converse of Theorem 4.66 holds in case $k = \chi(G) = 1$ or 2. That is, any 1-colorable graph is uniquely colorable, as in any connected 2-colorable graph. However the converse of Theorem 4.66 does not hold in general. This can be seen from following figure , which pictures a 3-chromatic graph admitting the distinct 3-colorings $\{1,2\}, \{3,4\}, \{5,6\}$ and $\{1,6\}, \{2,3\}, \{4,5\}$. Each of these coloring has the property that the subgraph induced by the union of any two color classes is connected.




**Theorem 4.67** *(Harary, Hedetniemi and Robinson) For all $n \geq 3$, there is a uniquely $n$-colorable graph which contains no subgraph isomorphic to $K_n$*

For $n = 3$ the graph in the following figure, a uniquely 3-colorable graph having no triangles, illustrates the theorem.



**Theorem 4.68** *Every uniquely $k$-colorable graph is $(k-1)$-connected.*

There are also some theorems presented by Bollobas which are based on lower bounds of minimum degree of $G$, for a uniquely $k$-colorable graph.

**Theorem 4.69** *Let $G$ be a $k$-colorable $(k \geq 2)$ graph of order $n$ such that* $\delta(G) > ((3k-5)/(3k-2))n$.
*Then $G$ is uniquely colorable.*

**Theorem 4.70** *Let $G$ be a graph of order $n-k$ having a $k$-coloring $(k \geq 2)$ satisfying the following conditions:*
*any subgraph induced by the union of any two color classes of $k$-coloring is connected. If*
$\delta(G) > (1-(1/(k-1)))n$
*then $G$ is uniquely colorable.*

The proof of the following theorem can be found in [33] and [30].

**Theorem 4.71** *If $G$ is uniquely vertex colorable graph, then*
$|E(G)| \geq (\chi(G)-1)|V(G)| - \binom{\chi(G)}{2}$.
*Moreover, this bound is best possible.*

All uniquely vertex colorable graphs constructed with minimum numbers of edges given in Theorem 4.71 contain a $K_k$ as a subgraph, where $k = \chi(G)$. This is the motivation of Shaoji conjecture.

**Conjecture**
If $G$ is uniquely vertex colorable graph with size $|E(G)| = (\chi(G)-1)|V(G)| - \binom{\chi(G)}{2}$, then $G$ contains a $K_k$ as a subgraph, where $k = \chi(G)$.

For each $k$, Daneshgar [13] has constructed uniquely vertex colorable graph, such that $\chi(G) = k$ and $|E(G)| = (\chi(G)-1)|V(G)| - \binom{\chi(G)}{2} + 1$, which contains no $K_k$.

### 4.9.2 Uniquely edge colorable graphs

**Definition 4.22** *A graph $G$ is called **uniquely** $k$-**edge**-colorable if any two proper $k$-edge-coloring of $G$ induce the same partition of $E(G)$.*

**Theorem 4.72** *(Greenwell, Kronk) Every uniquely 3-edge-colorable 3-regular graph is Hamiltonian.*

It is not difficult to observe that every uniquely edge colorable , simple bipartite graph $G$ with edges colored with colors $1, 2, \ldots, \Delta(G)$ has the property that each 2-colored subgraph is connected , i.e. is a path or an even cycle. Using this property it is not difficult to characterize uniquely colorable, simple bipartite graphs.

**Proposition 4.73** *A simple bipartite graph is uniquely edge colorable if and only if it is either a path, an even cycle or a star.*

The following important theorem was conjectured by R. J. Wilson in 1967 and proved by A. G. Thomason in 1978. The proof can be found in [6].

**Theorem 4.74** *For $k \geq 4$, the only uniquely $k$-edge colorable graphs are the stars, $K_{1,k}$.*

### 4.9.3 Uniquely total colorable graphs

**Definition 4.23** *A graph $G$ is called* ***uniquely*** *total colorable if $V(G) \cup E(G)$ can be partitioned into $\chi''(G)$ color classes in exactly one way.*

For example, cycle of size $n$, $n \equiv 0 \pmod 3$, empty graph, and paths are uniquely total colorable. Some other classes of graphs have been examined, and several results have been obtained by S. Akbari, M. Behzad, H. Hajiabolhasan and E. S. Mahmoodian, [?] which confirm the following conjecture.

**Conjecture**
Aside from the cycles of order $n$, $n \equiv 0 \pmod 3$, empty graph, and paths there is no graph which is uniquely total colorable.

The following theorem can be found in [1],

**Theorem 4.75** *If a graph $G \neq K_2$ is uniquely total colorable, then $\chi''(G) = \Delta(G) + 1$.*

## 4.10 Connection between different kind of graph colorings

One can observe from the definition that it is not so easy to give a proper total coloring for arbitrary graph $G$. In this section we introduce some algorithms which simplify it, i.e. by using other type of coloring.

### 4.10.1 Total coloring through vertex coloring

**Theorem 4.76** *Let $G$ be a graph and $G_T$ the total graph of $G$. Then for each total coloring of $G$ with $k \in \mathbf{N}$ colors, there is a corresponding $k$-vertex coloring of $G_T$ and vice versa.*

**Proof :**

(i) Let $f : V(G) \cup E(G) \to \{1, \dots, k\}$ be a total coloring of $G$ with $k$ colors. Then by definition:
$f(u) \neq f(v)$ for adjacent vertices $u, v \in V(G)$,
$f(v) \neq f(e)$ for incident vertex $v \in V(G)$ and edge $e \in E(G)$,
$f(e) \neq f(h)$ for adjacent edges $e, h \in E(G)$.
We define $g : V(G_T) \to \mathbf{N}$ as follows:
$g(v) := f(v)$ for $v \in V(G)$
$g(v) := f(e)$ for $v \in V(G_T) - V(G)$, where $e$ is the $v$ corresponding edge with $v$ of $G$.
We have $g(u) \neq g(v)$ for adjacent vertices $u, v \in V(G_T)$ and $g(V(G_T)) \subset \{1, \dots, k\}$.
Therefore $g$ is a $k$-vertx coloring of $G_T$.
(ii) Let $g : V(G_T) \to \{1, \dots, k\}$ be a vertex coloring of $G_T$. i.e. $g(u) \neq g(v)$ for adjacent vertices $u, v \in V(G_T)$.
We define $f : V(G) \cup E(G) \to \mathbf{N}$ as follows:
$f(v) := g(v)$ for $v \in V(G)$
$f(e) := g(v)$ for $e \in E(G)$, where $v$ is the $e$ corresponding vertex of $G_T$.
$f$ satisfies the conditions (*) and $f(V(G) \cup E(G)) \subset \{1, \dots, k\}$. Therfore $f$ is a total coloring of $G$ with $k$ colors.

**Corollary 4.77** *For a graph $G$ and its total graph $G_T$ :*
$\chi''(G) = \chi(G_T)$.

**Proof :**
Let $f$ be a total coloring of $G$ with $\chi''(G)$ colors. Then by Theorem 4.76, there is a $\chi''(G)$)-vertex coloring of $G_T$. Hence $\chi''(G) \geq \chi(G_T)$.
Let $g$ be a $\chi(G_T)$-vertex coloring of $G_T$. Then there is total coloring of $G$ with $\chi(G_T)$ colors. Hence $\chi''(G) \leq \chi(G_T)$.

So one can
(i) use the knowledge of properties of a total graph to determine the total chromatic number of its corresponding graph.
(ii) use the vertex coloring algorithms for total graph coloring.

An algorithm for vertex coloring is described by Christofides . This is an implicit enumeration method.

**Algorithm 4.4**

Input: A graph $G$ with $V(G) = \{v_1, \ldots, v_{n(G)}\}$.
Output: A vertex coloring $g : V(G) \to \{1, \ldots, \chi(G)\}$.
Suppose that the vertices are orderd in some way and are renumbered so that $v_i$ is the ith vertex in this ordering. An initial feasible coloring can be obtained as follows:
(i) Color $v_i$ with color 1.
(ii) Color each remaining vertices sequentially so that a vertex $v_i$ is colored with the lowest-numbered color that is feasible (i.e. which has not been used so far to color any vertices adjacent to $v_i$).
Let $q$ be the number of colors required by the above coloring. If a coloring using $q-1$ colors exists, then all vertices colored with $q$ must be recolored with $j < q$. If $v_t$ is the first vertex in the vertex ordering which has been colored $q$, and since (from (ii) above) it has been so colored because it could not be colored with any of colors $j < q$, this vertex can only be recolored with $j < q$ if at least one of its adjacent vertices is also recolored. Thus, a backtracking step from $v_t$ can be taken as follows.
Of these vertices $v_1, \ldots, v_{t-1}$ which are adjacent to $v_t$ find the last one in the vertex ordering (i.e. the one with the largest index) and let this be $v_k$. If $v_k$ is colored with $j_k$, recolor $x_k$ with the lowest-numbered feasible alternative color $j'_k, j'_k \geq j_k$.
If $j'_k < q$ continue by recoloring sequentially all the vertices $v_{k+1}$ to $v_n$ using method (ii) above, and provided that color $q$ is not needed. If this is possible a new better coloring using less than $q$ colors has been found, otherwise if a vertex is encountered which requires color $q$, then backtracking can again take place from such a vertex.
If $j'_k = q$, or no alternative feasible color $j'_k$ exists, then backtracking can take place immediately from vertex $v_k$. The algorithm terminates when backtracking reaches vertex $v_1$.

