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An Intersection-Theoretic Approach to Correspondence Problems in Tropical Geometry

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For Mary

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Contents

Introduction

The process of tropicalization transforms algebro-geometric objects into polyhedral objects. For example, the tropicalization of a smooth marked algebraic curve over a valued field is a marked metric graph, obtained by taking the dual graph of the special fiber of the stable reduction of the curve. From the perspective of enumerative geometry, the fact that tropicalizations of curves are 1-dimensional again, tropical curves so to speak, hints at a relation between algebraic and tropical enumerative problems. More precisely, we can ask what happens if we replace the well-studied problem of counting algebraic plane curves of given degree and genus through an appropriate number of generic points by the problem of counting tropical plane curves of the same degree and genus through the same number of generic points. This was answered by Mikhalkin in his well-known correspondence theorem [Mik05], which states that the solutions of the two corresponding problems are in fact equal. Note, however, that in the tropical problem we have to count curves with certain multiplicities. The analogous result is also known for rational curves in arbitrary dimension [NS06].

In genus greater than 0 and dimension greater than 2 several problems occur. First of all, the moduli spaces of algebraic curves usually have larger dimension than expected in this case. Secondly, there exist so-called superabundant combinatorial types of tropical curves, that is combinatorial types for which the dimension of the corresponding stratum in the moduli space of tropical curves is too large. Finally, not every tropical curve is realizable, that is equal to the tropicalization of an algebraic curve [Spe14]. The first problem can be solved by considering Gromov-Witten invariants instead of actually counting algebraic curves. This is also the preferred solution to the second problem. Unfortunately, tropical Gromov-Witten theory is still in its infancy. Moduli spaces of tropical stable maps are easily constructed as abstract polyhedral complexes, yet they rarely have enough structure to allow for a reasonable intersection theory. In the genus 0 case, however, they do admit an embedding as a tropical cycle into some $\mathbb{R}^N$ [SS04, GKM09] and therefore support the tropical intersection theory developed in [AR10, FR13, Sha13]. This allows to define tropical genus 0 descendant Gromov-Witten invariants [Rau08, KM09, MR09]. These are equal to the genus 0 logarithmic descendant Gromov-Witten invariants of toric varieties, as we will see later.

The goal of this thesis is to increase our understanding of the classical/tropical correspondence theorems by relating the algebraic and tropical Gromov-Witten theories. The difference to other approaches as for instance in [NS06] is that instead of tropicalizing single curves and then studying the fibers of the tropicalization, we tropicalize a suitable moduli space and then relate its intersection theory to that of its tropicalization. This approach frequently confronts us with one or several of the following four tasks:

1. Define a tropicalization (map) for more general types of algebraic varieties.
2. Extend the scope of tropical intersection theory to work for more general notions of tropical varieties.
3. Relate the intersection theory of an algebraic variety (as in 1.) to the intersection theory of its tropicalization (as in 2.).
4. Apply known results about the interplay of tropicalization and intersection theory to obtain a classical/tropical correspondence.

In the case of rational curves, the intersection theory on the tropical moduli spaces is well-developed, and we can use the intensely studied tropicalization map for subvarieties of algebraic tori [Kat09, ST08, Tev07, Gub13, OP13]. Therefore, we can skip 1. through 3. and directly turn to 4. Here, several details have to be worked out, but eventually we can reprove Nishinou and Siebert's correspondence theorem [NS06]. This has been published in [Gro14] and makes up Chapter 1 of this thesis. The same methods cannot be used to obtain a correspondence between genus 0 descendant Gromov-Witten invariants because they only work for non-compact (even very affine) moduli spaces.

In order to tropicalize the intersection products occurring in the definition of descendant Gromov-Witten invariants, we need to tropicalize cycle classes instead of mere cycles. Motivated by their appearance in several recent papers [ACP15, CMR14, CMR14, Ran15, Thu07, Uli15a] we develop a toolkit for toroidal embeddings (1.). We also develop the fundamental intersection-theoretic concepts for the tropicalizations of toroidal embeddings (2.), and show how they are related to their algebraic counterparts (3.). We then apply our tools to an appropriate proper moduli space of genus 0 maps to toric varieties and obtain a correspondence theorem for genus 0 descendant invariants (4.). Based on the article [Gro15], we treat this in Chapter 2 .

Given the fact that toroidal embeddings are special instances of logarithmic schemes, and in view of the promising results relating logarithmic and tropical geometry [Uli13, Ran15], we partially extend our toolkit to work for logarithmic schemes in Chapter 3. However, while with good prospects, at this stage we have not yet used these tools to obtain more general correspondences.

Results of this Thesis

Numbered according to the three chapters of this thesis, we give a short overview of our main results. More details can be found at the beginning of the respective chapters.

1. We show that the embedding of the moduli space of tropical stable maps given in [SS04, GKM09] is compatible with a certain embedding of the moduli space of so-called generic logarithmic stable maps. By compatible we mean that

for every generic logarithmic stable map the tropical curve corresponding to the tropicalization of its associated point in the moduli space is equal to its intrinsic tropicalization. As a consequence, we see that tropicalization commutes with evaluation. A detailed study of Osserman and Payne's [OP13] genericity conditions to ensure that intersection commutes with tropicalization then yields a new proof of Nishinou and Siebert's correspondence theorem for rational curves satisfying toric constraints [NS06].

2. We develop the foundations of an intersection theory on the cone complex associated to a strict toroidal embedding and construct a tropicalization morphism mapping cycle classes on the toroidal embedding to cycle classes on its cone complex. This tropicalization map is shown to respect intersections with boundary divisors and push-forwards along toroidal morphisms. Applying this to the space of genus 0 logarithmic stable maps to toric varieties yields a correspondence theorem for genus 0 descendant Gromov-Witten invariants.

3. We use the notions of logarithmic schemes and Kato fans to clarify and partly generalize our results in the toroidal case. To do so we exhibit a duality between certain intersection-theoretic constructions on cone complexes and Kato fans. As the intersection theory of Kato fans is remarkably similar to that of algebraic varieties, this yields further insight into the connections between algebraic and tropical intersection theory. Afterwards, we show that under an appropriate condition we can define a pull-back from the Chow group of the associated Kato fan of a log scheme to the Chow group of the underlying scheme. We then interpret the tropicalization introduced for toroidal varieties as the dual of this pull-back.

Note again that the material of Chapter 1 has been published in [Gro14], and the material of Chapter 2 is part of the article [Gro15].

1 A Correspondence Theorem for Generic Conditions

Rational tropical curves are surely among the most well-behaved objects of tropical geometry. Their moduli space is known to have a universal family [FH13], it supports the tropical intersection theory of [AR10, FR13, Sha13], and every tropical rational curve is the tropicalization of an algebraic rational curve [Spe14]. In this sense it is not surprising that there is a correspondence theorem for rational curves in any dimension [NS06]. However, the proof of Nishinou and Siebert does not use the pleasant qualities of the tropical moduli space, but instead uses only finitely many tropical curves to construct a toric degeneration. What we want to do in this chapter is to reprove the correspondence theorem by studying the tropicalizations of subvarieties of a suitable moduli space which are given by toric constraints.

Let us clarify what we mean by this in the case of degree d rational curves to $\mathbb{P}^n$ satisfying point conditions. The tropical moduli space $\mathrm{TSM}_n^\circ(r,d)$ of n-marked degree d rational tropical stable maps to $\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$ can be embedded into some $\mathbb{R}^N$ such that its image, equipped with weight 1 everywhere, is a tropical cycle [GKM09, SS04]. The correct algebraic analogue of $\mathrm{TSM}_n^\circ(r,d)$ turns out to be the moduli space $\mathrm{LSM}_n^\circ(r,d)$ of what we call generic logarithmic stable maps. These are precisely the n-marked stable degree d maps to $\mathbb{P}^r$ with irreducible domain which are rigid with respect to the toric boundary of $\mathbb{P}^r$ (cf. [FP95]). We provide an explicit embedding $\mathrm{LSM}_n^\circ(r,d) \to \mathbb{G}_m^N$ with respect to which the tropicalization satisfies

$$\mathrm{Trop}(\mathrm{LSM}_n^\circ(r,d)) = \mathrm{TSM}_n^\circ(r,d).$$

There are so-called evaluation morphisms $\mathrm{ev}_i\colon \mathrm{LSM}_n^\circ(r,d) \to \mathbb{P}^r$ for every mark $1 \leq i \leq n$ which assign to a map $(C,(p_j),f\colon C \to \mathbb{P}^r)$ the image $f(p_i)$ of the i-th marked point. They can be used to impose conditions on the curves in the moduli space. For example, if $(P_i)_{1\leq i\leq n}$ is a family of points in $\mathbb{P}^r$, we can consider the closed subvariety

$$\mathrm{ev}_1^{-1}\{P_1\} \cap \ldots \cap \mathrm{ev}_n^{-1}\{P_n\} \subseteq \mathrm{LSM}_n^\circ(r,d).$$

It only contains curves which pass through the n given points. Analogously, there also exist tropical evaluation morphisms $\mathrm{ev}_i^{\mathrm{t}}\colon \mathrm{TSM}_n^\circ(r,d) \to \mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$, and for a family $(P_i^t)_{1\leq i\leq n}$ of points in $\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$ we can consider the tropical cycle

$$\mathrm{ev}_1^{\mathrm{t}\,*}[P_1^t]\cdots\mathrm{ev}_n^{\mathrm{t}\,*}[P_n^t]\cdot\mathrm{TSM}_n^\circ(r,d).$$

Let us point out that this is, unlike in the algebraic case, not a set-theoretic intersection but a tropical intersection product. In view of the similarity of the algebraic and tropical setup, the obvious guess is that

$$\mathrm{Trop}\left(\mathrm{ev}_1^{-1}\{P_1\}\cap\ldots\cap\mathrm{ev}_n^{-1}\{P_n\}\right)=\mathrm{ev}_1^{\mathrm{t}\,*}[P_1^t]\cdots\mathrm{ev}_n^{\mathrm{t}\,*}[P_n^t]\cdot\mathrm{TSM}_n^\circ(r,d),$$

at least for generic P_i tropicalizing to P_i^t. This would also explain the presence of the intersection product on the tropical side; it might happen that several of the algebraic curves passing through the P_i have the same tropicalization. Accordingly, we should count their tropicalization with weight greater than 1.

After giving more details about the moduli spaces and embeddings involved in Section 1.1, we will prove the correctness of the above equation. Its proof has two essential ingredients. First we need to show that the embeddings of $\mathrm{LSM}_n^\circ(r,d)$ and $\mathrm{TSM}_n^\circ(r,d)$ are compatible. As already mentioned, they are chosen in a way that $\mathrm{Trop}(\mathrm{LSM}_n^\circ(r,d))=\mathrm{TSM}_n^\circ(r,d)$, which is essential. But to ensure that algebraic and tropical point conditions are related, we also need tropicalization to commute with evaluation, that is $\mathrm{trop}\circ\mathrm{ev}_i=\mathrm{ev}_i^{\mathrm{t}}\circ\mathrm{trop}$ for all i. We prove this in the first part of Section 1.2. The second part carefully analyzes which genericity conditions we have to require for the P_i.

In Section 1.3 we generalize our results for maps to $\mathbb{P}^r$ to allow arbitrary toric target. The main difference to the case of projective spaces – aside from the increased amount of notation – is that we also allow evaluations at the points which are mapped to the boundary. We conclude this chapter with a brief discussion of what our methods imply about tropical lines in smooth cubic surfaces.

The material of this chapter has been published in [Gro14].

1.1 Preliminaries

In this section we will review some results about moduli spaces of curves, tropical intersection theory, and the tropicalization map which we need for the correspondence theorem. In this whole chapter we will work over an algebraically closed valued field (K,ν) of characteristic 0, with valuation ring $(R,\mathfrak{m})$, residue field $\kappa=R/\mathfrak{m}$, and value group $G=\nu(K^*)$. For a set S we will denote the set of pairs of distinct elements of S by $D(S)$.

1.1.1 Tropical Varieties and their Intersection Theory

Let N be a lattice, that is a finitely generated free abelian group, with dual lattice $M=\mathrm{Hom}(N,\mathbb{Z})$. We denote $N_\mathbb{R}=N\otimes_\mathbb{Z}\mathbb{R}$ and $N_G=N\otimes_\mathbb{Z}G$. Following [AR10] we define a *weighted complex* in $N_\mathbb{R}$ as a pure-dimensional rational polyhedral complex, together with a weight function assigning an integer to each of its inclusion-maximal cells. Here, rational means that each of its cells is a finite intersection of half-spaces $\{w\in N_\mathbb{R}\mid\langle m,w\rangle\le a\}$, where $m\in M$ and $a\in\mathbb{R}$. Assume σ is a rational polyhedron in $N_\mathbb{R}$, and $w\in\sigma$. Then the subspace $U=\mathrm{Span}(\sigma-w)$ is independent of the choice of w. Thus, it makes sense to define $N_\sigma:=N\cap U$. It is a saturated sublattice of N of rank $\dim(\sigma)$. If τ is a face of σ, then N_σ/N_τ is a lattice of rank 1 and therefore has exactly two generators. The *lattice normal vector* $u_{\sigma/\tau}$ is defined as that one of them pointing in the direction of σ. A *tropical complex* A is a weighted complex satisfying the so-called *balancing condition*. This condition requires that whenever τ is a codimension-1 cell of A, and $\sigma_1, \dots, \sigma_k$ are the maximal cells containing τ, with weights $\omega_1, \dots, \omega_k$, respectively, then

$$\sum_{i=1}^{k} \omega_i u_{\sigma_i/\tau} = 0 \quad \text{in } N/N_\tau.$$

The *support* $|A|$ of A is defined as the union of all its maximal cells with nonzero weight.

Tropical cycles are tropical complexes modulo refinements. Of course, refinements have to respect weights so that the support of a cycle is well-defined. For $d \in \mathbb{N}$ let $Z_d(N_\mathbb{R})$ be the set of d-dimensional tropical cycles in $N_\mathbb{R}$. There is a naturally defined addition of tropical cycles of the same dimension, making $Z_d(N_\mathbb{R})$ an abelian group. Given a cycle A in $N_\mathbb{R}$, the set of d-dimensional cycles $B \in Z_d(N_\mathbb{R})$ with $|B| \subseteq |A|$ is a subgroup of $Z_d(N_\mathbb{R})$, denoted by $Z_d(A)$. The graded group $\bigoplus_d Z_d(A)$ is denoted by $Z_*(A)$. A *tropical variety* is a tropical cycle with positive weights only.

Let A and B be tropical varieties in $N_\mathbb{R}$ and $N'_\mathbb{R}$, respectively. Then the *morphisms* from A to B are all maps $|A| \to |B|$ which are induced by maps $N_\mathbb{R} \to N'_\mathbb{R}$ of the form $v \mapsto \varphi_\mathbb{R}(v) + w$, where $\varphi\colon N \to N'$ is a morphism of abelian groups and $w \in N'_\mathbb{R}$.

Tropical cycles are the main protagonists in the intersection theory of $N_\mathbb{R}$. Unlike in the algebraic case, one does not need a concept of rational equivalence to define intersection products for cycles; there is a bilinear map

$$Z_*(N_\mathbb{R}) \times Z_*(N_\mathbb{R}) \to Z_*(N_\mathbb{R}),$$

usually denoted by "$\cdot$", making it into a commutative ring with unity $N_\mathbb{R}$ (considered as a tropical cycle by giving it weight 1 everywhere) [AR10]. Altering the grading on $Z_*(N_\mathbb{R})$ via $d \leftrightarrow \operatorname{rk} N - d$ makes it a graded ring. We denote it by $Z^*(N_\mathbb{R})$.

Quite similar to the construction of the intersection products on smooth algebraic varieties, the construction of the intersection product on $N_\mathbb{R}$ employs reduction to the diagonal. The diagonal is cut out by so-called *rational functions*, which are piecewise affine linear functions. Locally taking products of these rational functions one arrives at the notion of *tropical cocycles* [Fra13]. Intersections with these cocycles make sense on an arbitrary tropical variety A. That is, if we denote the ring of cocycles on A by $C^*(A)$, then $Z_*(A)$ is a $C^*(A)$-module. In the case where $A = N_\mathbb{R}$ the map

$$C^*(N_\mathbb{R}) \to Z^*(N_\mathbb{R}), \quad \varphi \mapsto \varphi \cdot N_\mathbb{R}$$

is a ring isomorphism. Since cocycles can be pulled back along arbitrary morphism, it makes sense to write $f^*B \cdot C$ when we are given a morphism $f\colon A \to N_\mathbb{R}$ of tropical varieties, and cycles $B \in Z_*(N_\mathbb{R})$ and $C \in Z_*(A)$.

Given an arbitrary morphism $f\colon A \to B$ of tropical varieties, there also is a *push-forward* $f_*\colon Z_*(A) \to Z_*(B)$ [GKM09]. This is a graded morphism of $C^*(B)$-modules, or, in other words, the projection formula holds.

Even though not needed to define intersection products, there is a notion of *rational equivalence* [AHR16] for tropical varieties which is compatible with intersection products and push-forwards. In the case of $N_{\mathbb{R}}$ every rational equivalence class in $Z_*(N_{\mathbb{R}})$ has a unique representative which is a *fan cycle*, that is it is given by a tropical complex whose underlying polyhedral complex is a fan. Given a cycle $A \in Z_*(N_{\mathbb{R}})$, the unique fan cycle rationally equivalent to A can be obtained by shrinking all bounded polyhedra in a polyhedral structure of $|A|$ to 0.

Remark 1.1.1. Intersection products cannot only be defined for cycles in affine space, but more generally for cycles in tropical varieties which are locally isomorphic to the Bergman fan of a matroid. This can be done using reduction to the diagonal [FR13] or with tropical modifications [Sha13].

1.1.2 The Moduli Space of Tropical Stable Maps

A (rational) *tropical curve* is a connected metric graph $\Gamma = (V, E, \ell)$ of genus 0, V being the set of vertices, E the set of edges, and $\ell : E \to \mathbb{R}_{>0} \cup \{\infty\}$ a length function. The edges adjacent to a 1-valent vertex are called *legs*, and we require that the legs are exactly those edges with infinite lengths. All other edges are called *bounded edges*. The 1-valent vertices should be imagined as the points at infinity of the unique leg incident to them and will be called *feet*. All other vertices will be referred to as *inner vertices*. The interior part Γ° of Γ will denote the set of points of Γ not equal to a foot. As every leg should only have one foot, we exclude the pathological case of graphs consisting of exactly 2 vertices. In that way, every leg $e \in E$ is incident to a unique inner vertex v_e. If a tropical curve has a 2-valent vertex, we can replace it and the two edges incident to it by one edge of adequate length. The resulting curve should obviously be considered as isomorphic to the original one. For this reason, we restrict our attention to tropical curves without 2-valent vertices. The advantage of this is that in this way the category of tropical curves, whose morphisms are morphisms of the underlying graphs respecting the lengths, induces the "correct" notion of isomorphisms.

For a nonempty finite set I we define an *I-marked tropical curve* to be a tropical curve together with an injection $I \to E,\ i \mapsto e_i$ assigning a leg to every element of I. Of course, morphisms of marked curves have to respect the markings. We denote by $M_{0,I}^{\text{trop}}$ the set of all I-marked tropical curves with exactly $|I|$ legs, modulo isomorphisms.

The set $M_{0,I}^{\text{trop}}$ can be identified with a tropical variety in the following way. For every tropical curve $(\Gamma, (e_i)_{i\in I}) \in M_{0,I}^{\text{trop}}$ and pair of distinct indices $i, j \in I$ there is a well-defined distance $\text{dist}_\Gamma(i, j)$ between the vertices v_{e_i} and v_{e_j}. We use these distances to define the *tropical Plücker embedding* which maps Γ to the point

$$\text{pl}^{\text{t}}(\Gamma) := (-\operatorname{dist}_\Gamma(i, j)/2)_{(i,j)} \in \mathbb{R}^{D(I)}/\mathbb{R}^I,$$

where $\mathbb{R}^I$ acts on $\mathbb{R}^{D(I)}$ via the morphism

$$\mathbb{R}^I \to \mathbb{R}^{D(I)}, (x_i)_i \mapsto (x_i + x_j)_{(i,j)}.$$

It follows from the results of [SS04, Thm. 4.2] and [GKM09, Thm. 3.7] that pl^{t} is an embedding and that its image is the support of a pure-dimensional fan satisfying the balancing condition after giving weight 1 to all inclusion-maximal cones. We will frequently identify $M_{0,I}^{\mathrm{trop}}$ with this tropical cycle. Note that both, Speyer and Sturmfels, and Gathmann, Kerber, and Markwig, consider slightly different embeddings. Namely, they consider unordered pairs instead of ordered ones, and they leave out the factor $-1/2$ in front of the distances. Our reason to work with ordered pairs is to ease the notation when we compare it with the corresponding algebraic moduli space. The factor $-1/2$ is included because the lattice in $\mathbb{R}^{D(I)}/\mathbb{R}^{I}$ used in [GKM09] is not the canonical one, which is induced by $\mathbb{Z}^{D(I)} \subseteq \mathbb{R}^{D(I)}$. It follows from [FR13, Example 7.2] that they instead use exactly 2 times the canonical lattice.

Now we want to define tropical stable maps. To do this, we first need the notion of degree. Let N be a lattice, and J a finite set (disjoint from I). Then every map $\Delta : J \to N$ such that $\sum_{j \in J} \Delta(j) = 0$ is called a *degree* (in N with index set J). An *I-marked (rational) tropical stable map to $N_{\mathbb{R}}$ of degree Δ* is a triple $(\Gamma, (e_\lambda), h)$ consisting of an $L_0 := I \cup J$-marked curve $(\Gamma, (e_\lambda))$ and a map $\Gamma^\circ \to N_{\mathbb{R}}$ which is linear on the edges (that is affine linear after identifying an edge e with $[0, \ell(e)]$ if it is bounded or with $[0, \infty)$ in case it is a leg) and satisfies the additional requirements stated in the following. For each inner vertex v incident to an edge e there is a well-defined direction $\mathrm{v}(v, e)$, which is the derivative of $h|_e$ when choosing the parametrization of e starting at v. In case e is a leg, we also denote $\mathrm{v}(v, e)$ by $\mathrm{v}(e)$. We require that all these direction vectors are in N, and, furthermore, that at every inner vertex v the *balancing condition* is fulfilled, that is

$$\sum_{e : v \in e} \mathrm{v}(v, e) = 0.$$

Finally, we require that $\mathrm{v}(e_j) = \Delta(j)$ for all $j \in J$, and $\mathrm{v}(e_i) = 0$ for all $i \in I$. Morphisms of stable maps should, of course, respect the maps into $N_{\mathbb{R}}$. We denote by $\mathrm{TSM}_I^\circ(N_{\mathbb{R}}, \Delta)$ the set of all I-marked tropical stable maps to $N_{\mathbb{R}}$ of degree Δ having exactly $|L_0|$ legs, modulo isomorphisms.

For every $i \in I$ there is an evaluation map $\mathrm{ev}_i^{\mathrm{t}} : \mathrm{TSM}_I^\circ(N_{\mathbb{R}}, \Delta) \to N_{\mathbb{R}}$ which assigns to a tropical stable map the unique point in $N_{\mathbb{R}}$ to which e_i is mapped.

Let $\mathrm{ft}^{\mathrm{t}} \colon \mathrm{TSM}_I^\circ(N_{\mathbb{R}}, \Delta) \to M_{0,L_0}^{\mathrm{trop}}$ be the *forgetful map* which assigns to a tropical stable map its underlying L_0-marked curve. Fix $i_0 \in I$. By [GKM09, Prop. 4.7], the map

$$\mathrm{ft}^{\mathrm{t}} \times \mathrm{ev}_{i_0}^{\mathrm{t}} \colon \mathrm{TSM}_I^\circ(N_{\mathbb{R}}, \Delta) \to M_{0,L_0}^{\mathrm{trop}} \times N_{\mathbb{R}}$$

is bijective. In particular we can use the tropical Plücker embedding to identify $\mathrm{TSM}_I^\circ(N_{\mathbb{R}}, \Delta)$ with a tropical variety; the morphism

$$(\mathrm{pl}^{\mathrm{t}} \circ \mathrm{ft}^{\mathrm{t}}) \times \mathrm{ev}_{i_0}^{\mathrm{t}} : \mathrm{TSM}_I^\circ(N_{\mathbb{R}}, \Delta) \to \mathbb{R}^{D(L_0)}/\mathbb{R}^{L_0} \times N_{\mathbb{R}}$$

is an embedding whose image is the support of a pure-dimensional polyhedral fan which defines a tropical variety after assigning weight 1 to all its maximal cones. So by construction, we have $\mathrm{TSM}_I^\circ(N_{\mathbb{R}}, \Delta) = M_{0,L_0}^{\mathrm{trop}} \times N_{\mathbb{R}}$ as tropical cycles.

We are now able to formulate the tropical enumerative problem for counting tropical stable maps using intersection theory. The goal is to compute cycles of the form

$$\prod_{i \in I} \mathrm{ev}_i^{\mathrm{t}\,*}[w_i] \cdot \mathrm{TSM}_I^{\circ}(N_{\mathbb{R}}, \Delta), \tag{1.1.1}$$

where $w_i \in N_{\mathbb{R}}$.

The most important situation for us is when $N = \mathbb{Z}^{r+1}/\mathbb{Z}\mathbf{1}$. In this case, we obtain a degree Δ_d for every $d \in \mathbb{N}$ by setting $J = \{0, \dots, r\} \times \{1, \dots, d\}$ and $\Delta((i,j)) = s_i$, where s_i denotes the image of the i-th standard basis vector of $\mathbb{Z}^{r+1}$ in $\mathbb{Z}^{r+1}/\mathbb{Z}\mathbf{1}$. By abuse of notation, we usually just write d instead of Δ_d, and we abbreviate $\mathrm{TSM}_I^{\circ}(\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}, \Delta_d) = \mathrm{TSM}_I^{\circ}(r, d)$. The reason why this case is of special importance is that $\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$ is the tropicalization of the big open torus of $\mathbb{P}^r$, so that the tropicalization of an algebraic curve of degree d in $\mathbb{P}^r$ which intersects the coordinate hyperplanes generically should intuitively be some tropical curve of degree d in $\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$.

1.1.3 Tropicalizations of Subvarieties of Tori

We want to relate moduli of algebraic and tropical curves using tropicalizations of subvarieties of algebraic tori. Let us briefly recall how these tropicalizations are defined. Let T be an algebraic torus, let N be its lattice of one-parameter subgroups (1-psgs), and let M be its lattice of characters, which is canonically dual to N. Then there is a canonical isomorphism $T \cong \mathrm{Spec}(K[M])$. Any K-rational point $x \in T(K)$ induces a group homomorphism $M \to K^*$. Composing this with the valuation $\nu : K^* \to \mathbb{R}$ we obtain an element in $N_{\mathbb{R}} \cong \mathrm{Hom}(M, \mathbb{R})$ called the *tropicalization* of x. By this procedure we obtain a map

$$\mathrm{trop}_T : T(K) \to N_{\mathbb{R}}.$$

We shorten the notation to trop if T is clear from the context. It follows immediately from the definition that the tropicalization map factors through N_G.

Example 1.1.2. Let T be the big open torus of $\mathbb{P}^n$, considered with its standard toric structure. We will denote $T = \mathbb{T}^n$. It is equal to the complement of the coordinate hyperplanes in $\mathbb{P}^n$. The lattice N of 1-psgs of $\mathbb{T}^n$ is equal to $\mathbb{Z}^{n+1}/\mathbb{Z}\mathbf{1}$, where $\mathbf{1}$ denotes the vector whose coordinates all equal 1. Its dual lattice M is the subgroup of $\mathbb{Z}^{n+1}$ defined by the condition that the coordinates sum up to 0. The tropicalization map assigns to an element $(a_1 : \dots : a_n) \in \mathbb{T}^n$ given in homogeneous coordinates the class of $\big(\nu(a_1), \dots, \nu(a_n)\big)$ in $N_{\mathbb{R}} = \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Now let X be a pure-dimensional subscheme of T and $w \in N_G$. Then there exists $t \in T$ with $\mathrm{trop}(t) = w$. The *initial degeneration* $\mathrm{in}_w X$ is defined as the special fiber of the scheme-theoretic closure of $t^{-1}X$ in the canonical R-model $\mathcal{T} = \mathrm{Spec}(R[M])$ of T. It is well-defined up to translation by a point in $T_\kappa = \mathrm{Spec}(\kappa[M])$. If w is not in N_G we can use the same definition after a suitable extension of valued fields. Of course, this depends on the choice of the field extensions, but it can be shown that for every two of those extensions L and L' there is a common field extension $\overline{L}$ such that the initial degenerations defined over L and L' become translates of each other after passing to $\overline{L}$ [Gub13, Section 5]. We remark that it is also possible to define initial degenerations at general $w \in N_{\mathbb{R}}$ using the tilted group rings $R[M]^w$ [OP13]. With this definition they are well-defined without the choices of a field extension and a $t \in T$.

The *tropicalization* $\operatorname{Trop}(X)$ of X can be defined as a tropical cycle in $N_{\mathbb{R}}$ whose underlying set $|\operatorname{Trop}(X)|$ consists of the $w \in N_{\mathbb{R}}$ with $\operatorname{in}_w X \neq \emptyset$. This is indeed a purely $\dim(X)$-dimensional polyhedral complex [BG84], and if the valuation is nontrivial it is the closure of $\operatorname{trop}(X(K))$ [Dra08, OP13, Pay09b, SS04]. For any point $w \in |\operatorname{Trop}(X)|$, the sum of the multiplicities of the components of $\operatorname{in}_w X$ is well-defined and called the *tropical multiplicity* of X at w. The multiplicity is constant on the relative interiors of the maximal cells of any polyhedral structure on $|\operatorname{Trop}(X)|$ and satisfies the balancing condition [Gub13, Spe05, ST08]. Thus, we get a well-defined tropical cycle $\operatorname{Trop}(X)$. Extending by linearity, we obtain a morphism $\operatorname{Trop}\colon Z_*(T) \to Z_*(N_{\mathbb{R}})$ (the tropicalization of a subscheme of X is the weighted sum of the tropicalizations of its components [Gub13, Prop. 13.6]).

The fiber of 0 with respect to the tropicalization map trop can be described algebraically. It is exactly the image of the inclusion $\mathcal{T}(R) \hookrightarrow T(K)$, which maps an R-valued point $\operatorname{Spec} R \to \mathcal{T}$ to the induced morphism $\operatorname{Spec} K \to T$ on the generic fibers. For this reason we call points in $T(K)$ tropicalizing to 0 the *R-integral points* of T. This identification can be used to give $\operatorname{trop}^{-1}\{0\}$ a topology. Namely, we give it the initial topology with respect to the reduction map $\mathcal{T}(R) \to T_\kappa(\kappa)$, where we consider $T_\kappa(\kappa)$ with the Zariski topology. This makes $\operatorname{trop}^{-1}\{0\}$ a topological group. Note that if the valuation is trivial, we have $T(K) = \mathcal{T}(R) = T_\kappa(\kappa)$ and the topology just defined is the ordinary Zariski topology on $T(K)$.

If $w \in N_G$, then the fiber $\operatorname{trop}^{-1}\{w\}$ is a $\operatorname{trop}^{-1}\{0\}$-torsor and therefore has a well-defined topology induced by the topology on $\operatorname{trop}^{-1}\{0\}$. It could also be defined using reduction by identifying $\operatorname{trop}^{-1}\{w\}$ with $(\operatorname{Spec} R[M]^w)(R)$.

In general, the tropicalization of the intersection of two subschemes X and X' of T is not equal to the intersection of $\operatorname{Trop}(X)$ and $\operatorname{Trop}(X')$. But for generic intersections this in fact holds. More precisely, we have

$$\operatorname{Trop}(X \cap tX') = \operatorname{Trop}(X) \cdot \operatorname{Trop}(X'),$$

where t can be chosen in a nonempty open subset of $\operatorname{trop}^{-1}\{0\}$ [OP13, Thm. 5.3.3]. This is the central fact about tropicalization that we will need to prove our correspondence theorem. Note that its proof uses a Bertini argument and hence requires $\operatorname{char}(K)$ to be 0.

Tropicalization is also compatible with push forwards. Whenever $f\colon T \to T'$ is a morphism of tori with lattices N and N' of 1-psgs, and $g\colon N_{\mathbb{R}} \to N'_{\mathbb{R}}$ is the induced morphism, we have

$$\operatorname{Trop}(f_*[X]) = g_* \operatorname{Trop}(X)$$

for every subscheme X of T [BPR15, OP13, Gub13]. We will denote g by $\operatorname{Trop}(f)$.

1.1.4 The Moduli Space of Generic Logarithmic Stable Maps

In this chapter, an (algebraic) curve will always be a smooth curve which is isomorphic to $\mathbb{P}^1$. For a nonempty finite set I, an *I-marked curve* is constituted by a curve C together with an injective map $I \to C(K)$. We denote the moduli space of I-marked curves by $M_{0,I}$. The algebraic analogues of tropical stable maps are logarithmic stable maps. If the target is toric, then any tropical degree gives rise to a condition for logarithmic stable maps that specifies the contact order with the toric boundary. In this chapter, we restrict ourselves to target $X = \mathbb{P}^r$ and degree $d \in \mathbb{N}$. Let $J = \{0, \dots, r\} \times \{1, \dots, d\}$, and let I be a nonempty finite set disjoint from J. An *I-marked generic logarithmic stable map to $\mathbb{P}^r$ of degree d* is a tuple $\mathcal{C} = (C, (p_\lambda)_{\lambda \in L_0}, f)$ such that $(C, (p_\lambda))$ is an $L_0 = I \cup J$-marked curve, and $f\colon C \to \mathbb{P}^r$ is a morphism such that $f(C)$ is not contained in any of the coordinate hyperplanes H_i of $\mathbb{P}^r$ and

$$f^* H_i = \sum_{j=1}^{d} p_{ij}$$

for all $0 \leq i \leq r$. An isomorphism of two generic logarithmic stable maps is an isomorphism of the underlying curves that respects the markings and the maps to $\mathbb{P}^r$. We denote the set of isomorphism classes of generic logarithmic stable maps of degree d to $\mathbb{P}^r$ by $\mathrm{LSM}_I^\circ(r, d)$.

Remark 1.1.3. As indicated by the "generic" in the terminology and the $\circ$ in the notation, the space $\mathrm{LSM}_I^\circ(r, d)$ that we consider in this chapter is a dense open subset of a larger, complete space $\mathrm{LSM}_I(r, d)$ of (possible non-generic) logarithmic stable maps. More precisely, it is the open subset $\mathrm{LSM}_I(r, d)$ where the logarithmic structure is trivial. Non-generic logarithmic stable maps may have reducible domain and may even be mapped into the boundary of $\mathbb{P}^r$. We refer to Section 3.3.1 for a brief introduction to logarithmic geometry, and to [Ogu06] for a thorough one. Details about logarithmic stable maps can be found in [GS13, Che14, Ran15], where the last mentioned paper is most relevant to our setting as it considers genus 0 logarithmic maps to toric varieties.

For every $i \in I$, there is an evaluation map $\mathrm{ev}_i\colon \mathrm{LSM}_I^\circ(r, d) \to \mathbb{T}^r$ assigning to a generic logarithmic stable map $\mathcal{C} = (C, (p_\lambda)_\lambda, f)$ the point $\mathrm{ev}_i(\mathcal{C}) = f(p_i)$. Using these evaluation maps, the algebraic enumerative problem we consider can be stated as the computation of the cardinality (if finite) of sets of the form

$$\bigcap_{i \in I} \mathrm{ev}_i^{-1}\{P_i\}$$

for $P_i \in \mathbb{T}^r$ generic. As tropicalization commutes with generic intersections, we expect the tropicalization of this intersection to be equal to the expression (1.1.1) for suitable w_i. For this tropicalization to make sense we have to identify $\mathrm{LSM}_I^\circ(r, d)$ with a subvariety of an algebraic torus. Similarly as for tropical stable maps, we do this in two steps. First, we show that for fixed $i_0 \in I$ the map

$$\mathrm{ft} \times \mathrm{ev}_{i_0}\colon \mathrm{LSM}_I^\circ(r, d) \to M_{0,L_0} \times \mathbb{T}^r,$$

where ft is the *forgetful map* which forgets the map to $\mathbb{P}^r$, is bijective. Let $\mathcal{C} = (C, (p_\lambda)_{\lambda \in L_0}, f) \in \mathrm{LSM}_I^\circ(r,d)$. As C is only well defined up to isomorphisms we can assume $C = \mathbb{P}^1$, in which case we can write $p_\lambda = (b_\lambda : a_\lambda)$ in homogeneous coordinates for appropriate $a_\lambda, b_\lambda \in K$. Since the curve has degree d, the morphism f is given by $r+1$ homogeneous polynomials $f_0, \ldots, f_r$ of degree d. The fact that $f^{-1}H_i = \{p_{i1}, \ldots, p_{id}\}$ implies that f_i is of the form $f_i = c_i \prod_{1 \leq j \leq d}(a_{ij}x - b_{ij}y)$ for some $c_i \in K^*$. The marked point p_{i_0} is not mapped to any H_i, so it follows that f is completely determined by the p_{ij} and $\mathrm{ev}_{i_0}(\mathcal{C})$. In particular, $\mathcal{C}$ is determined by $\mathrm{ft} \times \mathrm{ev}_{i_0}(\mathcal{C})$. On the other hand, the description of f from above can be used to construct an inverse image of any point in $M_{0,L_0} \times \mathbb{T}^r$ under $\mathrm{ft} \times \mathrm{ev}_{i_0}$, that is the map is indeed bijective. In particular, we have identified $\mathrm{LSM}_I^\circ(r,d)$ with an algebraic variety.

Remark 1.1.4. Using the description of f in terms of the marked points and the image of p_{i_0}, it is easy to write down a universal family over $\mathrm{LSM}_I^\circ(r,d)$. That is, $\mathrm{ft} \times \mathrm{ev}_{i_0}$ does not only identify $\mathrm{LSM}_I^\circ(r,d)$ with an algebraic variety, but also makes it a fine moduli space for the appropriate moduli functor.