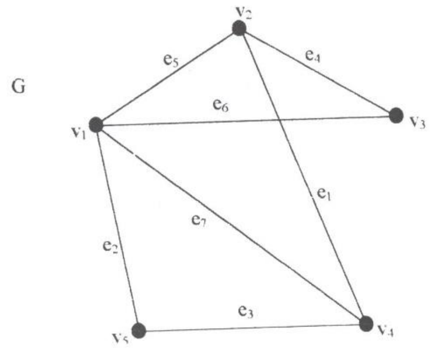
The following observations can help to speed up the above implicit enumeration procedure.
(a) Whatever the vertex ordering, the feasible colors $j$ for vertex $v_i$ are $j \leq i$ (provided $i < q$). This is apparent since only $i-1$ vertices precede $v_i$ in the vertex ordering and hence colors $j > i$ will never be needed. Thus for vertex $v_1$ the only feasible color is 1, for vetex $v_2$ the feasible colors are 1 and 2 (unless $v_2$ is adjacent to $v_1$ in which case only color 2 is feasible) etc.
(b) From (a) it is apparent that it would be computationally beneficial to order the vertices in such way so that the first $\rho$ (say) vertices form the largest clique of $G$. This would imply that each vertex $v_i$ $(1 \leq i \leq \rho)$ has only one feasible color, i.e. color $i$, and the algorithm can stop earlier when backtracking reaches vertex $v_\rho$.

A total coloring of a graph $G$ can be obtained in the way that, at first we find a vertex coloring of its total graph $G_T$ by Algorithm 4.4 and then as in part(ii) of

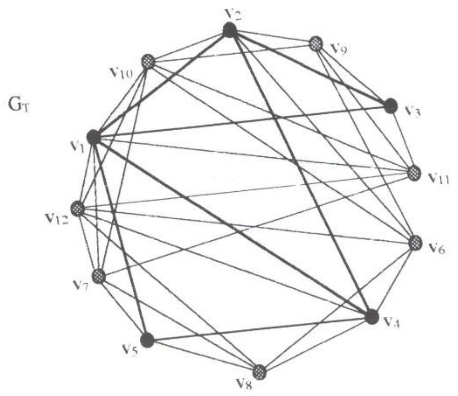
Theorem 4.76 determine the corresponding total coloring of $G$.

**Remark 4.14** *If the chromatic number of a graph $G$ is known, then the Algorithm 4.4 can be stopped when $q = \chi(G)$.*

**Example 21** *For the following graph $G$ with $V(G) = \{v_1, \ldots, v_5\}$ and $E(G) = \{e_1, \ldots, e_7\}$, we find the total coloring $f : V(G) \cup E(G) \to \{1, \ldots, \chi''(G)\}$.*




*The graph $G$ satisfies the conditions of Theorem 4.51, therefore $\chi''(G) = 5$.*
*We construct the total graph $G_T$ of $G$ according to Algorithm 4.2,i.e. $V(G_T) = \{v_1, \ldots, v_{12}\}$ and for $6 \leq i \leq 12$, $v_i$ is corresponding vertex with $e_{i-5} \in E(G)$.*

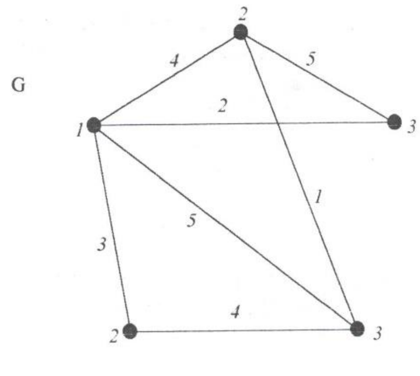


*Now we apply the Algorithm 4.4 on the graph $G_T$:*

(1) $k := 0$
(2) $g(v_1) := 1$
$g(v_2) := 2$
$g(v_3) := 3$
$g(v_4) := 3$
$g(v_5) := 2$
$g(v_6) := 1$
$g(v_7) := 3$
$g(v_8) := 4$
$g(v_9) := 4$
$g(v_{10}) := 5$
$g(v_{11}) := 2$
$g(v_{12}) := 6$
(3) $q := 6$
(4) $t := 12$
(5) $k := 11$
(6) $g(v_{11}) := 6$
(2) $g(v_{12}) := 2$
(3) $q := 6$
(4) $t := 11$
(5) $k := 10$
(6) $g(v_{10}) := 6$
(2) $g(v_{11}) := 2$
$g(v_{12}) := 5$
(3) $q := 6$
(4) $t := 10$
(5) $k := 9$
(6) $g(v_9) := 5$
(2) $g(v_{10}) := 4$
$g(v_{12}) := 5$
$f(v_{10}) := 3$
(3) $q := 5$

*According to above remark and Corollary 4.77, the algorithm can be stopped hier.*

*The following figure shows a total coloring of $G$, which is delivered from vertex coloring of $G_T$, as mentioned in Theorem 4.76.*




### 4.10.2 Total coloring through edge coloring

Another possibilty to determine a total colring of a graph is, doing that through edge coloring which is much easier. In this section we develope an algorithm which can be applied for total coloring a graph through edge coloring.

By the way of proof of Theorem 4.49 presented by Zhang, Zhang and Wang we can apply an edge coloring algorithm to find a total coloring of a graph $G$ (with in general more than $\chi''(G)$ colors).

**Algorithm 4.5**
Input: A graph $G$.
Output: A map $f : V(G) \cup E(G) \to \mathbf{N}$
(1) Let $u \notin V(G)$.
Construct function $f : V(G) \cup E(G) \to \mathbf{N}$,
put $E(H) = E(G) \cup \{uv | v \in V(G)\}$.
(2) Find an edge coloring of $H$ with $\chi'(H)$, $h : E(H) \to \{1, \ldots, \chi'(H)\}$.
(3) Define $f : V(G) \cup E(G) \to \mathbf{N}$ as follows:
$f(v) := h(uv)$ for $v \in V(G) \subset V(H)$,
$f(e) := h(e)$ for $e \in E(G) \subset E(H)$.

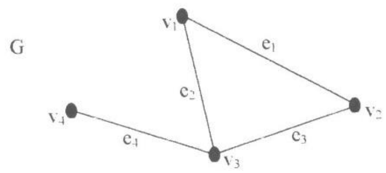
**Theorem 4.78** *The function $f : V(G) \cup E(G) \to \mathbf{N}$ obtained in Algorithm 4.5 is a total coloring of graph $G$ with $n(G)$ or $n(G) + 1$ colors.*

**Proof :**
(i) Since $h$ is an edge coloring $f(e) \neq f(e')$ for adjacent edges $e, e' \in E(G)$.
All edges $uv, v \in V(G)$ are adjacent in $H$, therefore $f(v) \neq f(v')$ for all vertices $v, v' \in V(G)$.
Since for all $v \in V(G)$ the edge $uv$ in $H$ is incident with all with $v$ incident edges $e \in E(G), f(v) \neq f(e)$ for $v \in V(G)$ and all with $v$ incident edges $e \in E(G)$.
(ii) We know $\chi'(H) \in \{\Delta(H), \Delta(H)+1\}$. But we have $\Delta(H) = n(G)$. Hence $max\{f(x)|x \in V(G) \cup E(G)\} \in \{n(G), n(G)+1\}$.

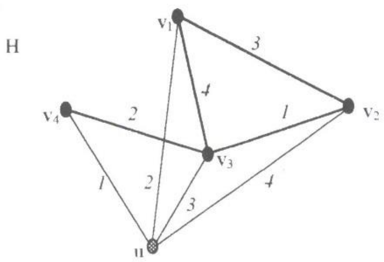
**Corollary 4.79** *Let $G$ be a graph with $\Delta(G) = n(G) - 1$, then $\chi''(G) \leq \Delta(G) + 2$.*

**Proof :**
From above theorem follows $\chi''(G) \leq n(G) + 1$.
Since $\Delta(G) = n(G) - 1$, we have $\chi''(G) \leq \Delta(G) + 2$.