In the second step we identify M_{0,L_0} with a subvariety of a torus. A standard torus embedding of M_{0,L_0} first uses the Gelfand-MacPherson correspondence [GM82] to identify $M_{0,I}$ with a quotient of an open subset of the Grassmannian $\mathbb{G}(2,L_0)$ of lines in $\mathbb{P}^{L_0}$, and then uses a quotient of the Plücker embedding. Let us give some details. Let $U \subset (\mathbb{P}^1)^{L_0}$ be the complement of the big diagonal, that is the set of L_0-tuples of distinct points in $\mathbb{P}^1$. Then the obvious surjection $U \to M_{0,L_0}$ identifies the moduli space of L_0-marked curves with the quotient $U/\mathrm{GL}(2)$, where $\mathrm{GL}(2)$ acts diagonally. The Gelfand-MacPherson correspondence takes the class of a point configuration $((a_\lambda : b_\lambda))_{\lambda \in L_0}$ writes the homogeneous coordinates of these points into the columns of an $L_0 \times 2$ matrix, and then takes the orbit of the thus defined line in $\mathbb{P}^{L_0}$ with respect to the action of $\mathbb{G}_m^{L_0}$ (considered as the subgroup of diagonal matrices of $\mathrm{GL}(K^{L_0})$). This procedure identifies M_{0,L_0} with the quotient $\mathbb{G}^0(2,L_0)/\mathbb{G}_m^{L_0}$, where $\mathbb{G}^0(2,L_0)$ is the open subset of $\mathbb{G}(2,L_0)$ consisting of all lines that do not pass through the intersection of two coordinate hypersurfaces.

Now consider the Plücker embedding of $\mathbb{G}(2,L_0)$ into $\mathbb{P}^{D(L_0)}$. Notice that we do not use the usual Plücker embedding, which involves only one coordinate per minor, but instead get each minor twice with opposite signs. This is redundant and not really necessary, but avoids the need to choose signs for the minors. The $\mathbb{G}_m^{L_0}$-action on $\mathbb{G}(2,L_0)$ can be extended to $\mathbb{P}^{D(L_0)}$ by setting $(x_\lambda).(p_{(\lambda,\mu)}) = (x_\lambda x_\mu p_{(\lambda,\mu)})$. The big open torus $\mathbb{T}^{D(L_0)}$ of $\mathbb{P}^{D(L_0)}$ is invariant with respect to this action, and the $\mathbb{G}_m^{L_0}$-action on it is induced by the morphism of tori given by $(x_\lambda)_\lambda \mapsto (x_\lambda x_\mu)_{(\lambda,\mu)}$. As the quotient $\mathbb{T}^{D(L_0)}/\mathbb{G}_m^{L_0} = \mathbb{G}_m^{D(L_0)}/\mathbb{G}_m^{L_0}$ is an algebraic torus again, we have thus seen that the Gelfand-MacPherson correspondence and the Plücker embedding produce a torus embedding

$$\mathrm{pl}\colon M_{0,I} \to \mathbb{G}_m^{D(L_0)}/\mathbb{G}_m^{L_0},$$

which we call the *Plücker embedding* (even though it should probably be called Gelfand-

MacPherson–Plücker embedding). Combining this with the first step, we obtain a torus embedding

$$(\mathrm{pl} \circ \mathrm{ft}) \times \mathrm{ev}_{i_0}\colon\ \mathrm{LSM}^{\circ}_{I}(r,d) \to \mathbb{G}_m^{D(L_0)}/\mathbb{G}_m^{L_0} \times \mathbb{T}^r.$$

Notice the analogy to the embedding of $\mathrm{TSM}^{\circ}_{I}(r,d)$.

1.2 The Correspondence Theorem for $\mathbb{P}^r$

In the previous section we have introduced embeddings

$$(\mathrm{pl}^{\mathrm{t}} \circ \mathrm{ft}^{\mathrm{t}}) \times \mathrm{ev}^{\mathrm{t}}_{i_0}\colon\ \mathrm{TSM}^{\circ}_{I}(r,d) \to \mathbb{R}^{D(L_0)}/\mathbb{R}^{L_0} \quad \text{, and}$$
$$(\mathrm{pl} \circ \mathrm{ft}) \times \mathrm{ev}_{i_0}\colon\ \mathrm{LSM}^{\circ}_{I}(r,d) \to \mathbb{G}_m^{D(L_0)}/\mathbb{G}_m^{L_0} \times \mathbb{T}^r.$$

The quotient $\mathbb{Z}^{D(L_0)}/\mathbb{Z}^{L_0}$ is the lattice of 1-psgs of the torus $\mathbb{G}_m^{D(L_0)}/\mathbb{G}_m^{L_0}$, so it is reasonable to expect that with these embeddings we have

$$\mathrm{Trop}(\mathrm{LSM}^{\circ}_{I}(r,d)) = \mathrm{TSM}^{\circ}_{I}(r,d).$$

And indeed, this follows directly from a well-known result from Speyer and Sturmfels who computed tropicalizations of Grassmannians [SS04, Thm. 3.4].

An immediate consequence of our choice of embeddings is that tropicalization commutes with evaluation at the point marked by i_0. That this is also true for all other marks $i \in I$ is the main statement of the first part of this section. The second part then uses this fact to show that tropicalizations of subvarieties of $\mathrm{TSM}^{\circ}_{I}(r,d)$ given by generic point conditions are equal to tropical intersection cycles in $\mathrm{LSM}^{\circ}_{I}(r,d)$ given by tropical point conditions. This reproves the correspondence theorem for rational curves in $\mathbb{P}^r$.

1.2.1 Tropicalization and Evaluation

We begin by constructing a natural candidate for the tropicalization of a given curve $\mathcal{C} = (C,(p_\lambda)_{\lambda\in L_0}, f)$ in $\mathrm{LSM}^{\circ}_{I}(r,d)$, which will be seen to be the tropical curve corresponding to $\mathrm{trop}(\mathcal{C})$. For this we assume that we are given a representative of $\mathcal{C}$ in *standard form*, that is that $C = \mathbb{P}^1$ and $p_{i_0} = (0:1)$. Our construction is a generalization of [Och13, Constr. 2.2.20] to general valued fields. It works for generic logarithmic stable maps to arbitrary toric varieties, as explained in Section 1.3, yet we will restrict ourselves to maps to projective space here to reduce the amount of notation needed.

Construction 1.2.1. Let $\mathcal{C} = (\mathbb{P}^1,(p_\lambda)_{\lambda\in L_0}, f) \in \mathrm{LSM}^{\circ}_{I}(r,d)$ be in standard form. Recall that $L_0 = I \cup J$, where $J = \{0,\dots,r\} \times \{1,\dots,d\}$. We will construct an I-marked tropical stable map to $\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$ of degree d as a candidate for the tropicalization of $\mathcal{C}$. Because all marked points are distinct and $p_{i_0} = (0:1)$ by assumption, there are unique $a_\lambda \in K^*$ such that $p_\lambda = (1 : a_\lambda)$ for $\lambda \neq i_0$. Furthermore, assume that $\mathrm{ev}_{i_0}(\mathcal{C}) = f(p_{i_0}) = (c_0 : \cdots : c_r)$.

For each nonempty $A \in \mathcal{P}(L) \setminus \{\emptyset\}$ in the power set of $L := L_0 \setminus \{i_0\}$ we define $\nu(A) = \min\{\nu(a_\mu - a_\lambda) \mid \lambda, \mu \in A\} \in \mathbb{R} \cup \{\infty\}$. We define a partial order on $\mathcal{P}(L) \setminus \{\emptyset\}$ by letting $A \preceq B$ if and only if $A \subseteq B$ and $\nu(A) = \nu(B)$. Let $V \subseteq \mathcal{P}(L) \setminus \{\emptyset\}$ be the set of all maximal elements with respect to this order. Note that all singletons belong to V. Ordering V by inclusion, we obtain another partially ordered set. We make it into the underlying set of a graph Γ by connecting two elements of V if and only if one covers the other. As L is greater than every other element of V, there is a path in Γ between any vertex $v \in V$ and L. This shows that Γ is connected. Even more, Γ is a tree: if A and B are two subsets of L with nonempty intersection, then it follows from the valuation properties that $\nu(A \cup B) = \min\{\nu(A), \nu(B)\}$. Thus we either have $A \preceq A \cup B$ or $B \preceq A \cup B$. If both A and B are in V then it follows from the definitions that either $A \subseteq B$ or $B \subseteq A$. In particular, every vertex $v \neq L$ is covered by precisely one element in V. This shows that Γ has exactly $|V \setminus \{L\}| = |V| - 1$ edges, which implies that it is a tree.

We make Γ a tropical curve by giving an edge $e = \{v \subset w\}$ the length $\ell(e) = \nu(v) - \nu(w)$ and attaching an extra foot $\{i_0\}$ along a leg to L. This curve is L_0-marked in a canonical way. We use Γ to construct an I-marked tropical stable map to $\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$ of degree d. All there is left to do is to define the morphism $h : \Gamma^\circ \to \mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$. To do so, let us introduce some more notation. Let $\pi : J \to \{0, \dots, r\}$ be the projection onto the first coordinate. Furthermore, denote by s_i the image of the i-th standard basis vector of $\mathbb{R}^{r+1}$ in $\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$. For every subset $A \subset L_0$ we define

$$s_A := \sum_{i=0}^{r} |\pi^{-1}\{i\} \cap A| s_i. \tag{1.2.1}$$

Let $v \in V$ be an inner vertex, and let $L = v_0, v_1, \dots, v_k = v$ be the vertices passed by the unique path from L to v. We define

$$q_v := \sum_{i=0}^{r} \nu(c_i) s_i + \sum_{i=1}^{k} \ell(\{v_i, v_{i-1}\}) s_{v_i}$$

and use this to define h as the map $\Gamma^\circ \to \mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$ which sends an inner vertex v to q_v, interpolates linearly on the bounded edges, and has direction s_v on a leg with foot v. Let us check the balancing condition. It is immediate that for any edge $e = \{v \subset w\}$ in Γ we have $\mathrm{v}(w, e) = s_v$. If v is an inner vertex, and $v_0, \dots, v_k$ are the vertices adjacent to v, where v_0 is the unique vertix covering v, then v is the disjoint union of $v_1, \dots, v_k$. This yields

$$\sum_{i=0}^{k} \mathrm{v}(v, \{v, v_i\}) = -s_v + \sum_{i=1}^{k} s_{v_i} = -s_v + s_{\bigcup_i v_i} = 0,$$

that is the balancing condition is satisfied at v. For $v = L$ there is no v_0, but instead we can use $s_L = 0$ to see that the balancing condition is satisfied here as well. As our map has degree d by construction, we really obtained an I-marked tropical stable map of degree d. We denote it by $\Gamma_{\mathcal{C}}$ and call it the *intrinsic tropicalization of* $\mathcal{C}$. ♢

Example 1.2.2. Let $K = \overline{\mathbb{C}((t))} = \bigcup_{n\in\mathbb{Z}} \mathbb{C}((t^{\frac{1}{n}}))$ be the field of Puiseux series. Furthermore, let $I = \{1,2\}$, $i_0 = 1$, and $d = r = 2$. We want to compute the intrinsic tropicalization of the generic logarithmic stable map $\mathcal{C}$ in standard form which is given by

$$\begin{array}{llll} c_0 = 1 + t & a_{01} = t^{-2} + 1 & a_{11} = t^{-1} & a_{21} = t^{-2} \\ c_1 = 2t^{-2} + 3t & a_{02} = 2 & a_{12} = 2 + t + 4t^3 & a_{22} = 2 + t \\ c_2 = t^{-1} + t & a_2 \ = 2 + t + 4t^3 - t^4 & & \end{array}$$

in the notation of the preceding construction. We can easily determine the elements of V using the following three observations. First, the real numbers which are of the form $\nu(v)$ for some $v \in V$ are exactly those which are equal to $\nu(a_\lambda - a_\mu)$ for some $\lambda, \mu \in L$. Secondly, whenever $v \in V$ and $\lambda \in v$ we can recover v from λ and $\nu(v)$ via the equality $v = \{\mu \in L \mid \nu(a_\lambda - a_\mu) \geq \nu(v)\}$. Finally, for every $r \in \mathbb{R} \cup \{\infty\}$ and $\lambda \in L$ we have $\{\mu \in L \mid \nu(a_\lambda - a_\mu) \geq r\} \in V$.

We begin by computing the set $R := \{\nu(a_\lambda - a_\mu) \mid \lambda, \mu \in L\}$. In our example case it is equal to $\{-2, -1, 0, 1, 3, 4, \infty\}$. We will obtain all elements of V if we determine the equivalence classes under the equivalence relations

$$\lambda \sim_r \mu :\Leftrightarrow \nu(a_\lambda - a_\mu) \geq r$$

for $r \in R$. Of course, it is advisable not to treat the computations of $L/{\sim_r}$ for different r as independent problems. Sorting the elements of R in an ascending order and computing the valuations of the vertices along the way will speed up the process considerably. In our example, we start with $r = -2$ and, of course, obtain the vertex L which has valuation -2. Then we continue with $r = -1$, where we obtain the two vertices $\{(0,1), (2,1)\}$ and $\{(1,1), (0,2), (1,2), (2,2), 2\}$ with valuations 0 and -1, respectively. That $\{(0,1), (2,1)\}$ has valuation 0 tells us that it will also appear as equivalence class for $r = 0$, the others being $\{(1,1)\}$ and $\{(0,2), (1,2), (2,2), 2\}$ with valuations ∞ and 1, respectively. Continuing like this, we see that we get the tropical curve depicted in Figure 1.1 on the left. The lengths of the bounded edges can be computed directly from the valuations of their vertices. The directions of the edges can be read off from their vertices. The image of $\Gamma_{\mathcal{C}}$ is also depicted in Figure 1.1. Interestingly, it looks like a union of two tropical lines, even though it is the tropicalization of an irreducible curve. In fact, the tropical stable map $\Gamma_{\mathcal{C}}$ can be considered as two 1-marked tropical stable maps of degree 1, glued together along a contracted edge.

Remark 1.2.3. Our construction of the intrinsic tropicalization $\Gamma_{\mathcal{C}}$ of a generic logarithmic stable map $\mathcal{C} = (C, (p_\lambda), f)$ is very explicit and well-suited for computations. Another approach to defining $\mathcal{C}$ is to use the theory of skeleta of Berkovich analytifications of curves developed in [BPR15]. In this setting one can define the underlying

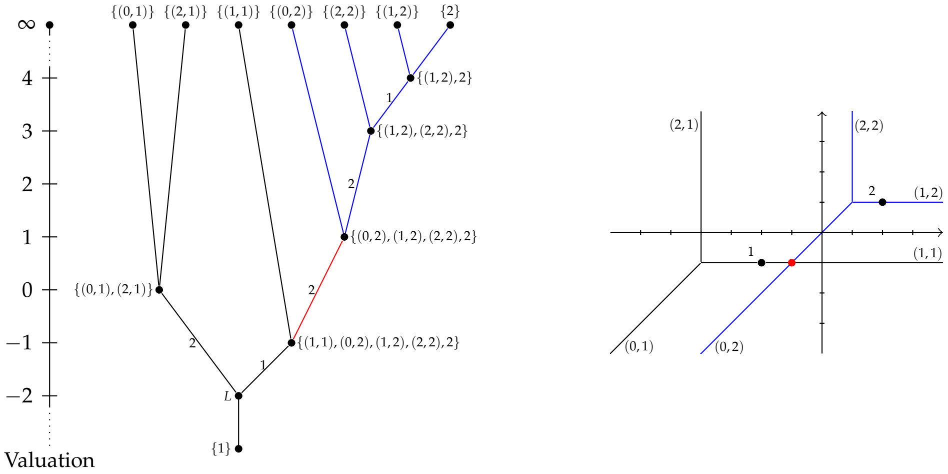


Figure 1.1: The underlying tropical curve of $\Gamma_{\mathcal{C}}$ and its image in $\{0\} \times \mathbb{R}^2 \cong \mathbb{R}^{3+1}/\mathbb{R}\mathbf{1}$

tropical curve of $\Gamma_{\mathcal{C}}$ as the closure in C^{an} of the minimal skeleton of the punctured curve $C^\circ = C \setminus \{p_\lambda \mid \lambda \in L_0\}$. The map to $\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$ is then obtained by taking the composite

$$\Gamma_C^\circ \hookrightarrow (C^\circ)^{\mathrm{an}} \to (\mathbb{T}^r)^{\mathrm{an}} \to \mathbb{R}^{r+1}/\mathbb{R}\mathbf{1},$$

where the second map is the analytification f^{an} of f, and the last map is the tropicalization map. This definition can be shown to coincide with ours.

Proposition 1.2.4. *Let $\mathcal{C} = (\mathbb{P}^1, (p_\lambda)_{\lambda \in L_0}, f) \in \mathrm{LSM}_I^\circ(r,d)$ be in standard form. Then the point in $\mathrm{TSM}_I^\circ(r,d)$ corresponding to $\Gamma_{\mathcal{C}}$ is the tropicalization of the point in $\mathrm{LSM}_I^\circ(r,d)$ corresponding to $\mathcal{C}$.*

Proof. We will use the same notation as in Construction 1.2.1. Let Γ' denote the tropical stable map corresponding to $\mathrm{trop}(\mathcal{C})$. Clearly, we have $\mathrm{ev}_{i_0}^{\mathrm{t}}(\Gamma') = \mathrm{trop}(\mathrm{ev}_{i_0}(\mathcal{C})) = \mathrm{ev}_{i_0}^{\mathrm{t}}(\Gamma_C)$, so it suffices to show that $\mathrm{ft}^{\mathrm{t}}(\Gamma') = \mathrm{ft}^{\mathrm{t}}(\Gamma_{\mathcal{C}})$. The tropical Plücker coordinates of $\mathrm{ft}^{\mathrm{t}}(\Gamma')$ are the tropicalizations of the algebraic Plücker coordinates of $\mathrm{ft}(\mathcal{C})$. So we need to consider the Plücker coordinates corresponding to the $2 \times L_0$ matrix whose λ-th column is equal to $(0,1)^T$ if $\lambda = i_0$, and $(1, a_\lambda)^T$ else. The minor $d_{(\lambda,\mu)}$ corresponding to a pair of distinct indices $\lambda, \mu \in L_0$ is equal to ± 1 if λ or μ is equal to i_0, and equal to $a_\mu - a_\lambda$ else. With this notation, $\mathrm{pl}^{\mathrm{t}}(\mathrm{ft}^{\mathrm{t}}(\Gamma'))$ is equal the class of $(\nu(d_{(\lambda,\mu)}))_{(\lambda,\mu)}$ in $\mathbb{R}^{D(L_0)}/\mathbb{R}^{L_0}$.

We compare this to the point of $\mathbb{R}^{D(L_0)}/\mathbb{R}^{L_0}$ corresponding to $\mathrm{ft}^{\mathrm{t}}(\Gamma_{\mathcal{C}}) = (\Gamma, (e_\lambda))$. For the computation of the distances of the legs of $\mathrm{ft}^{\mathrm{t}}(\Gamma_{\mathcal{C}})$ we use the notation $v_\lambda := v_{e_\lambda}$ for the inner vertex incident to e_λ. For every pair of distinct indices (λ, μ) the distance $\mathrm{dist}_\Gamma(\lambda, \mu)$ is, by definition, equal to the distance between v_λ and v_μ. If $\lambda = i_0$, we have $v_\lambda = L$ and thus $\mathrm{dist}_\Gamma(\lambda,\mu) = \nu(v_\mu) - \nu(L)$. Similarly, we get that

$\mathrm{dist}_\Gamma(\lambda, \mu) = \nu(v_\lambda) - \nu(L)$ if $\mu = i_0$. Else, let $v_{\{\lambda,\mu\}}$ be the inclusion-minimal vertex containing $\{\lambda, \mu\}$. Then

$$\mathrm{dist}_\Gamma(\lambda, \mu) = \nu(v_\lambda) + \nu(v_\mu) - 2\nu(v_{\{\lambda,\mu\}}) = \nu(v_\lambda) + \nu(v_\mu) - 2\nu(a_\mu - a_\lambda)$$

So if we define

$$d^{\mathrm{t}}_{(\lambda,\mu)} = \begin{cases} \left(\nu(L) - \nu(v_\mu)\right)/2 & \text{, if } \lambda = i_0 \\ \left(\nu(L) - \nu(v_\lambda)\right)/2 & \text{, if } \mu = i_0 \\ \nu(a_\mu - a_\lambda) - \left(\nu(v_\lambda) + \nu(v_\mu)\right)/2 & \text{, else ,} \end{cases}$$

then $\Gamma_{\mathcal{C}}$ is represented by the class of $(d^{\mathrm{t}}_{(\lambda,\mu)})$. We may replace $(d^{\mathrm{t}}_{(\lambda,\mu)})$ by an element in its $\mathbb{R}^{L_0}$ orbit without changing the represented curve. In particular, we can add $(x_\lambda)_{\lambda \in L_0}$, where

$$x_\lambda = \begin{cases} -\nu(L)/2 & \text{, if } \lambda = i_0 \\ \nu(v_\lambda)/2 & \text{, else .} \end{cases}$$

The (λ, μ)-th coordinate of the resulting representative of $\mathrm{pl}^{\mathrm{t}}(\mathrm{ft}^{\mathrm{t}}(\Gamma_{\mathcal{C}}))$ is equal to $0 = \nu(d_{(\lambda,\mu)})$, if $\lambda = i_0$ or $\mu = i_0$, and equal to $\nu(a_\mu - a_\lambda) = \nu(d_{(\lambda,\mu)})$, else. We conclude that $\Gamma_{\mathcal{C}} = \Gamma'$. □

Next we will prove that evaluation commutes with tropicalization. This will be the key step in making generic logarithmic stable maps accessible to the methods developed in the second part of this section.

Proposition 1.2.5. *Let $\mathcal{C} = (\mathbb{P}^1, (p_\lambda)_{\lambda \in L_0}, f)$ be a curve in $\mathrm{LSM}^\circ_I(r, d)$ in standard form with intrinsic tropicalization $\Gamma_{\mathcal{C}} = (\Gamma, (e_\lambda), h)$. Then $h(\Gamma^\circ) \cap G^{r+1}/G\mathbf{1} = \mathrm{trop}(f(C) \cap \mathbb{T}^r)$ and for every $i \in I$ we have $\mathrm{ev}^{\mathrm{t}}_i(\Gamma_{\mathcal{C}}) = \mathrm{trop}(\mathrm{ev}_i(\mathcal{C}))$.*

Proof. Both parts of the statement are instances of the same question, namely how to give a nice description of the tropicalization of a point $f(1 : a)$ for $a \in K$. We will acquire such a description while proving that $\mathrm{trop}(f(C) \cap \mathbb{T}^r)$ is a subset of $h(\Gamma^\circ) \cap G^{r+1}/G\mathbf{1}$. Let $p \in \mathbb{P}^1$ be not equal to any of the p_{ij}. If $p = p_{i_0} = (0 : 1)$, then $\mathrm{trop}(f(p)) = h(L)$. Thus, we may assume that $p = (1 : a)$ for some $a \in K$. The image $f(1 : a)$ of p in $\mathbb{P}^r$ is represented by

$$\left(c_0 \prod_{i=1}^{d} (a - a_{0i}), \ldots, c_r \prod_{i=1}^{d} (a - a_{ri}) \right),$$

where we use the same notation as in Construction 1.2.1. Therefore, the tropicalization $\mathrm{trop}(f(p))$ is represented by

$$\left(\nu(c_0) + \sum_{i=1}^{d} \nu(a - a_{0i}), \ldots, \nu(c_r) + \sum_{i=1}^{d} \nu(a - a_{ri}) \right).$$

Let $r_1 < \cdots < r_k$ be the elements of $\{\nu(a - a_\lambda) \mid \lambda \in J\}$, and for each $1 \le i \le k$ define $D_i = \{\lambda \in J \mid \nu(a - a_\lambda) \ge r_i\}$. Then the expression above can be written as

$$\sum_{i=0}^{r} \nu(c_i) s_i + \sum_{i=2}^{k} (r_i - r_{i-1}) s_{D_i}, + \underbrace{r_1 s_{D_1}}_{=0}, \tag{1.2.2}$$

where $s_{D_1} = 0$ because it is represented by d-times the all one vector. Let $v_i \in V$ be the inclusion-minimal vertex containing D_i. In order to prove that $\operatorname{trop}(f(p))$ is the image of a point of Γ on the path from L to v_k we will collect some properties of the v_i. First of all, the choice of v_i ensures that $v_i \supseteq v_j$ for $i \le j$. Moreover, it follows immediately from the definition of V that $\nu(v_i) = \nu(D_i)$ for every $1 \le i \le k$. Using the valuation properties of ν we also see directly that $\nu(D_i) \ge r_i$ for all i. We claim that for $i < k$ we even have equality. Assume the opposite, in which case $\nu(D_i) > r_i$. Let $\lambda \in D_i$ such that $\nu(a - a_\lambda) = r_i$, and let $\mu \in D_k$. Then on the one hand we have

$$\nu(a_\lambda - a_\mu) \ge \nu(D_i) > r_i = \nu(a - a_\lambda),$$

and hence $\nu(a - a_\mu) = r_i$ by the valuation properties, whereas on the other hand we have $\nu(a - a_\mu) = r_k$ by choice of μ, which obviously contradicts the fact that $r_i \ne r_k$.

Next we show that $v_i \cap J = D_i$. One inclusion is clear, so let $\lambda \in v_i \cap J$, and choose an arbitrary $\mu \in D_i$. We have

$$\nu(a - a_\lambda) = \nu((a - a_\mu) + (a_\mu - a_\lambda)) \ge \min\{\nu(a - a_\mu), \nu(a_\mu - a_\lambda)\}.$$

As $\nu(a - a_\mu) \ge r_i$ by the definition of D_i, and $\nu(a_\mu - a_\lambda) \ge \nu(v_i) = \nu(D_i) \ge r_i$, this minimum is greater or equal to r_i, proving that $\lambda \in D_i$.

Now we show that for any $1 \le i < k$, there does not exist a vertex $v \in V$ such that $v_i \supsetneq v \supseteq v_{i+1}$ and $v \cap J \supsetneq v_{i+1} \cap J = D_{i+1}$. We assume the opposite again. Let $\lambda \in v \cap J \setminus v_{i+1}$, and let $\mu \in D_{i+1}$ arbitrary. By choice of λ, we have $\lambda \in D_i \setminus D_{i+1}$, and hence $\nu(a_\lambda - a) = r_i$. Furthermore, $\nu(a - a_\mu) \ge r_{i+1} > r_i$. Thus, $\nu(a_\lambda - a_\mu) = r_i$ by the valuation properties. But both λ and μ are in v, leading to the contradiction

$$r_i = \nu(a_\lambda - a_\mu) \ge \nu(v) > \nu(v_i) = r_i.$$

Let $L = w_0 \supsetneq w_1 \supsetneq \cdots \supsetneq w_l = v_k$ be the path from L to v_k, and for $1 \le i \le k$ let j_i denote the index such that $w_{j_i} = v_i$. Furthermore, let s be the maximal index such that $\nu(w_s) < r_k$. Since $\nu(v_k) \ge r_k$, and $\nu(v_{k-1}) < r_k$ we have $j_{k-1} \le s < l$ and

$$0 < r_k - \nu(w_s) \le \nu(w_{s+1}) - \nu(w_s) = \ell(\{w_s, w_{s+1}\}).$$

Using all of this we can rewrite the second summand in Expression 1.2.2 as

$$\begin{aligned}\sum_{i=2}^{k} (r_i - r_{i-1}) s_{D_i} &= \sum_{i=2}^{k-1} (\nu(v_i) - \nu(v_{i-1})) s_{D_i} + (r_k - \nu(v_{k-1})) s_{D_k} \\ &= \sum_{i=2}^{k-1} \sum_{j=j_{i-1}+1}^{j_i} \ell(\{w_{j-1}, w_j\}) s_{w_i} + \sum_{j=j_{k-1}+1}^{s} \ell(\{w_{j-1}, w_j\}) s_{w_j} \\ &\quad + (r_k - \nu(w_s)) s_{w_{s+1}}.\end{aligned}$$

Adding $\sum_i \nu(c_i)s_i$ to both sides we get that

$$\operatorname{trop}(f(p)) = q_{w_s} + (r_k - \nu(w_s))\, \mathrm{v}(w_s, \{w_s, w_{s+1}\}),$$

which clearly is the image under h of a point on the edge from w_s to w_{s+1}.

We have just seen how to construct a point $x \in \Gamma^\circ$ for a given point $p \in \mathbb{P}^1$ such that $\operatorname{trop}(f(p)) = h(x)$. Let us apply this construction to the case when $p = p_\iota$ for some $\iota \in I \setminus \{i_0\}$ before proving the other inclusion $h(\Gamma^\circ) \cap G^{r+1}/G\mathbf{1} \subseteq \operatorname{trop}(f(C) \cap \mathbb{T}^r)$. With the notations as above, let $\lambda \in D_k$, and let $v \in V$ be the inclusion-minimal vertex containing $\{\iota, \lambda\}$. It has valuation $\nu(v) = \nu(\{\iota, \lambda\}) = r_k$, and since $w_{s+1} \cap v \neq \emptyset$ we either have $w_{s+1} \subseteq v$ or $v \subseteq w_{s+1}$. Because $\nu(w_{s+1}) \geq r_k$ by construction, we must have $w_{s+1} \subseteq v$. If v strictly contained w_{s+1}, then it would also contain w_s and therefore satisfy $r_k = \nu(v) \leq \nu(w_s) < r_k$, a contradiction! We deduce that $v = w_{s+1}$ and hence that $\nu(w_{s+1}) = r_k$ and $\iota \in w_{s+1}$. Furthermore, we claim that w_{s+1} is inclusion-minimal with the property of containing $\{\iota\}$ and having nonempty intersection with J. Assume the opposite, that is that there exists a vertex $\iota \in u \in V$ with $u \subsetneq w_{s+1}$ and $u \cap J \neq \emptyset$. Then for every $\mu \in u \cap J$ we have $\nu(a_\iota - a_\mu) \geq \nu(u) > \nu(w_{s+1}) = r_k$, which contradicts the definition of r_k! This proves that

$$\operatorname{ev}_\iota^{\mathrm{t}}(\Gamma_{\mathcal{C}}) = h(w_{s+1}) = \operatorname{trop}(f(p_\iota)) = \operatorname{trop}(\operatorname{ev}_\iota(\mathcal{C}))$$

for $\iota \in I \setminus \{i_0\}$, and since it is trivially true for $\iota = i_0$ we even get the equality $\operatorname{ev}_\iota^{\mathrm{t}}(\Gamma) = \operatorname{trop}(\operatorname{ev}_\iota(\mathcal{C}))$ for all $\iota \in I$.

We continue to prove the equality $h(\Gamma^\circ) \cap G^{r+1}/G\mathbf{1} = \operatorname{trop}(f(C) \cap \mathbb{T}^r)$. Let x be a point in $h(\Gamma^\circ) \cap G^{r+1}/G\mathbf{1}$. Then it is of the form $x = q_v + t\,\mathrm{v}(v, \{v, w\})$ for an appropriate edge $\{v \supset w\}$ and some $t \in G$ with $0 < t \leq \nu(w) - \nu(v)$. If $J \subseteq w$, then $x = \operatorname{trop}(f(p_{i_0}))$ and we are done, so assume $w \cap J \subsetneq J$. After replacing w with some vertex on the path from w to L we can also assume that $w \cap J \neq \emptyset$.

In the proof of the other inclusion we have seen that we have to construct an $a \in K$ such that $r := \max\{\nu(a - a_\lambda) \mid \lambda \in J\}$ is equal to $\nu(v) + t$, and the set $D := \{\lambda \in J \mid \nu(a - a_\lambda) \geq r\}$ is contained in w. To do this, choose $\mu \in w \cap J$. Because $\nu(a_\mu - a_\lambda) \geq \nu(v) + t$ for all $\lambda \in w \cap J$, and κ is infinite, there exists $\overline{a} \in K$ such that $\nu(\overline{a} + a_\mu - a_\lambda) = \nu(v) + t$ for all $\lambda \in w \cap J$. We claim that $a := \overline{a} + a_\mu$ has the desired properties. By construction of a, we have

$$\nu(a - a_\lambda) = \nu(\overline{a} + a_\mu - a_\lambda) = \nu(v) + t$$

for all $\lambda \in w \cap J$. On the other hand, if $\lambda \in J \setminus w$, then $\nu(a_\mu - a_\lambda) = \nu(w \cup \{\lambda\})$. This is equal to the valuation of a vertex which strictly contains w, and therefore contains v. In particular, we have $\nu(a_\mu - a_\lambda) \leq \nu(v) < \nu(v) + t$, and thus

$$\nu(a - a_\lambda) = \nu(\overline{a} + a_\mu - a_\lambda) < \nu(v) + t.$$

As a consequence, we have $r = \nu(v) + t$ and $D = w \cap J \subseteq w$. With what we saw in the proof of the other inclusion this yields $x = \operatorname{trop}(f(1 : a))$.

$\square$

Example 1.2.6. We want to apply the method described at the beginning of the preceding proof to compute a point x of the intrinsic tropicalization of the curve of Example 1.2.2 such that $h(x) = \operatorname{trop}(f(1 : 2+t+t^2))$. We have

$$\{\nu(2+t+t^2-a_\lambda) \mid \lambda \in J\} = \{-2,-1,1,2\},$$

that is in the notation of the preceding proof we have $k = 4$ and r_i, D_i and v_i as follows:

i	1	2	3	4
r_i	-2	-1	1	2
D_i	J	$\{(1,1),(0,2),(1,2),(2,2)\}$	$\{(0,2),(1,2),(2,2)\}$	$\{(1,2),(2,2)\}$
v_i	$L = J \cup \{2\}$	$D_2 \cup \{2\}$	$D_3 \cup \{2\}$	$D_4 \cup \{2\}$
$\nu(v_i)$	-2	-1	1	3

Note that we can easily read off the v_i from Figure 1.1. The path $L = w_0 \supsetneq \cdots \supsetneq w_l = v_k$ just consists of the v_i in this example, that is $l = 3$ and $w_i = v_{i+1}$. The largest index s such that $\nu(w_s) < r_k$ is equal to 2. We conclude that the construction yields a point on the edge from v_3 to v_4. More precisely, because $r_k - \nu(v_3) = 1 = \frac{1}{2}\ell(\{v_3, v_4\})$ the point is exactly in the middle of that edge and, as can be seen in Figure 1.1, is mapped to $(0 : 0 : 0)$.

Because every algebraic curve is equivalent to one in standard form, Propositions 1.2.4 and 1.2.5 yield the following corollary.

Corollary 1.2.7. *For every $i \in I$ the diagram*

$$\begin{array}{ccc} \mathrm{LSM}_I^\circ(r,d) & \xrightarrow{\mathrm{ev}_i} & \mathbb{T}^r \\ \big\downarrow \mathrm{trop} & & \big\downarrow \mathrm{trop} \\ \mathrm{TSM}_I^\circ(r,d) & \xrightarrow{\mathrm{ev}_i^t} & \mathbb{R}^{r+1}/\mathbb{R}\mathbf{1} \end{array}$$

is commutative.

We finish this section with another result on the evaluation maps.

Proposition 1.2.8. *For every $\iota \in I$ the evaluation map $\mathrm{ev}_\iota : \mathrm{LSM}_I^\circ(r,d) \to \mathbb{T}^r$ can be extended to a morphism of tori $\overline{\mathrm{ev}}_\iota : \mathbb{G}_m^{D(L_0)}/\mathbb{G}_m^{L_0} \times \mathbb{T}^r \to \mathbb{T}^r$.*

Proof. The morphism of tori

$$\begin{aligned} \mathbb{G}_m^{D(L_0)} \times \mathbb{T}^r &\to \mathbb{T}^r \\ \left((d_{(\lambda,\mu)}), (c_i)_{0\le i\le r}\right) &\mapsto \left(c_i \prod_{j=1}^{d} \frac{d_{((i,j),\iota)}}{d_{((i,j),\iota_0)}}\right)_{0\le i\le r} \end{aligned}$$

obviously induces a morphism $\overline{\text{ev}}_\iota : \mathbb{G}_m^{D(L_0)}/\mathbb{G}_m^{L_0} \times \mathbb{T}^r \to \mathbb{T}^r$. All there is left to show is that it coincides with ev_ι on $\text{LSM}_I^\circ(r,d)$. So let $\mathcal{C} = (\mathbb{P}^1, (p_\lambda), f)$ be a representative in standard form of a point in $\text{LSM}_I^\circ(r,d)$. Using the notation of Construction 1.2.1, the proof of Proposition 1.2.4 shows us that $\mathcal{C}$ is represented by the point $((d_{(\lambda,\mu)}), (c_i)) \in \mathbb{G}_m^{D(L_0)} \times \mathbb{T}^r$, where $d_{(\lambda,\mu)}$ is equal to -1 if $\lambda = i_0$, equal to 1 if $\mu = i_0$, and equal to $a_\mu - a_\lambda$ else. Thus $\overline{\text{ev}}_\iota(\mathcal{C})$ is represented by $\left(c_i \prod_{j=1}^d (a_\iota - a_{ij})\right)_{0\le i\le r}$. This is exactly $f(1 : a_\iota)$, which is equal to $\text{ev}_\iota(\mathcal{C})$ by definition. □

1.2.2 Tropicalizing Generic Conditions

Knowing that evaluation commutes with tropicalization, we are a big step closer to proving that

$$\text{Trop}\left(\text{ev}_1^{-1}\{P_1\} \cap \ldots \cap \text{ev}_n^{-1}\{P_n\}\right) = \text{ev}_1^{\text{t}*}[P_1^{\text{t}}] \cdots \text{ev}_n^{\text{t}*}[P_n^{\text{t}}] \cdot \text{TSM}_I^\circ(r,d)$$

for generic points P_i tropicalizing to P_i^{t}. What is still left to do is to show that taking pull-backs of points which are generic in an appropriate sense commutes with taking tropicalizations. We will formulate the results of this subsection without reference to $\text{LSM}_I^\circ(r,d)$ or $\text{TSM}_I^\circ(r,d)$ so that we can also apply them in similar situations, for example when considering maps to general toric target. Our setup is as follows. Let V be a subvariety of an algebraic torus T, and let $V^{\text{t}} = \text{Trop}(V) \subseteq N_\mathbb{R}$, where N is the lattice of 1-psgs of T. Let $f_i\colon V \to T_i$ for $1 \le i \le n$ be morphisms to algebraic tori T_i which are induced by dominant morphisms $\overline{f}_i\colon T \to T_i$ of algebraic tori. Tropicalizing these morphisms we obtain integral linear maps $\overline{f}_i^{\text{t}}\colon N_\mathbb{R} \to (N_i)_\mathbb{R}$, where N_i is the lattice of 1-psgs of T_i. Restricting them to V^{t} yields morphisms $f_i^{\text{t}}\colon V^{\text{t}} \to (N_i)_\mathbb{R}$.