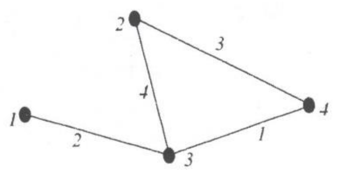
**Example 22** *Consider the following graph $G$. Here we find a total coloring of $G$ by applying the Algorithm 4.5.*




*Construct the graph $H$ according to Algorithm 4.5 and color its edges with $\chi'(H)$ colors. The following figure shows a 4-edge coloring of $H$, and we know $\chi'(H) \geq \Delta(H)$, therefore $\chi'(H) = 4$.*



*The total coloring of G obtained by Algorithm 4.5 step (3) is shown in following figure*




*Since the graph G satisfies the conditions of Theorem 4.52, we have* $\chi''(G) = 4$ *and G is colored with* $\chi''(G)$ *colors.*

### 4.10.3 Edge coloring through vertex coloring

As we mentioned earlier for every edge coloring of a graph $G$ there is a vertex coloring of $L(G)$ and vice versa. So we can complete the connections between different kind of graph colorings. To find an edge coloring of a graph $G$ we determine a vertex coloring of the line graph of $G$, $L(G)$, by known algorithm, e.g. Algorithm 4.4. Then this vertex coloring of $L(G)$ is the correspondent coloring to an edge coloring of $G$.

**Algorithm 4.6** Input: A graph $G$.
Output: A map $f : E(G) \to \mathbf{N}$
(1) put $V(H) = E(G)$
$Findavertexcoloringof$ H$with\chi'(H)$ colors, $h : V(H) \to \{1, \ldots, \chi'(H)\}$.
(3) Define $f : E(G) \to \mathbf{N}$ as follows:
$f(e) := h(uv)$ for $e = uv \in E(G) = V(H)$.

# 5 Defining sets in graph colorings

## 5.1 Defining set in vertex coloring of graphs

**Definition 5.1** *In a given graph $G$, a set of vertices $D_v$ with an assignment of colors is said to be a* ***defining set*** *of the vertex coloring of $G$, if there exists a unique extension of the colors $D_v$ to a $\chi(G)$-coloring of the vertices of $G$.*

**Definition 5.2** *A defining set with minimum cardinality is called* ***minimum*** *defining set (of vertex coloring) and its cardinality is denoted by $d_v(G)$.*

To emphasize the previous definitions, we consider the following example;

**Example 23** *Consider the path of length $n$, $P_n$, a vertex coloring for $P_n$ is given in the following figure.*


*The defining set of this coloring contains only one vertex, the circled one. With having the color of this single vertex we can give the color classes of $P_n$, at first we find the maximal independent set which contains this vertex, this is one color class, say $V_1$, then because of the fact $\chi(P_n) = 2$, $V_2 = V(P_n) - V_1$ is the second desired color class. So we could uniquely determine the color classes of $P_n$ with knowing the color of only one of its vertex. This means $d_v(P_n) = 1$*

**Proposition 5.1** *The minimum defining set of vertex coloring of $K_{m,n}$ has cardinality 1.*

On the defining set of vertex coloring, the following results are obtained. Proofs may be found in [24].

**Theorem 5.2** *For any graph $G$ we have,*

$$d_v(G) \geq |V(G)| - \frac{|E(G)|}{\chi(G)-1}.$$

**Corollary 5.3** *For an odd cycle of order $2n+1$, we have*

$$d_v(C_{2n+1}) = 2n + 1.$$

**Theorem 5.4** *If $G$ is the cartesian product of $K_2$ by $C_{2n+1}$ then, $d_v(G) = n + 1$*

**Theorem 5.5** *For any graph $G$ with $\chi(G) \leq n$, we have*

$$d_v(G \times K_n) \geq |V(G)|(n-1) - 2|E(G)|.$$

If in theorem 5.5 we let $G = K_m$, we obtain;

**Corollary 5.6**

$$d_v(K_m \times K_n) \geq m(n-m) \text{ for } n \geq m.$$

The following theorem shows that equality is possible in the above corollary.

**Theorem 5.7** *If $n \geq m^2$, then*

$$d_v(K_m \times K_n) = m(n-m).$$

**Theorem 5.8** *For $n \geq 6$ we have,*

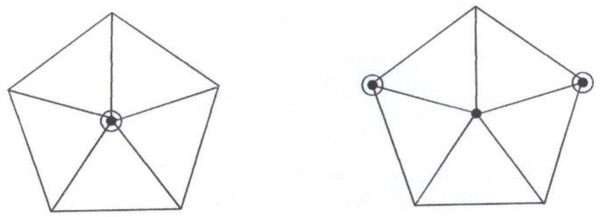
$$d_v(C_m \times K_n) = m(n-3).$$

### 5.1.1 Independent sets

**Definition 5.3** *A subset $I$ of $V$ is called an **independent** set of $G$ if no two vertices of $I$ are adjacent in $G$. An independent set $I$ is **maximum** if $G$ has no independent set $I'$ with $|I'| > |I|$. And $I$ is **maximal** when there is no other independent set that contains $I$.*

In this work we consider the maximal independent sets.

Examples of maximal independent sets are shown in the following figures:



### 5.1.2 Defining sets of maximal independent sets

Similar to defining set of vertex coloring one can define the defining set of a maximal (maximum) independent set of a graph.

**Definition 5.4** *A* ***maximal*** *independent set in a graph is an independent set which cannot be enlarged by adding any vertex.*

**Definition 5.5** *$D \subset V(G)$ is called a* ***defining set*** *of a maximal independent set $I$, if for every maximal independent set $I'$ with $D \subset I'$ it follows $I = I'$.*

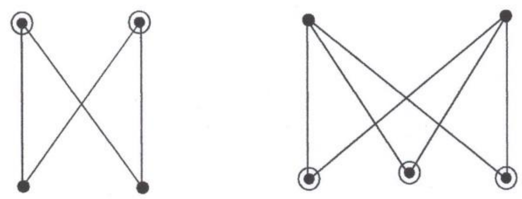
**Definition 5.6** *A defining set with minimum cardinality is called* ***minimum*** *defining set ( of a maximal independent set ) and its cardinality is denoted by $d_i(I)$.*

**Remark 5.1**
$d_i(P_n) = 1.$
$d_i(C_n) = 1.$

**Proof :**
The maximal independent set of a path $P_n$ is the set of vertices, in which from every two adjacent two vertices in $P_n$, exactly one is included.

The circled vertices construct a maximal independent set in $P_n$ (here n is even). To determine this maximal independent set uniquely, we need only one vertex.
We have the same construction for a maximal independent set in cycle, hence here the minimum defining set have also only one vertex.

**Remark 5.2** *Consider the bipartite graphs $K_{m,n}$, here we have only two maximal independent sets, the vertices in upper layer or in lower layer, hence $d_i(K_{m,n}) = 1$.*



The chromatic number of a graph is the least number of independent sets into which its vertex set can be partitioned. Because every independent set is a subset of a maximal independent set, it suffices to determine all the maximal independent sets in a graph in order to find a vertex coloring for it.

Moreover we can obtain the following:

**Theorem 5.9** *Let $D_j$ be a minimum defining set of maximal independent set $I_j$, $j = 1, \ldots, k$ of a graph $G$, with $\chi(G) = k$, so that $V(G) = \cup_{I_j} V(I_j)$.*
*The union of $D_j, j = 1, \ldots, k$ is a defining set of a vertex coloring of $G$ and $d_v(G) \leq \sum_{I_j} d_i(I_j) - min_{I_j} d_i(I_j)$.*

**Proof :**
Without lose of generality we can rearrange the indices of these sets such that $|D_1| \leq |D_2| \leq \ldots \leq |D_{k-1}| \leq |D_k|$. Now we determine maximal independent sets $I_1, \ldots I_{k-1}$ with help of its minimal defining sets.
Now consider $A = V(G) - \cup_{j \leq k-1} V(I_j)$. we claim that $A$ is a maximal independent set with defining set $D_k$. For, by the assumption $I_j, j = 1, \ldots, k$ is a partition of $V(G)$, this implies that $I_k = V(G) - \cup_{i \leq k-1} V(I_i)$, hence $I_k = A$. This partition

gives us a vertex coloring of G, and for this partition we used $\sum_{i\leq k-1}|D_i|$. But $d_v(G)$ is minimum number which determines the $\chi$-coloring of $G$, hence $d_v(G) \leq \sum_{I_j} d_i(I_j) - min_{I_j} d_i(I_j)$.
By considering the above theorem we can construct an algorithm, which gives a coloring of a graph with help of minimal defining sets of its maximal independent sets.