Note that to assume that the $\overline{f}_i^{\text{t}}$ are dominant is not really a restriction as we can always replace the T_i by suitable subtori. Let us also point out that for a morphism of tori f the notions of dominance, surjectivity, and flatness are all equivalent to the fact that $\text{Trop}(f)$ is surjective.

What we want to prove in this more abstract setting is that

$$\text{Trop}\left(f_1^{-1}\{X_1\} \cap \ldots \cap f_n^{-1}\{X_n\}\right) = f_1^{\text{t}*}[X_1^{\text{t}}] \cdots f_n^{\text{t}*}[X_n^{\text{t}}] \cdot V^{\text{t}} \tag{1.2.3}$$

for generic $X_i \subseteq T^i$ tropicalizing to $X_i^{\text{t}} \subseteq (N_i)_\mathbb{R}$. As a first step, we prove this for $n = 1$ and $V = T$. It turns out that no genericity condition is needed in this case.

Proposition 1.2.9. *Let $f : T \to T'$ be a dominant morphism of algebraic tori, let N and N' be the lattices of 1-psgs of T and T', respectively, and let $f^{\text{t}} = \text{Trop}(f) : N_\mathbb{R} \to N'_\mathbb{R}$ be the tropicalization of f. Furthermore, let $X \subseteq T'$ be a pure-dimensional subscheme of T'. Then*

$$\text{Trop}(f^{-1}X) = f^{\text{t}*}\,\text{Trop}(X),$$

that is tropicalization commutes with flat pull-back.

Proof. Let M and M' denote the character lattices of T and T', respectively. Since f is dominant, the induced morphism $f^*: M' \to M$ is injective and we can assume that M' is a subgroup of M. Let $\widetilde{M}$ denote the saturation of M' in M. Then the inclusion $M' \hookrightarrow M$ factors through $\widetilde{M}$. Denoting the torus associated to $\widetilde{M}$ by $\widetilde{T} = \mathrm{Spec}(K[\widetilde{M}])$, we see that we can write f as the composite of two dominant morphism $T \to \widetilde{T}$ and $\widetilde{T} \to T'$ of which the first one splits and the second one is finite. As both the algebraic and the tropical pull-backs are functorial, it suffices to assume that f is either finite or splits.

First assume that f splits. In this case, we can assume that $T = T' \times \overline{T}$ for some torus $\overline{T}$, and f is the projection onto the first coordinate. Then $f^{-1}X = X \times \overline{T}$ and, denoting the lattice of 1-psgs of $\overline{T}$ by $\overline{N}$, it is immediate that the underlying set of its tropicalizations is equal to $|\operatorname{Trop}(X)| \times \overline{N}_{\mathbb{R}}$, which is the underlying set of the pull-back of $\operatorname{Trop}(X)$. As for multiplicities, let $w = (w', \overline{w}) \in N'_{\mathbb{R}} \times \overline{N}_{\mathbb{R}}$. After an appropriate extension of scalars we may assume that there is a $t = (t', \overline{t}) \in T' \times \overline{T}$ that tropicalizes to w. The pull-back of $t'^{-1}X$ along f is equal to $t^{-1}(f^{-1}X)$. Together with the flatness of f this implies $\mathrm{in}_w(f^{-1}X) \cong f_\kappa^{-1}(\mathrm{in}_{w'}(X))$ by [Gub13, Cor. 4.5], where f_κ is the induced morphism $T_\kappa \to T'_\kappa$ between the initial degenerations of the tori. Because flat pull-backs are compatible with pulling back subschemes ([Ful98, Lemma 1.7.1]), the sum of the multiplicities of the components of $\mathrm{in}_w(f^{-1}X)$ is equal to the sum of the multiplicities of the components of $\mathrm{in}_{w'}(X)$. Since these sums are, by definition, the multiplicities of the tropicalizations of $f^{-1}X$ and X at w and w', respectively, we get the desired result that $f^{\mathrm{t}*}\operatorname{Trop}(X) = \operatorname{Trop}(X) \times \overline{N}_{\mathbb{R}} = \operatorname{Trop}(f^{-1}X)$ as tropical cycles.

Now suppose that f is finite. Then its degree is equal to the index $[M : M']$. By [Ful98, Ex. 1.7.4], the push-forward of the cycle associated to $f^{-1}X$ is equal to $f_*[f^{-1}X] = f_*(f^*[X]) = [M : M'] \cdot [X]$. Now we tropicalize this equation. As the tropicalization of a torus subscheme is equal to the tropicalization of its associated cycle [Gub13, Remark 13.12] and tropicalization commutes with push-forward [Gub13, Thm. 13.17] we get $f_*^{\mathrm{t}}\operatorname{Trop}(f^{-1}X) = [M : M'] \cdot \operatorname{Trop}(X)$. On the other hand, using tropical intersection theory we obtain the equation

$$
\begin{aligned}
f_*^{\mathrm{t}} f^{\mathrm{t}*}\operatorname{Trop}(X) = f_*^{\mathrm{t}}(f^{\mathrm{t}*}\operatorname{Trop}(X) \cdot N_{\mathbb{R}}) = \operatorname{Trop}(X) \cdot f_*^{\mathrm{t}} N_{\mathbb{R}} = \\
= \operatorname{Trop}(X) \cdot ([M : M'] \cdot N'_{\mathbb{R}}) = [M : M'] \cdot \operatorname{Trop}(X),
\end{aligned}
$$

where the second equality uses the projection formula [FR13, Thm. 8.3]. We see that $\operatorname{Trop}(f^{-1}X)$ and $f^{\mathrm{t}*}(\operatorname{Trop}(X))$ have the same push-forward under f^{t}. But since f^{t} is an isomorphism of $\mathbb{R}$-vector spaces, the push-forward $f_*^{\mathrm{t}}: Z_*(N_{\mathbb{R}}) \to Z_*(N'_{\mathbb{R}})$ is injective. We conclude that $\operatorname{Trop}(f^{-1}X) = f^{\mathrm{t}*}(\operatorname{Trop}(X))$, finishing the proof. □

When $V \neq T$ and we consider more than one morphism, Equation (1.2.3) will not hold for arbitrary X_i. This is caused by the fact that the tropicalization of the intersection of two torus subschemes does not coincide with the stable intersection of their tropicalizations in general. But as mentioned earlier, they do coincide when we translate one of the two schemes by a general point tropicalizing to 0, that is with any point in some nonempty open subset of $\mathrm{trop}^{-1}\{0\}$. To work with this sense of generality we will need the following two lemmas.

Lemma 1.2.10. *Let $f : T \to T'$ be a dominant morphism of tori. Then the induced map*

$$T(K) \supseteq \mathrm{trop}_T^{-1}\{0\} \to \mathrm{trop}_{T'}^{-1}\{0\} \subseteq T'(K)$$

is open.

Proof. Recall from Section 1.1.3 that the topology on $\mathrm{trop}_T^{-1}\{0\}$ is defined by identifying it with the set $\mathcal{T}(R)$ of R-valued points of the canonical R-model $\mathcal{T}$ of T, and then taking the initial topology with respect to the reduction map $\mathcal{T}(R) \to T_\kappa(\kappa)$. Let M and M' denote the character lattices of T and T', respectively. Then the set $\mathcal{T}(R)$ can be naturally identified with $\mathrm{Hom}(M, R^*)$ and, analogously, $\mathcal{T}'(R)$ with $\mathrm{Hom}(M', R^*)$, where $\mathcal{T}'$ denotes the canonical R-model of T'. Further identifying $T_\kappa(\kappa)$ and $T'_\kappa(\kappa)$ with $\mathrm{Hom}(M, \kappa^*)$ and $\mathrm{Hom}(M', \kappa^*)$, respectively, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(M, R^*) & \xrightarrow{f_R} & \mathrm{Hom}(M', R^*) \\ \downarrow r & & \downarrow r' \\ \mathrm{Hom}(M, \kappa^*) & \xrightarrow{f_\kappa} & \mathrm{Hom}(M', \kappa^*), \end{array}$$

the vertical morphisms r and r' being the reduction maps. We observe that to show the openness of f_R, it is sufficient to prove that f_κ is open and that $f_R(r^{-1}U) = (r')^{-1}f_\kappa(U)$ for all open sets $U \subseteq T_\kappa(\kappa)$. The first of these assertions follows immediately from f_κ being a dominant morphism of tori over an algebraically closed field. For the latter one it suffices to show that the map from $\mathrm{Hom}(M, R^*)$ into the set-theoretic fiber product of $\mathrm{Hom}(M, \kappa^*)$ and $\mathrm{Hom}(M', R^*)$ over $\mathrm{Hom}(M', \kappa^*)$ is surjective. To show this consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(M, 1+\mathfrak{m}) & \longrightarrow & \mathrm{Hom}(M, R^*) & \longrightarrow & \mathrm{Hom}(M, \kappa^*) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}(M', 1+\mathfrak{m}) & \longrightarrow & \mathrm{Hom}(M', R^*) & \longrightarrow & \mathrm{Hom}(M', \kappa^*) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0. & & \end{array}$$

It has exact rows because M and M' are free, and the last two columns are exact since R^* and κ^* are divisible. If we can show that the first column is exact as well, the rest of the proof is just an easy diagram chase. Thus, it suffices to show that $1 + \mathfrak{m}$ is divisible. Let $a \in 1 + \mathfrak{m}$ and $n \in \mathbb{N}$. Then the polynomial $x^n - a \in R[x]$ factors in $R[x]$ as $(x - b_1)\cdots(x - b_n)$ for appropriate $b_i \in R$ because K is algebraically closed and R is integrally closed in K. Reducing modulo $\mathfrak{m}$ yields $x^n - 1 = (x - b_1)\cdots(x - b_n)$ in

$\kappa[x]$. Plugging in 1 we get that $b_i \in 1 + \mathfrak{m}$ for some i. Hence, a has an n-th root in $1 + \mathfrak{m}$, proving the desired divisibility. □

Lemma 1.2.11. *Let $X_1, \ldots, X_n$ be subschemes of the torus T. Then for general R-integral points $(t_2, \ldots, t_n)$ of T^{n-1} we have*

$$\operatorname{Trop}(X_1 \cap t_2 X_2 \cap \cdots \cap t_n X_n) = \operatorname{Trop}(X_1) \cdots \operatorname{Trop}(X_n).$$

Proof. For $n = 2$ this is the statement of [OP13, Thm. 5.3.3]. To prove the general case let $\Delta : T \to T^n$ be the diagonal embedding, which makes T a subscheme of T^n. It corresponds to the diagonal embedding $\Delta^{\mathrm{t}} : N_{\mathbb{R}} \to N_{\mathbb{R}}^n$ on the tropical side. By the $n = 2$ case, for every general R-integral point $t = (t_1, \ldots, t_n) \in T^n(K)$ we have

$$\operatorname{Trop}(\Delta \cap t(X_1 \times \cdots \times X_n)) = \operatorname{Trop}(\Delta) \cdot \operatorname{Trop}(X_1 \times \cdots \times X_n).$$

Since $\Delta \cap t(X_1 \times \cdots \times X_n)$ is the push-forward of $t_1 X_1 \cap \cdots \cap t_n X_n$, and tropicalization commutes with push-forwards, the term on the left hand side is equal to $\Delta^{\mathrm{t}}_*(\operatorname{Trop}(t_1 X_1 \cap \cdots \cap t_n X_n))$. On the other hand, by [OP13, Prop. 5.2.2] the term on the right is equal to $\Delta^{\mathrm{t}}_*(\operatorname{Trop}(X_1) \cdots \operatorname{Trop}(X_n))$. Because Δ^{t}_* is one-to-one, it follows that $\operatorname{Trop}(t_1 X_1 \cap \cdots \cap t_n X_n) = \operatorname{Trop}(X_1) \cdots \operatorname{Trop}(X_n)$. After multiplying by t_1^{-1} we can assume that $t_1 = 1$, and Lemma 1.2.10 ensures that the resulting R-rational point $(t_2, \ldots, t_n) \in T^{n-1}(K)$ still is generic. □

We are now ready to prove the main theorem of this section.

Theorem 1.2.12. *With the notation as before, let X_i for $1 \le i \le n$ be a pure-dimensional closed subscheme of T_i with tropicalization $\operatorname{Trop}(X_i) = X_i^{\mathrm{t}}$. Then for generic R-integral points $(t_1, \ldots, t_k) \in \prod_i T_i(K)$ we have*

$$\operatorname{Trop}(f_1^{-1}(t_1 X_1) \cap \cdots \cap f_n^{-1}(t_n X_n)) = f_1^{\mathrm{t}*}(X_1^{\mathrm{t}}) \cdots f_n^{\mathrm{t}*}(X_n^{\mathrm{t}}) \cdot V^{\mathrm{t}}.$$

In particular, if $P^{\mathrm{t}} = (P_1^{\mathrm{t}}, \ldots, P_n^{\mathrm{t}}) \in \prod_i (N_i)_G$, then for generic points $P = (P_1, \ldots, P_n)$ of $\operatorname{trop}^{-1}\{P^{\mathrm{t}}\}$ we have

$$\operatorname{Trop}\left(f_1^{-1}\{P_1\} \cap \cdots \cap f_n^{-1}\{P_n\}\right) = f_1^{\mathrm{t}*}[P_1^{\mathrm{t}}] \cdots f_n^{\mathrm{t}*}[P_n^{\mathrm{t}}] \cdot V^{\mathrm{t}}.$$

Proof. Proposition 1.2.9 and Lemma 1.2.11 tell us that for a generic R-integral point $(s_1, \ldots, s_n) \in T^n(K)$ we have

$$\operatorname{Trop}\left(s_1 \cdot \overline{f}_1^{-1}(X_1) \cap \cdots \cap s_n \cdot \overline{f}_n^{-1}(X_n) \cap V\right) = \overline{f}_1^{\mathrm{t}*}(X_1^{\mathrm{t}}) \cdots \overline{f}_n^{\mathrm{t}*}(X_n^{\mathrm{t}}) \cdot V^{\mathrm{t}}. \tag{1.2.4}$$

Let $t = (t_1, \ldots, t_n) = (\overline{f}_1(s_1), \ldots, \overline{f}_n(s_n)) \in \prod_i T_i(K)$. As all evaluation maps are dominant, we can apply Lemma 1.2.10 and see that t is again generic. Furthermore, the left-hand side of Equation 1.2.4 is equal to

$$\operatorname{Trop}\left(\overline{f}_1^{-1}(t_1 X_1) \cap \cdots \cap \overline{f}_n^{-1}(t_n X_n) \cap V\right) = \operatorname{Trop}\left(f_1^{-1}(t_1 X_1) \cap \cdots \cap f_n^{-1}(t_n X_n)\right).$$

As the right-hand side of Equation 1.2.4 is equal to $f_1^{t*}(X_1^t)\cdots f_n^{t*}(X_n^t)\cdot V^t$, this proves the main statement of the theorem.

For the "in particular" statement, let $P^t = (P_1^t,\dots,P_n^t) \in \prod_i (N_i)_G$, and choose an arbitrary $P = (P_1,\dots,P_n) \in \mathrm{trop}^{-1}\{P^t\}$. By what we have already proved, we have

$$\mathrm{Trop}\left(f_1^{-1}\{t_1P_1\}\cap\cdots\cap f_n^{-1}\{t_nP_n\}\right) = f_1^{t*}[P_1^t]\cdots f_n^{t*}[P_n^t]\cdot V^t$$

for generic $t=(t_1,\dots,t_n)\in\mathrm{trop}^{-1}\{0\}$. Since $\mathrm{trop}^{-1}\{P^t\}$ is a $\mathrm{trop}^{-1}\{0\}$-torsor, the elements of the form tP for generic R-rational points t are generic in $\mathrm{trop}^{-1}\{P^t\}$. $\square$

An important special case of Theorem 1.2.12 is when all X_i are points and the set $f^{-1}X_1\cap\ldots\cap f^{-n}X_n$ is finite. Then the tropicalization map gives an interpretation of the multiplicities of the tropical cycle $f_1^{t*}(X_1^t)\cdots f_n^{t*}(X_n^t)\cdot V^t$.

Corollary 1.2.13. *With the notation as before, assume that V is integral, and that $\dim(V) = \sum_i \dim(T_i)$. Then for every $P^t = (P_i^t) \in \prod_i (N_i)_G$, there exists a $P = (P_i) \in \prod_i T_i(K)$ with $\mathrm{trop}(P) = P^t$ such that $f_1^{-1}\{P_1\}\cap\cdots\cap f_n^{-1}\{P_n\}$ is reduced and*

$$|f_1^{-1}\{P_1\}\cap\cdots\cap f_n^{-1}\{P_n\}| = \deg(f_1^{t*}[P_1^t]\cdots f_n^{t*}[P_n^t]\cdot V^t),$$

and such that the multiplicity of every point $Q^t \in \left|f_1^{t*}[P_1^t]\cdots f_n^{t*}[P_n^t]\cdot V^t\right|$ is equal to the number of points in $f_1^{-1}\{P_1\}\cap\cdots\cap f_n^{-1}\{P_n\}$ tropicalizing to it. Furthermore, the set of such P is dense in $\prod_i T_i(K)$. More precisely, P can be any point in the intersection of a nonempty open subset of $\mathrm{trop}^{-1}\{P^t\}$ and a nonempty Zariski open subset of $\prod_i T_i$.

Proof. Let $P^t = (P_i^t)\in\prod_i (N_i)_G$. Then by Theorem 1.2.12 there is a nonempty open subset $U\subseteq \mathrm{trop}^{-1}\{P^t\}$ such that

$$\mathrm{Trop}(f_1^{-1}\{P_1\}\cap\cdots\cap f_n^{-1}\{P_n\}) = f_1^{t*}[P_1^t]\cdots f_n^{t*}[P_n^t]\cdot V^t$$

for every $(P_1,\dots,P_n)\in U$. Using a Bertini-type argument, we find a dense open subset $U'\subseteq\prod_i T_i$ such that $f_1^{-1}\{P_1\}\cap\cdots\cap f_n^{-1}\{P_n\}$ is a reduced finite subscheme of V for all $(P_1,\dots,P_n)\in U'$. By [OP13, Thm. 4.2.5], every nonempty open subset of a fiber of the tropicalization map $\mathrm{trop}:\prod T_i\to\prod(N_i)_{\mathbb{R}}$ is Zariski dense in $\prod T_i$. Hence, $U\cap U'$ is dense in $\prod T_i$, as well. Observing that the tropicalization of a reduced finite subscheme of a torus is nothing but the union of the tropicalizations of its points, where each point has as multiplicity the number of points tropicalizing to it, finishes the proof. $\square$

Remark 1.2.14. The condition on the point P in the formulation of Corollary 1.2.13 asks that P is chosen in the intersection of a nonempty open subset of the fiber $\mathrm{trop}^{-1}\{P^t\}$ and a nonempty Zariski open subset of $\prod_i T_i$. A nonempty open subset of $\mathrm{trop}^{-1}\{P^t\}$ is not Zariski open, so even though such a set is very large (and Zariski dense) this does *not* mean that P can be chosen generically in $\prod_i T_i$ in the usual sense. However, Corollary 1.2.13 is also true when the valuation on K is trivial. In this case 0 is the only point of $\prod_i (N_i)_G$ and $\mathrm{trop}^{-1}\{0\}$ is equal to $\prod_i T_i(K)$. Furthermore, $K = R = \kappa$

so that the topology on $\operatorname{trop}^{-1}\{0\}$ induced by the reduction map coincides with the Zariski topology on $\prod_i T_i(K)$. We conclude that in this situation we have

$$|f_1^{-1}\{P_1\} \cap \dots \cap f_n^{-1}\{P_n\}| = \deg(f_1^{\mathrm{t}*}[0] \cdots f_n^{\mathrm{t}*}[0] \cdot V^{\mathrm{t}}).$$

for $(P_i)_i \in \prod_i T_i(K)$ generic in the usual sense. In particular, the number of points $|f_1^{-1}\{P_1\} \cap \dots \cap f_n^{-1}\{P_n\}|$ is invariant from the choice of the P_i (as long as they are generic).

Example 1.2.15. Let d, r, and n be positive integers. Consider the case where $V = \mathrm{LSM}_n^{\circ}(r,d)$ with the embedding given in Subsection 1.1.4, and $f_i = \mathrm{ev}_i$. Then we have seen that $V^{\mathrm{t}} = \mathrm{Trop}(\mathrm{LSM}_n^{\circ}(r,d)) = \mathrm{TSM}_n^{\circ}(r,d)$, embedded as in Subsection 1.1.2. Each of the evaluation maps is dominant as we can freely choose where one of the marked points is mapped to, and by Proposition 1.2.8 they can be extended to morphisms $\overline{\mathrm{ev}}_i$ of algebraic tori. By Proposition 1.2.5, the morphisms f_i^{t} are equal to the tropical evaluation maps $\mathrm{ev}_i^{\mathrm{t}}$. Therefore, we have

$$\mathrm{Trop}\left(\mathrm{ev}_1^{-1}\{P_1\} \cap \ldots \cap \mathrm{ev}_n^{-1}\{P_n\}\right) = \mathrm{ev}_1^{\mathrm{t}*}[P_1^{\mathrm{t}}] \cdots \mathrm{ev}_n^{\mathrm{t}*}[P_n^{\mathrm{t}}] \cdot \mathrm{TSM}_n^{\circ}(r,d)$$

for generic points $(P_i)_i \in (\mathbb{T}^r(K))^n$ tropicalizing to $(P_i^{\mathrm{t}})_i \in (G^{r+1}/G\mathbf{1})^n$ by Theorem 1.2.12. Furthermore, if $nr = n + r + rd + d - 3$, then with a slightly stronger genericity assumption on the P_i we get that $\mathrm{ev}_1^{-1}\{P_1\} \cap \ldots \cap \mathrm{ev}_n^{-1}\{P_n\}$ is finite, and

$$|\,\mathrm{ev}_1^{-1}\{P_1\} \cap \dots \cap \mathrm{ev}_n^{-1}\{P_n\}| = \deg(\mathrm{ev}_1^{\mathrm{t}*}[P_1^{\mathrm{t}}] \cdots \mathrm{ev}_n^{\mathrm{t}*}[P_n^{\mathrm{t}}] \cdot \mathrm{TSM}_n^{\circ}(r,d))$$

by Corollary 1.2.13. This means that the number of generic logarithmic stable maps of degree d through the points P_i is equal to the number of tropical stable maps of degree d through the points P_i^{t}, counted with multiplicities. Moreover, the multiplicity of a tropical curve Γ through the P_i^{t} is equal to the number of algebraic curves through the P_i which tropicalize to Γ.

Example 1.2.16. Corollary 1.2.13 is only applicable to point conditions. For all other conditions, arising for example when we want to count curves passing through lines, planes, e.t.c., we have to use Theorem 1.2.12. We illustrate this at the example of counting lines with nonempty intersection with four given lines in $\mathbb{P}^3$. Let $\overline{L}_1, \dots, \overline{L}_4$ be four lines in $\mathbb{G}^0(2,4)$ (the part of the Grassmannian where all Plücker-coordinates are nonzero). Define $L_i := \overline{L}_i \cap \mathbb{T}^3$. Then Theorem 1.2.12 tells us that for generic R-integral points $t_i \in \mathbb{T}^3(K)$ we have

$$\mathrm{Trop}\left(\mathrm{ev}_1^{-1}(t_1 L_1^{\mathrm{t}}) \cap \dots \cap \mathrm{ev}_4^{-1}(t_4 L_4)\right) = \mathrm{ev}_1^{\mathrm{t}*}(L_1^{\mathrm{t}}) \cdots \mathrm{ev}_4^{\mathrm{t}*}(L_4^{\mathrm{t}}) \cdot \mathrm{TSM}_4^{\circ}(3,1),$$

where $L_i^{\mathrm{t}} = \mathrm{Trop}(L_i)$. Proceeding as in the proof of Corollary 1.2.13, a Bertini-type argument shows that for generic t_i, the subscheme $\mathrm{ev}_1^{-1}(t_1 L_1) \cap \dots \cap \mathrm{ev}_4^{-1}(t_4 L_4)$ of $\mathrm{LSM}_4^{\circ}(3,1)$ is reduced and finite. This yields the equality

$$\left|\mathrm{ev}_1^{-1}(t_1 L_1) \cap \dots \cap \mathrm{ev}_4^{-1}(t_4 L_4)\right| = \deg\left(\mathrm{ev}_1^{\mathrm{t}*}(L_1^{\mathrm{t}}) \cdots \mathrm{ev}_4^{\mathrm{t}*}(L_4^{\mathrm{t}}) \cdot \mathrm{TSM}_4^{\circ}(3,1)\right).$$

If we define a tropical line to be the tropicalization of an algebraic line in $\mathbb{G}^0(2,4)$, we see that the number of tropical lines through 4 given tropical lines, counted with multiplicities, is the same as the number of algebraic lines passing through 4 given algebraic lines, that is 2. Note that for this conclusion it is essential that translates of lines are lines again.

1.3 General Toric Target and Lines in Cubics

As the formulation of our results of Subsection 1.2.2 was not specific to the correspondence problem for projective space, we may also apply them to other enumerative problems in which the conditions are given as fibers of suitable morphisms. This includes, for example, the problem of counting generic logarithmic stable maps to toric varieties which we will study in the first part of this section. At its end we will have a new proof of Nishinou and Siebert's correspondence theorem [NS06]. In the second part, we will study the problem of giving a tropical analogue of the well-known algebraic statement of the existence of precisely 27 lines on a smooth cubic surface.

1.3.1 General Toric Target

Let N be a lattice and let Σ be a (rational, polyhedral, strongly convex) fan in $N_\mathbb{R}$. We will assume that the rays of Σ span $N_\mathbb{R}$. Let us consider rational curves in the toric variety $X = X_\Sigma$ associated to Σ which intersect all torus orbits in the expected dimension. Such curves do not intersect the torus orbits of codimension greater than 1, so we may assume that Σ only consists of its set of rays $\Sigma_{(1)}$ and $\{0\}$. In particular, X is smooth and the boundary divisors D_ρ for $\rho \in \Sigma_{(1)}$ are Cartier. Let I and J be nonempty disjoint finite sets. For every $j \in J$ we specify one of the boundary divisors, that is a ray of $\rho_j \in \Sigma_{(1)}$, and an intersection multiplicity $\omega_j \in \mathbb{N}_+$. This information is encoded by the map

$$\Delta \colon J \to N, \;\; j \mapsto \omega_j u_{\rho_j},$$

where u_ρ denotes the primitive generator of ρ; the ray ρ_j is the unique ray of Σ such that $\Delta(j) \in \rho_j$, and ω_j is equal to $|\Delta(j)| := |(\mathbb{R}\Delta(j) \cap N)/\mathbb{Z}\Delta(j)|$.

Definition 1.3.1. Let $\Delta \colon J \to N \setminus \{0\}$ be a map such that $\Delta(j) \in |\Sigma|$ for all $j \in J$. Furthermore, let $L_0 = I \cup J$. An *I-marked generic logarithmic stable map to X of contact order Δ* is a triple $\mathcal{C} = (C, (p_\lambda)_{\lambda \in L_0}, f)$ such that $(C, (p_\lambda))$ is an L_0-marked rational curve and $h \colon C \to X$ is a morphism that does not map C into any boundary component of X and satisfies

$$f^*(D_\rho) = \sum_{\lambda \colon \Delta(\lambda) \in \rho} |\Delta(\lambda)| p_\lambda.$$

Two generic logarithmic stable maps to X are isomorphic if there is an isomorphism of their underlying curves respecting the marks and the maps. We denote the set of all of their isomorphism classes by $\mathrm{LSM}^\circ_I(X, \Delta)$

Observe that if $(C, (p_\lambda), f)$ is a generic logarithmic stable map of contact order Δ, then the relations of the boundary divisors D_ρ in $\mathrm{Pic}(X)$ force $\sum_{j \in J} \Delta(j)$ to be 0. Hence we may assume that Δ is a tropical degree, as defined in Section 1.1.2.

We can identify $\mathrm{LSM}^\circ_I(X, \Delta)$ with a variety and then embed it into a torus in a way analogous to how we did it for $X = \mathbb{P}^r$. Let $\mathrm{ft}\colon \mathrm{LSM}^\circ_I(X, \Delta) \to M_{0,L_0}$ be the forgetful map, and fix $i_0 \in I$. We claim that

$$\mathrm{ft} \times \mathrm{ev}_{i_0}\colon \mathrm{LSM}^\circ_I(X, \Delta) \to M_{0,L_0} \times T$$

is a bijection, where the evaluation map ev_{i_0} is defined similarly as in the case $X = \mathbb{P}^r$, and T denotes the big open torus of X. To see this, let $\mathcal{C} = (C, (p_\lambda), f)$ be a generic logarithmic stable map to X of contact order Δ. We may assume that $C = \mathbb{P}^1$. By [Cox95, Thm. 2.1] every morphism $\mathbb{P}^1 \to X$ of contact order Δ is determined by a family of homogeneous polynomials $(f_\rho)_{\rho \in \Sigma_{(1)}}$ such that f_ρ has degree $\sum_{\Delta(j) \in \rho} |\Delta(j)|$ for all $\rho \in \Sigma_{(1)}$, and no two of them vanish simultaneously. The morphism $\mathbb{P}^1 \to X$ can then be recovered from (f_ρ) by considering the natural identification of X with the quotient of $U := \mathbb{A}^{\Sigma_{(1)}} \setminus Z$ by the algebraic torus $H := \mathrm{Hom}_\mathbb{Z}(\mathrm{Pic}(X), \mathbb{G}_m)$ [CLS11, Thm. 5.1.11], where Z is the subset where at least two coordinates are 0. Therefore, the points of $\mathbb{P}^1$ mapped to D_ρ are precisely those for which f_ρ vanishes. In particular, every polynomial f_ρ is determined by the family (p_λ) up to a constant factor. If we also specify $\mathrm{ev}_{i_0}(\mathcal{C})$ then (f_ρ) is determined up to the action of H. Since two families of polynomials yield the same morphism to X if and only if they lie in the same H-orbit, we get the desired result.

We consider $\mathrm{LSM}^\circ_I(X, \Delta)$ with the torus embedding $(\mathrm{pl} \circ \mathrm{ft}) \times \mathrm{ev}_{i_0}$. The same argument as in Section 1.2.1 shows that the tropicalization of $\mathrm{LSM}^\circ_I(X, \Delta)$ is equal to the tropical moduli space $\mathrm{TSM}^\circ_I(N_\mathbb{R}, \Delta)$. The reader will not be surprised to learn that all constructions and results from Section 1.2.1 can easily be carried over to the more general situation. In fact, looking back at the construction of the intrinsic tropicalization of a generic logarithmic stable map in standard form (Construction 1.2.1), we see that all we need to change is the definition of the vectors s_A in Equation 1.2.1, namely by defining

$$s_A := \sum_{j \in A \cap J} \Delta(j).$$

The remaining results of Section 1.2.1 can then be copied almost literally to prove the analogous results in the more general situation. In particular, we get

Corollary 1.3.2. *For every $i \in I$ the diagram*

$$\begin{array}{ccc} \mathrm{LSM}^\circ_I(X, \Delta) & \xrightarrow{\mathrm{ev}_i} & T \\ \downarrow{\scriptstyle \mathrm{trop}} & & \downarrow{\scriptstyle \mathrm{trop}} \\ \mathrm{TSM}^\circ_I(N_\mathbb{R}, \Delta) & \xrightarrow{\mathrm{ev}^\mathrm{t}_i} & N_\mathbb{R} \end{array}$$

is commutative. Furthermore, the evaluation morphisms ev_i can be extended to morphisms $\overline{\mathrm{ev}}_i : \mathbb{G}_m^{D(L_0)} / \mathbb{G}_m^{L_0} \times T \to T$ of algebraic tori.

So far, we have only considered the evaluation maps at the marked points which are mapped to the big torus T of X. However, one also wishes to evaluate at the marked points which are mapped into the boundary. This can be done algebraically as well as tropically. In algebraic geometry, for every $j \in J$ and curve $(C, (p_\lambda), f) \in \mathrm{LSM}^\circ_I(X, \Delta)$ the image $f(p_j)$ is a well-defined point in the torus orbit $O(j) := O(\mathbb{R}_{\geq 0}\Delta(j))$ corresponding to $\mathbb{R}_{\geq 0}\Delta(j) \in \Sigma_{(1)}$. This yields a map

$$\mathrm{ev}_j : \mathrm{LSM}^\circ_I(X, \Delta) \to O(j),$$

the evaluation map at j. As in the case of evaluations at marked points, these evaluation maps are restrictions of morphisms of tori:

Proposition 1.3.3. *For every $j \in J$ the map $\mathrm{ev}_j : \mathrm{LSM}^\circ_I(X, \Delta) \to O(j)$ can be extended to a morphism $\overline{\mathrm{ev}}_j : \mathbb{G}_m^{D(L_0)}/\mathbb{G}_m^{L_0} \times T \to O(j)$ of algebraic tori. In particular, the evaluation maps are morphisms.*

Proof. The natural toric morphism $U = \mathbb{A}^{\Sigma_{(1)}} \setminus Z \to X$ induces a torus morphism from the torus orbit $O_U(j)$ of U corresponding to $\mathbb{R}_{\geq 0}\Delta(j)$ (or rather to the cone generated by the basis vector of $\mathbb{R}^{\Sigma_{(1)}}$ corresponding to $\mathbb{R}_{\geq 0}\Delta(j)$) to the torus orbit $O(j)$ of X. The orbit $O_U(j)$ can naturally be identified with $\mathbb{G}_m^{\Sigma_{(1)} \setminus \{\mathbb{R}_{\geq 0}\Delta(j)\}}$ and therefore it makes sense to define the morphism of algebraic tori

$$\begin{aligned} \mathbb{G}_m^{D(L_0)} \times \mathbb{G}_m^{\Sigma_{(1)}} &\to \quad O_U(j) \\ \Big((d_{(\lambda,\mu)}), (c_\rho)\Big) &\mapsto \left(c_\rho \prod_{\Delta(\lambda) \in \rho} \left(\frac{d_{(\lambda,j)}}{d_{(\lambda,i_0)}} \right)^{|\Delta(\lambda)|} \right)_{\rho\colon \Delta(j) \notin \rho} . \end{aligned}$$

Taking the composite of this morphism with the morphism $O_U(j) \to O(j)$ we obtain a morphism $\mathbb{G}_m^{D(L_0)} \times \mathbb{G}_m^{\Sigma_{(1)}} \to O(j)$ of algebraic tori which obviously contains the naturally embedded $\mathbb{G}_m^{L_0} \times H$ (remember that $H = \mathrm{Hom}_\mathbb{Z}(\mathrm{Pic}(X), \mathbb{G}_m)$) in its kernel. Hence, it induces a morphism

$$\overline{\mathrm{ev}}_j : \mathbb{G}_m^{D(L_0)}/\mathbb{G}_m^{L_0} \times T \to O(j)$$

of algebraic tori. That this morphism coincides with ev_j on $\mathrm{LSM}^\circ_I(X, \Delta)$ can be seen in the same way as in Proposition 1.2.8. □

In the tropical world, the situation is similar. Given $j \in J$ and a tropical curve $(\Gamma, (e_\lambda), h) \in \mathrm{TSM}^\circ_I(N_\mathbb{R}, \Delta)$, the images of the points of e_j all differ by an element in $\mathbb{R}\Delta(j)$. Hence, we get a well-defined point in $O^{\mathrm{t}}(j) = O^{\mathrm{t}}(\mathbb{R}_{\geq 0}\Delta(j)) := N_\mathbb{R}/\mathbb{R}\Delta(j)$, yielding a map

$$\mathrm{ev}^{\mathrm{t}}_j : \mathrm{TSM}^\circ_I(N_\mathbb{R}, \Delta) \to O^{\mathrm{t}}(j),$$

the evaluation map at j. The vector space $O^{\mathrm{t}}(j)$ is canonically isomorphic to $N(\lambda) \otimes_\mathbb{Z} \mathbb{R}$, where $N(\lambda)$ is the image of N under the projection map $N_\mathbb{R} \to O^{\mathrm{t}}(j)$.

Because $N(\lambda)$ can naturally be identified with the lattice of 1-psgs of $O(j)$, there is a tropicalization map

$$\mathrm{trop}_{\mathrm{O}(j)}\colon\ \mathrm{O}(j) \to O^{\mathrm{t}}(j).$$

Therefore, we obtain a commutative diagram analogous to the one in Corollary 1.3.2:

Proposition 1.3.4. *For every $j \in J$ the diagram*

$$\begin{array}{ccc} \mathrm{LSM}^{\circ}_{I}(X,\Delta) & \xrightarrow{\mathrm{ev}_j} & O(j) \\ \downarrow{\scriptstyle \mathrm{trop}} & & \downarrow{\scriptstyle \mathrm{trop}} \\ \mathrm{TSM}^{\circ}_{I}(N_{\mathbb{R}},\Delta) & \xrightarrow{\mathrm{ev}^{\mathrm{t}}_j} & O^{\mathrm{t}}(j) \end{array}$$

is commutative.

Proof. Let $\mathcal{C} = (\mathbb{P}^1, (p_\lambda), f)$ be in standard form. We use the notation of Construction 1.2.1, that is $p_\lambda = (1 : a_\lambda)$ for $\lambda \neq i_0$, and ev_{i_0} is represented by $(c_\rho)_{\rho\in\Sigma_{(1)}}$. The image of the point p_j under f is represented by the point

$$\left(c_\rho \prod_{\Delta(\lambda)\in\rho} (a_j - a_\lambda)^{|\Delta(\lambda)|} \right)_{\rho\colon\, \Delta(j)\notin\rho}$$

in the torus orbit $\mathbb{G}_m^{\Sigma_{(1)}\setminus\{\mathbb{R}_{\geq 0}\Delta(j)\}}$ of U corresponding to $\mathbb{R}_{\geq 0}\Delta(j)$ (recall that U is the subset of $\mathbb{A}^{\Sigma_{(1)}}$ where at most one coordinate vanishes). Therefore, $\mathrm{trop}\left(\mathrm{ev}_j(\mathcal{C})\right)$ is represented by

$$\sum_{\rho\in\Sigma_{(1)}} \nu(c_\rho)u_\rho + \sum_{\lambda\in J\setminus\{j\}} \nu(a_j - a_\lambda)\Delta(\lambda). \tag{1.3.1}$$

Note that multiples of $\Delta(\lambda)$ for λ with $\mathbb{R}_{\geq 0}\Delta(\lambda) = \mathbb{R}_{\geq 0}\Delta(j)$ do not change the represented vector. As in the proof of Proposition 1.2.5, let $r_1 < \cdots < r_k$ be the elements of $\{\nu(a_j - a_\lambda) \mid \lambda \in J\}$, and let $D_i = \{\lambda \in J \mid \nu(a_j - a_\lambda) \geq r_i\}$ for $1 \leq i \leq k$. Furthermore, let v_i be the inclusion-minimal vertex of $\Gamma_{\mathcal{C}}$ containing D_i. The new thing here is that now $r_k = \infty$ is infinite. But the techniques of Proposition 1.2.5 still work, showing that $\nu(v_i) = r_i$ for $1 \leq i < k$, that $v_i \cap J = D_i$ for all i, and that there is no vertex v strictly between v_i and v_{i+1} such that $v \cap J \neq D_{i+1}$. With this notation, $\mathrm{trop}(\mathrm{ev}_j(\mathcal{C}))$ is represented by

$$\sum_{\rho\in\Sigma_{(1)}} \nu(c_\rho)u_\rho + \sum_{i=2}^{k-1} \left(\nu(v_i) - \nu(v_{i-1})\right) s_{v_i},$$

which is easily seen to be equal to $q_{v_{k-1}}$. Because $v_k = \{j\}$, the properties of the v_i imply that $v \cap J = \{j\}$ for every vertex strictly between v_{k-1} and $\{j\}$. We conclude that $q_{v_{k-1}}$ differs from the images of the points on e_j only by a multiple of $\Delta(j)$, which finishes the proof. □

Let us point out the geometric meaning of the tropical evaluation maps $\mathrm{ev}_j^{\mathrm{t}}$ for $j \in J$. Given a fan Θ in $N_{\mathbb{R}}$, the construction of the toric variety X_Θ can be mimicked in tropical geometry [Kaj08, Pay09a]: for every cone $\theta \in \Theta$ one can define an "affine" tropical variety $\overline{\theta} = \mathrm{Hom}_{\mathrm{sg}}(\theta^\vee \cap M, \overline{\mathbb{R}})$, where $M = \mathrm{Hom}(N, \mathbb{Z})$, and give it the topology of pointwise convergence. These topological spaces can be glued, as in the algebraic situation, to the "tropical toric variety" $\overline{\Theta}$. Again as in toric geometry, there is a natural stratification $\overline{\Theta} = \coprod_\theta O^{\mathrm{t}}(\theta)$, where $O^{\mathrm{t}}(\theta) = N_{\mathbb{R}} / \operatorname{Span}(\theta)$ is defined as before. The algebraic and tropical constructions can be connected via the so-called extended tropicalization $\mathrm{trop} : X_\Theta \to \overline{\Theta}$ which is equal to the tropicalization $\mathrm{trop}_{O(\theta)} : O(\theta) \to O^{\mathrm{t}}(\theta)$ on each stratum. Returning to our situation, the evaluation map $\mathrm{ev}_j^{\mathrm{t}}$ assigns to a map $(\Gamma, (e_\lambda), h)$ the topological limit of the ray e_j in $\overline{\Sigma}$. This limit point lies in the boundary of $\overline{\Sigma}$, in the stratum $O^{\mathrm{t}}(j)$ to be precise, and could be considered as the image point of the foot of e_j. Thus, the fibers of $\mathrm{ev}_j^{\mathrm{t}}$ consist precisely of the curves whose foot with mark j is mapped to a fixed point in the boundary, that is they have the same interpretation as their algebraic counterparts.

The following corollary summarizes our results on generic logarithmic stable maps by reproving Nishinou and Siebert's correspondence theorem. To shorten notation, we define $O(i) := T$ and $O^{\mathrm{t}}(i) := N_{\mathbb{R}}$ for $i \in I$.

Corollary 1.3.5 ([NS06, Thm. 8.3])**.** *For every $\lambda \in L_0$ let A_λ be a rational affine linear subspace of $O^{\mathrm{t}}(\lambda)$ of codimension d_λ with difference space L_λ. Furthermore, assume that*

$$\sum_{\lambda \in L_0} d_\lambda = |I| + |J| - 3 + \dim(N_{\mathbb{R}}).$$

Then for a generic family of points $(p_\lambda) \in \left(\prod_\lambda O(\lambda)\right)(K)$ with $\mathrm{trop}(p_\lambda) \in A_\lambda$ for all $\lambda \in L_0$ we have

$$\left| \bigcap_{\lambda \in L_0} \mathrm{ev}_\lambda^{-1} \left(\mathbb{G}(L_\lambda \cap N(\lambda)) p_\lambda \right) \right| = \deg \left(\prod_{\lambda \in L_0} \mathrm{ev}_\lambda^{\mathrm{t}*}(A_\lambda) \cdot \mathrm{TSM}_I^\circ(N_{\mathbb{R}}, \Delta) \right),$$

where $\mathbb{G}(L_\lambda \cap N(\lambda))$ denotes the subtorus of $O(\lambda)$ generated by the 1-psgs in $L_\lambda \cap N(\lambda)$.

Proof. The proof is analogous to that of Corollary 1.2.13. □

1.3.2 Lines on Cubic Surfaces

It is a well-known theorem by Cayley and Salmon that every smooth algebraic cubic surface in projective three-space over an algebraically closed field contains exactly 27 straight lines. The strong analogy between algebraic and tropical geometry makes it promising to prove a similar result in tropical geometry. However, it has already been discovered that the exact analogue of the algebraic statement is simply wrong in the tropical world. It has been shown in [Vig10] that there are smooth tropical cubic surfaces that contain infinitely many lines. On the other hand, there are combinatorial types of tropical cubic surfaces such that every surface of that type contains exactly

27 tropical lines [Vig07]. This encourages to believe that it is still possible to prove a slightly modified version of the statement, for example by allowing the lines on tropical cubic surfaces to have multiplicities. These multiplicities should, of course, be 0 for all but finitely many lines on a given cubic surface. Also, they should be defined purely tropically, for example by intersection-theoretical considerations in some moduli space. Finally, it would be nice if the multiplicities would reflect the relative realizability of the lines, that is whenever a tropical cubic A is the tropicalization of a generic algebraic cubic X, the multiplicity of each tropical line in A should be equal to the number of algebraic lines in X tropicalizing to it. One way of defining these multiplicities, namely by constructing moduli spaces for tropical lines in given tropical surfaces, was suggested in [Och13, Section 3.3]. In an example class of cubics, these multiplicities indeed add up to 27. Unfortunately, this has not been proven for general cubics, and it is difficult to decide whether the lines in question are relatively realizable or not. For this reason, we chose a different approach. Namely, we tropicalize the algebraic incidence correspondence of lines and cubics.

But first, let us review some results about tropical hypersurfaces. Assume we are given a tropical polynomial $h = \sum_{m\in\mathbb{Z}^n} r_m x^m$, where the coefficients are in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and only finitely many of them are not equal to ∞. The vanishing set $|V_a^{\mathrm{t}}(h)|$ of a tropical polynomial h is equal to all points $w \in \mathbb{R}^n$ such that the minimum

$$h(w) = \min_{m\in\mathbb{Z}^n} \{r_m + \langle m, w\rangle\}$$

is obtained at least twice. This set has a polyhedral structure whose cells are in dimension-reversing correspondence to the cells of the Newton subdivision of the Newton polytope $\mathrm{Newt}(h) = \mathrm{conv}\{m \mid r_m \neq \infty\}$ [MS15, Prop. 3.1.6]. It can be made a tropical variety $V_a^{\mathrm{t}}(h)$ by assigning to a facet of $|V_a^{\mathrm{t}}(h)|$ the lattice length of its corresponding edge in the Newton subdivision. This construction is compatible with tropicalization, that is if $g = \sum_{m\in\mathbb{Z}^n} a_m x^m$ is a polynomial in the coordinate ring of $\mathbb{G}_m^n$, and we tropicalize its coefficients to obtain $h = \sum_{m\in\mathbb{Z}^n} \nu(a_m) x^m$, then we have $\mathrm{Trop}\,(V(g)) = V_a^{\mathrm{t}}(h)$ [MS15, Lemma 3.4.6]. Now given a point $w \in \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ with representative $x \in \mathbb{R}^{n+1}$, and a homogeneous polynomial $h \in \overline{\mathbb{R}}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$, the value $h(x)$ is not independent of the choice of x. Nevertheless, it does not depend on x whether or not the minimum is obtained twice. Thus, we get a well-defined set $|V_p^{\mathrm{t}}(h)| \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$, which can be made into a tropical variety $V_p^{\mathrm{t}}(h)$ in the same way as in the affine case.

Unfortunately, unlike in the algebraic case, there is no bijective correspondence between tropical hypersurfaces in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ and homogeneous polynomials in $\overline{\mathbb{R}}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$ modulo multiplication by a monomial. But for a given hypersurface $A \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$, the Newton polytopes and subdivisions of the tropical polynomials which have A as their vanishing sets are translates of each other. Therefore, it makes sense to talk of the Newton polytope and subdivision of A and to call A smooth if the Newton subdivision is unimodular.

Following [Vig10] we define tropical cubic surfaces in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$ as the vanishing sets of tropical polynomials in the tropical polynomial ring $\overline{\mathbb{R}}[x_0, \dots, x_3]$ whose Newton polytopes are equal to three times the standard simplex, that is equal to

$3\,\mathrm{conv}\{t_0,\ldots,t_3\}$, where $(t_i)_i$ denotes the standard basis. Also recall that a tropical line in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ is the tropicalization of a line in $\mathbb{G}^0(2,4)$

For the rest of this section we will work over a field K with surjective valuation. For example, over the field $\mathbb{C}((t^{\mathbb{R}}))$ of generalized power series. As indicated earlier, we want to tropicalize the algebraic incidence correspondence V of lines and cubic surfaces. This variety parametrizes all pairs (L,X) of lines L and cubic surfaces X such that $L \subseteq X$. Lines in $\mathbb{P}^3$ are parametrized by the Grassmannian $\mathbb{G}(2,4)$, and cubics are uniquely determined by the 20 coefficients of their defining homogeneous polynomials of degree 3. Therefore, the incidence correspondence V is a subvariety of $\mathbb{G}(2,4) \times \mathbb{P}^{19}$. Since only lines whose Plücker coordinates are all nonzero tropicalize to tropical lines in our sense, and only cubics whose defining polynomials contain all monomials of degree 3 can possibly tropicalize to smooth tropical cubics, we can replace V by the part of the incidence correspondence contained in $\mathbb{G}^0(2,4) \times \mathbb{T}^{19}$.

Consider $\mathbb{G}^0(2,4)$ in its Plücker embedding in $\mathbb{T}^5$. By [SS04, Thm. 3.8], its tropicalization $G^{\mathrm{t}}(2,4)$ parametrizes the tropical lines in a way respecting the tropicalization. The situation is not quite as nice for cubics: A given point $g \in \mathbb{P}^{19}$, representing a cubic polynomial modulo constant factor, tropicalizes to a point $h = \mathrm{trop}(g) \in \mathbb{R}^{20}/\mathbb{R}\mathbf{1}$, determining a tropical cubic polynomial modulo constant summand. Of course, the vanishing set of a tropical polynomial does not change when adding a constant summand. Thus, as in the algebraic setting, there is a well-defined tropical hypersurface $V_p^{\mathrm{t}}(h)$, which is clearly equal to $\mathrm{Trop}\left(V_p(g)\right)$. However, as mentioned above, $\mathbb{R}^{20}/\mathbb{R}\mathbf{1}$ does not parametrize the set of tropical cubics because the map assigning to $h \in \mathbb{R}^{20}/\mathbb{R}\mathbf{1}$ its associated cubic $V_p^{\mathrm{t}}(h)$ is not injective. Fortunately, if A is a smooth cubic, there actually is a unique $h \in \mathbb{R}^{20}/\mathbb{R}\mathbf{1}$ such that $V_p^{\mathrm{t}}(h) = A$.

Let us apply our results from Section 1.2.2 to the 19-dimensional subvariety V of $T = \mathbb{T}^5 \times \mathbb{T}^{19}$. We only consider one morphism $f = f_1$, namely the restriction to V of the projection $\overline{f} = \mathrm{pr} : T \to \mathbb{T}^{19}$. Of course, $\overline{f}$ is dominant, and $\mathbb{T}^{19}$ has the correct dimension to apply Corollary 1.2.13. By Cayley and Salmon's theorem, the fibers $\mathrm{ev}^{-1}\{g\}$ consists of exactly 27 points for all g in the open subset of $\mathbb{T}^{19}$ of smooth cubic surfaces. So if we let $V^{\mathrm{t}} = \mathrm{Trop}(V)$, and let f^{t} be the restriction of $\overline{f}^{\mathrm{t}} = \mathrm{Trop}(\overline{f})$ to V^{t}, Corollary 1.2.13 tells us that $\deg(f^{\mathrm{t}\,*}[h] \cdot V^{\mathrm{t}}) = 27$ for every $h \in \mathbb{R}^{20}/\mathbb{R}\mathbf{1}$. Every point $(\Gamma, h) \in V^{\mathrm{t}} \subseteq G^{\mathrm{t}}(2,4) \times \mathbb{R}^{20}/\mathbb{R}\mathbf{1}$ is the tropicalization of a pair $(L,g) \in V$. In particular, we have $\Gamma \subseteq V_p^{\mathrm{t}}(h)$. We see that for every tropical cubic polynomial h we get multiplicities on the tropical lines contained in $V_p^{\mathrm{t}}(h)$ that add up to 27 and respect tropicalization. Since a smooth tropical cubic corresponds to a unique tropical cubic polynomial, we obtain the following corollary:

Corollary 1.3.6. *Let A be a smooth tropical cubic. Then there are intersection-theoretically defined multiplicities* $\mathrm{mult}(\Gamma)$ *for the lines* $\Gamma \subseteq A$ *that are compatible with tropicalization. More precisely, if X is a generic smooth cubic surface tropicalizing to A, then the multiplicity* $\mathrm{mult}(\Gamma)$ *of a tropical line* $\Gamma \subseteq A$ *is equal to the number of lines of X tropicalizing to* Γ*. In particular, only finitely many tropical lines in A have nonzero weight, the weights are all nonnegative, and they add up to* 27*.*

2 Correspondences for Toroidal Embeddings

In Chapter 1 we have seen that the moduli space of generic logarithmic stable maps tropicalizes to the moduli space of tropical stable maps, and that the tropicalization map between them can be used to relate algebraic and enumerative problems. Our methods used that the algebraic moduli space is very affine, that is that it can be embedded into an algebraic torus, in an essential way. In particular, we had to work with a non-complete moduli space. While this can certainly be seen as a feature – after all it is usually a very difficult problem to construct reasonable compactifications – it also has its drawbacks. For example, we had to use intersection theory on the tropical side, but could not use it on the algebraic side because the Chow groups of $\mathrm{LSM}_n^\circ(r,d)$ are trivial. It is the goal of this chapter to remedy this. Of course, the first step is to replace $\mathrm{LSM}_n^\circ(r,d)$ by a complete moduli space. As implicit in notation and terminology, the most suitable compactification is the moduli space $\mathrm{LSM}_n(r,d)$ of logarithmic stable maps [AC14, GS13]. We aim to define a tropicalization morphism for cycle classes on $\mathrm{LSM}_n(r,d)$ that gives a meaning to the equation

$$\mathrm{Trop}\left(\prod_{i=1}^n \mathrm{ev}_i^*[P_i]\cdot[\mathrm{LSM}_n(r,d)]\right) = \prod_{i=1}^n \mathrm{ev}_i^{\mathrm{t}\,*}[\mathrm{trop}(P_i)]\cdot[\mathrm{TSM}_n(r,d)],$$

where (P_i) is a family of points in $\mathbb{P}^r$, and $\mathrm{TSM}_n(r,d)$ is a canonical compactification of $\mathrm{TSM}_n^\circ(r,d)$. Of course, we also want to show that equality holds. At the end of this chapter, we will almost have done this. We have to replace the P_i by intersections of r divisors though because so far we cannot pull-back cycles of codimension greater 1.

We will develop the methods to achieve this correspondence on an intersection-theoretic level not only for $\mathrm{TSM}_n(r,d)$, but more generally for toroidal embeddings. This type of varieties plays an increasingly important role in tropical geometry. Building on results of Thuillier [Thu07] – and of course on [KKMSD73] – toroidal methods have recently been employed to tropicalize several moduli spaces. Examples include the moduli spaces of stable curves [ACP15], Hassett's moduli spaces of weighted stable curves [Uli15b], the moduli spaces of rational stable maps to $\mathbb{P}^1$ relative to 2 points [CMR14], the moduli spaces of admissible covers [CMR16], and, of special importance for us, the moduli spaces of genus 0 logarithmic stable maps to toric varieties [Ran15]. Note, however, that only few of these toroidal embeddings are strict, a condition that we will assume in this chapter.

In Section 2.1 we review the definitions and basic properties of strict toroidal embeddings. We recall basic material about cone complexes and the definition of the associated cone complex of a strict toroidal embedding. We also define the central tropical objects of this chapter, so-called weakly embedded cone complexes. These are cone complexes, together with a piecewise integral linear function to $N_{\mathbb{R}}$ for some lattice N.

In Section 2.2 we develop the foundations of an intersection theory on weakly embedded cone complexes. We begin by adapting basic construction of the theory for $\mathbb{R}^n$ of [AR10] to our more general setting. Afterwards, we introduce an intersection product with Cartier divisors that uses the boundary of the canonical compactification of a cone complex. We use it to define rational equivalence and show that all of the operations we consider respect rational equivalence.

Section 2.3 is devoted to the construction of a tropicalization map for subvarieties of toroidal embeddings. We first define a tropicalization of cocycles and then use it to define a tropicalization for cycles. We show that this tropicalization respects rational equivalence, allowing us to tropicalize cycle classes. Then we prove that tropicalization commutes with taking intersections with certain divisors, which is of special importance for our applications. We conclude the section with a comparison with the "classical" tropicalization map for subvarieties of algebraic tori, in case the interior of the toroidal embedding is very affine.

Finally, Section 2.4 works out the details to prove the desired equality from above. In fact, we prove a little more: we also allow the presence of ψ-classes in the intersection products, thus obtaining a correspondence theorem for descendant invariants.

Note that the material of this chapter is part of the article [Gro15].

2.1 Cone Complexes and Toroidal Embeddings

The purpose of this section is to briefly recall the definitions of cone complexes and extended cone complexes. We refer to [KKMSD73, Section II.1] and [ACP15, Section 2] for further details. We will also introduce weakly embedded cone complexes, the main objects of study in the next section, and show how to obtain them from toroidal embeddings.

2.1.1 Cone Complexes

An *(integral) cone* is a pair (σ, M) consisting of a topological space σ and a lattice M of real-valued continuous functions on σ such that the product map $\sigma \to \operatorname{Hom}(M, \mathbb{R})$ is a homeomorphism onto a strongly convex rational polyhedral cone. By abuse of notation we usually just write σ for the cone (σ, M) and refer to M as M^σ. Furthermore, we will write $N^\sigma = \operatorname{Hom}(M, \mathbb{Z})$ so that σ is identified with a rational cone in $N^\sigma_{\mathbb{R}} = N^\sigma \otimes_{\mathbb{Z}} \mathbb{R}$. We will also use the notation M^σ_+ for the semigroup of nonnegative functions in M, and N^σ_+ for $\sigma \cap N^\sigma$. Giving an integral cone is equivalent to specifying a lattice N and a full-dimensional strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. A subspace $\tau \subseteq \sigma$ is called a *subcone* of σ if the restrictions of the functions in M^σ to τ make it a cone. Identifying σ with its image in $N^\sigma_{\mathbb{R}}$, this is the case if and only if τ itself is a rational polyhedral cone in $N^\sigma_{\mathbb{R}}$. We say that σ is *simplicial* if the primitive vectors of its rays form a basis for the vector space $N^\sigma_{\mathbb{R}}$. It is called *strictly simplicial* if they even form a basis for the lattice N^σ. A *morphism* between two integral cones σ and τ is a continuous map $f\colon \sigma \to \tau$ such that $m \circ f \in M^\sigma$ for all $m \in M^\tau$. The product $\sigma \times \tau$ of two cones σ and τ in the category of cones is given by the topological space $\sigma \times \tau$, together with the functions

$$m + m' \colon \sigma \times \tau \to \mathbb{R}, \quad (x, y) \mapsto m(x) + m'(y)$$

for $m \in M^\sigma$ and $m' \in M^\tau$. The notation already suggests that we can identify $M^\sigma \oplus M^\tau$ with $M^{\sigma\times\tau}$ via the isomorphism $(m, m') \mapsto m + m'$.

An *(integral) cone complex* $(\Sigma, |\Sigma|)$ consists of a topological space $|\Sigma|$ and a finite set Σ of integral cones which are closed subspaces of $|\Sigma|$, such that their union is $|\Sigma|$, every face of a cone in Σ is in Σ again, and the intersection of two cones in Σ is a union of common faces. A subset $\tau \subseteq |\Sigma|$ is called a *subcone* of Σ if there exists some $\sigma \in \Sigma$ such that τ is a subcone of σ. We say that Σ is *(strictly) simplicial* if all of its cones are (strictly) simplicial. A *morphism* $f\colon \Sigma \to \Delta$ of cone complexes is a continuous map $|\Sigma| \to \ |\Delta|$— such that for every $\sigma \in \Sigma$ there exists $\delta \in \Delta$ such that $f(\sigma) \subseteq \delta$, and the restriction $f|_\sigma \colon \sigma \to \delta$ is a morphism of cones.

The reason why the tropical intersection theory developed in [AR10] fails to generalize to cone complexes is that when gluing two cones σ and σ' along a common face, there is no natural lattice containing both N^σ and $N^{\sigma'}$, hence making it impossible to speak about balancing. To remedy this we make the following definition that interpolates between fans and cone complexes:

Definition 2.1.1. A *weakly embedded cone complex* is a cone complex Σ, together with a lattice N^Σ, and a continuous map $\varphi_\Sigma \colon |\Sigma| \to N^\Sigma_\mathbb{R}$ which is integral linear on every cone of Σ.

We recover the notion of fans by imposing the additional requirements that φ_Σ is injective, and the lattices spanned by $\varphi_\Sigma(N^\sigma_+)$ for $\sigma \in \Sigma$ are saturated in N^Σ. In this case we also call Σ an embedded cone complex. On the other hand, the notion of cone complexes is recovered by setting $N^\Sigma = 0$.

Given a weakly embedded cone complex Σ we write $M^\Sigma = \operatorname{Hom}(N^\Sigma, \mathbb{Z})$ for the dual of N^Σ, and for $\sigma \in \Sigma$ we write N^Σ_σ for what is usually denoted by $N^\Sigma_{\varphi_\Sigma(\sigma)}$ in toric geometry, that is $N^\Sigma_\sigma = N^\Sigma \cap \operatorname{Span} \varphi_\Sigma(\sigma)$. Furthermore, we will frequently abuse notation and write φ_Σ for the induced morphisms $N^\sigma \to N^\Sigma$ and $N^\sigma_\mathbb{R} \to N^\Sigma_\mathbb{R}$.

A *morphism* between two weakly embedded cone complexes Σ and Δ is comprised of a morphism $\Sigma \to \Delta$ of cone complexes and a morphism $N^\Sigma \to N^\Delta$ of lattices forming a commutative square with the weak embeddings.

A morphism $f\colon \Sigma \to \Delta$ is a called a *subdivision* if it identifies $|\Sigma|$ with a subspace of $|\Delta|$, in this identification every cone of Σ is a subcone of Δ, and N^Σ is identified with a saturated sublattice of N^Δ. If additionally f is surjective, we say that the subdivision is *proper*.

Let Σ be a cone complex, and let $\tau \in \Sigma$. For every $\sigma \in \Sigma$ containing τ let σ/τ be the image of σ in $N^\sigma_\mathbb{R}/N^\tau_\mathbb{R}$. Whenever σ and σ' are two cones containing τ such that σ' is a face of σ, the cone σ'/τ is a naturally identified with a face of σ/τ. Gluing the cones σ/τ for $\tau \preceq \sigma \in \Sigma$ along these face maps produces a new cone complex, the *star*

$\mathrm{S}_\Sigma(\tau)$ (or just $\mathrm{S}(\tau)$ if Σ is clear) of Σ at τ. If Σ comes with a weak embedding, then the star is naturally a weakly embedded complex again. Namely, for every cone σ of Σ containing τ there is an induced integral linear map $\sigma/\tau \to (N^\Sigma/N^\Sigma_\tau)_\mathbb{R} =: N_\mathbb{R}^{\mathrm{S}(\tau)}$, and these maps glue to a continuous map $\varphi_{\mathrm{S}(\tau)} \colon |\,\mathrm{S}(\tau)| \to N_\mathbb{R}^{\mathrm{S}(\tau)}$.

The product $\Sigma \times \Delta$ of two cone complexes Σ and Δ in the category of cone complexes is the cone complex with underlying space $|\Sigma| \times |\Delta|$ and cones $\sigma \times \delta$ for $\sigma \in \Sigma$ and $\delta \in \Delta$. If both Σ and Δ are weakly embedded, then their product in the category of weakly embedded cone complexes is the product of the cone complexes together with the weak embedding $\varphi_{\Sigma\times\Delta} = \varphi_\Sigma \times \varphi_\Delta$.

2.1.2 Extended Cone Complexes

In the image of an integral cone σ under its canonical embedding into $\mathrm{Hom}(M^\sigma, \mathbb{R})$ are exactly those morphisms $M^\sigma \to \mathbb{R}$ that are nonnegative on M^σ_+. Therefore, σ is canonically identified with the set $\mathrm{Hom}(M^\sigma_+, \mathbb{R}_{\geq 0})$ of morphisms of monoids. This identification motivates the definition of the *extended cone* $\overline{\sigma} = \mathrm{Hom}(M^\sigma_+, \overline{\mathbb{R}}_{\geq 0})$ of σ, where $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$, and the topology on $\overline{\sigma}$ is that of pointwise convergence. The cone σ is an open dense subset of $\overline{\sigma}$, and $\overline{\sigma}$ is compact by Tychonoff's theorem. If $v \in \overline{\sigma}$ is an element of this compactification, the sets $\{m \in M^\sigma_+ \mid \langle m, v\rangle \in \mathbb{R}_{\geq 0}\}$ and $\{m \in M^\sigma_+ \mid \langle m, v\rangle = 0\}$ generate two faces of $\sigma^\vee = \mathbb{R}_{\geq 0} M^\sigma_+ \subseteq M^\sigma_\mathbb{R}$, and these are dual to two comparable faces of σ. In this way we obtain a stratification of $\overline{\sigma}$, the stratum corresponding to a pair $\tau \supseteq \tau'$ of faces of σ being

$$F^\circ_\sigma(\tau, \tau') = \left\{ v \in \overline{\sigma} \;\middle|\; \begin{matrix} \langle m, v\rangle \in \mathbb{R}_{\geq 0} \Leftrightarrow m \in (\tau')^\perp \cap M^\sigma_+, \\ \langle m, v\rangle = 0 \Leftrightarrow m \in \tau^\perp \cap M^\sigma_+ \end{matrix} \right\}.$$

Denote by $F_\sigma(\tau, \tau')$ the subset of $\overline{\sigma}$ obtained by relaxing the second condition in the definition of $F^\circ_\sigma(\tau, \tau')$ and allowing the vanishing locus of v to be possibly larger than $\tau^\perp \cap M^\sigma_+$. Since $(\tau')^\perp \cap M^\sigma$ is canonically identified with the dual lattice of $N^\sigma/N^\sigma_{\tau'}$, we can identify $F_\sigma(\tau, \tau')$ with the image τ/τ' of τ in $(N^\sigma/N^\sigma_{\tau'})_\mathbb{R}$. This gives $F_\sigma(\tau, \tau')$ the structure of an integral cone.

Every morphism $f \colon \tau \to \sigma$ of cones induces a morphism $\overline{f} \colon \overline{\tau} \to \overline{\sigma}$ of extended cones, and if f identifies τ with a face of σ, this extended morphism maps $\overline{\tau}$ homeomorphically onto $\coprod_{\tau' \preceq \tau} F_\sigma(\tau, \tau') = \overline{F(\tau, 0)}$. Every subset of $\overline{\sigma}$ occurring like this is called an extended face of $\overline{\sigma}$.

Given a cone complex Σ we can glue the extensions of its cones along their extended faces according to the inclusion relation on Σ and obtain a compactification $\overline{\Sigma}$ of Σ, which is called the *extended cone complex associated to* Σ. For a cone $\tau \in \Sigma$, the cones $\sigma/\tau = F_\sigma(\sigma, \tau)$, where $\sigma \in \Sigma$ with $\tau \subseteq \sigma$, are glued in $\overline{\Sigma}$ exactly as in the construction of $\mathrm{S}(\tau)$, and therefore there is an identification of $\mathrm{S}(\tau)$ with a locally closed subset of $\overline{\Sigma}$ which extends naturally to an identification of the extended cone complex $\overline{\mathrm{S}(\tau)}$ with a closed subset of $\overline{\Sigma}$. With these identifications, we see that $\overline{\Sigma}$ is stratified by the stars of Σ at its various cones.

For every morphism $f\colon \Sigma \to \Delta$ between two cone complexes, there is an induced map $\overline{f}\colon \overline{\Sigma} \to \overline{\Delta}$, and we call any map arising that way a *dominant* morphism of extended cone complexes. If both Σ and Δ are weakly embedded we require f to respect these embeddings. Whenever $\sigma \in \Sigma$, and $\delta \in \Delta$ is the minimal cone of Δ containing $f(\sigma)$, there is an induced morphism $\mathrm{S}_f(\sigma)\colon \mathrm{S}_\Sigma(\sigma) \to \mathrm{S}_\Delta(\delta)$ of cone complexes. It is easily checked that this describes $\overline{f}$ on the stratum $\mathrm{S}_\Sigma(\sigma)$, that is that the diagram

$$\begin{array}{ccccc} \mathrm{S}_\Sigma(\sigma) & \longrightarrow & \overline{\mathrm{S}_\Sigma(\sigma)} & \longrightarrow & \overline{\Sigma} \\ \downarrow{\scriptstyle \mathrm{S}_f(\sigma)} & & \downarrow{\scriptstyle \overline{\mathrm{S}_f(\sigma)}} & & \downarrow{\scriptstyle \overline{f}} \\ \mathrm{S}_\Delta(\delta) & \longrightarrow & \overline{\mathrm{S}_\Delta(\delta)} & \longrightarrow & \overline{\Delta} \end{array}$$

is commutative. If Σ and Δ are weakly embedded, and f respects the weak embeddings, then so does $S_f(\sigma)$.

In general, we define a *morphism of extended cone complexes* between $\overline{\Sigma}$ and $\overline{\Delta}$ to be a map $\overline{\Sigma} \to \overline{\Delta}$ which factors through a dominant morphism to $\overline{\mathrm{S}_\Delta(\delta)}$ for some $\delta \in \Delta$. If Σ and Δ are weakly embedded, we additionally require the dominant morphism to respect the weak embeddings.

2.1.3 Toroidal Embeddings

Before we start with the development of an intersection theory on weakly embedded cone complexes, we want to point out how to obtain them from toroidal embeddings as this will be our primary source of motivation and intuition. A *toroidal embedding* is a pair (X_0, X) consisting of a normal variety X and a dense open subset $X_0 \subseteq X$ such that the open immersion $X_0 \to X$ formally locally looks like the inclusion of an algebraic torus T into a T-toric variety. More precisely, this means that for every closed point $x \in X$ there exists an affine toric variety Z, a closed point $z \in Z$, and an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Z,z}$ over the ground field k which identifies the ideal of $X \setminus X_0$ with that of $Z \setminus Z_0$, where Z_0 denotes the open orbit of Z. A toroidal embedding is called strict, if all components of $X \setminus X_0$ are normal. In this thesis, we will only consider strict toroidal embeddings and therefore omit the "strict".

Every toroidal embedding X has a canonical stratification. If $E_1, \dots, E_n$ are the components of $X \setminus X_0$, then the strata are given by the connected components of the sets

$$\bigcap_{i \in I} E_i \setminus \bigcup_{j \notin I} E_j,$$

where I is a subset of $\{1, \dots, n\}$.

The *combinatorial open subset* $X(Y)$ of X associated to a stratum Y is the union of all strata of X containing Y in their closure. This defines an open subset of X, as it is constructible and closed under generalizations. Furthermore, the stratum Y defines the following lattices, semigroups, and cones:

$$\begin{aligned} M^Y &= \{\text{Cartier divisors on } X(Y) \text{ supported on } X(Y) \setminus X_0\} \\ N^Y &= \operatorname{Hom}(M^Y, \mathbb{Z}) \\ M^Y_+ &= \{\text{Effective Cartier divisors in } M^Y\} \\ N^Y_{\mathbb{R}} \supseteq \sigma^Y &= (\mathbb{R}_{\geq 0} M^Y_+)^\vee. \end{aligned}$$

If X is unclear, we write $M^Y(X)$, $N^Y(X)$, $M^Y_+(X)$, and σ^Y_X. Whenever a stratum Y is contained in the closure of a stratum Y' the morphism $N^{Y'} \to N^Y$ induced by the restriction of divisors maps injectively onto a saturated sublattice of N^Y and identifies $\sigma^{Y'}$ with a face of σ^Y. Gluing along these identifications produces the cone complex $\Sigma(X)$. We refer to [KKMSD73, Section II.1] for details. In accordance with the toric case we will write $\mathrm{O}(\sigma)$ for the stratum of X corresponding to a cone $\sigma \in \Sigma(X)$. Its closure will be denoted by $\mathrm{V}(\sigma)$, and we will abbreviate $X(\mathrm{O}(\sigma))$ to $X(\sigma)$.

Define $M^X = \Gamma(X_0, \mathcal{O}^*_X)/k^*$, and let $N^X = \operatorname{Hom}(M^X, \mathbb{Z})$ be its dual. For every stratum Y of X we have a morphism

$$M^X \to M^Y, \;\; f \mapsto \operatorname{div}(f)|_{X(Y)},$$

which induces an integral linear map $\sigma^Y \to N^X_{\mathbb{R}}$. Obviously, these maps glue to a continuous function $\varphi_X \colon |\Sigma(X)| \to N^X_{\mathbb{R}}$ which is integral linear on the cones of $\Sigma(X)$. In other words, we obtain a weakly embedded cone complex naturally associated to X, which we again denote by $\Sigma(X)$.

Example 2.1.2.

a) Let Σ be a fan in $N_{\mathbb{R}}$ for some lattice N, and let X be the associated normal toric variety. Let $M = \operatorname{Hom}(N, \mathbb{Z})$ be the dual of N and $T = \operatorname{Spec} k[M]$ the associated algebraic torus. By definition, $T \subseteq X$ is a toroidal embedding. The components of the boundary $X \setminus T$ are the T-invariant divisors D_ρ corresponding to the rays $\rho \in \Sigma_{(1)}$, and the strata of X are the T-orbits $O(\sigma)$ corresponding to the cones $\sigma \in \Sigma$. The combinatorial open subsets of X are precisely its T-invariant affine opens. For every $\tau \in \Sigma$ the isomorphism

$$M/(M \cap \tau^\perp) \to M^{O(\tau)}, \;\; [m] \mapsto \operatorname{div}(\chi^m),$$

where χ^m denotes the character associated to m, induces identifications of $N^{O(\tau)}$ with N_τ and $\sigma^{O(\tau)}$ with τ. After identifying M and M^X via the isomorphism

$$M \to M^X = \Gamma(T, \mathcal{O}^*_X)/k^*, \;\; m \mapsto \chi^m$$

we see that the image of $\sigma^{O(\tau)}$ in $N_{\mathbb{R}}$ under φ_X is precisely τ. We conclude that $\Sigma(X)$ is an embedded cone complex, naturally isomorphic to Σ.

b) For a non-toric example consider $X = \mathbb{P}^2$ with open part $X_0 = X \setminus H_1 \cup H_2$, where $H_i = \{x_i = 0\}$. Since X is smooth, $\Sigma(X)$ is naturally identified with the

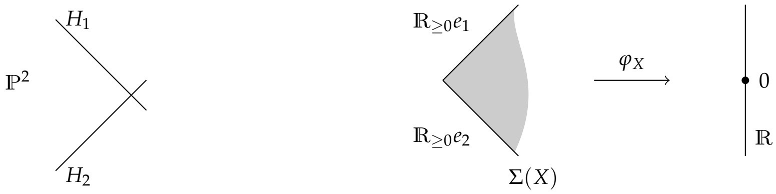


Figure 2.1: The toroidal embedding $\mathbb{P}^2 \setminus (H_1 \cup H_2) \subseteq \mathbb{P}^2$ and its weakly embedded cone complex

orthant $(\mathbb{R}_{\geq 0})^2$, whose rays $\mathbb{R}_{\geq 0}e_1$ and $\mathbb{R}_{\geq 0}e_2$ correspond to the divisors H_1 and H_2. The lattice M^X is generated by $\frac{x_1}{x_2}$, and using that generator to identify M^X with $\mathbb{Z}$ the weak embedding φ_X sends e_1 to 1 and e_2 to -1, as depicted in Figure 2.1.

It follows directly from the definitions that whenever Y is a stratum of a toroidal embedding X, the embedding $Y \subseteq \overline{Y}$ is toroidal again. In the following lemma we compare the weakly embedded cone complex of $\overline{Y}$ with the star of $\Sigma(X)$ at σ^Y.

Lemma 2.1.3. *Let $X_0 \subseteq X$ be a toroidal embedding, and let Y and Y' be two strata of X such that $Y' \subseteq \overline{Y}$. Then the sublattice $M^{Y'}(X) \cap (\sigma_X^Y)^\perp$ consists precisely of the divisors in $M^{Y'}(X)$ whose support does not contain Y. The natural restriction map*

$$M^{Y'}(X) \cap (\sigma_X^Y)^\perp \to M^{Y'}(\overline{Y})$$

is an isomorphism identifying $M_+^{Y'}(X) \cap (\sigma_X^Y)^\perp$ with $M_+^{Y'}(\overline{Y})$. Therefore, it induces an isomorphism $\sigma_{\overline{Y}}^{Y'} \xrightarrow{\cong} \sigma_X^{Y'}/\sigma_X^Y$ of cones. These isomorphisms glue and give an identification of $\Sigma(\overline{Y})$ with $\mathrm{S}_{\Sigma(X)}(\sigma_X^Y)$. Furthermore, there is a natural morphism $N^{\Sigma(\overline{Y})} \to N^{\mathrm{S}_{\Sigma(X)}(\sigma_X^Y)}$ respecting the weak embeddings.