**Algorithm 5.1** *Input: A class of minimal defining sets $D_1, \dots, D_k$ in graph $G$ of maximal independent sets $I_1, \dots, I_k$ such that $V(G) = \cup_{I_j} V(I_j)$*
*Output: A vertex coloring of $G$*
*(1) Choose the $D_j$ with maximum cardinality,*
*(2) Determine the correspondent maximal independent set $I_j$ with help of its defining set $D_j$,*
*(3) If $j < k$ go to (1),*
*(4) else put $I_k = V(G) - \cup_{j\leq k-1} V(I_j)$*
*(5) Stop, $I_1, \dots, I_k$ are the color classes of a $k$-coloring of $G$.*

## 5.2 Defining set in edge coloring of graphs

***Definition 5.7*** *In a given graph $G$ a set of edges $D_e$ with an assignment of colors is said to be an* ***edge defining set****, or simply defining set, of edge coloring of $G$, if there exists a unique extension for the colors of $D_e$ to a $\chi'(G)$-edge-coloring of $G$.*

***Definition 5.8*** *A defining set with minimum cardinality is called a* ***minimum defining set*** *(of edge coloring) and its cardinality is denoted by $d_e(G)$.*

*To illustrate the previous definitions, we consider the following example;*

***Example 24*** *In the following figure an edge coloring of $P_6$ is given:*
*We are interested in finding a set of edges with color assignment which can be*


*extended to this edge coloring of $P_6$. The defining set of this edge coloring contains only one edge, whose color is circled. With having the color of this single edge we can give the color classes of $P_6$, at first we find the maximal matching which contains this edge, this is one color class, say $E_1$, then because of the fact $\chi'(P_n) = 2$, it* *means we have only two maximal matchings, hence $E_2 = E(P_n) - E_1$ is the second desired edge color class. So we could uniquely determine the edge color classes of $P_6$ with knowing the color of only one of its edge. This means $d_e(P_6) = 1$*

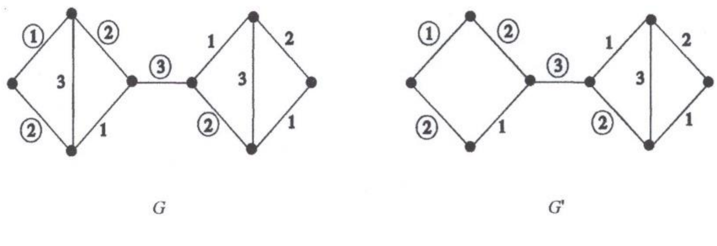
*The following results are obtained from the definition of the edge defining set:*

**Remark 5.3** 1. *For any graph $G$ the set $E(G)$ is obviously a defining set of any edge coloring of $G$. So $d_e(G)$ exists and $d_e(G) \geq 0$. In fact, $d_e(G) = 0$ if and only if either $G = K_2$ (we do not consider the null graph) or $G$ has no edges.*

2. *A minimum defining set of an edge coloring of a graph $G$ is not necessarily unique.*
3. $d_e(G) \geq \chi'(G) - 1$
4. *Let $e = uv$ be an edge of $G$ with edge degree $d$, i.e. the degree of $e$ in line graph $L(G)$ is $d = deg_G u + deg_G v - 2$. If $d < \chi'(G) - 1$, then $e$ is necessarily a member of any edge defining set of $G$. For, let $e$ be such an edge and not in the defining set, say $D_e$, assume $e$ has not colored yet and the other edges in $G$ have been colored. Since $d < \chi'(G) - 1$, we can color this edge at least in to ways, which is a contradiction to defenition of defining set.*

**Problem:** *Is it true that, if $G'$ be an arbitrary subgraph of $G \Rightarrow d_e(G') \leq d_e(G)$ ?*

*The answer is no.*
*The following example shows that, in general, $d_e(G') \leq d_e(G)$ when $G'$ is a subgraph of $G$, is not true; even if $\chi'(G) = \chi'(G')$.*




*In above figure the set of edges with circled color number in graph $G$ and $G'$ are defining sets for $G$ and respectively $G'$. So $3 = d_e(G) < d_e(G') = 5$, although $G'$ is a subgraph of $G$ and $\chi'(G) = \chi'(G') = 3$.*

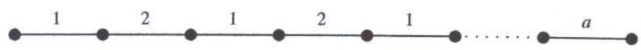
*The following definition and theorem due to Mahdian and Mahmoodian are very useful in some of our results.*

**Definition 5.9** *A graph $G$, with $v$ vertices has the property $M(2)$, if for any list of colors $S_1, S_2, \cdots, S_v$ ( $S_i$ is a list of colors available at vertex $i$ ), with $|S_i| \geq 2$; having a proper vertex coloring for $G$ implies that there exists also a different proper vertex coloring for $G$.*

**Theorem 5.10** *A graph has the property $M(2)$ if and only if every block of which is either a cycle, a complete graph, or a complete bipartite graph.*

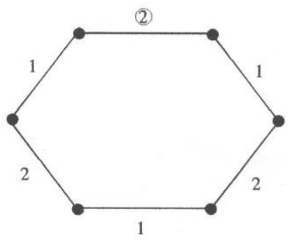
**Theorem 5.11**
$d_e(P_n) = 1$,
$d_e(C_{2n}) = 1$,
$d_e(C_{2n+1}) = n + 1$

**Proof :**
*Consider the paths of length $n$, $P_n$, we have $\chi'(P_n) = 2$,*




*If $n = 2k \Rightarrow a = 2$ and if $n = 2k + 1 \Rightarrow a = 1$.*
*Here we have the same argument as in Example 24. It means if we have the color of only one edge of $P_n$ then we can extend it to this edge coloring of $P_n$ uniquely.*

*Now consider the even cycles $C_{2n}$, we have $\chi'(C_{2n}) = 2$. It is easy to observe that $d_e(C_{2n}) = 1$. In the following figure, the edge with circled color number is an edge defining set for $C_6$.*




*But for odd cycles it is different,*

$\chi'(C_{2n+1}) = 3$.

*We claim that from every two adjacent edges in* $C_{2n+1}$*, one must be included in any defining set. For, assume that two adjacent edges have not been colored yet, and the other edges have been colored. The set of colors available at each of these edge is of size 2. By exchanging these colors we obtain two different edge colorings for* $C_{2n+1}$*. Hence,* $d_e(C_{2n+1}) = n + 1$

**Theorem 5.12** *For every simple graph* $G$ *we have*

$$d_e \geq |E(G)| - \frac{\sum_{v\in V(G)} \binom{d_G(v)}{2}}{\chi'(G)-1}$$

**Proof :**

*As we mentioned earlier every edge coloring of a graph* $G$ *is a vertex coloring of its line graph* $L(G)$*. By using Theorem 5.2 and considering that* $|V(L(G))| = |E(G)|$*, and the fact that* $|E(L(G))| = \sum_{v\in V(G)} \binom{d_G(v)}{2}$ *for line graphs, the assertion follows. In the following there are some results for edge defining set of trees:*

1. $\chi'(T) = \Delta(T)$,

2. $\Delta(T) = 1$ *then* $T$ *is* $K_2$*, and* $\chi'(T) = 1$ *and* $d_e = 0$,

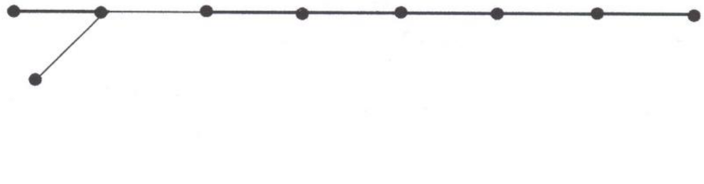
3. $\Delta(T) = 2$ *then* $T$ *is* $P_n$*, and* $\chi'(T) = 2$ *and* $d_e = 1$,

4. $\Delta(T) = n$ *and* $|V(T)| = n+1$ *then* $T$ *is star* $K_{1,n}$*, and* $\chi'(T) = n$ *and* $d_e = n-1$

**Theorem 5.13** *Let* $T$ *be a tree with* $\Delta(T) = 3$, $|V(T)| = n+1 \geq 4$ *and* $T$ *contains only one vertex of degree* 3 *then* $d_e = n - 2$.

**Proof :**

*It is clear that* $\chi'(T) = 3$*, and* $T$ *is* $K_{1,3} + P_{n-4}$ *as shown in following figure:*



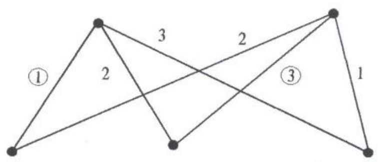
*We have $|E(T)| = n$, now we claim that among the edges of $T$ at least $n-2$ of them must belong to any edge defining set. For, assume that 3 of the edges have not been colored yet, and other edges have been colored. Because of the form of our tree at least two of these edges have edge degree 1 and the set of colors availabe at each of these edges has size 2. By Theorem 5.10 we obtain two different edge coloring for $T$. Therefore $d_e(T) \geq n-2$. In the above figure the set of edges marked bold forms a defining set, this gives the equality.*

$d_e(T) = \lceil \frac{n}{2} + 1 \rceil$.

***Example 25***

*We have in general $\chi'(K_{m,n}) = max\{m, n\}$, in the following an edge coloring for $K_{2,3}$ is given:*

$\chi'(K_{2,3}) = 3$ *and* $d_e(K_{2,3}) = 2$




***Theorem 5.14*** *For $n \geq m$,*

$d_e(K_{m,n}) \geq m(n-m)$

**Proof :**

*As we mentioned earlier, $\chi(G_L) = \chi'(G)$, where $G_L$ is the line graph of $G$. And we have also $L(K_{m,n}) \simeq K_m \times K_n$.*

*Hence by applying the Theorem 5.6 we obtain:*

$d_e(K_{m,n}) = d_v(K_m \times K_n) \geq m(n-m)$ *for* $n \geq m$

***Theorem 5.15*** *If $n \geq m^2$,then*

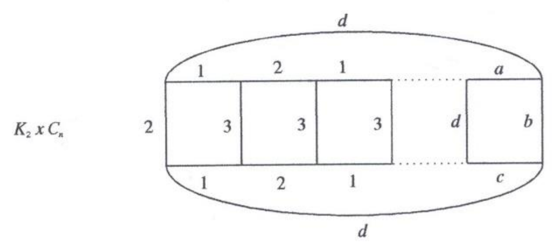
$d_e(K_{m,n}) = d_v(K_m \times K_n) = m(n-m)$.