Proof. Let $D_1, \ldots, D_s$ be the boundary divisors containing Y', and assume that they are labeled such that $D_1, \ldots, D_r$ are the ones containing Y. Let u_i be the primitive generator of the ray σ^{D_i}. For a Cartier divisor $D = \sum_i a_i D_i \in M^{Y'}(X)$ we have $\langle D, u_i \rangle = a_i$ by [KKMSD73, p. 63]. It follows that $M^{Y'}(X) \cap (\sigma^Y)^\perp$ contains exactly those divisors of $M^{Y'}(X)$ with $a_1 = \cdots = a_r = 0$, which are precisely those whose support does not contain Y. These divisors can be restricted to the combinatorial open subset $\overline{Y}(Y')$ of $\overline{Y}$, yielding divisors in $M^{Y'}(\overline{Y})$. To show that the restriction map is an isomorphism we may reduce to the toric case by choosing a local toric model at a closed point of Y and using [KKMSD73, II, §1, Lemma 1]. So assume $X = U_\sigma$ is an affine toric variety defined by a cone $\sigma \subset N_\mathbb{R}$ in a lattice N, the stratum $Y' = O(\sigma)$ is the closed orbit, and $Y = O(\tau)$ for a face $\tau \prec \sigma$. Further reducing to the case in which X has no torus factors we may assume that σ is full-dimensional. Let M be the dual of N. The isomorphism

$$M \to M^{Y'}(X), \;\; m \mapsto \mathrm{div}(\chi^m),$$

where χ^m is the character associated to m, induces a commutative diagram

$$\begin{array}{ccccc} M & \supseteq & M \cap \tau^\perp & & \\ \cong\big\downarrow & & \cong\big\downarrow & \searrow & \\ M^{Y'}(X) & \supseteq & M^{Y'}(X) \cap (\sigma^Y)^\perp & \longrightarrow & M^{Y'}(\overline{Y}) \end{array}$$

It is well known that the character lattice of the open torus embedded in the toric variety $\overline{Y}$ can be naturally identified with $M \cap \tau^\perp$. With this identification, the upper right morphism in the diagram sends a character in $M \cap \tau^\perp$ to its associated principal divisor. Thus, it is an isomorphism which implies that the restriction map $M^{Y'}(X) \cap (\sigma^Y)^\perp \to M^{Y'}(\overline{Y})$ is an isomorphism as well. Both $M_+^{Y'}(X) \cap (\sigma_X^Y)^\perp$ and $M_+^{Y'}(\overline{Y})$ correspond to $M \cap \tau^\perp \cap \sigma^\vee$, hence they get identified by the restriction map. Dualization induces an isomorphism of the dual cone $\sigma_{\overline{Y}}^{Y'}$ of $M_+^{Y'}(\overline{Y})$ and the dual cone $\sigma_X^{Y'}/\sigma_X^Y$ of $M_+^Y(X) \cap (\sigma_X^Y)^\perp$. If Y'' is a third stratum of X such that $Y'' \subseteq \overline{Y'}$, then the diagram

$$\begin{array}{ccc} M^{Y''}(X) \cap (\sigma_X^Y)^\perp & \longrightarrow & M^{Y''}(\overline{Y}) \\ \big\downarrow & & \big\downarrow \\ M^{Y'}(X) \cap (\sigma_X^Y)^\perp & \longrightarrow & M^{Y'}(\overline{Y}) \end{array}$$

is commutative because all maps involved are restrictions to open or closed subschemes. It follows that the isomorphisms of cones glue to an isomorphism $\Sigma(\overline{Y}) \cong \mathrm{S}_{\Sigma(X)}(\sigma_X^Y)$. To see that this isomorphism respects the weak embeddings, note that $M^{\mathrm{S}_{\Sigma(X)}(\sigma_X^Y)}$ is equal to the sublattice $(N^X_{\sigma_X^Y})^\perp$ of M^X by definition. It consists exactly of those rational functions that are invertible on $X(Y)$. Hence, they can be restricted to $\overline{Y}$, giving a morphism

$$M^{\mathrm{S}_{\Sigma(X)}(\sigma_X^Y)} \to M^{\overline{Y}}.$$

For any rational function f on X that is invertible on Y we have $\mathrm{div}(f|_{\overline{Y}}) = \mathrm{div}(f)|_{\overline{Y}}$. It follows directly from this equality that the identification $\Sigma(\overline{Y}) \to \mathrm{S}_{\Sigma(X)}(\sigma_X^Y)$ is in fact a morphism of weakly embedded cone complexes. □

A *dominant toroidal* morphism between two toroidal embeddings X and Y is a dominant morphism $X \to Y$ of varieties which can be described by toric morphisms in local toric models (see [AK00] for details). A dominant toroidal morphism $f\colon X \to Y$ induces a dominant morphism $\mathrm{Trop}(f)\colon \overline{\Sigma}(X) \to \overline{\Sigma}(Y)$ of extended cone complexes. The restrictions of $\mathrm{Trop}(f)$ to the cones of $\Sigma(X)$ are dual to pulling back Cartier divisors. From this we easily see that $\mathrm{Trop}(f)$ can be considered as a morphism of weakly embedded extended cone complexes by adding to it the data of the linear map $N^X \to N^Y$ dual to the pull-back $\Gamma(Y_0, \mathcal{O}_Y^*) \to \Gamma(X_0, \mathcal{O}_X^*)$. We call a morphism $f\colon X \to Y$ *toroidal* if it factors as $X \xrightarrow{f'} \mathrm{V}(\sigma) \xrightarrow{i} Y$, where f' is dominant toroidal, and $\sigma \in \Sigma(Y)$. By Lemma 2.1.3 the closed immersion i induces a morphism $\mathrm{Trop}(i) \colon \overline{\Sigma}(\mathrm{V}(\sigma)) \to \overline{\Sigma}(Y)$ of weakly embedded extended cone complexes, namely the composite of the canonical morphism $\overline{\Sigma}(\mathrm{V}(\sigma)) \to \overline{\mathrm{S}_{\Sigma(Y)}(\sigma)}$ and the inclusion of $\overline{\mathrm{S}_{\Sigma(Y)}(\sigma)}$ in $\overline{\Sigma}(Y)$. Thus we can define $\mathrm{Trop}(f) \colon \overline{\Sigma}(X) \to \overline{\Sigma}(Y)$ as the composite $\mathrm{Trop}(i) \circ \mathrm{Trop}(f')$.

A special class of toroidal morphisms is given by *toroidal modifications*. They are the analogues of the toric morphisms resulting from refinements of fans in toric geometry. For every subdivision Σ' of the cone complex Σ of a toroidal embedding X there is a unique toroidal modification $X \times_\Sigma \Sigma' \to X$ whose tropicalization is the subdivision $\Sigma' \to \Sigma$ [KKMSD73]. This modification maps $(X \times_\Sigma \Sigma')_0$ isomorphically onto X_0. Modifications are compatible with toroidal morphisms in the sense that if $f \colon X \to Y$ is dominant toroidal, and Σ' and Δ' are subdivisions of $\Sigma(X)$ and $\Sigma(Y)$, respectively, such that $\mathrm{Trop}(f)$ induces a morphism $\Sigma' \to \Delta'$, then f lifts to a toroidal morphism $f' \colon X \times_{\Sigma(X)} \Sigma' \to Y \times_{\Sigma(Y)} \Delta$ [AK00, Lemma 1.11].

Remark 2.1.4. Our usage of the term toroidal morphism is nonstandard. Non-dominant toroidal morphisms in our sense are called subtoroidal in [ACP15]. The notation for toroidal modifications is due to Kazuya Kato [Kat94] and is not as abusive as it may seem [GR14, Prop. 10.6.14].

2.2 Intersection Theory on Weakly Embedded Cone Complexes

In what follows we will develop the foundations of a tropical intersection theory on weakly embedded cone complexes. Our constructions are motivated by the relation of algebraic and tropical intersection theory as well as by the well-known constructions for the embedded case studied in [AR10] (see Subsection 1.1.1 for a brief survey). In fact, those of our constructions that work primarily in the finite part of the cone complex are natural generalizations of the corresponding constructions for embedded complexes. Intersection theoretical constructions involving boundary components at infinity have been studied in the setup of tropical manifolds [Mik06, Sha11] and for Kajiwara's and Payne's tropical toric varieties [Mey11]. The latter is closer to our setup, yet definitions and proofs vary significantly from ours.

2.2.1 Minkowski Weights, Tropical Cycles, and Tropical Divisors

For the definitions of Minkowski weights and tropical cycles we need the notion of lattice normal vectors. Let τ be a codimension 1 face of a cone σ of a weakly embedded cone complex Σ. We denote by $u_{\sigma/\tau}$ the image under the morphism

$$N^\sigma / N^\tau \to N^\Sigma / N^\Sigma_\tau$$

induced by the weak embedding of the generator of N^σ/N^τ which is contained in the image of $\sigma \cap N^\sigma$ under the projection $N^\sigma \to N^\sigma/N^\tau$, and call it the *lattice normal vector of σ relative to τ*. Note that lattice normal vectors may be equal to 0.

Definition 2.2.1. Let Σ be a weakly embedded cone complex. A *k-dimensional Minkowski weight* on Σ is a map $c\colon \Sigma_{(k)} \to \mathbb{Z}$ from the k-dimensional cones of Σ to the integers such that it satisfies the *balancing condition* around every $(k-1)$-dimensional cone $\tau \in \Sigma$: if $\sigma_1, \dots, \sigma_n$ are the k-dimensional cones containing τ, then

$$\sum_{i=1}^{n} c(\sigma_i) u_{\sigma_i/\tau} = 0 \qquad \text{in } N^{\Sigma}/N^{\Sigma}_{\tau}.$$

The k-dimensional Minkowski weights naturally form an abelian group, which we denote by $\mathrm{M}_k(\Sigma)$.

We define the group of *tropical k-cycles* on Σ by $Z_k(\Sigma) = \varinjlim \mathrm{M}_k(\Sigma')$, where Σ' runs over all proper subdivisions of Σ. If c is a k-dimensional Minkowski weight on a proper subdivision of Σ, we denote by $[c]$ its image in $Z_k(\Sigma)$. The group of tropical k-cycles on the extended complex $\overline{\Sigma}$ is defined by $Z_k(\overline{\Sigma}) = \bigoplus_{\sigma \in \Sigma} Z_k(\mathrm{S}(\sigma))$. We will write $\mathrm{M}_*(\Sigma) = \bigoplus_k \mathrm{M}_k(\Sigma)$ for the graded group of Minkowski weights, and similarly $Z_*(\Sigma)$ and $Z_*(\overline{\Sigma})$ for the graded groups of tropical cycles on Σ and $\overline{\Sigma}$, respectively.

The support $|A|$ of a cycle $A = [c] \in Z_k(\Sigma)$ is the union of all k-dimensional cones on which c has nonzero weight. This is easily seen to be independent of the choice of c.

Remark 2.2.2. In the definition of cycles we implicitly used that a proper subdivision Σ' of a weakly embedded cone complex Σ induces a morphism $\mathrm{M}_k(\Sigma) \to \mathrm{M}_k(\Sigma')$. The definition of this morphism is clear, yet a small argument is needed to show that balancing is preserved. But this can be seen similarly as in the embedded case [AR10, Lemma 2.11].

Remark 2.2.3. Minkowski weights are meant as the analogues of cocycles on toroidal embeddings, whereas tropical cycles are the analogues of cycles. To see the analogy consider the toric case. There, the cohomology group of a normal toric variety X corresponding to a fan Σ is canonically isomorphic to the group of Minkowski weights on Σ [FS97]. On the other hand, for a subvariety Z of X this isomorphism cannot be used to assign a Minkowski weight to the cycle $[Z]$ unless X is smooth. In general, we first have to take a proper transform of Z to a smooth toric modification of X, and therefore only obtain a Minkowski weight up to refinements of fans, i.e. a tropical cycle. That this tropical cycle is well-defined has been shown e.g. in [ST08, Kat09].

Example 2.2.4. Consider the weakly embedded cone complex from Example 2.1.2 b) and let us denote it by Σ. Let σ be its maximal cone, and $\rho_1 = \mathbb{R}_{\geq 0} e_1$ and $\rho_2 = \mathbb{R}_{\geq 0} e_2$ its two rays. The two lattice normal vectors u_{σ/ρ_1} and u_{σ/ρ_2} are equal to 0. Therefore, the balancing condition for 2-dimensional Minkowski weights is trivial and we have $\mathrm{M}_2(\Sigma) = \mathbb{Z}$. In dimension 1, we have $u_{\rho_1/0} = 1$ and $u_{\rho_2/0} = -1$. Hence, every balanced 1-dimensional weight must have equal weights on the two rays, which implies $\mathrm{M}_1(\Sigma) = \mathbb{Z}$. In dimension 0 there is, of course, no balancing to check and we have $\mathrm{M}_0(\Sigma) = \mathbb{Z}$ as well.

Cocycles on a toroidal embedding can be pulled back to closures of strata in its boundary. This works for Minkowski weights as well:

Construction 2.2.5 (Pull-backs of Minkowski weights)**.** Let Σ be a weakly embedded cone complex, and let γ be a cone of Σ. Whenever we are given an inclusion $\tau \preceq \sigma$ of two cones of Σ which contain γ, we obtain an inclusion $\tau/\gamma \preceq \sigma/\gamma$ of cones in $\mathrm{S}(\gamma)$ of the same codimension. By construction, the restriction of the weak embedding $\varphi_{\mathrm{S}(\gamma)}$ to σ/γ is equal to the map $\sigma/\gamma \to (N^\Sigma/N^\Sigma_\gamma)_\mathbb{R} = N^{\mathrm{S}(\gamma)}_\mathbb{R}$ induced by φ_Σ. It follows that there is a natural isomorphism $N^\Sigma/N^\Sigma_\tau \xrightarrow{\cong} N^{\mathrm{S}(\gamma)}/N^{\mathrm{S}(\gamma)}_{\tau/\gamma}$. Since $N^{\sigma/\gamma} = N^\sigma/N^\gamma$ by construction, we also have a canonical isomorphism $N^\sigma/N^\tau \xrightarrow{\cong} N^{\sigma/\gamma}/N^{\tau/\gamma}$. These isomorphisms fit into a commutative diagram

$$\begin{array}{ccc} N^\sigma/N^\tau & \longrightarrow & N^\Sigma/N^\Sigma_\tau \\ \downarrow \cong & & \downarrow \cong \\ N^{\sigma/\gamma}/N^{\tau/\gamma} & \longrightarrow & N^{\mathrm{S}(\gamma)}/N^{\mathrm{S}(\gamma)}_{\tau/\gamma}. \end{array}$$

If τ has codimension 1 in σ, we see that the image of the lattice normal vector $u_{\sigma/\tau}$ in $N^{\mathrm{S}(\gamma)}/N^{\mathrm{S}(\gamma)}_{\tau/\gamma}$ is the lattice normal vector $u_{(\sigma/\gamma)/(\tau/\gamma)}$. Now suppose we are given a Minkowski weight $c \in \mathrm{M}_k(\Sigma)$ with $k \geq \dim \gamma$. It follows immediately from the considerations above that the induced weight on $\mathrm{S}(\gamma)$ assigning $c(\sigma)$ to σ/γ for $\sigma \in \Sigma_{(k)}$ with $\gamma \preceq \sigma$ satisfies the balancing condition. Hence it defines a Minkowski weight in $\mathrm{M}_{k-\dim\gamma}(\mathrm{S}(\gamma))$. If $i\colon \mathrm{S}(\gamma) \to \overline{\Sigma}$ is the inclusion map, we denote it by i^*c. In case $k < \dim \gamma$ we set $i^*c = 0$. $\Diamond$

Definition 2.2.6. Let Σ be a weakly embedded cone complex. A *Cartier divisor* on Σ is a continuous function $\psi\colon |\Sigma| \to \mathbb{R}$ which is integral linear on all cones of Σ. We denote the group of Cartier divisors by $\mathrm{Div}(\Sigma)$. A Cartier divisor ψ is said to be *combinatorially principal* (cp for short) if ψ is induced by a function in M^Σ on each cone. The subgroup of $\mathrm{Div}(\Sigma)$ consisting of cp-divisors is denoted by $\mathrm{CP}(\Sigma)$. Two divisors are called *linearly equivalent* if their difference is induced by a function in M^Σ. We denote by $\mathrm{ClCP}(\Sigma)$ the group of cp-divisors modulo linear equivalence.

Remark 2.2.7. Note that our usage of the term Cartier divisors is nonstandard. Usually, piecewise integral linear functions on $\mathbb{R}^n$, or more generally embedded cone complexes, are called rational functions. This is because they arise naturally as "quotients", that is differences, of tropical "polynomials", that is functions of the form $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto \min\{\langle m, x\rangle \mid m \in \Delta\}$ for some finite set $\Delta \subseteq \mathbb{Z}^n$. As we lack the embedding, we chose a different analogy: assuming that the cone complex Σ comes from a toroidal embedding X, integer linear functions on a cone σ correspond to divisors on $X(\sigma)$ supported on $X(\sigma) \setminus X_0$. Since the cones in Σ and the combinatorial open subsets of X are glued accordingly, this induces a correspondence between $\mathrm{Div}(\Sigma)$ and the group of Cartier divisors on X supported away from X_0. Unlike in the toric case, a Cartier divisor on a combinatorial open subset $X(\sigma)$ supported away from X_0 does not need to be principal. In case it is, it is defined by a rational function in $\Gamma(X_0, \mathcal{O}^*_X)$. This means that the associated linear function on σ is the pull-back of a function in M^X, explaining the terminology combinatorially principal. Finally, linear equivalence is defined precisely in such a way that two divisors are linearly equivalent if and only if their associated divisors on X are.

The functorial behavior of tropical Cartier divisors is as one would expect from algebraic geometry. They can be pulled back along dominant morphisms of weakly embedded extended cone complexes, and along arbitrary morphisms if we pass to linear equivalence. In the latter case, however, we need to restrict ourselves to cp-divisors.

Construction 2.2.8 (Pull-backs of Cartier divisors)**.** Let $f\colon \overline{\Sigma} \to \overline{\Delta}$ be a morphism of weakly embedded extended cone complexes, and let ψ be a divisor on Δ. If f is dominant, then it follows directly from the definitions that $f^*\psi = \psi \circ f$ is a divisor on Σ. We call it the *pull-back* of ψ along f. Moreover, if ψ is combinatorially principal, then so is $f^*\psi$. The map $f^*\colon \mathrm{CP}(\Delta) \to \mathrm{CP}(\Sigma)$ clearly is a morphism of abelian groups, and it preserves linear equivalence. Therefore, it induces a morphism $\mathrm{ClCP}(\Delta) \to \mathrm{ClCP}(\Sigma)$ which we again denote by f^*. If f is arbitrary, it factors uniquely as $\overline{\Sigma} \xrightarrow{f'} \overline{\mathrm{S}_\Delta(\delta)} \xrightarrow{i} \overline{\Delta}$, where $\delta \in \Delta$, f' is dominant, and i is the inclusion map. Hence, to define a pull-back $f^* : \mathrm{ClCP}(\Delta) \to \mathrm{ClCP}(\Sigma)$ it suffices to construct pull-backs of divisor classes on Δ to $\mathrm{S}_\Delta(\delta)$. So let ψ be a cp-divisor on Δ. Choose $\psi_\delta \in M^\Delta$ such that
$\sigma \in \Delta$ with $\delta \preceq \sigma$ there is an induced integral linear map $\overline{\psi}'_\sigma\colon \sigma/\delta \to \mathbb{R}$. These maps patch together to a cp-divisor $\overline{\psi}'$ on $\mathrm{S}_\Delta(\delta)$. Clearly, $\overline{\psi}'$ depends on the choice of ψ_δ, but different choices produce linearly equivalent divisors. Thus, we obtain a well-defined element $i^*\psi \in \mathrm{ClCP}(\mathrm{S}_\Delta(\delta))$. By construction, divisors linearly equivalent to 0 are in the kernel of i^*, so there is an induced morphism $\mathrm{ClCP}(\Delta) \to \mathrm{ClCP}(\mathrm{S}_\Delta(\delta))$ which we again denote by i^*. Finally, we define the pull-back along f by $f^* := (f')^* \circ i^*$. $\diamondsuit$

2.2.2 Push-forwards

Now we want to construct push-forwards of tropical cycles. To do so we need to know that cone complexes allow sufficiently fine proper subdivisions with respect to given subcones.

Lemma 2.2.9. *Let Σ be a cone complex and $\mathcal{A}$ a finite set of subcones of Σ. Then there exists a proper subdivision Σ' of Σ such that every cone in $\mathcal{A}$ is a union of cones in Σ'.*

Proof. If Σ is a fan in $N_\mathbb{R}$ for some lattice N, it is well-known how to obtain a suitable subdivision. Namely, we choose linear inequalities for every $\tau \in \mathcal{A}$ and intersect the cones in Σ with the half-spaces defined by these inequalities. In the general case we may assume that Σ is strictly simplicial by choosing a strictly simplicial proper subdivision Σ' and intersecting all cones in $\mathcal{A}$ with the cones in Σ'. Let $\varphi\colon |\Sigma| \to \mathbb{R}^{\Sigma_{(1)}}$ be the weak embedding sending the primitive generator of a ray $\rho \in \Sigma_{(1)}$ to the generator e_ρ of $\mathbb{R}^{\Sigma_{(1)}}$ corresponding to ρ. Let Δ be a complete fan in $\mathbb{R}^{\Sigma_{(1)}}$ such that $\varphi(\sigma)$ is a union of cones of Δ for every $\sigma \in \Sigma \cup \mathcal{A}$. We define Σ' as the proper subdivision of Σ consisting of the cones $\varphi^{-1}\delta \cap \sigma$ for $\delta \in \Delta$ and

$\sigma \in \Sigma$ and claim that it satisfies the desired property. Let $\tau \in \mathcal{A}$, and let $\sigma \in \Sigma$ be a cone containing it. Then by construction there exist cones $\delta_1, \ldots, \delta_k$ of Δ such that $\varphi(\tau) = \bigcup_i \delta_i$. Each δ_i is in the image of σ, hence $\varphi(\varphi^{-1}\delta_i \cap \sigma) = \delta_i$. With this it follows that $\varphi(\tau) = \varphi\left(\bigcup_i (\varphi^{-1}\delta_i \cap \sigma)\right)$, and hence that τ is a union of cones in Σ' since φ is injective on σ by construction. □

Construction 2.2.10 (Push-forwards)**.** Let $f\colon \overline{\Sigma} \to \overline{\Delta}$ be a morphism of weakly embedded extended cone complexes, and let $A \in Z_k(\overline{\Sigma})$ be a tropical cycle. We construct a push-forward $f_*A \in Z_k(\overline{\Delta})$ similarly as in the embedded case [GKM09]. First suppose that f is dominant, and that A is represented by a Minkowski weight c on a proper subdivision Σ' of Σ. We further reduce to the case where the images of cones in Σ' are cones in Δ: for every $\sigma' \in \Sigma'$ the image $f(\sigma')$ is a subcone of Δ. By Lemma 2.2.9 we can find a proper subdivision Δ' of Δ such that each of these subcones is union of cones of Δ'. The cones $\sigma' \cap f^{-1}\delta'$ for $\sigma' \in \Sigma'$ and $\delta' \in \Delta'$ form a proper subdivision Σ'' of Σ', and standard arguments show that $f(\sigma'') \in \Delta'$ for all $\sigma'' \in \Sigma''$. As c induces a Minkowski weight on Σ'', and Minkowski weights on Δ' define tropical cycles on Δ we assume $\Sigma = \Sigma''$ and $\Delta = \Delta'$ and continue to construct a Minkowski weight f_*c on Δ. For a k-dimensional cone $\delta \in \Delta$ we define

$$f_*c(\delta) = \sum_{\sigma \mapsto \delta} [N^\delta : f(N^\sigma)] c(\sigma),$$

where the sum is over all k-dimensional cones of Σ with image equal to δ. A slight modification of the argument for the embedded case [GKM09, Prop. 2.25] shows that f_*c is balanced. We define the cycle $f_*A \in Z_k(\Delta)$ as the tropical cycle defined by f_*c. This definition is independent of the choices of subdivisions, as can be seen as follows. The support of f_*c is equal to the union of the images of the k-dimensional cones contained in $|A|$ on which f is injective, hence independent of all choices. Let us denote it by $|f_*A|$. Outside a $(k-1)$-dimensional subset of $|f_*A|$, namely the image of the union of all cones of Σ of dimension less or equal to k whose image has dimension strictly less than k, every point of $|f_*A|$ has finitely many preimages in $|A|$, and they all lie in the relative interiors of k-dimensional cones of Σ on which f is injective. Thus, locally, the number of summands occurring in the definition of f_*c does not depend on the chosen subdivisions. Since the occurring lattice indices are local in the same sense, the independence of f_*A of all choices follows. It is immediate from the construction that pushing forward is linear in A.

Now suppose that f is arbitrary, and A is contained in $Z_k(\mathrm{S}_\Sigma(\sigma))$ for some $\sigma \in \Sigma$. The restriction $f|_{\overline{\mathrm{S}_\Sigma(\sigma)}}$ defines a dominant morphism to $\overline{\mathrm{S}_\Delta(\delta)}$ for some $\delta \in \Delta$. Using the construction for the dominant case we define

$$f_*A := \left(f|_{\overline{\mathrm{S}_\Sigma(\sigma)}}\right)_* A \in Z_k(\mathrm{S}_\Delta(\delta)) \subseteq Z_k(\overline{\Delta})$$

and call it the *push-forward* of A. Extending by linearity the push-forward defines a graded morphism $f_*\colon Z_*(\overline{\Sigma}) \to Z_*(\overline{\Delta})$. ◊

Proposition 2.2.11. *Let* $f\colon \overline{\Sigma} \to \overline{\Delta}$ *and* $g\colon \overline{\Delta} \to \overline{\Theta}$ *be morphisms of weakly embedded extended cone complexes. Then*

$$(g \circ f)_* = g_* \circ f_*.$$

Proof. By construction it suffices to prove that if f and g are dominant, and $A \in Z_k(\Sigma)$, then $(g \circ f)_* A = g_*(f_* A)$. But this works analogously to the embedded case [Rau09, Remark 1.3.9]. □

2.2.3 Intersecting with Divisors

Next we construct an intersection product $\psi \cdot A$ for a divisor ψ and a tropical cycle $A \in Z_k(\Sigma)$. Note that when $\Sigma = \Sigma(X)$ for a toroidal embedding X, the divisor ψ corresponds to a Cartier divisor D on X supported away from X_0. We can only expect an intersection product $D \cdot Z$ with a subvariety Z of X to be defined without passing to rational equivalence if Z intersects X_0 nontrivially. In the tropical world this is reflected by the requirement of A living in the interior of $\overline{\Sigma}$.

Construction 2.2.12 (Tropical intersection products)**.** Let Σ be a weakly embedded cone complex, $A \in Z_k(\Sigma)$ a tropical cycle, and ψ a Cartier divisor on $\overline{\Sigma}$. We construct the intersection product $\psi \cdot A \in Z_{k-1}(\overline{\Sigma})$. First suppose that A is defined by a Minkowski weight $c \in \mathrm{M}_k(\Sigma)$. Writing $i_\rho\colon \mathrm{S}(\rho) \to \overline{\Sigma}$ for the inclusion map for all rays $\rho \in \Sigma_{(1)}$, we define

$$\psi \cdot A = \sum_{\rho \in \Sigma_{(1)}} \psi(u_\rho)[i_\rho^* c] \in Z_{k-1}(\overline{\Sigma}),$$

where u_ρ is the primitive generator of ρ.

In general, A is represented by a Minkowski weight on a proper subdivision Σ' of Σ. Let $f\colon \overline{\Sigma}' \to \overline{\Sigma}$ be the morphism of weakly embedded extended cone complexes induced by the subdivision. Then we define

$$\psi \cdot A = f_*(\psi \cdot_{\Sigma'} A),$$

where $\psi \cdot_{\Sigma'} A$ denotes the cycle in $Z_{k-1}(\overline{\Sigma}')$ constructed above.

To verify that this is independent of the choice of Σ' it suffices to prove that in case A is already represented by a Minkowski weight on Σ, this gives the same definition as before. Let $c' \in \mathrm{M}_k(\Sigma')$ and $c \in \mathrm{M}_k(\Sigma)$ be the Minkowski weights representing A. For a ray $\rho \in \Sigma$, which is automatically a ray in Σ', it is not hard to see that the complex $\mathrm{S}_{\Sigma'}(\rho)$ is a proper subdivision of $\mathrm{S}_\Sigma(\rho)$. Writing $i'_\rho\colon S_{\Sigma'}(\rho) \to \overline{\Sigma}'$ for the inclusion we see that $f_*[(i'_\rho)^* c'] = [i_\rho^* c]$. For a ray $\rho' \in \Sigma' \setminus \Sigma$ there exists a minimal cone $\tau \in \Sigma$ containing it, which has dimension at least 2. Every k-dimensional cone $\sigma' \in \Sigma'$ containing ρ' with $c'(\sigma') \neq 0$ is contained in a k-dimensional cone $\sigma \in \Sigma$. Then τ must be a face of σ and hence the image of $\sigma'/\rho' \subseteq \mathrm{S}_{\Sigma'}(\rho')$ in $\mathrm{S}_\Sigma(\tau)$ has dimension at most $\dim(\sigma/\tau)$, which is strictly less than k. This shows that $f_*[(i'_{\rho'})^* c'] = 0$ and finishes the prove of the independence of choices. ♢

Remark 2.2.13. In the construction of $\psi \cdot A$ for a cycle A represented by a Minkowski weight c on a proper subdivision Σ' of Σ, we can compute the contributions of the weights of c on the cones of Σ' explicitly. Assume that σ' is a k-dimensional cone of Σ'. We want to determine its contribution to the component of $\psi \cdot A$ in $\mathrm{S}_\Sigma(\tau)$ for some $\tau \in \Sigma$. We can only have such a contribution if σ' has a ray $\rho \in \sigma'_{(1)}$ intersecting the relative interior of τ. In this case, let $\delta \in \Sigma$ be the smallest cone of Σ containing σ'. As $\sigma' \cap \operatorname{relint}(\tau) \neq \emptyset$ we must have $\tau \preceq \delta$. The image of $\sigma'/\rho \subseteq \mathrm{S}_{\Sigma'}(\rho)$ in $\delta/\tau \subseteq \mathrm{S}_\Sigma(\tau)$ is the image of σ' under the canonical projection $\delta \to \delta/\tau$. This is $(k-1)$-dimensional if and only if the sublattice $N^{\sigma'} \cap N^\tau$ of N^δ has rank 1. In this case, we have $\sigma' \cap \tau = \rho$, and the contribution of σ' to the component of $\psi \cdot A$ in $\mathrm{S}_\Sigma(\tau)$ is the cone $(\sigma' + \tau)/\tau$ with weight

$$\psi(u_\rho) \operatorname{index}\left(N^\tau + N^{\sigma'}\right) c(\sigma'),$$

where u_ρ is the primitive generator of ρ, and the occurring index is the index of $N^\tau + N^{\sigma'}$ in its saturation in N^δ.

We see that σ' contributes exactly to those boundary components of $\psi \cdot A$ which we would expect from topology. This is because "the part at infinity" $\overline{\sigma}' \setminus \sigma'$ of σ' in $\overline{\Sigma}$ intersects $\mathrm{S}_\Sigma(\tau)$ if and only if $\operatorname{relint}(\tau) \cap \sigma' \neq \emptyset$, and the intersection is $(k-1)$-dimensional if and only if $N^{\sigma'} \cap N^\tau$ has rank 1.

As mentioned earlier, Minkowski weights on the cone complex $\Sigma(X)$ of a toroidal embedding X correspond to cocycles on X. For an (algebraic) Cartier divisor D on X supported away from X_0 and a cocycle c in $A^*(X)$ the natural "intersection" is the cup product $c_1(\mathcal{O}_X(D)) \cup c$. It turns out that we can describe the associated Minkowski weight of the product by a tropical cup product in case the associated tropical divisor of D is combinatorially principal (cf. Proposition 2.3.13).

Construction 2.2.14 (Tropical cup products)**.** Let Σ be a weakly embedded cone complex, $c \in \mathrm{M}_k(\Sigma)$ a Minkowski weight, and ψ a cp-divisor on Σ. We construct a Minkowski weight $\psi \cup c \in \mathrm{M}_{k-1}(\Sigma)$. For every $\sigma \in \Sigma$ choose an integral linear function $\psi_\sigma \in M^\Sigma$ such that $\psi_\sigma \circ \varphi_\Sigma|_\sigma = \psi|_\sigma$. Note that ψ_σ is not uniquely defined by this property, but its restriction $\psi_\sigma|_{N^\Sigma_\sigma}$ is. Whenever we have an inclusion $\tau \preceq \sigma$ of cones of Σ, the function $\psi_\sigma - \psi_\tau$ vanishes on N^Σ_τ and hence defines a morphism $N^\Sigma / N^\Sigma_\tau \to \mathbb{Z}$. Therefore, we can define a weight on the $(k-1)$-dimensional cones of Σ by

$$\psi \cup c \colon \tau \mapsto \sum_{\sigma : \tau \prec \sigma} (\psi_\sigma - \psi_\tau)(c(\sigma) u_{\sigma/\tau}),$$

where the sum is taken over all k-dimensional cones of Σ containing τ. The balancing condition for c ensures that this weight is independent of the choices involved. It has been proven in [AR10, Prop. 3.7a] for the embedded case that $\psi \cup c$ is a Minkowski weight again, and a slight modification of the proof in loc. cit. works in our setting as well.

The cup-product can be extended to apply to tropical cycles as well. For $A \in Z_k(\Sigma)$ there is a subdivision Σ' of Σ such that A is represented by a Minkowski weight $c \in \mathrm{M}_k(\Sigma')$. It is easy to see that the tropical cycle $[\psi \cup c]$ does not depend on the choice of Σ'. Thus, we can define $\psi \cup A := [\psi \cup c]$. For details we again refer to [AR10].

It is immediate from the definition that the cup-product is independent of the linear equivalence class of the divisor. Therefore, we obtain a pairing

$$\mathrm{ClCP}(\Sigma) \times Z_*(\Sigma) \to Z_*(\Sigma),$$

which is easily seen to be bilinear. It can be extended to include tropical cycles in the boundary of $\overline{\Sigma}$. Let $\tau \in \Sigma$, and denote the inclusion $i : \mathrm{S}(\tau) \to \overline{\Sigma}$ by i. For a tropical cycle $A \in \mathrm{S}_\Sigma(\tau)$ and a divisor class $\overline{\psi} \in \mathrm{ClCP}(\Sigma)$ we define $\overline{\psi} \cup A := i^*\overline{\psi} \cup A$. In this way we obtain a bilinear map

$$\mathrm{ClCP}(\Sigma) \times Z_*(\overline{\Sigma}) \to Z_*(\overline{\Sigma}). \qquad \lozenge$$

In algebraic geometry, intersecting a cycle with multiple Cartier divisors does not depend on the order of the divisors. We would like to prove an analogous result for tropical intersection products on cone complexes. However, to define multiple intersections we need rational equivalence, which we will only introduce in Definition 2.2.18. For cup-products on the other hand there is no problem in defining multiple intersections and they do, in fact, not depend on the order of the divisors:

Proposition 2.2.15. *Let Σ be a weakly embedded cone complex, $A \in Z_*(\overline{\Sigma})$ a tropical cycle, and $\psi, \chi \in \mathrm{CP}(\Sigma)$ two cp-divisors. Then we have*

$$\psi \cup (\chi \cup A) = \chi \cup (\psi \cup A).$$

Proof. This can be proven similarly as in the embedded case [AR10, Prop. 3.7]. □

Next we will consider the projection formula for weakly embedded cone complexes. We will prove two versions, one for "$\cdot$"-products and one for "$\cup$"-products.

Proposition 2.2.16. *Let $f : \overline{\Sigma} \to \overline{\Delta}$ be a morphism between weakly embedded cone complexes.*

a) *If f is dominant, $A \in Z_*(\Sigma)$, and ψ is a divisor on Δ, then*

$$f_*(f^*\psi \cdot A) = \psi \cdot f_*A.$$

b) *For $\overline{\psi} \in \mathrm{ClCP}(\Delta)$ and $A \in Z_*(\overline{\Sigma})$ we have*

$$f_*(f^*\overline{\psi} \cup A) = \overline{\psi} \cup f_*A.$$

Proof. In both cases both sides of the equality we wish to prove are linear in A. Therefore, we may assume that A is a k-dimensional tropical cycle. For part a) choose proper subdivisions Σ' and Δ' of Σ and Δ, respectively, such that A is defined by a Minkowski weight $c \in \mathrm{M}_k(\Sigma')$ and f maps cones of Σ' onto cones of Δ'. Consider the commutative diagram

$$\begin{array}{ccc} \overline{\Sigma}' & \xrightarrow{f'} & \overline{\Delta}' \\ \Big\downarrow p & & \Big\downarrow q \\ \overline{\Sigma} & \xrightarrow{f} & \overline{\Delta}. \end{array}$$

By construction of the intersection product, we have $f^*\psi \cdot A = p_*[f^*\psi \cdot c]$, where we identify $f^*\psi$ with $p^*(f^*\psi)$. Therefore, we have

$$f_*(f^*\psi \cdot A) = (f \circ p)_*[f^*\psi \cdot c] = q_*\big(f'_*[f^*\psi \cdot c]\big).$$

On the other hand, by construction of push-forward and intersection product, we have

$$\psi \cdot f_*A = q_*[\psi \cdot f'_*c].$$

This reduces the proof to the case where images of cones in Σ are cones in Δ, and A is represented by a Minkowski weight $c \in \mathrm{M}_k(\Sigma)$. Let $\rho \in \Delta$ be a ray and $\delta \in \Delta$ a k-dimensional cone containing it. Every cone $\sigma \in \Sigma$ mapping injectively onto δ contains exactly one ray ρ' mapping onto ρ. The contribution of the cone σ in $f_*(i^*_{\rho'}c)$, where $i_{\rho'} \colon \mathrm{S}_\Sigma(\rho') \to \overline{\Sigma}$ is the inclusion, is a weight $[N^{\delta/\rho} : f(N^{\sigma/\rho'})]c(\sigma)$ on the cone δ/ρ. The lattice index is equal to

$$[N^{\delta/\rho} : f(N^{\sigma/\rho'})] = [N^\delta/N^\rho : (f(N^\sigma) + N^\rho)/N^\rho] = [N^\delta : f(N^\sigma)]/[N^\rho : f(N^{\rho'})].$$

The index $[N^\rho : f(N^{\rho'})]$ is also defined by the fact that $f(u_{\rho'}) = [N^\rho : f(N^{\rho'})]u_\rho$, where u_ρ and $u_{\rho'}$ denote the primitive generators of ρ and ρ', respectively. Combined, we see that the weight of the component of $f_*(f^*\psi \cdot c)$ in $\mathrm{S}_\Delta(\rho)$ at δ/ρ is equal to

$$\sum_{\rho' \mapsto \rho} \sum_{\substack{\rho' \prec \sigma, \\ \sigma \mapsto \delta}} \psi(f(u_{\rho'}))[N^\delta : f(N^\sigma)]/[N^\rho : f(N^{\rho'})]c(\sigma) = \psi(u_\rho) \sum_{\sigma \mapsto \delta} [N^\delta : f(N^\sigma)]c(\delta),$$

which is precisely the multiplicity of $\psi \cdot f_*c$ at δ/ρ. Because the $\mathrm{S}_\Delta(\delta)$-component is 0 for both sides of the projection formula if δ is not a ray, we have proven part a).