**Proof :**
*With the same argumentation as in previous theorem and applying Theorem 5.7, the statement will be obtained.*

***Theorem 5.16*** *The size of minimum edeg defining set of $K_2 \times C_n$ is*

$$d_e(K_2 \times C_n) = [\tfrac{n}{2}]$$

**Proof :**
*Let $G = K_2 \times C_n$, in the following an edge coloring of $G$ is given:*




*if $n = 2k \Rightarrow \{a = c = 1, b = 2, d = 3\}$ and if $n = 2k+1 \Rightarrow \{a = c = 2, b = 1, d = 3\}$. $\chi'(K_2 \times C_n) = 3$.*
*Consider the paths $P_3$ which have one edge in upper copy of $C_n$, one edge in lower copy of $C_n$, and the other edge is a copy of $K_2$. We claim that from every two consecutive paths with this construction, two edges of one of them must be included in any defining set. For, assume that $P_1 = v_1v_2v_3v_4$ and $P_2 = u_1u_2u_3u_4$ be two such paths which have not been colored yet and the other edges have been colored. Because of the form of these paths $u_2 = v_1$ and $u_3 = v_2$. Now it is not difficult to see that the sets of availabe colors at each of the edges of $P_1$ and $P_2$ have the size 2, and by Theorem 5.10 we obtain two different edge colorings for $G$.t Since $K_2 \times C_n$ has $[\frac{n}{2}]$ such paths, therefore $d_e(K_2 \times C_n) = [\frac{n}{2}]$. To show the equality, we give a edge defining set of size $[\frac{n}{2}]$ in following picture (for odd $n$).*

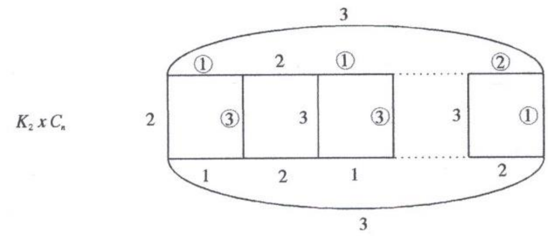



Figure 2: A defining set for $K_2 \times C_n$

*Similar to Theorem 5.9 for maximal independent sets, we can formulate the following theorem;*

***Theorem 5.17*** *Let $D_j$ be a minimum defining set of maximal matching $M_j$, $j = 1, \ldots, k$ of a graph $G$, with $\chi'(G) = k$, so that $E(G) = \cup_{M_j} E(M_j)$. The union of $D_j, j = 1, \ldots, k$ is a defining set of an edge coloring of $G$ and*
$d_e(G) \leq \sum_{M_j} d_m(M_j) - min_{M_j} d_m(M_j)$.

**Proof :**
*Similar to the proof of Theorem 5.9.*

*Here we can introduce a similar algorithm to find an edge coloring of $G$ with help of defining sets of maximal matching.*

***Algorithm 5.2*** *Input: A class of minimal defining sets $D_1, \ldots, D_k$ in graph $G$ of maximal matchings $M_1, \ldots, M_k$ such that $E(G) = \cup_M V(M)$*
*Output: An edge coloring of $G$*
*(1) Choose the $D_i$ with maximum cardinality,*
*(2) Determine the correspondent maximal matching $M_i$ with help of its defining set $D_i$,*
*(3) If $i < k$ go to (1),*
*(4) else put $M_k = E(G) - \cup_{i \leq k-1} E(M_i)$*
*(5) Stop, $M_1, \ldots, M_k$ are the color classes of a $k$-edge-coloring of $G$.*

***Theorem 5.18*** *Every edge-coloring of $K_{2n-1}$ can be extended to an edge-coloring for $K_{2n}$.*

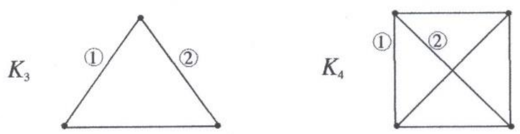
**Proof :**
*We consider an arbitrary edge coloring of $K_{2n-1}$ we have $\chi'(k_{2n-1}) = 2n-1$ and $d_G(v) = 2n-2 \quad \forall v \in E(K_{2n-1})$. Hence for each $i$, $1 \leq i \leq 2n-1$, there is one color of the given $2n-1$ colors which is not allocated to the edge with the end point $v_i$. Now we claim that these colors are distant. For, assume for example color 1 in vertices u and v is not allocated, then the number of all not allocated colors in other vertices is $2n-3$ and the total number of such colors is $2n-2$. It means there is a color, for example, color $2n-1$ which is allocated to an edge for any end point. Therefore we have $\frac{2n-1}{2}$ edges with $2n-1$ colors, which is not possible, because the number of vertices is odd. Now we consider graph $K_{2n-1} + \{v_{2n}\}$ which is $K_{2n}$ and color each edge $v_{2n}v_i$ with missing color at $v_i$, this gives an edge coloring for $K_{2n}$.*

***Theorem 5.19*** *Every edge defining set for $K_{2n-1}$ is an edge defining set for $K_{2n}$, as well.*

**Proof :** *Let $D$ be an edge defining set for $K_{2n-1}$. By definition of $D$ we can extend it uniquely to an edge coloring of $K_{2n-1}$. Now by previous theorem we can extend this edge coloring to an edge coloring of $K_{2n}$ uniquly. This means $D$ is a defining set for $K_{2n}$, too.*

***Definition 5.10*** *A **wheel** of length n, denoted by $W_n$, is a graph obtained from $C_n$ by adding a new vertex and edges joining it to all the vertices of the cycle.*

***Example 26*** *Consider $K_3$. It is obvious that $d_e(K_3) = 2$, then by Theorem 5.19 we have $d_e(K_4) = 2$, too.*
*In the following picture the minimum edge defining set of $K_3$ and $K_4$ are given:*




**Proof** :
*By applying Theorem 5.19 and previous example the assertion follows.*

***Theorem 5.20*** *$d_e(W_n) = 2n - 3$ for $n \geq 3$, for all $n \geq 3$.*

**Proof :** *It can be easily seen that $\chi'(W_n) = n$, for all $n$. As it is earlier pointed out, if $e = uv$ be an edge with degree $d = deg_G u + deg_G v - 2$ in $G$ and $d < \chi'(G) - 1$, then $e$ is necessarily a member of any edge defining set of $G$. Now consider the line graph of $W_n$, it contains the subgraph $C_n$ and each line on $C_n$ has degree less than $\chi'(G)$, then every edge on $C_n$ must belong to any defining set and there are $n$ such edges. Now we claim that among $n$ remaining edges of $W_n$ at least $n - 3$ edges must belong to any edge defining set of $W_n$. For, assume that four of these edges have not been colored yet, and the other edges of $W_n$ have been colored. These 4 edges constitute a complete graph $K_4$ in line graph of $W_n$. And the set of colors available at each of its vertices are of size 2 each. By Theorem 5.10 we obtain two different edge colorings for $W_n$. Therefore $d(W_n, \chi') \geq 2n - 3$. To show the equality, we give an edge defining set of size $2n - 3$ in the following figure.*

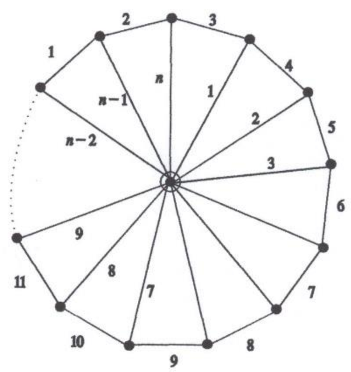



Figure 3: A minimum edge defining set for $W_n$

***Remark 5.4*** *For $n = 3$, $W_3 = K_4$, and $\chi'(K_4) = 3$, in this case $d_e(W_3) = 2$.*

*Here we study the relation between critical sets for latin squares and defining set of edge coloring of complete bipartite graphs.*

### 5.2.1 Latin squares

*Latin squares arise in Eulers "thirty-six officers" problem. The concept of arranging numbers that satisfy certain properties into arrays is ages old. Well-known examples of this practice include group tables, matrices and Magic Squares. In 1779, Euler posed a recreational problem, involving the arrangement of 36 particular military officers into* $6 \times 6$ *array. the officers were chosen from six distinct regiments with precisely six officers selected from each. Further, the set of six officers from each regiment represented six distinct ranks from a given set of six ranks. For each officer, Euler used Latin letters to represent the officer's regiment and Greek letters for the officer's rank. Eulers problem was to arrange these 36 officers into a* $6 \times 6$ *array so that each row and column contained one officer from each regiment, each of different rank. This type of array was hence referred to as a Graeco-Latin square. Euler found a solution for the equivalent problem on 16 officers ( four each from four different regiments and four distinct ranks ) and conjectured that the problem of the 36 officers was impossible to solve.*

*In Eulers problem, both Latin and Greek letters were used in the array. An array containing just the latin letters was often referred to at the time as an*

***Eulerian*** *square. This array is now more commonly known as a* ***Latin*** *square.*

***Definition 5.11*** *A* ***Latin square*** *of order* $n$ *is an* $n \times n$ *array or matrix with entries taken from the set* $\{1, \ldots, n\}$, *with the property that each entry occurs exactly once in each row and column.*

*For convenience, a Latin square will sometimes be represented as a set of ordered triples* $(i, j; k)$, *this is read to mean that element* $k$ *occurs in cell* $(i, j)$ *of the Latin square.*

***Definition 5.12*** *For* $m < n$, *a* ***Latin rectangle*** *of order* $m \times n$, *is an* $m \times n$ *array with entries chosen from* $\{1, \ldots, n\}$, *having the property that each entry occurs exactly once in each row and at most once in each column.*

***Definition 5.13*** *For* $m \leq n$, *a* ***back circulant*** *latin rectangle of order* $m \times n$, *is an* $m \times n$ *array with entries chosen from* $\{1, \ldots, n\}$, *having the property that each entry in the position* $(i, j) \equiv i + j - 1 \pmod{n}$.

***Definition 5.14*** *Let* $L$ *be a Latin square of order* $n$. *If* $n - s$ *rows of* $L$ *can be deleted, and* $n - s$ *columns of* $L$ *can be deleted to leave* $s^2$ *elements of* $L$ *which form a Latin square* $S$ *of order* $s$ *then* $S$ *is Latin* ***subsquare****, or simply subsquare of* $L$.

***Definition 5.15*** *A* ***partial*** *Latin square* $P$ *of order* $n$ *is an* $n \times n$ *array with entries chosen from a set* $N$ *of size* $n$, *such that each element of* $N$ *occurs at most once in each row and column.*

***Definition 5.16*** *A partial Latin square P of order n, is said to be* ***uniquely completable*** *(or P has (UC)) if for given set of possible entries, N, there is one and only one Latin square , L, of order n which has element k in position $(i,j)$ for each $(i,j;k) \in P$.*

*It is sometimes said that P completes uniquely to L.*

***Definition 5.17*** *A critical set in a Latin square L is a partial square which has a unique completion to L and all proper subsets of the partial Latin square complete to at least two distinct latin squares. Formally, a critical set, in a Latin square L of order n is a set $C = \{(i,j;k) \quad | \, i,j \in \{1,\ldots,n\}$ and $k \in N\}$ such that,*

1. *L is the only Latin square of order n which has element k in position $(i,j)$, for each $(i,j;k) \in C$;*
2. *no proper subset of C satisfies 1.*

***Example 27*** *Consider $C_6$, the back circulant latin square of order 6, and let: $E_6 = \{(0,0;0),(0,1;1),(0,2;2),(1,0;1),(1,1;2),(2,0;2),(4,5;3),(5,4;3),(5,5;4)\}$. The set $E_6$ is a critical set in $C_6$. The latin square $C_6$ and the critical set $E_6$ are as follows.*

$$C_6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 0 & 1 & 2 & * & * & * \\ 1 & 2 & * & * & * & * \\ 2 & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & 3 \\ * & * & * & * & 3 & 4 \end{pmatrix}$$

*Colbourn, Colbourn and Stinson [10] make the observation that the although the recognition of critical sets in "special cases" (where the unique completion of the partial latin square is relatively easy to verify) is "straightforward", it is "not the case in general". They prove that "deciding whether a partial latin square has more than one completion" is NP- complete, even if one completion is given as part of the problem description.*

***Lemma 5.21*** *(Curran and van Rees [12]) The set*
$E_n = \{(i, j; i + j) | i = 0, \ldots, \frac{n}{2} - 1 \text{ and } j = 0, \ldots, \frac{n}{2} - 1 - i\} \cup$
$\{(i, j; i + j) | i = \frac{n}{2} + 1, \ldots, n - 1 \text{ and } j = (\frac{n}{2} - i)(mod n), \ldots, n - 1\}$
*of cardinality* $\frac{n^2}{4}$*, is a critical set in the back circulant latin square of even order* $n$.

*The set in Example 27 is just such a critical set in the back circulant latin square of order 6.*

*Curran and van Rees actually showed that a certain partial latin square which is isotopic to* $E_n$ *is a critical set in* $C_n$ *for* $n$ *even.*

***Definition 5.18*** *Two latin squares* $L$ *and* $M$ *(both of order* $n$*) are said to be isotopic if there exists an ordered triple* $(\phi, \psi, \chi)$ *of one-to-one mappings such that map the rows, columns and entries, respectively, of* $L$ *onto* $M$.

*Then* $M = \{(r\phi, c\psi; e\chi) | (r, c; e) \in L\}$.
*That is, two latin squares are* ***isotopic*** *if one can be transformed into the other by rearranging rows, rearranging columns and renaming entries.*

***Definition 5.19*** *The two latin squares are said to be* ***isomorphic*** *if the one-to-one mappings* $\phi, \psi$, *and* $\chi$ *are equal.*

*One interesting application of critical sets of latin squares is in secret sharing schemes;*

*In information based systems, the integrity of the information is commonly provided for by requiring that certain operations can be carried out only by one or more participants who have access rights. Access is gained through a secure key , password or token, and governed by a secure management scheme. Shared security systems are used in financial institutions, in communication networks, in computing systems serving educational institutions and distribution environments. However, the best known examples of applications of shared security systems are in the military: for instance, in activating a nuclear weapon, several senior officers must concur before the necessary password can be reconstructed. A secret sharing scheme is a method whereby n pieces of information called shares or shadows are assigned to a secret key* $K$. *The shares have the property that certain authorized groups of shares can be used to reconstruct the secret key. The secret cannot be reconstructed from an unauthorized group of shares. The problem of finding critical sets with minimum cardinality in latin squares is the similar concept of finding defining set for edge coloring of some special graphs.*

***Remark 5.5*** *It is easy to check that an* $m \times n$ *latin rectangle ,* $m \leq n$*, is equivalent to a vertex coloring of the graph* $K_m \times K_n$.

*As we mentioned earlier:*
$L(K_{m,n}) \simeq K_m \times K_n$, *hence the concept of critical sets in latin squares is the same as minimum defining set for edge coloring of* $K_{m,n}$. *Now by using the results in [25] we obtain the following results:*

***Remark 5.6*** *If* $n \geq m^2$, *then*

$$d_e(K_{m,n}) = m(n-m)$$

***Theorem 5.22*** *Let* $L$ *be an* $m \times n$ *back circulant latin rectangle, where* $2m \leq n$. *Then* $L$ *contains a critical set of size* $m(n-m) + \lfloor \frac{(m-1)^2}{4} \rfloor$, *which is the smallest critical set for such a latin rectangle.*

***Corollary 5.23*** *Consider bipartite graph* $K_{m,n}$, *where* $2m \leq n$. *Then* $K_{m,n}$ *has a minimum defining edge set of size* $m(n-m) + \lfloor \frac{(m-1)^2}{4} \rfloor$, *for a corresponding edge coloring of* $K_{m,n}$.

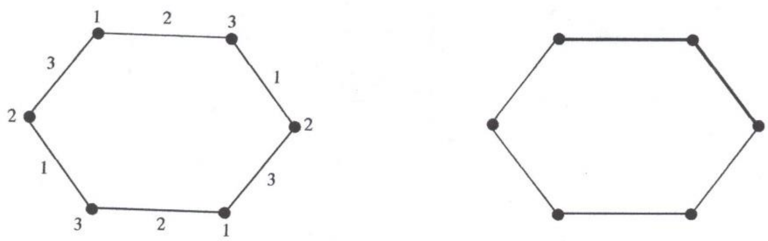
## 5.3 Defining set in total coloring of graphs

***Definition 5.20*** *In a given graph* $G$, *a set of element (vertices and edges)* $D_t$ *with an assignment of colors is said to be a **defining set** of the total coloring of* $G$, *if there exists a unique extension of the colors* $D_t$ *to a* $\chi''(G)$*-coloring of the elements of* $G$.

***Definition 5.21*** *A defining set with minimum cardinality is called **minimum total defining set** and its cardinality is denoted by* $d_t(G)$.