For part b) we may reduce to the case that f is dominant and $A \in Z_k(\Sigma)$. It also suffices to prove the equation for divisors instead of divisor classes. Analogous to part a) we then reduce to the case where A is given by a Minkowski weight on Σ, and images of cones of Σ are cones of Δ. In this case, a slight variation of the proof of the projection formula for embedded complexes [AR10, Prop. 4.8] applies. □

Proposition 2.2.17. *Let Σ be a weakly embedded cone complex.*

a) *For every Minkowski weight $c \in \mathrm{M}_k(\Sigma)$, cp-divisor $\psi \in \mathrm{CP}(\Sigma)$, and cone $\tau \in \Sigma$ we have*

$$i^*(\psi \cup c) = i^*\psi \cup i^*c,$$

where $i \colon \mathrm{S}(\tau) \to \overline{\Sigma}$ denotes the inclusion map.

b) *If* $A \in Z_*(\Sigma)$, $\chi \in \operatorname{Div}(\Sigma)$, *and* $\overline{\psi} \in \operatorname{ClCP}(\Sigma)$, *then*

$$\overline{\psi} \cup (\chi \cdot A) = \chi \cdot (\overline{\psi} \cup A).$$

Proof. We begin with part a). If $\dim(\tau) \geq k$, then both sides of the equation are equal to 0. So assume $\dim(\tau) < k$. Let $\sigma \in \Sigma$ be a $(k-1)$-dimensional cone containing τ, and let $\gamma_1, \dots, \gamma_l$ be the k-dimensional cones of Σ containing σ. To compute the weights of the two sides of the equation at σ/τ, we may assume that ψ vanishes on σ. Let $\psi_i \in M^\Sigma$ be linear functions such that $\psi_i \circ \varphi_\Sigma|_{\gamma_i} = \psi|_{\gamma_i}$. Then if $v_1, \dots, v_l$ are representatives in N^Σ of the lattice normal vectors $u_{\gamma_1/\sigma}, \dots, u_{\gamma_l/\sigma}$, the weight of $\psi \cup c$ at σ, and hence the weight of $i^*(\psi \cup c)$ at σ/τ, is given by

$$\sum_{i=1}^{l} \psi_i(v_i) c(\gamma_i).$$

Since ψ vanishes on σ, each ψ_i vanishes on N^Σ_τ. Thus for every i we have an induced map $\overline{\psi}_i \in \operatorname{Hom}(N^\Sigma / N^\Sigma_\tau, \mathbb{Z}) = M^{\mathrm{S}(\tau)}$. By construction of the pull-backs of divisors, these functions define $i^*\psi$ around σ/τ. We saw in Construction 2.2.5 that the lattice normal vectors $u_{(\gamma_i/\tau)/(\sigma/\tau)}$ are the images of the lattice normal vectors $u_{\gamma_i/\sigma}$ under the quotient map $N^\Sigma / N^\Sigma_\tau \to N^{\mathrm{S}(\tau)} / N^{\mathrm{S}(\tau)}_{\sigma/\tau}$. This implies that the image $\overline{v}_i$ of v_i under the quotient map $N^\Sigma \to N^{\mathrm{S}(\tau)}$ represents $u_{(\gamma_i/\tau)/(\sigma/\tau)}$. Hence, the weight of $i^*\psi \cup i^*c$ at σ/τ is

$$\sum_{i=1}^{l} \overline{\psi}_i(\overline{v}_i) i^*c(\gamma_i/\tau) = \sum_{i=1}^{l} \psi_i(v_i) c(\gamma_i),$$

proving the desired equality.

For part b) we my assume that A is pure-dimensional. We first treat the case in which it is given by a Minkowski weight c on Σ. Using the definition of the intersection product and the bilinearity of the cup-product we get

$$\overline{\psi} \cup (\chi \cdot A) = \left[\sum_{\rho \in \Sigma_{(1)}} \chi(u_\rho) \left(\overline{\psi} \cup i^*_\rho c \right) \right] \overset{a)}{=} \left[\sum_{\rho \in \Sigma_{(1)}} \chi(u_\rho) i^*_\rho \left(\overline{\psi} \cup c \right) \right] = \chi \cdot (\overline{\psi} \cup c),$$

where $i_\rho \colon \mathrm{S}(\rho) \to \overline{\Sigma}$ denotes the inclusion map. The general case follows by considering a suitable proper subdivision of Σ and using the projection formulas. □

2.2.4 Rational Equivalence

We denote by Π^1 the weakly embedded cone complex associated to $\mathbb{P}^1$ equipped with the usual toric structure. It is the unique complete fan in $\mathbb{R}$, its maximal cones being $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$. We identify the extended cone complex $\overline{\Pi}^1$ with $\mathbb{R} \cup \{\infty, -\infty\}$, where the boundary points ∞ and $-\infty$ of $\overline{\Pi}^1$ correspond to the boundary points 0 and ∞ of $\mathbb{P}^1$, respectively. The identity function on $\mathbb{R}$ defines a cp-divisor on Π^1 which we denote by ψ_{Π^1}. Its associated cycle $\psi_{\Pi^1} \cdot [\overline{\Pi}^1]$, where $[\overline{\Pi}^1]$ is the 1-cycle on Π^1 with weight 1 on both of its cones, is $[\infty] - [-\infty]$. We use $\overline{\Pi}^1$ to define rational equivalence on cone complexes in the same way as $\mathbb{P}^1$ is used to define rational equivalence on algebraic varieties.

Definition 2.2.18. Let Σ be a weakly embedded cone complex, and let $p\colon \Sigma \times \Pi^1 \to \Sigma$ and $q\colon \Sigma \times \Pi^1 \to \Pi^1$ be the projections onto the first and second coordinate. For $k \in \mathbb{N}$ we define $R_k(\overline{\Sigma})$ as the subgroup of $Z_k(\overline{\Sigma})$ generated by the cycles $p_*(q^*\psi_{\Pi^1} \cdot A)$, where $A \in Z_{k+1}(\overline{\mathrm{S}(\sigma \times 0)})$ and $\sigma \in \Sigma$. We call two cycles in $Z_k(\overline{\Sigma})$ *rationally equivalent* if their difference lies in $R_k(\overline{\Sigma})$. The k-th *(tropical) Chow group* $A_k(\overline{\Sigma})$ of $\overline{\Sigma}$ is defined as the group of k-cycles modulo rational equivalence, that is $A_k(\overline{\Sigma}) = Z_k(\overline{\Sigma})/R_k(\overline{\Sigma})$. We refer to the graded group $A_*(\overline{\Sigma}) = \bigoplus_{k\in\mathbb{N}} A_k(\overline{\Sigma})$ as the (total) Chow group of $\overline{\Sigma}$.

Remark 2.2.19. Note that $q^*(\psi_{\Pi^1}) \cdot A$ is strictly speaking not defined if A is not contained in $Z_*(\Sigma \times \Pi^1) = Z_*(\mathrm{S}(0 \times 0))$. However, as $q^*(\psi_{\Pi^1})$ vanishes on $\sigma \times 0$ for $\sigma \in \Sigma$, there is a canonically defined pull-back to $\mathrm{S}(\sigma \times 0)$ which we can intersect with cycles in $Z_*(\mathrm{S}(\sigma \times 0))$. Taking the canonical identification of $\mathrm{S}(\sigma \times 0)$ with $\mathrm{S}(\sigma) \times \Pi^1$, this linear function is nothing but the pull-back of ψ_{Π^1} via the projection onto Π^1. So if we define $R_*(\Sigma)$ as the subgroup of $Z_*(\overline{\Sigma})$ generated by $p_*(q^*\psi_{\Pi^1} \cdot A)$ for $A \in Z_*(\Sigma \times \Pi^1)$, then $R_*(\overline{\Sigma})$ is generated by the push-forwards of $R_*(\mathrm{S}(\sigma))$ for $\sigma \in \Sigma$. In particular, it follows immediately from the definition that the inclusions $i\colon \overline{\mathrm{S}(\sigma)} \to \overline{\Sigma}$ induce push-forwards $i_*\colon A_*(\overline{\mathrm{S}(\sigma)}) \to A_*(\overline{\Sigma})$ on the level of Chow groups.

Remark 2.2.20. Our definition of tropical rational equivalence is analogous to the definition of rational equivalence in algebraic geometry given for example in [Ful98, Section 1.6]: two cycles on an algebraic scheme X are rationally equivalent if and only if they differ by a sum of elements of the form $p_*([q^{-1}\{0\} \cap V] - [q^{-1}\{\infty\} \cap V])$, where p is the projection from $X \times \mathbb{P}^1$ onto X, and V is a subvariety of $X \times \mathbb{P}^1$ mapped dominantly to $\mathbb{P}^1$ by the projection $q\colon X \times \mathbb{P}^1 \to \mathbb{P}^1$. Rewriting $[q^{-1}\{0\} \cap V] - [q^{-1}\{\infty\} \cap V]$ as $q^* \operatorname{div}(x) \cdot [V]$, where x denotes the identity on $\mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{G}_m$, we see that we obtain the tropical definition from the algebraic one by replacing $\mathbb{P}^1$ by its associated cone complex and V by a cycle $A \in Z_*(\mathrm{S}(\sigma \times 0))$. That we do not allow A to be in $Z_*(\mathrm{S}(\sigma \times \tau))$ for $\tau \neq 0$ corresponds to the dominance of V over $\mathbb{P}^1$.

Proposition 2.2.21. *Let $f\colon \overline{\Sigma} \to \overline{\Delta}$ be a morphism between weakly embedded extended cone complexes. Then taking push-forwards passes to rational equivalence, that is f_* induces a graded morphism $A_*(\overline{\Sigma}) \to A_*(\overline{\Delta})$, which we again denote by f_*.*

Proof. First assume that f is dominant. Let p, p' be the projections to the first coordinates of $\Sigma \times \Pi$ and $\Delta \times \Pi$, respectively, and similarly let q, q' be the projections to the second coordinates. For $A \in Z_*(\mathrm{S}_{\Sigma\times\Pi^1}(\sigma \times 0))$ we have

$$f_*p_*(q^*\psi_{\Pi^1} \cdot A) = p'_*(f \times \mathrm{id})_*\Big(((f \times \mathrm{id})^*q'^*\psi_{\Pi^1}) \cdot A\Big) = p'_*\Big(q'^*\psi_{\Pi^1} \cdot ((f \times \mathrm{id})_*A)\Big),$$

which is in $R_*(\overline{\Delta})$. We see $R_*(\overline{\Sigma})$ is mapped to 0 by the composite $Z_*(\overline{\Sigma}) \to Z_*(\overline{\Delta}) \to A_*(\overline{\Delta})$, which proves the assertion.

If f is not dominant, then f can be factored as a dominant morphism followed by an inclusion $\overline{S_\Delta(\delta)} \to \overline{\Delta}$ for some δ in Δ. For both of these morphisms the push-forward respects rational equivalence, and hence it does the same for f. □

Let Π^0 denote the embedded cone complex associated to the trivial fan in the zero lattice 0. It is the tropical analogue of the one point scheme $\mathbb{P}^0$ in the sense that it is the associated cone complex of $\mathbb{P}^0$, as well as in the sense of being the terminal object in the category of weakly embedded (extended) cone complexes. Note that $\overline{\Pi}^0 = \Pi^0$. Since all 1-cycles on $\Pi^0 \times \Pi^1 = \Pi^1$ are of the form $k \cdot [\overline{\Pi}^1]$ for some $k \in \mathbb{Z}$ there is a natural identification $A_*(\overline{\Pi}^0) = Z_*(\overline{\Pi}^0) = \mathbb{Z}$.

Definition 2.2.22. Let Σ be a weakly embedded cone complex and $f\colon \overline{\Sigma} \to \overline{\Pi}^0$ the unique morphism to $\overline{\Pi}^0$. We define the degree map $\deg\colon Z_*(\overline{\Sigma}) \to \mathbb{Z}$ as the composite of f_* and the identification $Z_*(\overline{\Pi}^0) = \mathbb{Z}$. By Proposition 2.2.21 the degree map respects rational equivalence, i.e. there is an induced morphism $A_*(\overline{\Sigma}) \to \mathbb{Z}$, which we again denote by deg.

Proposition 2.2.23. *Let Σ be a weakly embedded cone complex, let $A \in Z_*(\Sigma)$, and let $\psi \in \mathrm{CP}(\Sigma)$. Then*

$$\psi \cup A = \psi \cdot A \quad \text{in } A_*(\overline{\Sigma}).$$

Proof. We may assume that A is purely k-dimensional, as both sides are linear in A. We relate the intersection product and the cup product via the "full graph" of ψ, which has been used in the embedded case e.g. in [Mik05, AR10]. Let Σ' be a proper subdivision of Σ such that A is represented by a Minkowski weight c on Σ', and ψ does not change sings on the cones of Σ'. Let $\Gamma_\psi\colon |\Sigma| \to |\Sigma \times \Pi^1|$ be the graph map of ψ. Then the full graph $\Gamma_\psi(A)$ of ψ on A is the tropical cycle in $\Sigma \times \Pi^1$ whose underlying subdivision consists of the subcones $\Gamma_\psi(\sigma')$ for $\sigma' \in \Sigma'_{(k)}$, and

$$\Gamma^+_{\leq\psi}(\tau) = \{(x,t) \mid x \in \tau, t \leq \Gamma_\psi(x)\} \cap (\tau \times \mathbb{R}_{\geq 0}) \quad \text{and}$$
$$\Gamma^-_{\leq\psi}(\tau) = \{(x,t) \mid x \in \tau, t \leq \Gamma_\psi(x)\} \cap (\tau \times \mathbb{R}_{\leq 0})$$

for $\tau \in \Sigma'_{(k-1)}$. The weights of these cones are $c(\sigma')$ on $\Gamma_\psi(\sigma')$ and $(\psi \cup c)(\tau)$ on $\Gamma^\pm_{\leq\psi}(\tau)$. A slight modification of the argument given in [AR10, Construction 3.3] shows that this is a well-defined cycle, i.e. satisfies the balancing condition.

Let p and q be the projections onto the first and second coordinate of $\Sigma \times \Pi^1$. We claim that

$$p_*(q^*\psi_{\Pi^1} \cdot \Gamma_\psi(A)) = \psi \cdot A \; - \psi \cup A.$$

To verify this let us analyze how the cones of $\Gamma_\psi(A)$ contribute to the left hand side of the equation using the computation of Remark 2.2.13. It is easily seen that the cones of the form $\Gamma^+_{\leq\psi}(\tau)$ do not contribute at all. The cones $\Gamma^-_{\leq\psi}(\tau)$ on the other hand contribute to the $S(0 \times \mathbb{R}_{\leq 0})$-component of $q^*\psi_{\Pi^1} \cdot \Gamma_\psi(A)$ with weight $-\psi \cup c(\tau)$ on the cone $\Gamma^-_{\leq\psi}(\tau)/(0 \times \mathbb{R}_{\leq 0})$. This cone is mapped injectively onto τ by the isomorphism $S(0 \times \mathbb{R}_{\leq 0}) \to \Sigma$ induced by p. Hence, the total contribution of the cones considered so far is $-\psi \cup A$.

Now let $\sigma' \in \Sigma'_{(k)}$. Depending on whether ψ is nonnegative or nonpositive on σ', the cone $\Gamma_\psi(\sigma')$ can have contributions only in components corresponding to cones in $|\Sigma| \times \mathbb{R}_{\geq 0}$ or $|\Sigma| \times \mathbb{R}_{\leq 0}$. Without loss of generality assume ψ is nonnegative on σ'. For every $\tau \in \Sigma$ we have $\Gamma_\psi(\sigma') \cap (\tau \times \mathbb{R}_{\geq 0}) = \Gamma_\psi(\sigma' \cap \tau)$. In particular, this is one dimensional with relative interior contained in relint$(\tau \times \mathbb{R}_{\geq 0})$ if and only if $\rho := \sigma' \cap \tau$ is one dimensional, intersects the relative interior of τ, and $\psi|_\rho$ is nonzero. In this case let $\delta \in \Sigma$ be a cone containing σ', and denote by $\psi_\delta \in M^\delta$ the linear function defining $\psi|_\delta$. We have the equality

$$\Gamma_{\psi_\delta}(N^{\sigma'}) + (N^\tau \times \mathbb{Z}) = (N^{\sigma'} + N^\tau) \times \mathbb{Z}$$

of sublattices of $N^{\delta \times \mathbb{R}_{\geq 0}} = N^\delta \times \mathbb{Z}$. Therefore, $\Gamma_\psi(\sigma')$ has a contribution in the $\mathrm{S}(\tau \times \mathbb{R}_{\geq 0})$-component of $q^*\psi_{\Pi^1} \cdot \Gamma_\psi(A)$ if and only σ' has a contribution in the $\mathrm{S}(\tau)$-component of $\psi \cdot A$. To see that they even contribute with the same weight we first notice the equality

$$\operatorname{index}\left(N^{\Gamma_\psi(\sigma')} + N^{\tau \times \mathbb{R}_{\geq 0}}\right) = \operatorname{index}\left(\Gamma_{\psi_\delta}(N^{\sigma'}) + (N^\tau \times \mathbb{Z})\right) = \operatorname{index}\left(N^\tau + N^{\sigma'}\right)$$

of indices. Since the primitive generator $u_{\Gamma_\psi(\rho)}$ of $\Gamma_\psi(\sigma') \cap (\tau \times \mathbb{R}_{\geq 0})$ is equal to the image $\Gamma_\psi(u_\rho)$ of the primitive generator u_ρ of ρ, we also see that

$$q^*\psi_{\Pi^1}(u_{\Gamma_\psi(\rho)}) = q(u_\rho, \psi(u_\rho)) = \psi(u_\rho).$$

Finally, the weight of $\Gamma_\psi(A)$ on $\Gamma_\psi(\sigma')$ is equal to that of A on σ' by definition. Hence, the contributions are equal. Combining this with the fact that $(\Gamma_\psi(\sigma') + \tau \times \mathbb{R}_{\geq 0})/(\tau \times \mathbb{R}_{\geq 0})$ is mapped injectively onto $(\sigma' + \tau)/\tau$ by the isomorphism $\mathrm{S}(\tau \times \mathbb{R}_{\geq 0}) \to \mathrm{S}(\tau)$ induced by p, we conclude that the total contribution of the cones of the form $\Gamma_\psi(\sigma')$ for $\sigma' \in \Sigma'_{(k)}$ to $p_*\big(q^*\psi_{\Pi^1} \cdot \Gamma_\psi(A)\big)$ is $\psi \cdot A$, and with this we have proven the desired equality. □

Proposition 2.2.24. *Let Σ be a weakly embedded cone complex. Then there is a well-defined bilinear map*

$$\mathrm{ClCP}(\Sigma) \times A_*(\overline{\Sigma}) \to A_*(\overline{\Sigma}), \quad (\overline{\psi}, [A]) \mapsto [\psi \cup A].$$

By abuse of notation we denote this pairing by "$\cdot$".

Proof. We need to show that $\psi \cup A = 0$ in $A_*(\overline{\Sigma})$ for all $\psi \in \mathrm{CP}(\Sigma)$ and $A \in R_*(\overline{\Sigma})$. It suffices to show this for $A = p_*(q^*\psi_{\Pi^1} \cdot B)$ for some $B \in Z_*(\Sigma \times \Pi^1)$, where p and q denote the projections to the first and second coordinate. In this case we have

$$\psi \cup A = p_*\big(p^*\psi \cup (q^*\psi_{\Pi^1} \cdot A)\big) = p_*\big(q^*\psi_{\Pi^1} \cdot (p^*\psi \cup A)\big) \in R_*(\overline{\Sigma}),$$

where the first equality uses the projection formula, and the second one uses Proposition 2.2.17. □

Remark 2.2.25. Recall that the intersection product $\psi \cdot A$ of Construction 2.2.12 is only defined when A is contained in the finite part of $\overline{\Sigma}$. That the bilinear pairing of the preceding proposition is its appropriate extension is justified by Proposition 2.2.23, which states that modulo rational equivalence the cup-product and the "$\cdot$"-product coincide.

Definition 2.2.26. Let Σ be a weakly embedded cone complex, let $\psi_1, \dots, \psi_k \in \mathrm{ClCP}(\Sigma)$, and let $A \in A_*(\overline{\Sigma})$. Then we define the cycle class

$$\psi_1 \cdots \psi_k \cdot A = \prod_{i=1}^{k} \psi_i \cdot A$$

inductively by $\prod_{i=1}^{k} \psi_i \cdot A = \psi_1 \cdot \left(\prod_{i=2}^{k} \psi_i \cdot A \right)$, where the base case is the pairing of Proposition 2.2.24.

2.3 Tropicalization

In this section we will tropicalize cocycles and cycles on toroidal embeddings. As already mentioned in Remark 2.2.3, cocycles will define Minkowski weights and cycles will define tropical cycles on the associated weakly embedded extended cone complex. Afterwards, we will show that tropicalization respects intersection theoretical constructions like push-forwards, intersections with Cartier divisors supported on the boundary, and rational equivalence. For the rest of this chapter we assume that all algebraic varieties are defined over an algebraically closed field k of characteristic 0.

2.3.1 Tropicalizing Cocycles

In the well-known paper [FS97], cocycles on complete toric varieties are described by Minkowski weights on the associated fans by recording the degrees of their intersections with the boundary strata. The same method yields Minkowski weights associated to cocycles on complete toroidal embeddings:

Proposition 2.3.1. *Let X be a complete n-dimensional toroidal embedding, and let $c \in A^k(X)$ be a cocycle. Then the $(n-k)$-dimensional weight*

$$\omega \colon \Sigma(X)_{(n-k)} \to \mathbb{Z}, \quad \sigma \mapsto \deg(c \cap [\mathrm{V}(\sigma)])$$

satisfies the balancing condition.

Proof. Let τ be an $(n-k-1)$-dimensional cone of $\Sigma(X)$, and let $\sigma_1, \dots, \sigma_n$ be the $(n-k)$-dimensional cones containing it. To prove that the balancing condition is fulfilled at τ, it suffices to show that

$$\sum_{i=1}^{n} \omega(\sigma_i) \langle f, u_{\sigma_i/\tau} \rangle = \langle f, \sum_{i=1}^{n} \omega(\sigma_i) u_{\sigma_i/\tau} \rangle = 0$$

for every $f \in (N^X/N_\tau^X)^* = (N_\tau^X)^\perp$. A rational function $f \in M^X$ is contained in $(N_\tau^X)^\perp$ if and only if it is an invertible regular function on $X(\tau)$. Therefore, for every $f \in (N_\tau^X)^\perp$ we have $\operatorname{div}(f)|_{X(\sigma_i)} \in M^{\sigma_i} \cap \tau^\perp$. The lattice $M^{\sigma_i} \cap \tau^\perp$ is the group of integral linear functions of the ray σ_i/τ, and $u_{\sigma_i/\tau}$ is by definition the image of the primitive generator of σ_i/τ in N^X/N_τ^X. It follows that $\langle f, u_{\sigma_i/\tau}\rangle$ is the pairing of $\operatorname{div}(f)|_{X(\sigma_i)}$ with the primitive generator of σ_i/τ. By Lemma 2.1.3 this is equal to the pairing of $\operatorname{div}(f)|_{\mathrm{V}(\tau)\cap X(\sigma_i)}$ with the primitive generator of the ray $\sigma_{\mathrm{V}(\tau)}^{\mathrm{O}(\sigma_i)}$, which in turn is equal to the multiplicity of $\operatorname{div}(f)|_{\mathrm{V}(\tau)}$ at $\mathrm{V}(\sigma_i)$. Using this we obtain

$$\sum_{i=1}^n \omega(\sigma_i)\langle f, u_{\sigma_i/\tau}\rangle = \sum_{i=1}^n \deg\left(c \cap [\mathrm{V}(\sigma_i)]\right)\langle f, u_{\sigma_i/\tau}\rangle =$$
$$= \deg\left(c \cap \sum_{i=1}^n \langle f, u_{\sigma_i/\tau}\rangle [\mathrm{V}(\sigma_i)]\right) = \deg\left(c \cap \left[\operatorname{div}(f)|_{\mathrm{V}(\sigma_i)}\right]\right) = 0,$$

which finishes the proof. □

Definition 2.3.2. Let X be a complete n-dimensional toroidal embedding with weakly embedded cone complex Σ. We define the tropicalization map

$$\operatorname{Trop}_X \colon A^k(X) \to \mathrm{M}_{n-k}(\Sigma),$$

by sending a cocycle on X to the weight defined as in the preceding proposition. This is clearly a morphism of abelian groups. For a cone $\sigma \in \Sigma$, and a cocycle $c \in A^k(\mathrm{V}(\sigma))$, we define $\operatorname{Trop}_X(c) \in \mathrm{M}_{n-\dim(\sigma)-k}(\mathrm{S}_\Sigma(\sigma))$ as the pushforward to $\mathrm{M}_*(\mathrm{S}_\Sigma(\sigma))$ of the Minkowski weight $\operatorname{Trop}_{\mathrm{V}(\sigma)}(c) \in \mathrm{M}_*(\Sigma(\mathrm{V}(\sigma)))$. We will just write $\operatorname{Trop}(c)$ when no confusion arises.

Proposition 2.3.3. *Let X be a complete toroidal embedding with weakly embedded cone complex Σ, and let $\tau \in \Sigma$. Furthermore, let $c \in A^*(X)$ be a cocycle on X, and let $i\colon \mathrm{V}(\tau) \to X$ and $j\colon \overline{\mathrm{S}(\tau)} \to \overline{\Sigma}$ be the inclusion maps. Then we have the equality*

$$\operatorname{Trop}_X(i^*c) = j^* \operatorname{Trop}_X(c)$$

of Minkowski weights on $\mathrm{S}(\tau)$.

Proof. By Lemma 2.1.3 we have $\mathrm{O}(\sigma/\tau) = \mathrm{O}(\sigma)$, hence $\mathrm{V}(\sigma/\tau) = \mathrm{V}(\sigma)$. The projection formula implies that the weight of $\operatorname{Trop}(i^*c)$ at σ/τ is equal to the weight of $\operatorname{Trop}(c)$ at σ, which is equal to the weight of $j^*\operatorname{Trop}(c)$ at σ/τ by Construction 2.2.5. □

Remark 2.3.4. If the canonical morphism $\Sigma(\mathrm{V}(\tau)) \to \mathrm{S}(\tau)$ is an isomorphism of weakly embedded cone complexes, the statement of Proposition 2.3.3 reads

$$\operatorname{Trop}(i^*c) = \operatorname{Trop}(i)^* \operatorname{Trop}(c).$$

However, in general there may be more invertible regular functions on $\mathrm{O}(\tau)$ than obtained as restrictions of rational functions in $(N_\tau^X)^\perp \subseteq M^X$. In this case, the pull-back of a Minkowski weight on Σ to $\mathrm{S}(\tau)$ may not be a Minkowski weight on $\Sigma(\mathrm{V}(\tau))$. In other words, the pull-back $\mathrm{Trop}(i)^*\colon \mathrm{M}_*(\Sigma(X)) \to \mathrm{M}_*(\Sigma(\mathrm{V}(\tau)))$ may be ill-defined. An example where this happens is when X is equal to the blowup of $\mathbb{P}^2$ in the singular point of a plane nodal cubic C, and its boundary is the union of the exceptional divisor E and the strict transform $\widetilde{C}$. The cone complex of X consists of two rays $\rho_{\widetilde{C}}$ and ρ_E, corresponding to the two boundary divisors, which span two distinct strictly simplicial cones, one for each point in $E \cap \widetilde{C}$. The lattice N^X is trivial, so that any two integers on the 2-dimensional cones define a Minkowski weight in $\mathrm{M}_2(\Sigma(X))$. But the pull-back of such a weight to $\mathrm{S}_{\Sigma(X)}(\rho_E)$ only defines a Minkowski weight in $\mathrm{M}_1(\Sigma(\mathrm{V}(\rho_E)))$ if the weights on its cones are equal. This is because the toroidal embedding $\mathrm{O}(\rho_E) \subseteq \mathrm{V}(\rho_E) = E$ is isomorphic to $\mathbb{P}^1$ with two points in the boundary, the weakly embedded cone complex of which consists of two rays embedded into $\mathbb{R}$.

2.3.2 Tropicalizing Cycles

Let X be a toroidal embedding with weakly embedded cone complex Σ. In [Uli13] Ulirsch constructs a map $\mathrm{trop}_X^{\mathrm{an}}\colon X^\beth \to \overline{\Sigma}$ as a special case of his tropicalization procedure for fine and saturated logarithmic schemes. We recall that, since X is separated, $X^\beth$ is the analytic domain of the Berkovich analytification X^{an} [Ber90] consisting of all points which can be represented by an R-integral point for some rank-1 valuation ring R extending k. The notation is due to Thuillier [Thu07]. The map $\mathrm{trop}_X^{\mathrm{an}}$ restricts to a map $X_0^{\mathrm{an}} \cap X^\beth \to \Sigma$ generalizing the "ord"-map $X_0\big(k(\!(t)\!)\big) \cap X\big(k[\![t]\!]\big) \to \Sigma$ from [KKMSD73]. Ulirsch's tropicalization map is also compatible with Thuillier's retraction map $\mathbf{p}_X\colon X^\beth \to X^\beth$ [Thu07] in the sense that the retraction factors through $\mathrm{trop}_X^{\mathrm{an}}$. The tropicalization map allows to define the set-theoretic tropicalization of a subvariety Z of X_0 as the subset $\mathrm{trop}_X(Z) := \mathrm{trop}_X^{\mathrm{an}}(Z^{\mathrm{an}} \cap X^\beth)$ of Σ. This set has been studied in greater detail in [Uli15a], where several parallels to tropicalizations of subvarieties of tori are exposed: first of all, $\mathrm{trop}_X(Z)$ can be given the structure of an at most $\dim(Z)$-dimensional cone complex whose position in Σ reflects the position of Z in X. Namely, there is a toroidal version of Tevelev's Lemma [Tev07, Lemma 2.2] stating that $\mathrm{trop}_X(Z)$ intersects the relative interior of a cone $\sigma \in \Sigma$ if and only if $\overline{Z}$ intersects $\mathrm{O}(\sigma)$. Furthermore, if $\mathrm{trop}_X(Z)$ is a union of cones of Σ, then $\overline{Z}$ intersects all strata properly.

When X is complete, we use this to give $\mathrm{trop}_X(Z)$ the structure of a tropical cycle in a similar way as done in [ST08] in the toric case. First, we choose a simplicial proper subdivision Σ' of Σ such that $\mathrm{trop}_X(Z)$ is a union of its cones. Since $\operatorname{char} k = 0$ by assumption and Σ' is simplicial, the toroidal modification $X' = X' \times_\Sigma \Sigma'$ is the coarse moduli space of a smooth Deligne-Mumford stack [Iwa09, Thm. 3.3] and thus has an intersection product on its Chow group $A_*(X)_\mathbb{Q}$ with rational coefficients [Vis89].

Definition 2.3.5. Let $d = \dim(Z)$ and $\sigma \in \Sigma'_{(d)}$. We define the multiplicity $\mathrm{mult}_Z(\sigma)$ of the cone σ by $\deg([\overline{Z}'] \cdot [\mathrm{V}(\sigma)])$, where $\overline{Z}'$ denotes the closure of Z in X'.

Remark 2.3.6. The multiplicity $\mathrm{mult}_Z(\sigma)$ is independent of the rest of the cone complex Σ'. This is because $\overline{Z}'$ intersects all strata properly so that $[\overline{Z}'] \cdot [\mathrm{V}(\sigma)]$ is a well-defined 0-cycle supported on $\overline{Z}' \cap \mathrm{O}(\sigma)$ and only depending on $X'(\sigma)$.

Proposition 2.3.7. *let Δ be a simplicial proper subdivision of Σ', and let $\delta \in \Delta_{(d)}$ be a d-dimensional cone contained in $\sigma \in \Sigma'_{(d)}$. Then $\mathrm{mult}_Z(\delta) = \mathrm{mult}_Z(\sigma)$. In particular, we have $\mathrm{mult}_Z(\sigma) \in \mathbb{Z}$ for all $\sigma \in \Sigma'$. Furthermore, the weight*

$$\Sigma'_{(d)} \to \mathbb{Z}, \ \ \sigma \mapsto \mathrm{mult}_Z(\sigma)$$

is balanced and the tropical cycle associated to it is independent of the choice of Σ'.

Proof. To ease the notation, we replace X and X' by $X \times_\Sigma \Sigma'$ and $X \times_\Sigma \Delta$, respectively, and denote the closures of Z in X and X' by $\overline{Z}$ and $\overline{Z}'$. Furthermore, we denote the toroidal modification induced by the subdivision $\Delta \to \Sigma'$ by $f \colon X' \to X$. Since X is an Alexander scheme in the sense of [Vis89], there is a pull-back morphism $f^* \colon A_*(X)_\mathbb{Q} \to A_*(X')_\mathbb{Q}$. The pull-back $f^*[\overline{Z}]$ is represented by a cycle $[X'] \cdot_f [\overline{Z}] \in A_d(f^{-1}\overline{Z})_\mathbb{Q}$. Since $\overline{Z}$ intersects all strata properly, and f locally looks like a toric morphism, the preimage $f^{-1}\overline{Z}$ is d-dimensional and $\overline{Z}'$ is its only d-dimensional component. Hence, $f^*[\overline{Z}]$ is a multiple of $[\overline{Z}']$. By the projection formula we have

$$f_* f^*[\overline{Z}] = f_*(f^*[\overline{Z}] \cdot [X']) = [\overline{Z}] \cdot [X] = [\overline{Z}],$$

showing that we, in fact, have $f^*[\overline{Z}] = [\overline{Z}']$. Again using the projection formula we obtain

$$\begin{aligned} \mathrm{mult}_Z(\delta) = \deg([\overline{Z}'] \cdot [\mathrm{V}(\delta)]) = \deg(f_*(f^*[\overline{Z}] \cdot [\mathrm{V}(\delta)])) = \\ = \deg([\overline{Z}] \cdot f_*[\mathrm{V}(\delta)]) = \deg([\overline{Z}] \cdot [\mathrm{V}(\sigma)]) = \mathrm{mult}_Z(\sigma). \end{aligned}$$

Since every complex has a strictly simplicial proper subdivision, we can choose Δ to be strictly simplicial. In this case, the multiplicity $\mathrm{mult}_Z(\delta)$ is defined by the ordinary intersection product on $A_*(X')$, and hence we have $\mathrm{mult}_Z(\sigma) = \mathrm{mult}_Z(\delta) \in \mathbb{Z}$. Similarly, to prove the balancing condition for the weight $\Sigma_{(d)} \ni \sigma \mapsto \mathrm{mult}_Z(\sigma) \in \mathbb{Z}$ it suffices to show that the induced weight on a strictly simplicial proper subdivision Δ is balanced. By what we just saw, this weight is equal to $\Delta_{(d)} \ni \delta \mapsto \mathrm{mult}_Z(\delta) \in \mathbb{Z}$. But this is nothing but the tropicalization of the cocycle corresponding to $[\overline{Z}']$ by Poincaré duality, which is balanced by Proposition 2.3.1. That the tropical cycle it defines is independent of all choices follows immediately from the fact that any two subdivisions have a common refinement. □

The previous result allows us to assign a tropical cycle to the subvariety Z of X_0. Its support will be contained in the set-theoretic tropicalization of Z.

Definition 2.3.8. Let X be a complete toroidal embedding with weakly embedded cone complex Σ, and let $Z \subseteq X_0$ be a d-dimensional subvariety. We define the *tropicalization* $\operatorname{Trop}_X(Z) \in Z_d(\Sigma)$ as the tropical cycle represented by the tropicalization of the cocycle corresponding to the closure $\overline{Z}'$ of Z in a smooth toroidal modification X' of X in which $\overline{Z}'$ intersects all strata properly. This is well-defined by Proposition 2.3.7. Extending by linearity, we obtain a tropicalization morphism

$$\operatorname{Trop}_X \colon Z_*(X) = \bigoplus_{\sigma\in\Sigma} Z_*(\mathrm{O}(\sigma)) \xrightarrow{\bigoplus_{\sigma\in\Sigma}\operatorname{Trop}_{\mathrm{V}(\sigma)}} \bigoplus_{\sigma\in\Sigma} Z_*(\mathrm{S}(\sigma)) = Z_*(\overline{\Sigma}).$$

Remark 2.3.9. Strictly speaking the "σ-th" coordinate of Trop_X is not $\operatorname{Trop}_{\mathrm{V}(\sigma)}$, but the composite $Z_*(\mathrm{O}(\sigma)) \xrightarrow{\operatorname{Trop}_{\mathrm{V}(\sigma)}} Z_*(\Sigma(\mathrm{V}(\sigma))) \to Z_*(\mathrm{S}(\sigma))$, where the second map is the push-forward induced by the identification of cone complexes of Lemma 2.1.3.

2.3.3 The Sturmfels-Tevelev Multiplicity Formula

We now give a proof of the Sturmfels-Tevelev multiplicity formula [ST08, Theorem 1.1] in the toroidal setting. It has its origin in tropical implicitization and states that push-forward commutes with tropicalization. A version for the embedded case over fields with nontrivial valuation has been proven in [BPR15], [OP13], and [Gub13].

Theorem 2.3.10. *Let $f \colon X \to Y$ be a toroidal morphism of complete toroidal embeddings, and let $\alpha \in Z_*(X)$. Then*

$$\operatorname{Trop}(f)_* \operatorname{Trop}_X(\alpha) = \operatorname{Trop}_Y(f_*\alpha).$$

Proof. Let Σ and Δ denote the weakly embedded cone complexes of X and Y, respectively. Since both sides of the equation are linear in α, we may assume that $\alpha = [\overline{Z}]$, where Z is a closed subvariety of $\mathrm{O}(\sigma)$ for some $\sigma \in \Sigma$, say of dimension $\dim(Z) = d$. The toroidal morphism $\mathrm{V}(\sigma) \to X \to Y$ factors through a dominant toroidal morphism $\mathrm{V}(\sigma) \to \mathrm{V}(\delta)$ for some $\delta \in \Delta$, and since tropicalization and push-forward commute for closed immersions of closures of strata by definition, this allows us to reduce to the case where f is dominant and $\mathrm{O}(\sigma) = X_0$.