*To emphasize the previous definitions, we consider the following example;*

***Example 28*** *Consider the cycle* $C_6$, $\chi''(C_6) = 3$. *In following picture left there is a total coloring of* $C_6$ *and in the right side the elements marked bold give a minimum total defining set for* $C_6$



*On the defining sets of total coloring, the following results are obtained. Proofs may be found in [4].*

***Theorem 5.24***
*(i)* $d_t(C_n) = 2$ *if* $n \equiv 0 \pmod 3$
*(ii)* $d_t(C_n) = \lceil \frac{2n}{3} \rceil$ *if* $n \equiv 2 \pmod 3$
*(iii)* $d_t(C_n) = \lceil \frac{2n}{3} \rceil + 1$ *if* $n \equiv 1 \pmod 3$

***Theorem 5.25*** *Every total defining set for the graph* $K_{2n}$, $n \in N$, *is a total defining set for graph* $K_{2n+1}$ *as well.*

***Theorem 5.26*** *For* $n \geq 6$, $d_t(W_n) = 3n - 4$.

# 6 Cliques

*A concept which is opposite of that of maximal (maximum) independent set is that of maximal (maximum) clique.*

***Definition 6.1*** *A subset $C$ of $V$ is called a* **clique** *of $G$ if every two vertices of $C$ are adjacent in $G$. A clique $C$ is* **maximum** *if $G$ has no clique $C'$ with $|C'| > |C|$. And $C$ is* **maximal** *when there is no other clique that contains $C$.*

*In contrast to a maximal (maximum) independent set for which no two vertices are adjacent, the set of vertices of a clique are all adjacent to each other. It is quite obvious, that the maximal (maximum) independent set of a graph $G$ corresponds to a maximal (maximum) clique of the graph $\overline{G}$ and vice versa, where $\overline{G}$ is the graph complementary of $G$.*

***Remark 6.1*** *The maximal (maximum) independent sets of complete graph, $K_n$ which is a clique has only one vertex and*
$d_i(K_n) = 1$

## 6.1 Defining sets of cliques

*It is obvious that every graph $G$ contains at least one maximal (maximum) clique.*

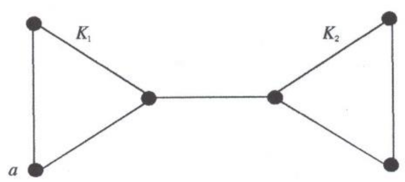
***Definition 6.2*** *A subset $D$ of vertices of a maximal (maximum) clique $C$ of $G$, is called a* **defining set** *if for all cliques $L$ with $D \subset L$, then $L = C$.*

***Definition 6.3*** *A defining set with minimum cardinality is called* **minimum** *defining set and its cardinality is denoted by $d_c(G)$.*

## 6.2 General examples

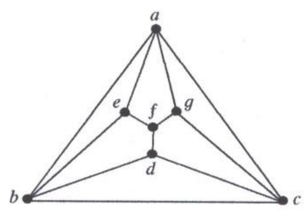
*In the following some examples are given to illustrate the perevious definitions and help us to have an idea for behaviours of maximal (maximum) cliques. :*

***Example 29***



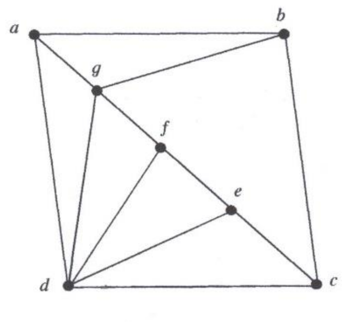
*Here the cliques $K_1$ and $K_2$ are maximal and at the same time they are maximum cliques. The defining sets of $K_1$ and $K_2$ have only one vertex each, say $a$ and $b$. So $d(G,c) = 1$.*

***Example 30***



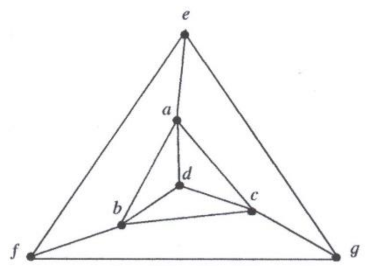

*For inside maximum cliques, "abcd" we have $d(G,c) = 1$, because only the vertex $d$ determines it. for maximal clique "$efg$" we need also only one vertex to determine it.*

***Example 31***



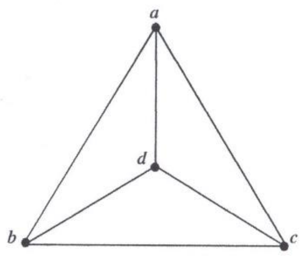

*Here we have $d(G,c) = 2$.*

***Example 32***



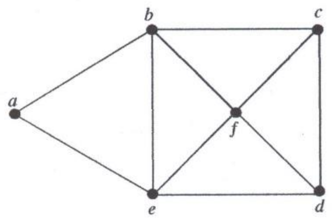

*Here for inside maximal (maximum) cliques, e.g. "aeb","agc" we need only one vertex to determine them and for "abc" we need all of its vertices. So* $d(G,c) = 1$.

***Example 33***



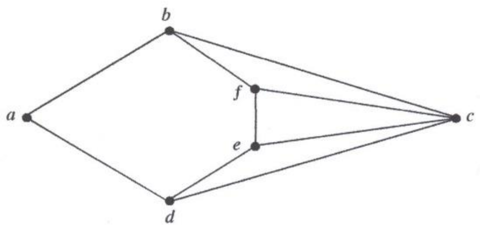

*Here we have* $d(K_4, c) = 0$.

***Example 34***



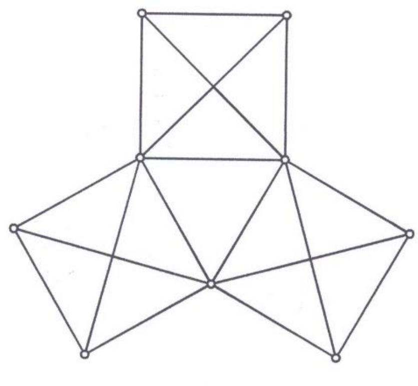
*The defining set of the maximal clique "abe" has only one vertex a and the defining set of all other maximal (maximum) cliques has two vertices. Hence* $d(G,c) = 1$.

***Example 35***



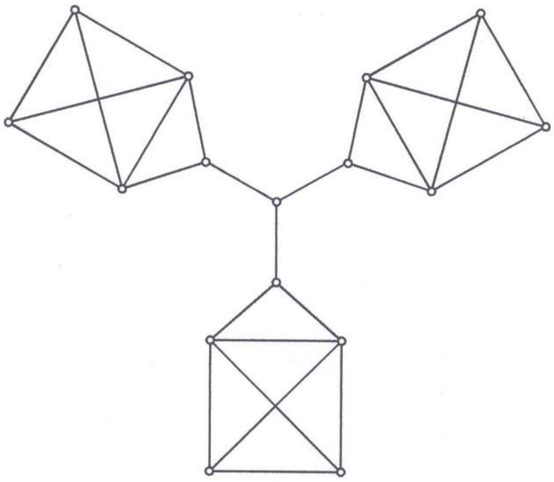

*Here we have* $d(G,c) = 1$.

***Example 36***



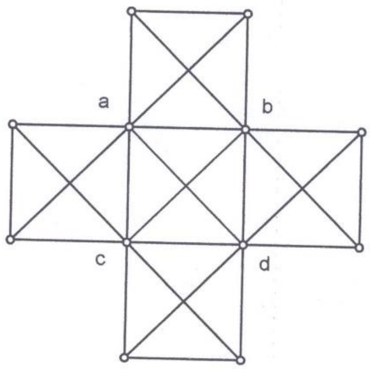
*Here we need only one vertex to determine each maximum clique* $K_4$, *and for the maximal clique "abc" we need all its vertices.*
$d(G,c) = 1$

***Example 37***



$d(G,c) = 1$

***Example 38***

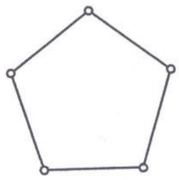


*The maximal (maximum) clique in the middle has one vertex in common with any other maximal cliques, so its defining set has three elements. For all other maximal (maximum) $K_4$ in $G$ we need only one of vertex for defining set and so $d(G, c) = 1$.*

## 6.3 Examples of Ramsey Graphs

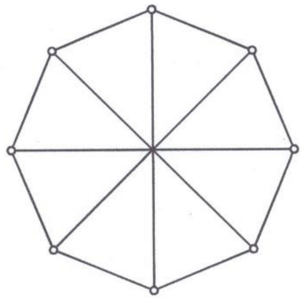
*In the following we consider some Ramsey graphs:*
*A $(k, l)$-Ramsey graph on $r(k, l) - 1$ vertices is a graph that contains neither a clique of $k$ vertices nor an independent set of $l$ vertices, where $r(k, l)$ is the smallest integer such that every graph on $r(k, l)$ vertices contains either a clique of $k$ vertices or an independent set of $l$ vertices.*
*For example it is easy to see that*
$r(1, l) = r(k, 1) = 1$, $r(2, l) = l$ *and* $r(k, 2) = k$
*By definition of $r(k, l)$, for any $k \geq 2$ and $l \geq 2$, there exists a $r(k, l)$- Ramsey graph.*

***Example 39***
*Consider the $r(3, 3)$-Ramsey graph which is $C_5$:*



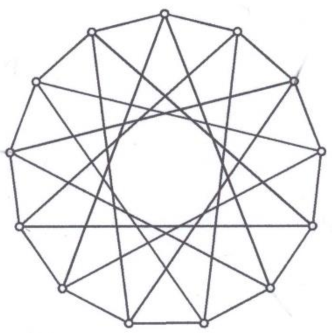
*Here the maximal cliques are edges, therefore every maximal clique has two vertices in its defining set and $d(r(3, 3), c) = 2$.*

***Example 40***
*Consider $r(3, 4)$-Ramsey graph:*



*Here the maximal cliques are, like in previous example, edges and therefore* $d(G, c) = 2$.