Assume that the dimension of $f(Z)$ is strictly smaller than that of Z. Since $\operatorname{Trop}(f)(\operatorname{trop}_X(Z))$ is equal to $\operatorname{trop}_Y(f(Z))$ (this follows from the surjectivity of $Z^{\mathrm{an}} \to f(Z)^{\mathrm{an}}$ [Ber90, Prop. 3.4.6] and the functoriality of $\operatorname{trop}^{\mathrm{an}}$ [Uli13, Prop. 6.2]), this implies that $\operatorname{Trop}(f)$ is not injective on any facet of $\operatorname{Trop}_X(Z)$, hence

$$\operatorname{Trop}(f)_* \operatorname{Trop}_X([\overline{Z}]) = 0 = \operatorname{Trop}_Y(0) = \operatorname{Trop}_Y(f_*[\overline{Z}]).$$

Now assume that $\dim(f(Z)) = \dim(Z) = d$. We subdivide Σ and Δ as follows. First, we take a proper subdivision Σ' of Σ such that $\operatorname{trop}_X(Z)$ is a union of cones of Σ'. Then we take a strictly simplicial proper subdivision Δ' of Δ such that the images of cones in Σ' are unions of cones of Δ'. Pulling back the cones of Δ', we obtain a proper subdivision Σ'' of Σ' whose cones map to cones in Δ'. By successively subdividing along rays (cf. [AK00, Remark 4.5]), we can achieve a proper subdivision of Σ'' whose cones are simplicial and mapped to cones of Δ by $\mathrm{Trop}(f)$. After renaming, we see that there are proper subdivisions Σ' and Δ' of Σ and Δ, respectively, such that Σ' is simplicial, Δ' is strictly simplicial, $\mathrm{Trop}(f)$ maps cones of Σ' onto cones of Δ', and $\mathrm{trop}_X(Z)$ is a union of cones in Σ'.

Let $X' = X \times_\Sigma \Sigma'$ and $Y' = Y \times_\Delta \Delta'$ the corresponding toroidal modifications. Then the induced toroidal morphism $f'\colon X' \to Y'$ is flat by [AK00, Remark 4.6]. As $\mathrm{trop}_Y(f(Z))$ is the image of $\mathrm{trop}_X(Z)$, it is a union of cones of Δ'. By construction of the tropicalization, the weight of $\mathrm{Trop}_Y(f_*[\overline{Z}])$ at a d-dimensional cone $\delta \in \Delta'$ is equal to

$$[K(\overline{Z}) : K(f(\overline{Z}))] \deg([f'(\overline{Z})] \cdot [\mathrm{V}(\delta)]) = \deg(f'_*[\overline{Z}'] \cdot [\mathrm{V}(\delta)]) = \deg([\overline{Z}'] \cdot f'^*[\mathrm{V}(\delta)]),$$

where the second equality uses the projection formula. The irreducible components of $f'^{-1}\,\mathrm{V}(\delta)$ are of the form $\mathrm{V}(\sigma)$, where σ is minimal among the cones of Σ' mapping onto δ. All of these cones σ are d-dimensional, and the multiplicity with which $\mathrm{V}(\sigma)$ occurs in $f'^{-1}\,\mathrm{V}(\delta)$ is $[N^\delta : \mathrm{Trop}(f)(N^\sigma)]$ as we see by comparing with the toric case using local toric charts. Combining this, we obtain

$$\deg([\overline{Z}'] \cdot f'^*[\mathrm{V}(\delta)]) = \sum_{\sigma \mapsto \delta} [N^\delta : \mathrm{Trop}(f)(N^\sigma)] \deg([\overline{Z}'] \cdot [\mathrm{V}(\sigma)]),$$

where the sum runs over all d-dimensional cones of Σ' mapping onto δ. Since $\deg([\overline{Z}'] \cdot [\mathrm{V}(\sigma)])$ is the weight of $\mathrm{Trop}_X(Z)$ at σ, the right hand side of this equation is precisely the multiplicity of $\mathrm{Trop}(f)_* \mathrm{Trop}_X(Z)$ at δ. □

2.3.4 Tropicalization and Intersections with Boundary Divisors

Let X be a toroidal embedding with weakly embedded cone complex Σ. By definition of Σ, the restriction of a Cartier divisor D on X which is supported away from X_0 to a combinatorial open subset $X(\sigma)$ for some $\sigma \in \Sigma$ is determined by an integral linear function on σ. Since the cones of Σ and the combinatorial open subsets of X are glued accordingly, D defines a continuous function $\psi\colon |\Sigma| \to \mathbb{R}$ which is integral linear on all cones of Σ. Conversely, all such functions define a Cartier divisor on X which is supported away from X_0. When X is not toric, the Picard groups of its combinatorial opens need not be trivial. Hence, the restriction $D|_{X(\sigma)}$ is not necessarily of the form $\mathrm{div}(f)|_{X(\sigma)}$ for some $f \in K(X)$. But if it is, the rational function f must be regular and invertible on X_0, and by definition of the weak embedding we then have $\psi|_\sigma = f \circ \varphi_X$. We see that D is principal on the combinatorial opens of X if and only if ψ is combinatorially principal. By the same argument we see that another such divisor D' with corresponding tropical cp-divisor ψ' is linearly equivalent to D if and only if ψ and ψ' are linearly equivalent tropical cp-divisors on Σ.

Definition 2.3.11. Let X be a toroidal embedding with weakly embedded cone complex Σ. We write $\mathrm{Div}_{X_0}(X)$ for the subgroup of $\mathrm{Div}(X)$ consisting of Cartier divisors which are supported on $X \setminus X_0$. We denote the tropical Cartier divisor on Σ

corresponding to $D \in \mathrm{Div}_{X_0}(X)$ by $\mathrm{Trop}_X(D)$. If D is principal on all combinatorial opens, or equivalently, if $\mathrm{Trop}_X(D) \in \mathrm{CP}(\Sigma)$, we say that D is combinatorially principal (cp). We write $\mathrm{CP}(X)$ for the group of cp-divisors on X. Furthermore, we write $\mathrm{ClCP}(X)$ for the image of $\mathrm{CP}(X)$ in $\mathrm{Pic}(X)$, and for $\mathcal{L} \in \mathrm{ClCP}(X)$ we write $\mathrm{Trop}_X(\mathcal{L})$ for its corresponding tropical divisor class in $\mathrm{ClCP}(\Sigma)$. If no confusion arises, we usually omit the reference to X.

Lemma 2.3.12. *Let $f\colon X \to Y$ be a morphism of toroidal embeddings, and let $\mathcal{L} \in \mathrm{ClCP}(Y)$. Then $f^*\mathcal{L} \in \mathrm{ClCP}(X)$, and*

$$\mathrm{Trop}_X(f^*\mathcal{L}) = \mathrm{Trop}(f)^*(\mathrm{Trop}_Y(\mathcal{L})).$$

Proof. It suffices to prove the lemma for dominant toroidal morphisms and closed immersions of closures of strata. The dominant case follows immediately from the definition of $\mathrm{Trop}(f)$. So we may assume that $X = \mathrm{V}(\delta)$ for some $\delta \in \Delta = \Sigma(Y)$, and $f\colon \mathrm{V}(\delta) \to Y$ is the inclusion. Let $D \in \mathrm{CP}(Y)$ be a representative for $\mathcal{L}$ and write $\psi = \mathrm{Trop}_Y(D)$. Since D is in $\mathrm{CP}(Y)$, there exists $g \in M^\Delta$ such that $\psi|_\delta = g \circ \varphi_\Delta|_\delta$. The tropicalization of the divisor $D - \mathrm{div}(g)$ is $\psi - g \circ \varphi_\Delta$. This vanishes on δ by construction, which means that $D - \mathrm{div}(g)$ is supported away from $\mathrm{O}(\delta)$. By Lemma 2.1.3 the tropicalization of the restriction of $D - \mathrm{div}(g)$ to $\mathrm{V}(\delta)$ is given by the tropical divisor in $\mathrm{Div}(\mathrm{S}(\delta)) = \mathrm{Div}(\Sigma(\mathrm{V}(\delta)))$ induced by $\psi - g \circ \varphi_\Delta$. This finishes the proof because $(D - \mathrm{div}(g))|_{\mathrm{V}(\delta)}$ represents $f^*\mathcal{L}$, and the divisor on $\mathrm{S}(\delta)$ induced by $\psi - g \circ \varphi_\Delta$ represents $\mathrm{Trop}(f)^*\psi$ by Construction 2.2.8. □

Proposition 2.3.13. *Let X be a complete toroidal embedding, let $\mathcal{L} \in \mathrm{ClCP}(X)$, and let $c \in A^k(X)$. Then*

$$\mathrm{Trop}(c_1(\mathcal{L}) \cup c) = \mathrm{Trop}(\mathcal{L}) \cup \mathrm{Trop}(c).$$

Proof. Let $n = \dim(X)$. Furthermore, let τ be a $(n-k-1)$-dimensional cone of $\Sigma = \Sigma(X)$, and let $\psi \in \mathrm{CP}(\Sigma)$ and $\psi_\tau \in \mathrm{CP}(\Sigma(\mathrm{V}(\tau)))$ be representatives for $\mathrm{Trop}(\mathcal{L})$ and $\mathrm{Trop}(i)^* \mathrm{Trop}(\mathcal{L})$, respectively, where $i\colon \mathrm{V}(\tau) \to X$ is the inclusion map. By Lemma 2.3.12 the tropicalization of $i^*\mathcal{L}$ is represented by ψ_τ, hence

$$\sum_{\tau \prec \sigma} \psi_\tau(u_{\sigma/\tau})[\mathrm{V}(\sigma)] = [i^*\mathcal{L}] = c_1(\mathcal{L}) \cap [\mathrm{V}(\tau)] \quad \text{in } A_{n-k-1}(X).$$

Therefore, the weight of $\mathrm{Trop}(c_1(\mathcal{L}) \cup c)$ at τ is

$$\deg((c_1(\mathcal{L}) \cup c) \cap [\mathrm{V}(\tau)]) = \sum_{\tau \prec \sigma} \psi_\tau(u_{\sigma/\tau}) \deg(c \cap [\mathrm{V}(\sigma)]).$$

Since $\deg(c \cap [\mathrm{V}(\sigma)])$ is the weight of $\mathrm{Trop}(c)$ at σ, this is equal to the weight of $\psi \cup \mathrm{Trop}(c)$ at τ by Construction 2.2.14. □

Theorem 2.3.14. *Let X be a complete toroidal embedding and let $D \in \mathrm{Div}_{X_0}(X)$. Then for every subvariety $Z \subseteq X_0$ we have*

$$\mathrm{Trop}(D \cdot [\overline{Z}]) = \mathrm{Trop}_X(D) \cdot \mathrm{Trop}(Z).$$

Proof. First note that D is supported on $X \setminus X_0$ and hence $D \cdot [\overline{Z}]$ is a well-defined cycle. Let Σ' be a strictly simplicial proper subdivision of $\Sigma = \Sigma(X)$ such that $\operatorname{trop}(Z)$ is a union of its cones, and let $f \colon X' := X \times_\Sigma \Sigma' \to X$ be the corresponding toroidal modification. The algebraic projection formula and Theorem 2.3.10 imply that

$$\operatorname{Trop}_X(D \cdot [\overline{Z}]) = \operatorname{Trop}_X(f_*(f^*D \cdot [\overline{Z}'])) = \operatorname{Trop}(f)_* \operatorname{Trop}_{X'}(f^*D \cdot [\overline{Z}']),$$

whereas the tropical projection formula and Lemma 2.3.12 imply that

$$\begin{aligned}\operatorname{Trop}_X(D) \cdot \operatorname{Trop}_X(Z) = \operatorname{Trop}_X(D) \cdot (\operatorname{Trop}(f)_* \operatorname{Trop}_{X'}(Z)) = \\ = \operatorname{Trop}(f)_*(\operatorname{Trop}_{X'}(f^*D) \cdot \operatorname{Trop}_{X'}(Z)).\end{aligned}$$

This reduces to the case where $X = X'$ and we may assume that Σ is strictly simplicial and $\overline{Z}$ intersects all boundary strata properly.

Denote $\operatorname{Trop}(D)$ by ψ, and let $\rho \in \Sigma_{(1)}$ be a ray of Σ. Since X is smooth, the boundary divisor $\mathrm{V}(\rho)$ is Cartier. Let $\mathrm{V}(\rho) \cdot [\overline{Z}] = [\mathrm{V}(\rho) \cap \overline{Z}] = \sum_{i=1}^k a_i [W_i]$. Because $\overline{Z}$ intersects all strata of X properly, each W_i meets $\mathrm{O}(\rho)$. For the same reason every W_i meets all strata of $\mathrm{V}(\rho)$ properly. It follows that $\operatorname{Trop}_X(W_i)$ is represented by the Minkowski weight on $\mathrm{S}(\rho)$ whose weight on a cone σ/ρ is equal to $\deg([\mathrm{V}(\sigma)] \cdot [W_i])$, where the intersection product is taken in $\mathrm{V}(\rho)$. Denote by i_ρ the inclusion of $\mathrm{V}(\rho)$ into X. Then $\mathrm{V}(\rho) \cdot [\overline{Z}] = i_\rho^*[\overline{Z}]$ in $A_*(\mathrm{V}(\sigma))$, and hence the weight of $\operatorname{Trop}_X(\mathrm{V}(\rho) \cdot [\overline{Z}])$ at σ/ρ is equal to

$$\sum_{i=1}^k a_i \deg([\mathrm{V}(\sigma)] \cdot [W_i]) = \deg([\mathrm{V}(\sigma)] \cdot i_\rho^*[\overline{Z}]) = \deg([\mathrm{V}(\sigma)] \cdot [\overline{Z}]),$$

where the first two intersection products are taken in $\mathrm{V}(\sigma)$, whereas the last one is taken in X, and the last equality follows from the projection formula. Since $\deg([\mathrm{V}(\sigma)] \cdot [\overline{Z}])$ is the weight of $\operatorname{Trop}_X(Z)$ at σ, this implies the equality

$$\operatorname{Trop}_X(\mathrm{V}(\rho) \cdot [\overline{Z}]) = j_\rho^* \operatorname{Trop}_X(Z),$$

where $j_\rho \colon \mathrm{S}(\rho) \to \overline{\Sigma}$ is the inclusion. Together with the fact that the divisor $\mathrm{V}(\rho)$ occurs in D with multiplicity $\psi(u_\rho)$, where u_ρ denotes the primitive generator of ρ, we obtain

$$\operatorname{Trop}_X(D \cdot [\overline{Z}]) = \sum_{\rho \in \Sigma_{(1)}} \psi(u_\rho) \operatorname{Trop}_X(\mathrm{V}(\rho) \cdot [\overline{Z}]) = \sum_{\rho \in \Sigma_{(1)}} \psi(u_\rho) j_\rho^* \operatorname{Trop}_X(Z),$$

which is equal to $\psi \cdot \operatorname{Trop}_X(Z)$ by Construction 2.2.12. □

2.3.5 Tropicalizing Cycle Classes

As already pointed out in Remark 2.2.20, rational equivalence for cycles on weakly embedded extended cone complexes is defined very similarly as rational equivalence in algebraic geometry. In fact, now that we know that tropicalization respects push-forwards and intersections with boundary divisors it is almost immediate that it respects rational equivalence as well. The only thing sill missing is to relate the weakly embedded cone complex $\Sigma(X \times Y)$ of the product of two toroidal embeddings X and Y to the product $\Sigma(X) \times \Sigma(Y)$ of their weakly embedded cone complexes. First note that by combining local toric charts for X and Y it is easy to see that $X_0 \times Y_0 \subseteq X \times Y$ really is a toroidal embedding. With the same method we see that the cone complexes $\Sigma(X \times Y)$ and $\Sigma(X) \times \Sigma(Y)$ are naturally isomorphic, the isomorphism being the product $\operatorname{Trop}(p) \times \operatorname{Trop}(q)$ of the tropicalizations of the projections p and q from $X \times Y$. That this even is an isomorphism of weakly embedded cone complexes, that is that $N^{X \times Y} \to N^X \times N^Y$ is an isomorphism as well, follows from a result by Rosenlicht which states that the canonical map

$$\Gamma(X_0, \mathcal{O}_X^*) \times \Gamma(Y_0, \mathcal{O}_Y^*) \to \Gamma(X_0 \times Y_0, \mathcal{O}_{X \times Y}^*)$$

is surjective [KKV89, Section 1].

Proposition 2.3.15. *Let X be a complete toroidal embedding with weakly embedded cone complex Σ. Then the tropicalization $\operatorname{Trop}_X \colon Z_*(X) \to Z_*(\overline{\Sigma})$ induces a morphism $A_*(X) \to A_*(\overline{\Sigma})$ between the Chow groups, which we again denote by Trop_X.*

Proof. By definition, the Chow group $A_k(X)$ is equal to the quotient of $Z_k(X)$ by the subgroup $R_k(X)$ generated by cycles of the form $p_*(q^*([0] - [\infty]) \cdot [W])$, where p and q are the first and second projection from the product $X \times \mathbb{P}^1$, and W is an irreducible subvariety of $X \times \mathbb{P}^1$ mapping dominantly to $\mathbb{P}^1$. Considering $\mathbb{P}^1$ with its standard toric structure, the projections are dominant toroidal morphisms. The boundary divisor $[0] - [\infty]$ is given by the identity on the cone complex Π^1 of $\mathbb{P}^1$ when considering its natural identification with $\mathbb{R}$. This tropical divisor was denoted by ψ_{Π^1} in Subsection 2.2.4. It follows that the tropicalization of the Cartier divisor $q^*([0] - [\infty])$ is equal to $\operatorname{Trop}(q)^*(\psi_{\Pi^1})$. Applying Theorems 2.3.10 and 2.3.14 we obtain that

$$\operatorname{Trop}_X(p_*(q^*([0] - [\infty]) \cdot [W])) = \operatorname{Trop}(p)_*(\operatorname{Trop}(q)^* \psi_{\Pi^1} \cdot \operatorname{Trop}_{X \times \mathbb{P}^1}(W)).$$

Noting that $\operatorname{Trop}(p)$ and $\operatorname{Trop}(q)$ are the projections from $\Sigma(X \times \mathbb{P}^1) = \Sigma \times \Pi^1$, and that the dominance of $W \to \mathbb{P}^1$ implies $\operatorname{Trop}(W) \in Z_k(S_{\Sigma(X \times \mathbb{P}^1)}(\sigma \times 0))$ for some $\sigma \in \Sigma$, we see that we have an expression exactly as given for the generators of $R_k(\overline{\Sigma})$ in Definition 2.2.18. Thus, the tropicalization Trop_X maps $R_k(X)$ to zero in $A_k(\overline{\Sigma})$ and the assertion follows. □

Example 2.3.16. Consider the toroidal embedding X of Example 2.1.2 b), that is $X = \mathbb{P}^2$, and the boundary is the union $H_1 \cup H_2$, where $H_i = V(x_i)$. Its weakly embedded cone complex Σ is shown in Figure 2.1. Let $L_1 = V(x_1 + x_2)$, and $L_2 = V(x_1 + x_2 - x_0)$. The line L_2 intersects all strata properly as it does not pass through the intersection point $P = (1 : 0 : 0)$ of H_1 and H_2. Because L_2 intersects H_1 and H_2 transversally, this implies that $\operatorname{Trop}(L_2)$ is represented by the Minkowski

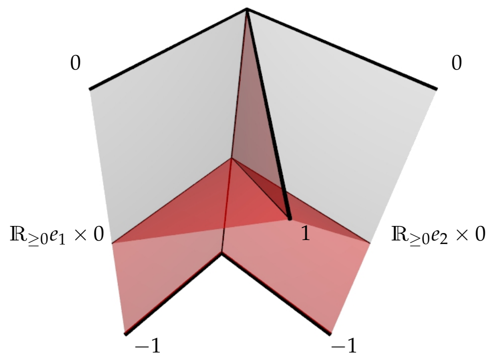


Figure 2.2: The tropicalization $\mathrm{Trop}_{X\times\mathbb{P}^1}(W)$ and its intersection with $\mathrm{pr}_2^* \psi_{\Pi^1}$

weight on Σ having weight 1 on both of its rays. In contrast, L_1 passes through P. Its strict transform in the blowup of X at P intersects the exceptional divisor, and the intersection is in fact transversal, but none of the strict transforms of H_1 or H_2. The blowup at P is the toroidal modification corresponding to the star subdivision of Σ. We see that $\mathrm{Trop}(L_2)$ is the tropical cycle given by a ray in the direction of $e_1 + e_2$ with multiplicity 1, where the e_i are as in Example 2.1.2 b). By Proposition 2.3.15 the two tropical cycles $\mathrm{Trop}(L_1)$ and $\mathrm{Trop}(L_2)$ are rationally equivalent. The proposition also tells us that the tropicalization of the closure of the graph of $(x_1 + x_2)/(x_1 + x_2 - x_0)$ in $X \times \mathbb{P}^1$ will give rise to the relation $\mathrm{Trop}(L_1) - \mathrm{Trop}(L_2) = 0$ in $A_*(\overline{\Sigma})$. Taking coordinates $x = x_1/x_0$ and $y = x_2/x_0$ on X, and a coordinate z on $\mathbb{P}^1 \setminus \{\infty\}$, the graph W is given by the equation $W = V(xz + yz - z - x - y)$. Using these coordinates, it is easy to compute the classical tropicalization of W when considered as subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ with its standard toric structure. It is not hard to see that $\mathrm{Trop}_{X\times\mathbb{P}^1}(W)$ can be obtained from the classical tropicalization by intersecting with $(\mathbb{R}_{\geq 0})^2 \times \mathbb{R}$. Its underlying set is depicted in Figure 2.2. The weights on its maximal faces are all 1.

Being able to tropicalize cycle classes we can formulate the compatibility statements of sections 2.3.3 and 2.3.4 modulo rational equivalence. That the tropicalization of cycle classes commutes with push-forwards follows immediately from Theorem 2.3.10. For intersections with cp-divisors the compatibility is subject of the following result.

Proposition 2.3.17. *Let X be a complete toroidal embedding, let $\mathcal{L} \in \mathrm{ClCP}(X)$, and let $\alpha \in A_*(X)$. Then we have*

$$\mathrm{Trop}(c_1(\mathcal{L}) \cap \alpha) = \mathrm{Trop}(\mathcal{L}) \cdot \mathrm{Trop}(\alpha)$$

Proof. Both sides are linear in α, so we may assume that $\alpha = [Z]$ for some irreducible subvariety Z of X. Let $\sigma \in \Sigma(X)$ be the cone such that $\mathrm{O}(\sigma)$ contains the generic point of Z, and denote by $i\colon \mathrm{V}(\sigma) \to X$ the inclusion map. Furthermore, let $D \in \mathrm{CP}(\mathrm{V}(\sigma))$ be a representative of $i^*\mathcal{L}$. By Theorem 2.3.10 and the projection formula we have

$$\mathrm{Trop}_X(c_1(\mathcal{L}) \cap [Z]) = \mathrm{Trop}(i)_* \mathrm{Trop}_{\mathrm{V}(\sigma)}(c_1(i^*\mathcal{L}) \cap [Z]) = \mathrm{Trop}(i)_* \mathrm{Trop}_{\mathrm{V}(\sigma)}(D \cdot [Z])$$

By Lemma 2.3.12 and Theorem 2.3.14 this is equal to

$$\operatorname{Trop}(i)_*\Big(\big(\operatorname{Trop}(i)^*\operatorname{Trop}(\mathcal{L})\big)\cdot\operatorname{Trop}_{V(\sigma)}\big(Z\cap O(\sigma)\big)\Big)$$

Propositions which is equal to $\operatorname{Trop}_X(\mathcal{L})\cdot\operatorname{Trop}_X([Z])$ by the tropical projection formula (Proposition 2.2.16) and the definition of the tropicalization. □

2.3.6 Comparison with Classical Tropicalization

Suppose we are given a complete toroidal embedding X, together with a closed immersion $\iota\colon X_0\to T$ into an algebraic torus T with lattice of 1-parameter subgroups Λ and character lattice $\Lambda^\vee$. Then we have two different ways to tropicalize X_0, either by taking the tropicalization $\operatorname{Trop}_X(X_0)=\operatorname{Trop}_X[X]$ in $\Sigma(X)$, or by taking the classical tropicalization in $\Lambda_\mathbb{R}$. Every morphism $\iota\colon X_0\to T$ induces a morphism

$$f\colon\Lambda^\vee\to M^X,\ \ m\mapsto\iota^\sharp\chi^m,$$

whose dualization gives rise to a weakly embedded cone complex $\Sigma_\iota(X)$ with cone complex $\Sigma(X)$ and weak embedding $\varphi_{X,\iota}=f^*_\mathbb{R}\circ\varphi_X$. We define the *tropicalization* $\operatorname{Trop}_\iota(X_0)$ *of* X_0 *with respect to* ι as the classical tropicalization of the cycle $\iota_*[X_0]$ in T. Its underlying set $|\operatorname{Trop}_\iota(X_0)|$ is the image of the composite map

$$X_0^{\mathrm{an}}\xrightarrow{\iota^{\mathrm{an}}}T^{\mathrm{an}}\xrightarrow{\operatorname{trop}_T^{\mathrm{an}}}\Lambda_\mathbb{R},$$

where $\operatorname{trop}_T^{\mathrm{an}}$ takes coordinate-wise minus-log-absolute values. As the next lemma shows, it is completely determined by $\Sigma_\iota(X)$:

Lemma 2.3.18. *Let X be a toroidal embedding, and let $\iota\colon X_0\to T$ be a morphism into the algebraic torus T. Then the diagram*

$$\begin{array}{ccc} X_0^{\mathrm{an}}\cap X^{\beth} & \xrightarrow{\operatorname{trop}_X^{\mathrm{an}}} & \Sigma_\iota(X)\\ \downarrow{\scriptstyle\iota^{\mathrm{an}}} & & \downarrow{\scriptstyle\varphi_{X,\iota}}\\ T^{\mathrm{an}} & \xrightarrow{\operatorname{trop}_T^{\mathrm{an}}} & \Lambda_\mathbb{R},\end{array}$$

is commutative. In particular, if X is complete we have $|\operatorname{Trop}_\iota(X_0)|=\varphi_{X,\iota}(|\Sigma_\iota(X)|)$.

Proof. Let $x\in X_0^{\mathrm{an}}\cap X^\beth$, and let $\sigma\in\Sigma_\iota(X)$ be the cone such that the reduction $r(x)$, that is the image of the closed point of the spectrum of a rank-1 valuation ring R under a representation $\operatorname{Spec}R\to X$ of x, is contained in $O(\sigma)$. By construction of $\operatorname{trop}_X^{\mathrm{an}}$ (cf. [Thu07, p. 424], [Uli13, Def. 6.1]), the point $\operatorname{trop}_X^{\mathrm{an}}(x)$ is contained in σ and its pairing with a divisor $D\in M^\sigma$ is equal to $-\log|f(x)|$ for an equation f for D around $r(x)$. It follows that the pairing of $\varphi_{X,\iota}(\operatorname{trop}_X^{\mathrm{an}}(x))$ with $m\in\Lambda^\vee$ is equal to $-\log|\iota^\sharp\chi^m(x)|$. This is clearly equal to $\langle m,\operatorname{trop}_T^{\mathrm{an}}(\iota^{\mathrm{an}}(x))\rangle$. The "in particular" statement follows, since for complete X we have $X^\beth=X^{\mathrm{an}}$, and the tropicalization map $\operatorname{trop}_X^{\mathrm{an}}$ is always surjective [Thu07, Prop. 3.11]. □

Remark 2.3.19. The set-theoretic equality of the "in particular"-part of the previous lemma is an instance of geometric tropicalization, that is a situation where one can read off the tropicalization of a subvariety of an algebraic torus from the structure of the boundary of a suitable compactification. It has already been pointed out in [LQ11] how the methods for geometric tropicalization developed in [HKT09] can be used to obtain the equality $|\operatorname{Trop}_\iota(X_0)| = \varphi_{X,\iota}(|\Sigma_\iota(X)|)$ for a complete toroidal embedding X.

The tropicalization $|\operatorname{Trop}_\iota(X_0)|$ is a union of cones and hence the underlying set of arbitrarily fine embedded cone complexes, that is fans, in $\Lambda_\mathbb{R}$. Similarly as in Construction 2.2.10 there exists such a fan Δ and a proper subdivision Σ' of $\Sigma_\iota(X)$ such that $\varphi_{X,\iota}$ induces a morphism $\Sigma' \to \Delta$ of weakly embedded cone complexes. Therefore, there is a push-forward morphism

$$Z_*(\Sigma(X)) \to Z_*(\Sigma_\iota(X)) = Z_*(\Sigma') \to Z_*(\Delta) = Z_*(\operatorname{Trop}_\iota(X_0)),$$

where $Z_*(\operatorname{Trop}_\iota(X_0))$ denotes the group of affine tropical cycles in $\operatorname{Trop}_\iota(X_0)$ [AR10, Def. 2.15]. Again as in Construction 2.2.10 we see that this morphism is independent of the choices of Σ' and Δ, and we denote it by $(\varphi_{X,\iota})_*$.

Theorem 2.3.20. *Let X be a complete toroidal embedding, and let $\iota\colon X_0 \to T$ be a morphism into the algebraic torus T. Then for every subvariety Z of X_0 we have*

$$(\varphi_{X,\iota})_* \operatorname{Trop}_X(Z) = \operatorname{Trop}_\iota(Z), \tag{2.3.1}$$

where $\operatorname{Trop}_\iota(Z)$ denotes the classical tropicalization of the cycle $\iota_*[Z]$ in T.

Proof. Since both sides of the equality are invariant under toroidal modifications, we may assume from the start that $\operatorname{trop}_X(Z)$ is a union of cones of $\Sigma(X)$. Let $d = \dim(Z)$. It follows from Lemma 2.3.18 that both sides of Equation 2.3.1 vanish if the dimension of $\iota(Z)$ is strictly smaller than d. Hence we may assume that $\dim(\iota(Z)) = d$. Let Δ be a complete strictly simplicial fan in $\Lambda_\mathbb{R}$ such that $|\operatorname{Trop}_\iota(Z)|$, as well as $\varphi_{X,\iota}(\sigma)$ for every $\sigma \in \Sigma_\iota(X)$, is a union of cones of Δ. Similarly as described in Construction 2.2.10 we obtain a simplicial proper subdivision Σ' of $\Sigma_\iota(X)$ such that images of cones of Σ' under $\varphi_{X,\iota}$ are cones in Δ. Namely, we can take suitable star-subdivisions of cones of the form $\varphi_{X,\iota}^{-1}\delta \cap \sigma$ for $\delta \in \Delta$ and $\sigma \in \Sigma_\iota(X)$ along rays. Let $X' = X \times_{\Sigma(X)} \Sigma'$, and let Y denote the toric variety associated to Δ. Whenever $\delta \in \Delta$ and $\sigma' \in \Sigma'$ such that $\varphi_{X,\iota}(\sigma') \subseteq \delta$, it follows form the definitions that $\operatorname{div}(\iota^\sharp \chi^m)|_{X'(\sigma')} \in M_+^{\sigma'}$ for every $m \in \delta^\vee \cap \Lambda^\vee$. In particular, $\iota^\sharp \chi^m$ is regular on $X'(\sigma')$, showing that ι extends to a morphism $X'(\sigma') \to U_\delta$, where U_δ is the affine toric variety associated to U_δ. These morphisms glue and give rise to an extension $\iota'\colon X' \to Y$ of ι. If $\sigma' \in \Sigma'$, and $\delta \in \Delta$ is minimal among the cones of Δ containing $\varphi_{X,\iota}(\sigma')$, then $\iota'(\mathrm{O}_{X'}(\sigma'))$ is contained in $\mathrm{O}_Y(\delta)$. Therefore, the preimages of torus orbits in Y are unions of strata of X. Let $\delta \in \Delta_{(d)}$. By [KP11, Lemma 2.3], the weight of $\operatorname{Trop}_\iota(Z)$ at δ is equal to $\deg([\mathrm{V}_Y(\delta)] \cdot \iota'_*[\overline{Z}'])$, where we denote by $\overline{Z}'$ the closure of Z in X'. Writing $D_1, \ldots, D_d$ for the boundary divisors associated to the rays $\rho_1, \ldots, \rho_d$ of δ, this can be written as

$$\deg(D_1 \cdots D_d \cdot \iota'_*[\overline{Z}']) = \deg(\iota'^* D_1 \cdots \iota'^* D_d \cdot [\overline{Z}'])$$

by the projection formula. By the combinatorics of ι', the support of $\iota'^* D_i$ is the union of all strata $O_{X'}(\sigma')$ associated to cones $\sigma' \in \Sigma'$ containing a ray which is mapped onto ρ_i. Hence, the intersection $|\iota'^* D_1| \cap \ldots \cap |\iota'^* D_d|$ has pure codimension d with components $V_{X'}(\sigma')$ corresponding to the d-dimensional cones of Σ' mapping onto δ. Let σ' be such a cone. The restriction $(\iota'^* D_i)|_{X'(\sigma')}$ is equal to $[N^\rho : \varphi_{X,\iota}(N^{\rho_i'})] \cdot V_{X'}(\rho_i')$, where ρ_i' is the ray of σ' mapping onto ρ_i. By comparing with a local toric model, we see that $V_{X'}(\rho_1') \cdots V_{X'}(\rho_d') = \mathrm{mult}(\sigma')^{-1}[V_{X'}(\sigma')]$ [CLS11, Lemma 12.5.2], where the multiplicity $\mathrm{mult}(\sigma')$ is the index of the sublattice of $N^{\sigma'}$ generated by the primitive generators of the rays $\rho_1', \ldots, \rho_d'$ of σ'. Thus, the multiplicity of $[V_{X'}(\sigma')]$ in $\iota'^* D_1 \cdots \iota'^* D_d$ is equal to

$$\frac{1}{\mathrm{mult}(\sigma')} \prod_{i=1}^{d} [N^\rho : \varphi_{X,\iota}(N^{\rho_i'})].$$

We easily convince ourselves that this is nothing but the index $[N^\delta : \varphi_{X,\iota}(N^{\sigma'})]$. It follows that the weight of $\mathrm{Trop}_\iota(Z)$ at δ is equal to

$$\sum_{\sigma' \mapsto \delta} [N^\delta : \varphi_{X,\iota}(N^{\sigma'})] \deg([V_{X'}(\sigma')] \cdot [\overline{Z}']),$$

where the sum is taken over all d-dimensional cones $\sigma' \in \Sigma'$ which are mapped onto δ by $\varphi_{X,\iota}$. Since $\deg([V_{X'}(\sigma')] \cdot [\overline{Z}'])$ is the weight of $\mathrm{Trop}_X(Z)$ at σ' by definition, the desired equality follows from the construction of the push-forward. □

Corollary 2.3.21. *Let X be a complete toroidal embedding, and let $\iota \colon X \to T$ be a closed immersion into the algebraic torus T. Then the weight of* $\mathrm{Trop}_\iota(X_0)$ *at a generic point w of its support $\varphi_{X,\iota}|\Sigma(X)|$ is equal to*

$$\sum_{\sigma} \mathrm{index}\left(\varphi_{X,\iota}(N^\sigma)\right),$$

where the sum is taken over all $\dim(X)$*-dimensional cones of $\Sigma(X)$ whose image under $\varphi_{X,\iota}$ contains w.*

Remark 2.3.22. Corollary 2.3.21 can also be proven using Cueto's multiplicity formula for geometric tropicalization [Cue12, Thm. 2.5].

2.4 Applications

In this section we apply our methods to obtain classical/tropical correspondences for genus 0 descendant Gromov-Witten invariants. But first, we need to understand the tropicalization map for the moduli space $\overline{M}_{0,n}$ of n-marked rational stable maps. In particular, we need to show that the tropicalizations of the ψ-classes on $\overline{M}_{0,n}$ recover the tropical ψ-classes on the tropical moduli space $M_{0,n}^{\mathrm{trop}}$ which have been studied in [Mik07, KM09, Rau08].

The boundary $\overline{M}_{0,n} \setminus M_{0,n}$ is a simple normal crossings divisor in the smooth variety $\overline{M}_{0,n}$. Therefore, the embedding $M_{0,n} \subseteq \overline{M}_{0,n}$ is strictly toroidal. We recall that the ψ-classes on $\overline{M}_{0,n}$ are elements of $A^1(\overline{M}_{0,n})$: for $1 \leq k \leq n$ the class ψ_k is defined as the first Chern class of the k-th cotangent line bundle on $\overline{M}_{0,n}$ (see [Koc] for details). By the smoothness of $\overline{M}_{0,n}$, we can consider the ψ-classes as elements of $\mathrm{Pic}(\overline{M}_{0,n})$. As $M_{0,n}$ is isomorphic to an open subset of $\mathbb{P}^{n-3}$, we have $\mathrm{Pic}(M_{0,n}) = 0$, and thus $\mathrm{Pic}(\overline{M}_{0,n}) = \mathrm{ClCP}(\overline{M}_{0,n})$. This allows us to tropicalize the ψ-classes, giving rise to elements $\mathrm{Trop}(\psi_k) \in \mathrm{ClCP}(\Sigma(\overline{M}_{0,n}))$. In order to compare them with the tropical ψ-classes, we need to show that $\Sigma(\overline{M}_{0,n})$ and $M_{0,n}^{\mathrm{trop}}$ are equal. We recall from Section 1.1.2 that we considered $M_{0,n}^{\mathrm{trop}}$ as a tropical cycle by identifying it with its image under the tropical Plücker embedding

$$\mathrm{pl}^{\mathrm{t}} \colon M_{0,n}^{\mathrm{trop}} \to \mathbb{R}^{D([n])}/\mathbb{R}^n,$$

where $D([n])$ denotes the set of pairs of distinct elements of $[n] = \{1, \ldots, n\}$. In this chapter, we want to consider $M_{0,n}^{\mathrm{trop}}$ as an embedded cone complex, and therefore have to specify a fan structure on this image. There is a canonical choice for its cones. Namely, we can take the closures of the images of all tropical curves of fixed combinatorial type [SS04, Thm. 4.2]. From now on, $M_{0,n}^{\mathrm{trop}}$ will denote this embedded cone complex. The dimension of a cone in $M_{0,n}^{\mathrm{trop}}$ is equal to the number of bounded edges in the corresponding combinatorial type. In particular, its rays correspond to the combinatorial types with exactly one unbounded edge. These are determined by the markings on one of its vertices. For $I \subseteq [n]$ with $2 \leq |I| \leq n-2$ we denote by $v_I = v_{I^c}$ (where $I^c = [n] \setminus I$) the primitive generator of the ray of $M_{0,n}^{\mathrm{trop}}$ corresponding to I. It is equal to the image under pl^{t} of the tropical curve of the corresponding combinatorial type whose bounded edge has length one, that is

$$v_I = -\frac{1}{2} \sum_{i \in I, j \notin I} \left(e_{(i,j)} + e_{(j,i)} \right),$$

where $e_{(i,j)}$ denotes the image of the (i,j)-th standard basis vector of $\mathbb{R}^{D([n])}$ in $\mathbb{R}^{D([n])}/\mathbb{R}^n$.