***Example 41***
*In ramsey graph of* $r(3, 5)$*:*



$d(G, c) = 2$.

***Example 42***
*In* $r(4, 4)$*-Ramsey graph the maximal (maximum) cliques are triangles, and because every* $C_3$ *has at least one vertex in common with another maximal (maximum) clique, the defining set of each clique has two vertices:*
$d(G, c) = 2$.

***Remark 6.2*** *By observation of these examples we obtain that if a maximal (maximum) clique has at least one vertex which is not in common with the others, then* $d(G, c) = 1$.

## 6.4 Finding a maximal (maximum) clique by its defining set

*The problem of generating all maximal (maximum) cliques ( maximal (maximum) independent sets ) of a given graph is fundemental in graph theory and is also one of the most important in terms of the application of graph theory. Here, we present an algorithm to find a certain maximal (maximum) clique ( maximal (maximum) independent set ) with help of its defining set.*

*The following algorithm can find the maximal (maximum) independent set given by its defining set $D$, in the way that we find the maximal (maximum) clique $C$ in $\overline{G}$, whose defining set is $D$. Then $C$ is the maximal (maximum) independent set in $G$.*

*In the following algorithm the set of all neighbours of a vertex $v_0$ is denoted by $A(v_0)$, i.e. the adjacency set which is defined:*
$A(v_0) = \{u \in V(G) | uv_0 \in E(G)\}$

***Algorithm 6.1***
*Input: A graph $G$ and a subset $D = \{v_0\} \subset V(G)$.*
*Output: A maximal clique $C$, whose defining set is $D$.*
*Step 1: Find the adjacency set of $v_0$, i.e. $A(v_0)$,*
*Step 2: Put $K := A(v_0)$,*
*Step 3: In this step we give label to vertices in $K$, which is defined as follows:*
*for all $u \in K$, $\quad l(u) := \#$ of disconnected vertices to $u$ in $K - \{u\}$*
*Step 4: If $l(u) = 0$ for all $u \in K$, then STOP*
*Step 5: Remove the vertex $v \in K$ with maximum label from $K$. Go to 3.*

*The induced subgraph of $G$ on $K \cup D$ ,$C$, is the desired maximal clique. It is a clique because by our algorithm all vertices in $K$ are adjacent to all vertices in $D$ and by definition of defining set, $C$ is the unique maximal clique determined by $D$.*

***Theorem 6.1*** *Let $D$ be a defining set of maximal clique $C$ consists of vertices. Then in step labeling of Algorithm 6.1, $K$ contains no vertex $u \in C$ with maximum label.*

**Proof :** *Here $|D| = 1$, $D = \{v_0\}$. It is clear that $v_0$ is on no other clique, by definition of defining set. Therefore $A(v_0)$ contains only the vertices, which are on the desired maximal clique or the vertices which are not connected together and not to all vertices on the maximal clique. Hence the maximum label comes from them.*

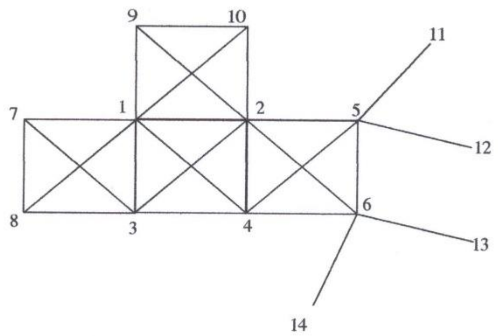
*For the case that $|D| > 1$ we use the following algorithm:*

***Algorithm 6.2***
*Input: A graph $G$ and a subset $D \subset V(G)$.*
*Output: A maximal clique $C$, whose defining set is $D$.*
*Step 1: Find the the vertex $v_0 \in D$ with minimum degree,*
*Step 2: Put $K := A(v_0) - D$,*
*Step 3: If $K = \emptyset$, then STOP*

*Step 4: Check the adjacency between all vertices $u \in A(v_0)$ and $v \in D - \{v_0\}$, if $uv \notin E(G)$, then $K := K - \{u\}$,*
*Step 5: (labeling)*
*for all $u \in K$, $l(u) := \#$ of disconnected vertices to $u$ in $K - \{u\}$*
*Step 6: If $l(u) = 0$ for all $u \in K$, then STOP*
*Step 7: Remove the vertex $v \in K$ with maximum label from $K$. Go to 5.*

*By the same argumentation as in Algorithm 6.1 the induced subgraph of $G$ on $K \cup D$ is the desired maximal clique.*

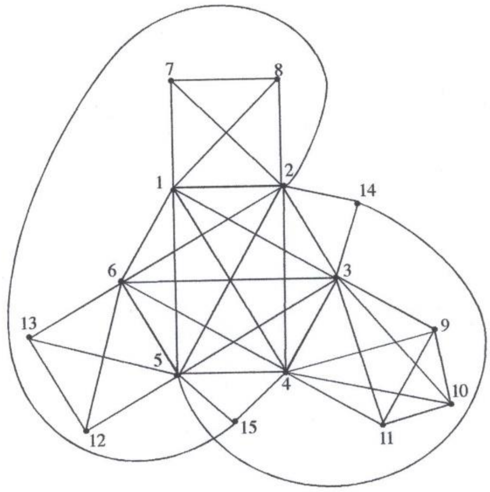
***Example 43***
*Consider the following graph $G$ and the defining set $D = \{v_6\}$. We apply the Algorithm 6.1 to find the maximal clique, whose defining set is $D$:*




*Step 1: $A(v_6) = \{v_2, v_4, v_5, v_{14}, v_{15}\}$*
*Step 2: $K = A(v_6)$*
*Step 3:*
$l(v_2) = 2,$
$l(v_4) = 2,$
$l(v_5) = 2,$
$l(v_{13}) = 3,$
$l(v_{14}) = 3,$
*Step 4: $l(u) \neq 0$ for some $u \in K \Rightarrow$ Continue,*
*Step 5: $K := K - \{v_{13}\} = \{v_2, v_4, v_5, v_{14}\}$,*
*Step 3:*
$l(v_2) = 1,$
$l(v_4) = 1,$

$l(v_5) = 1$,
$l(v_{14}) = 3$,
*Step 5:* $K := K - \{v_{14}\} = \{v_2, v_4, v_5\}$,
*Step 3:* $l(v_2) = l(v_4) = l(v_5) = 0$,
*Step 4:* *STOP*
$G|_{\{v_2,v_4,v_5,v_6\}}$ *is the desired maximal clique with defining set* $\{v_6\}$.

***Example 44***
*The graph G and the subset* $D = \{v_2, v_5\}$ *are given. Find the maximal clique with defining set D.*




*Here we apply the Algorithm 6.2:*
*Step 1:* $v_0 := v_2$,
*Step 2:* $K := \{v_1, v_3, v_4, v_5, v_6, v_7, v_8, v_{14}, v_{15}\}$ - $\{v_2, v_5\}$
$=\{v_1, v_3, v_4, v_6, v_7, v_8, v_{14}, v_{15}\}$
*Step 3:* $K \neq \emptyset \Rightarrow$ *Continue,*
*Step 4:* $v_7v_5 \notin E(G) \Rightarrow K := K - \{v_7\}$
$v_8v_5 \notin E(G) \Rightarrow K := K - \{v_8\}$
$K := \{v_1, v_3, v_4, v_6, v_{14}, v_{15}\}$
*Step 5:*
$l(v_1) = 2$
$l(v_3) = 1$
$l(v_4) = 1$

$l(v_6) = 2$
$l(v_{14}) = 4$
$l(v_{15}) = 4$
*Step 6:* $l(u) \neq 0$ *for some* $u \in K \Rightarrow$ *Continue,*
*Step 7:* $K := K - \{v_{15}\} = \{v_1, v_3, v_4, v_6, v_{14}\}$,
*Step 5:*
$l(v_1) = 1$
$l(v_3) = 0$
$l(v_4) = 1$
$l(v_6) = 1$
$l(v_{14}) = 3$
*Step 6:* $l(u) \neq 0$ *for some* $u \in K \Rightarrow$ *Continue,*
*Step 7:* $K := K - \{v_{14}\} = \{v_1, v_3, v_4, v_6\}$,
*Step 5:* $l(v_1) = l(v_3) = l(v_4) = l(v_6) = 0$,
*Step 6: STOP*
$G|_{\{v_1, v2, v_3, v_4, v_5, v_6)\}}$ *is the desired maximal clique with defining set* $\{v_2, v_5\}$.

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