We relate $\Sigma(\overline{M}_{0,n})$ to $M_{0,n}^{\mathrm{trop}}$ via the Plücker embedding

$$\mathrm{pl} \colon M_{0,n} \to \mathbb{G}_m^{D([n])}/\mathbb{G}_m^n$$

of Section 1.1.2. We have already observed in Section 1.2 that $\mathrm{Trop}_{\mathrm{pl}}(M_{0,n}) = [M_{0,n}^{\mathrm{trop}}]$, where $[M_{0,n}^{\mathrm{trop}}]$ is the tropical cycle having weight 1 on all maximal cones of $M_{0,n}^{\mathrm{trop}}$. By Lemma 2.3.18, this implies that the underlying set of $M_{0,n}^{\mathrm{trop}}$ is equal to the image of the weak embedding

$$\varphi_{\overline{M}_{0,n},\mathrm{pl}} \colon \Sigma(\overline{M}_{0,n}) \to \mathbb{R}^{D([n])}/\mathbb{R}^n.$$

What we want to show is that $\varphi_{\overline{M}_{0,n},\mathrm{pl}}$ is an embedding which maps the cones of $\Sigma(\overline{M}_{0,n})$ isomorphically onto the cones of $\overline{M}_{0,n}$. Note that the strata of $\overline{M}_{0,n}$, and hence the cones of $\Sigma(\overline{M}_{0,n})$, are in bijection with combinatorial types of tropical curves: for two curves in $\overline{M}_{0,n}$ the dual graph construction, which replaces the components of a stable curve by nodes, their intersection points by edges, and marked points by unbounded edges with the respective marks, yields the same combinatorial type of tropical curves if and only if they belong to the same stratum. Of course, $\varphi_{\overline{M}_{0,n},\mathrm{pl}}$ should identify a cone of $\Sigma(\overline{M}_{0,n})$ with a cone of $M_{0,n}^{\mathrm{trop}}$ if and only if they correspond to the same combinatorial type. Since both $\Sigma(\overline{M}_{0,n})$ and $M_{0,n}^{\mathrm{trop}}$ are strictly simplicial, and their cones are uniquely determined by their rays, it suffices to prove this for rays.

Let $I \subseteq [n]$ be a subset with $2 \leq |I| \leq n-2$. We write D_I for the boundary component of $M_{0,n}$ containing the curves with exactly two components which are marked by I and I^c, respectively. The image under $\varphi_{\overline{M}_{0,n},\mathrm{pl}}$ of the primitive generator of the corresponding ray $\sigma^{D_I} \in \Sigma(M_{0,n})$ can be computed explicitly, as demonstrated in [MS15]; it follows from [MS15, Prop. 6.5.14] that it is equal to

$$\sum_{(i,j)\in D(I)} e_{(i,j)}$$

which is equal to v_I (their natural representatives in $\mathbb{R}^{D([n])}$ only differ by the action of $\frac{1}{2}\sum_{i\in I} e_i \in \mathbb{R}^n$). In other words, the rays of $\Sigma(\overline{M}_{0,n})$ are mapped isomorphically onto the rays of $M_{0,n}^{\mathrm{trop}}$, which, as observed above, implies that $\varphi_{\overline{M}_{0,n},\mathrm{pl}}$ is an isomorphism between $\Sigma(\overline{M}_{0,n})$ and $M_{0,n}^{\mathrm{trop}}$.

Now that we know that $\mathrm{Trop}(\psi_k)$ can be seen as an element of $\mathrm{ClCP}(M_{0,n}^{\mathrm{trop}})$, we can compare it with the tropical ψ-classes of [Mik07] and [KM09]. In Mikhalkin's definition the tropical ψ-classes are codimension-1 tropical cycles on $M_{0,n}^{\mathrm{trop}}$, whereas Kerber and Markwig [KM09] define them as Cartier divisors. From our perspective, the correct definition is obtained by taking the images of Kerber and Markwig's Cartier divisors in $\mathrm{ClCP}(M_{0,n}^{\mathrm{trop}})$. We denote these images by ψ_k^{t} for $1 \leq k \leq n$.

Proposition 2.4.1 (cf. [KM09, Kat12])**.** *For every $1 \leq k \leq n$ we have*

$$\mathrm{Trop}_{\overline{M}_{0,n}}(\psi_k) = \psi_k^{\mathrm{t}}.$$

In particular, for arbitrary natural numbers $a_1, \dots, a_n$ we have

$$\mathrm{Trop}_{\overline{M}_{0,n}}\left(\prod_{k=1}^{n} \psi_k^{a_k} \cdot [\overline{M}_{0,n}]\right) = \prod_{k=1}^{n} (\psi_k^{\mathrm{t}})^{a_k} \cdot [\overline{M}_{0,n}^{\mathrm{trop}}] \quad \text{in } A_*(\overline{M}_{0,n}^{\mathrm{trop}}),$$

where $[\overline{M}_{0,n}^{\mathrm{trop}}] = \mathrm{Trop}_{\overline{M}_{0,n}}([\overline{M}_{0,n}])$ is the tropical cycle on the extended cone complex $\overline{M}_{0,n}^{\mathrm{trop}} = \overline{M_{0,n}^{\mathrm{trop}}}$ with weight 1 on all maximal cones, and the tropical intersection product is that of Definition 2.2.26.

Proof. For the first part of the statement we may assume that $k = 1$, the other cases follow by symmetry. We can write ψ_1 as a sum of boundary divisors, namely as

$$\psi_1 = \sum_{1 \in I;\, 2,3 \in J} D_I \quad \text{in } \mathrm{ClCP}(\overline{M}_{0,n}),$$

where the sum runs over all subsets $I \subseteq [n]$ with $2 \leq |I| \leq n-2$ which satisfy the given condition [Koc, 1.5.2]. Consequently, its tropicalization is represented by

$$\sum_{1 \in I;\, 2,3 \in J} \varphi_I,$$

where φ_I is the Cartier divisor on $M_{0,n}^{\mathrm{trop}}$ which has value 1 at v_I and 0 at v_J for all $I \neq J \neq I^c$. By [Rau08, Lemma 2.24], this is equal to ψ_1^{t} in $\mathrm{ClCP}(M_{0,n}^{\mathrm{trop}})$. The second part of the statement is an immediate consequence of Proposition 2.3.17. □

Now let us consider the moduli spaces of logarithmic stable maps. Let Σ be a complete fan in $N_{\mathbb{R}}$, where N is a lattice, and let $\Delta\colon J \to N$ be a tropical degree (cf. Section 1.1.2) that is supported on the rays of Σ. Then there is a modular compactification of the moduli space $\mathrm{LSM}_n^\circ(X,\Delta)$, namely the moduli space $\mathrm{LSM}_n(X,\Delta)$ of *n-marked genus 0 logarithmic stable maps to X with contact order Δ*. We refer to [AC14, GS13, Ran15] for details. The inclusion $\mathrm{LSM}_n^\circ(X,\Delta) \subseteq \mathrm{LSM}_n(X,\Delta)$ defines a strict toroidal embedding. In fact, $\mathrm{LSM}_n(X,\Delta)$ can be constructed as a toroidal modification of $\overline{M}_{0,n} \times X$. More precisely, the morphism

$$\mathrm{ft} \times \mathrm{ev}_1\colon\ \mathrm{LSM}_n(X,\Delta) \to \overline{M}_{0,L} \times X,$$

where ft is the morphism forgetting the map and $L = n + |J|$, is a toroidal modification by [Ran15, Thm. 4.2.4]. With what we have seen for $\overline{M}_{0,n}$, this yields a morphism

$$\Sigma(\mathrm{LSM}_n(X,\Delta)) \xrightarrow{\mathrm{Trop}(\mathrm{ft}\times\mathrm{ev}_1)} \Sigma(\overline{M}_{0,L} \times X) = \Sigma(\overline{M}_{0,L}) \times \Sigma \xrightarrow{\varphi_{\overline{M}_{0,L},\mathrm{pl}} \times \mathrm{id}} M_{0,L}^{\mathrm{trop}} \times \Sigma$$

of cone complexes which is a bijection on underlying sets. Note that the composite map is equal to the weak embedding $\varphi_{\mathrm{LSM}_n(X\Delta),(\mathrm{pl}\circ\mathrm{ft})\times\mathrm{ev}_1}$. The underlying set of $M_{0,L}^{\mathrm{trop}} \times \Sigma$ is equal to $\mathrm{TSM}_n^\circ(N_{\mathbb{R}},\Delta)$ embedded via

$$(\mathrm{pl}^{\mathrm{t}} \circ \mathrm{ft}^{\mathrm{t}}) \times \mathrm{ev}_1^{\mathrm{t}}\colon\ \mathrm{TSM}_n^\circ(N_{\mathbb{R}},\Delta) \to \mathbb{R}^{D([L])}/\mathbb{R}^L \times N_{\mathbb{R}}$$

as we have seen in Section 1.1.2. From now on we will consider $\mathrm{TSM}_n^\circ(N_{\mathbb{R}},\Delta)$ as an embedded cone complex whose fan structure is the subdivision of $M_{0,L}^{\mathrm{trop}} \times \Sigma$ induced by the bijection with $\Sigma(\mathrm{LSM}_n(X,\Delta))$. For a precise description of its cones we refer to [Ran15].

By construction, the tropicalization $\mathrm{Trop}(\mathrm{ft})$ of the forgetful map is equal to the tropical forgetful map

$$\Sigma(\mathrm{LSM}_n(X,\Delta)) = \mathrm{TSM}_n^\circ(N_{\mathbb{R}},\Delta) \xrightarrow{\mathrm{ft}^{\mathrm{t}}} M_{0,L}^{\mathrm{trop}} = \Sigma(\overline{M}_{0,L}).$$

In particular, if we define

$$\hat{\psi}_k = \mathrm{ft}^* \psi_k \in \mathrm{ClCP}(\mathrm{LSM}_n(X,\Delta)) \quad \text{, and}$$
$$\hat{\psi}_k^{\mathrm{t}} = \mathrm{ft}^{\mathrm{t}\,*} \psi_k^{\mathrm{t}} \in \mathrm{ClCP}(\mathrm{TSM}_n^{\circ}(N_{\mathbb{R}},\Delta)),$$

then $\mathrm{Trop}_{\mathrm{LSM}_n(X,\Delta)}(\hat{\psi}_k) = \hat{\psi}_k^{\mathrm{t}}$. Furthermore, we have $\mathrm{Trop}(\mathrm{ev}_i) = \mathrm{ev}_i^{\mathrm{t}}$ for all i by [Ran15, Prop. 5.0.2] (with a little effort this also follows from Proposition 1.2.5). This is all we need to apply our results of this chapter. We obtain the following correspondence theorem for genus 0 descendant Gromov-Witten invariants.

Corollary 2.4.2. *Let $a_1,\dots,a_n,b_1,\dots b_n$ be natural numbers, and for $1 \le i \le n$ and $1 \le j \le b_i$ let $D_{ij} \in \mathrm{Pic}(X) = \mathrm{ClCP}(X)$. Then we have the equality*

$$\mathrm{Trop}_{\mathrm{LSM}_n(X,\Delta)}\left(\prod_{i=1}^{n} \hat{\psi}_i^{a_i} \prod_{j=1}^{b_i} \mathrm{ev}_i^*(D_{ij}) \cdot [\mathrm{LSM}_n(X,\Delta)]\right) =$$
$$= \prod_{i=1}^{n} (\hat{\psi}_i^{\mathrm{t}})^{a_i} \prod_{j=1}^{b_i} \mathrm{ev}_i^{\mathrm{t}\,*}(\mathrm{Trop}_X(D_{ij})) \cdot [\mathrm{TSM}_n(N_{\mathbb{R}},\Delta)]$$

of cycle classes in $A_*(\mathrm{TSM}_n(N_{\mathbb{R}},\Delta))$, where $[\mathrm{TSM}_n(N_{\mathbb{R}},\Delta)]$ is the cycle on the extended cone complex $\mathrm{TSM}_n(N_{\mathbb{R}},\Delta) = \overline{\mathrm{TSM}_n^{\circ}(N_{\mathbb{R}},\Delta)}$ with weight 1 on all maximal cones.

Remark 2.4.3. In case we take top-dimensional intersections with $X = \mathbb{P}^r$, the tropical degree Δ containing each of the vectors $e_1,\dots,e_r,-\sum e_i$ exactly d times, and all the D_{ij} equal to classes of lines, we obtain the tropical descendant Gromov-Witten invariants of [MR09] and [Rau08] up to factor $(d!)^{r+1}$ on the tropical side.

Remark 2.4.4. We defined the ψ-classes $\hat{\psi}_k$ on $\mathrm{LSM}(X,\Delta)$ to be the analogues of the tropical ψ-classes on $\mathrm{TSM}_n^{\circ}(\mathbb{R}^r,\Delta)$ as defined in [MR09], that is as pull-backs of ψ-classes on $\overline{M}_{0,L}$. It has been shown in the upcoming paper [MR] that these are equal to the classes on $\mathrm{LSM}(X,\Delta)$ obtained by taking Chern classes of cotangent line bundles.

3 Generalizations to a Monoidal Setup

Let X be a toroidal embedding. We have seen in Chapter 2 that the tropical analogues of cocycles on X are Minkowski weights on its associated cone complex $\Sigma = \Sigma(X)$, and that the tropical analogues of $Z_*(X)$ and $A_*(X)$ are $Z_*(\Sigma)$ and $A_*(\Sigma)$, respectively. We have also seen that the tropicalization procedure respects intersections with cp-divisors. Yet, tropical cocycles behave somewhat peculiar in that we do not have natural cup and cap products

$$\begin{aligned} \mathrm{M}^*(\Sigma) \times \mathrm{M}^*(\Sigma) &\to \mathrm{M}^*(\Sigma) \quad \text{and} \\ \mathrm{M}^*(\Sigma) \times A_*(\Sigma) &\to A_*(\Sigma), \end{aligned}$$

where we write $\mathrm{M}^*(\Sigma)$ for $\mathrm{M}_*(\Sigma)$ with the dual grading, in general. To explain this behavior let us recall how tropicalization works for cocycles. Given a cocycle $c \in A^k(X)$ we apply Kronecker duality to obtain the morphism

$$A_k(X) \to \mathbb{Z}, \ \ \alpha \mapsto \deg(c \cap \alpha).$$

Pulling this morphism back via

$$\mathbb{Z}^{\Sigma^{(k)}} \to A_k(X), \ \ \sigma \mapsto [V(\sigma)]$$

we obtain $\mathrm{Trop}_X(c) \in \mathrm{M}^k(\Sigma)$. So the tropicalization map is equal to the composite

$$A^k(X) \to \mathrm{Hom}(A_k(X), \mathbb{Z}) \to \mathrm{M}^k(\Sigma).$$

The actual tropicalization happens at the second arrow, where morphisms $A_k(X) \to \mathbb{Z}$ are reduced to finitely many integers on the cones of Σ. The product structure of $A^k(X)$, however, is already lost at the first arrow: since the Kronecker duality map can be far from being injective or surjective in general, there is no induced product on $\bigoplus_{k\in\mathbb{N}} \mathrm{Hom}(A_k(X), \mathbb{Z})$.

The failure to preserve cap-products can be explained similarly. Let us recall how to tropicalize an $(n-k)$-dimensional subvariety Z of X_0 (where $n = \dim(X)$): we first choose a strictly simplicial proper subdivision Δ of Σ such that the closure $\overline{Z}$ of Z in $X \times_\Sigma \Delta$ intersects all strata properly, or equivalently, that $\mathrm{trop}_X(Z)$ is a union of cones of Δ. Then we take the Poincaré dual of $[\overline{Z}] \in A_*(X \times_\Sigma \Delta)$ and obtain an element of $A^*(X \times_\Sigma \Delta)$. We have seen in the proof of Proposition 2.3.7 that replacing Δ by a finer subdivision Δ' amounts to taking the pull-back

$$A^k(X \times_\Sigma \Delta) \to A^k(X \times_\Sigma \Delta')$$

of $[\overline{Z}]$ under the canonical map $X \times_\Sigma \Delta' \to X \times_\Sigma \Delta$. Since the set of strictly simplicial proper subdivisions Δ of Σ such that $\operatorname{trop}_X(Z)$ is a union of cones of Δ is cofinal in the directed system of proper subdivisions of Σ, we obtain a well-defined element of the direct limit $\varinjlim_\Delta A^k(X \times_\Sigma \Delta)$. Extending by linearity we obtain a morphism $Z_{n-k}(X_0) \to \varinjlim_\Delta A^k(X \times_\Sigma \Delta)$, and the tropicalization map is the composite

$$Z_{n-k}(X_0) \to \varinjlim_{\Delta} A^k(X \times_\Sigma \Delta) \to \varinjlim_{\Delta} \operatorname{Hom}(A_k(X \times_\Sigma \Delta), \mathbb{Z}) \to \varinjlim_{\Delta} M^k(\Delta) = Z_{n-k}(\Sigma).$$

The actual tropicalization happens at the last arrow, but similarly as for the cup-product, the cap-product is already lost at the second arrow when applying Kronecker duality.

We see that $\mathrm{M}^k(\Sigma)$ is not quite the tropical analogue of $A^k(X)$, but rather of $\operatorname{Hom}(A_k(X), \mathbb{Z})$. This brings up the question whether there is some suitably defined "tropical space" F associated to X such that for a suitably defined Chow group $A_k(F)$ we have $\mathrm{M}^k(\Sigma) = \operatorname{Hom}(A_k(F), \mathbb{Z})$. Even if we do not know how F should look like, it is obvious what $A_k(F)$ should be: it should be the free group on the set $\Sigma^{(k)}$ of codimension-k cones modulo all relations of the form

$$\sum_{\sigma : \tau < \sigma} \langle f, u_{\sigma/\tau} \rangle e_\sigma,$$

where $\tau \in \Sigma^{(k+1)}$, $f \in (N_\tau^X)^\perp$, and e_σ denotes the generator corresponding to σ. In the proof of Proposition 2.3.1 we have seen that these are exactly the relations which force a weight on $\Sigma^{(k)}$ to be balanced. Or in other words, the image of the monomorphism

$$\operatorname{Hom}(A_k(F), \mathbb{Z}) \to \operatorname{Hom}(\mathbb{Z}^{\Sigma^{(k)}}, \mathbb{Z}) = \mathbb{Z}^{\Sigma^{(k)}}$$

is equal to $\mathrm{M}^k(\Sigma)$. Furthermore, the proof showed that there is a morphism $A_k(F) \to A_k(X)$ sending e_σ to $[V(\sigma)]$, so that the tropicalization of an element in $\operatorname{Hom}(A_k(X), \mathbb{Z})$ is nothing but the pull-back with respect to this map. We want to think about $A_k(F) \to A_k(X)$ as being induced by a map $X \to F$. This also tells us what F should be. The points of F should correspond to the strata of X, and hence to the cones of Σ. The topology on F should respect the inclusion relation on Σ, that is $x \in F$ should specialize to $y \in F$ if and only if the cone corresponding to x is contained in the cone corresponding to y. In other words, we take F to be the subset of X consisting of the generic points of the strata and give it the induced topology. This space has a natural sheaf of monoids on it. Namely, we can define a sheaf of monoids $\mathcal{M}_X$ on X by

$$\Gamma(U, \mathcal{M}_X) = \{ f \in \Gamma(U, \mathcal{O}_X) \mid f|_{X_0 \cap U} \in \Gamma(X_0 \cap U, \mathcal{O}_X^*) \}$$

and then define $\mathcal{M}_F$ to be the pull-back of $\mathcal{M}_X / \mathcal{O}_X^*$ to F. The sections of $\mathcal{M}_X / \mathcal{O}_X^*$ are precisely the effective Cartier divisors supported in the boundary. Therefore, the stalk of $\mathcal{M}_F$ at the generic point of $\mathrm{O}(\sigma)$ is equal to M_+^σ, the monoid of nonnegative integral functions on σ. The space $(F, \mathcal{M}_F)$ has been studied in detail by Kazuya Kato in [Kat94], where he proved that it locally looks like the affine spectrum of a monoid. We will call such spaces Kato fans and aim to use their algebro-geometric structure to obtain a "copy" of the intersection theory for varieties for them. Unfortunately, the stalk of $\mathcal{M}_F$ at the generic point of an irreducible Kato fan F equals 0, so that there are no nontrivial rational functions of F which we could use to define rational equivalence. Similar as in the case of cone complexes, we remedy this by introducing weak embeddings for Kato fans.

Section 3.1 can be seen as an introduction to the theory of commutative monoids and monoidal spaces. Even though the theory is well-developed and very similar to the theory of rings and schemes, we feel that it is still nonstandard in algebraic geometry. For the sake of self-containedness we give a very detailed treatment, which is tailored to our needs. At the end of the section we introduce the notion of weakly embedded Kato fans and investigate their connection to weakly embedded cone complexes.

In Section 3.2 we develop the foundations for an intersection theory of weakly embedded Kato fans. The basic objects are the Chow groups, which are simply defined as cycle groups modulo rational equivalence. We then study their functorial behavior, that is we define push-forwards for subdivisions and pull-backs for locally exact morphisms, and show the correctness of expected identities like the projection formula. Finally, we reveal a duality between the intersection theory of a weakly embedded Kato fan and that of its associated weakly embedded cone complex.

As demonstrated in [Uli13], Kato fans cannot only be assigned to toroidal embeddings, but more generally to fine and saturated logarithmic schemes without monodromy. In Section 3.3 we will generalize this by relaxing the fine and saturated condition, but more importantly to include weak embeddings. We will then study how the canonical morphism $\overline{X} \to F_X$ from the sharpening of a log scheme X to its associated weakly embedded Kato fan relates the Chow groups of X and F_X. We show that if it has the correct fiber dimensions, then there is a pull-back $A_*(F_X) \to A_*(X)$. Afterwards, we prove that the functorial behavior of these pull-backs is analogous to that of flat morphisms in algebraic geometry. Finally, we use those results to define and study a tropicalization morphism for cycles in logarithmic schemes.

3.1 Monoids, Monoidal Spaces, and Kato Fans

This section provides a very detailed treatment of the theory of commutative monoids, locally monoidal spaces, and Kato fans which we need in the rest of this chapter. Excellent references for further details about these topics include [BG09, Ogu06, GR14], and also [Dei05] for Deitmar schemes and [Uli13] for the connection to cone complexes. An experienced reader might want to skip to Subsection 3.1.3, where we introduce weak embeddings for Kato fans.

3.1.1 Commutative Monoids

Recall that a monoid is a semi-group, that is a nonempty set with an associative binary operation, that has a neutral element. We will only consider commutative monoids and will usually write them additively. The neutral element of a monoid M will thus be denoted by 0, or 0_M if there is risk of confusion. We will only write a monoid multiplicatively if it is a submonoid of the multiplicative monoid of a ring, so no confusion will arise when we denote the neutral element of such a monoid by 1. A morphism between two monoids M and N is a map $f\colon M \to N$ respecting addition and satisfying $f(0) = 0$. We will denote the category of (commutative) monoids by $(\mathbf{Mon})$.

The category $(\mathbf{Ab})$ of abelian groups is a full subcategory. There are two natural functors $(\mathbf{Mon}) \to (\mathbf{Ab})$, which are left- and right adjoint to the inclusion $(\mathbf{Ab}) \to (\mathbf{Mon})$. The right adjoint assigns to a monoid M its group of *units*

$$M^* = \{m \in M \mid \exists m' \in M : m + m' = 0\},$$

whereas the left adjoint assigns to M its Grothendieck group, or *groupification*, M^{gp}. In other words, the group of units M^* is universal among all groups mapping into M. Hence, if $M^* = 0$, then all maps from groups into M are trivial, in which case we call M *sharp*. The groupification M^{gp} is universal among all groups being mapped into from M. It is a special case of *localization*: if $S \subseteq M$ is a submonoid, then we can define a monoid $M[-S]$ by $M \times S/\sim$, where $(m, s) \sim (m', s')$ if there exists $t \in S$ such that $t + s' + m = t + s + m'$, and addition is defined by $(m, s) + (m', s') = (m + m', s + s')$. We usually write $m - s$ for the class of (m, s). This construction looks much more familiar when written multiplicatively, because then it is the exact analogue of localization for rings. If $S = \mathbb{N} \cdot m$ for some $m \in M$ we write $M[-S] = M_m$. Similarly as for rings, the morphism $M \to M[-S]$, $m \to m$ is universal among all morphisms of monoids mapping S to units. It follows that $M^{\mathrm{gp}} = M[-M]$. A monoid is called *integral*, if $M \to M^{\mathrm{gp}}$ is injective. This is true if and only if $m + n = m + n'$ implies $n = n'$ for all $m, n, n' \in M$.

The category of monoids has free objects. For a set U the free monoid on U, denoted by $\mathbb{N}^{(U)}$, consists of all maps $U \to \mathbb{N}$ which map all but finitely many elements of U to 0. Addition on $\mathbb{N}^{(U)}$ is defined coordinatewise. For the construction of monoids represented by generators and relations, we need to take quotients. Unfortunately, a surjective morphism $f\colon M \to N$ of monoids is not determined by $f^{-1}\{0\}$, as it would be for groups. But of course, they are determined by the induced equivalence relation $R = \{(m, n) \in M \times M \mid f(m) = f(n)\}$ on M. The relations arising in this way are precisely those equivalence relations on M which are submonoids of $M \times M$ and are called *congruences*. Thus the monoid $\langle X \mid R \rangle$ represented by a set of generators X and relations $R \subseteq \mathbb{N}^{(X)} \times \mathbb{N}^{(X)}$ is equal to the quotient of $\mathbb{N}^{(X)}$ by the congruence generated by R. By the universal property of the groupification, the abelian group $\langle X \mid R \rangle^{\mathrm{gp}}$ is represented by X and R as well.

Example 3.1.1.

a) Let $M = \langle x \mid 2x = x \rangle$. Obviously, M has at most 2 elements, namely 0 and x. The element ∞ of the monoid $N = \{0, \infty\}$ (where $0 + n = n$ and $\infty + n = \infty$ for $n \in N$) satisfies $2\infty = \infty$, so the morphism $\mathbb{N} \to M$ sending 1 to ∞ induces an isomorphism $M \to N$. The preimage of 0 under the quotient map $\mathbb{N} \to M$ is trivial, which, of course, also holds for the identity $\mathrm{id}_{\mathbb{N}}$. Every element y of an abelian group satisfying $2y = y$ has to be 0. Therefore, $M^{\mathrm{gp}} = 0$.

b) Let $M = \langle a, b, c \mid a + b = 2c \rangle$. Then $M^{\mathrm{gp}} = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c / \langle a + b - 2c \rangle \cong \mathbb{Z}^2$, where we identify a, b, and c, with $(-1, 2)$, $(1, 0)$, and $(0, 1)$, respectively. Because of the relation $a + b = 2c$ all elements in M are either in $\mathbb{N}a + \mathbb{N}b$ or $\mathbb{N}a + \mathbb{N}b + c$. Both maps

$$\mathbb{N}^2 \to M^{\mathrm{gp}} = \mathbb{Z}^2, \quad (k, l) \mapsto ka + lb \quad \text{and}$$
$$\mathbb{N}^2 \to M^{\mathrm{gp}} = \mathbb{Z}^2, \quad (k, l) \mapsto ka + lb + c$$

are injective since $(-1, 2)$ and $(1, 0)$ are linearly independent in $\mathbb{Q}^2$. Their images are also disjoint: the first map only hits points whose second coordinate is even, whereas the second map only hits points with odd second coordinate. Thus, M is integral.

Let N be a submonoid of M. Two elements $m, m' \in M$ are equivalent in the congruence $\sim_N$ generated by $\{n \sim_N 0 \mid n \in N\}$ if and only if there are $n, n' \in N$ such that $m + n = m' + n'$. We denote the quotient monoid $M/\sim_N$ by M/N. The most important case will be $N = M^*$, where $m \sim_{M^*} m'$ if and only if $m = m' + u$ for some $u \in M^*$. Clearly, the quotient M/M^* is sharp. We denote it by $\overline{M}$ and call it the *sharpening* of M. The natural morphism $M \to \overline{M}$ is universal among all maps from M to a sharp monoid. In other words, the functor from (**Mon**) to its full subcategory (**sh** − **Mon**) of sharp monoids assigning to a monoid M its sharpening $\overline{M}$ is left adjoint to the inclusion functor.

A monoid M is called *finitely generated* if there is a surjective morphism $\mathbb{N}^{(U)} \to M$ for a finite set U. It is called *fine* if it is finitely generated and integral. For an integral monoid M, the set

$$M^{\mathrm{sat}} = \{m \in M^{\mathrm{gp}} \mid \exists k \in \mathbb{N}_+ : km \in M\}$$

is called the saturation of M. It is a submonoid of M^{gp} which clearly contains M. If $M = M^{\mathrm{sat}}$, we call M *saturated*. In any case, the morphism $M \to M^{\mathrm{sat}}$ is universal among all morphism to saturated monoids. The combination fine and saturated is sometimes abbreviated to *fs*.

Lemma 3.1.2. *Let M be a monoid. If M is integral, then so is $\overline{M}$. Furthermore, we have $(\overline{M})^{\mathrm{gp}} = M^{\mathrm{gp}}/M^*$, and M is saturated if and only if it is integral and $\overline{M}$ is saturated.*

Proof. Assume M is integral, and suppose $\overline{m} + \overline{n} = \overline{m} + \overline{n}'$ in $\overline{M}$. Then there exists $u \in M^*$ with $m + n + u = m + n'$ and hence $n + u = n'$. It follows that $\overline{n} = \overline{n}'$, showing that $\overline{M}$ is integral. The natural morphism $\pi\colon M^{\mathrm{gp}} \to (\overline{M})^{\mathrm{gp}}$ clearly is surjective with kernel containing M^*. Let $m - m'$ be an element in its kernel. Then $\overline{m} - \overline{m}' = 0$ in $(\overline{M})^{\mathrm{gp}}$, which is equivalent to $\overline{m} = \overline{m}'$ because $\overline{M}$ is integral. Thus, m and m' only differ by a unit, showing that $m - m' \in M^*$. Now assume that M is saturated, and let $\pi(y) = x \in (\overline{M})^{\mathrm{sat}}$. Then there exists $k \in \mathbb{N}_+$ such that $\pi(ky) = kx \in \overline{M}$. Let $m \in M$ with $\pi(m) = kx$. By what we showed above, we have $ky \in m + M^* \subseteq M$, hence $y \in M^{\mathrm{sat}} = M$ and $x = \pi(y) \in \overline{M}$. Conversely, assume that M is integral and $\overline{M}$ is saturated. Let $x \in M^{\mathrm{sat}}$. Then we clearly have $\pi(x) \in (\overline{M})^{\mathrm{sat}} = \overline{M}$ and therefore $x \in M + M^* = M$. □

We call M torsion-free if $km = km'$ implies $m = m'$ for all $k \in \mathbb{N}_+$, and $m, m' \in M$. An integral monoid M is torsion free if and only if M^{gp} is a torsion-free abelian group.

Lemma 3.1.3. *Let M be a saturated monoid. Then M is torsion-free if and only if $km = 0$ implies $m = 0$ for all $k \in \mathbb{N}_+$ and $m \in M$. In particular, sharp and saturated monoids are torsion-free.*

Proof. The necessity is obvious. For sufficiency assume that $km = 0$ implies $m = 0$, and let $m, n \in M$ and $k \in \mathbb{N}_+$ with $km = kn$. Then $k(m - n) = 0$ in M^{gp}, and hence $m - n \in M$ as M is saturated. Now the assumption implies that $m - n = 0$, that is $m = n$. If M is sharp, then $km = 0$ for $k \in \mathbb{N}_+$ implies $m \in M^* = \{0\}$. □

Example 3.1.4. Let M be the submonoid of $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by $(1, \overline{0})$ and $(1, \overline{1})$. It is certainly fine as it is a finitely generated submonoid of a group, and its groupification is equal to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus, M is not torsion-free. But if $km = 0$ for $k \in \mathbb{N}_+$ and $m \in M$, then the first coordinate of m is zero, and hence m is zero. By the previous lemma, M cannot be saturated. And indeed, $2 \cdot (0, \overline{1}) = (0, \overline{0}) \in M$, but $(0, \overline{1}) \notin M$.

A monoid is called *toric* if it is fine and saturated and torsion-free. Toric monoids are exactly those which are isomorphic to $\sigma \cap \mathbb{Z}^n$ for a rational polyhedral cone $\sigma \subseteq \mathbb{R}^n$. In fact, for every toric monoid we have $M = \mathbb{R}_{\geq 0} M \cap M^{\mathrm{gp}}$, where $\mathbb{R}_{\geq 0} M \subseteq M^{\mathrm{gp}}_{\mathbb{R}}$ is the rational polyhedral cone spanned by M.

Example 3.1.5. Consider the monoid $M = \langle a, b, c \mid a + b = 2c \rangle$ from Example 3.1.1 again. We saw that it can be realized as the submonoid of $M^{\mathrm{gp}} \cong \mathbb{Z}^2$ generated by $(-1, 2)$, $(1, 0)$, and $(0, 1)$. In particular it is fine and torsion-free. Let $\sigma = \mathrm{cone}\{(-1, 2), (1, 0)\} \subseteq \mathbb{R}^2$ be the rational polyhedral cone generated by M. We know that $\sigma \cap \mathbb{Z}^2$ is toric, and by Gordan's lemma (or rather its proof) it is generated by

$$\{(-1, 2), (1, 0)\} \cup \{\lambda(-1, 2) + \mu(1, 2) \mid \lambda, \mu \in [0, 1)\} \cap \mathbb{Z}^2.$$

This set is equal to $\{(-1, 2), (1, 0), (0, 1), 0\}$, and hence $\sigma \cap \mathbb{Z}^2 = M$ and M is toric.

A possibly empty subset I of a monoid M is called an *ideal* of M if $M + I \subseteq I$. A prime ideal is a proper ideal $I \subsetneq M$ for which $m + n \in I$ implies $m \in I$ or $n \in I$. It is immediate that $I \neq M$ if and only if $I \cap M^* = \emptyset$, so every proper ideal is contained in the complement of M^*. Thus, M has the unique maximal ideal $M \setminus M^*$, and this maximal ideal is prime. There also is a unique minimal prime ideal, the empty set. The *dimension* $\dim(M)$ of M is the supremum over all lengths of chains of prime ideals of M. We denote the set of prime ideals of M by $\operatorname{Spec} M$. If $f\colon M \to N$ is a morphism, and $I \subseteq N$ is an ideal of N, then $f^{-1}I$ is an ideal of M. If I is prime, then so is $f^{-1}I$. Hence, there is an induced map $\operatorname{Spec} f\colon \operatorname{Spec} N \to \operatorname{Spec} M$, making Spec a contravariant functor.

Obviously, prime ideals are in one-to-one correspondence with their complements. These are called the *faces* of M. A subset $F \subseteq M$ is a face if and only if it is nonempty and for all $m, n \in M$ we have $m + n \in F$ if and only if $m, n \in F$. If $\mathfrak{p}$ is a prime ideal of M with corresponding face $F = M \setminus \mathfrak{p}$, we denote $M_\mathfrak{p} = M[-F]$. The terminology "face" is justified by the following result:

Proposition 3.1.6. *Let $\sigma \subseteq \mathbb{R}^n$ be a rational polyhedral cone. Then the set of faces of the monoid $(\sigma, +)$ is equal to the set of faces of σ in the sense of discrete geometry. Furthermore, the natural map* $\operatorname{Spec}(\sigma) \to \operatorname{Spec}(\sigma \cap \mathbb{Z}^n)$ *is bijective.*

Proof. The first statement follows easily from [BG09, Prop. 1.13]. For the second one note that $\operatorname{Spec}(\sigma) \to \operatorname{Spec}(\sigma \cap \mathbb{Z}^n)$ is injective because all faces of σ are rational polyhedral cones and hence determined by their integral points. It is also surjective by [BG09, Prop. 2.36 b)]. □

Example 3.1.7. Let $M = \langle a, b, c | a + b = 2c \rangle$. We have seen in Example 3.1.5 that M is toric and that the cone generated by M can be identified with $\sigma = \operatorname{cone}\{(-1, 2), (1, 0)\}$. The faces of this cone are σ, $\operatorname{cone}\{(-1, 2)\}$, $\operatorname{cone}\{(1, 0)\}$, and $\{0\}$, and hence the faces of M are M, $\mathbb{N}a$, $\mathbb{N}b$, and $\{0\}$. The prime ideals of M are the complements of these faces.

3.1.2 Monoidal Spaces and Kato Fans

Let M be a monoid. As one would expect, the set $\operatorname{Spec} M$ can be made into a topological space by declaring the sets

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} M \mid I \subseteq \mathfrak{p}\}$$

for $I \subseteq M$ as the closed sets. The set $V(I)$ only depends on the ideal generated by I, and for two ideals $I, J \subseteq M$ we have $V(I \cap J) = V(I) \cup V(J)$. Furthermore, we clearly have $\bigcup_{\lambda\in\Lambda} V(I_\lambda) = V(\bigcup_{\lambda\in\Lambda} I_\lambda)$ for any family $(I_\lambda)_{\lambda\in\Lambda}$ of subsets of M. From now on, we will always consider $\operatorname{Spec} M$ with the thus defined topology, the *Zariski topology*. For $m \in M$ we define $D(m) = \operatorname{Spec} M \setminus V(m) = \{\mathfrak{p} \in \operatorname{Spec} M \mid m \notin \mathfrak{p}\}$. These are the so-called *distinguished open subsets* of $\operatorname{Spec} M$, and they form a basis for its topology. If $f\colon M \to N$ is a morphism of monoids and $\varphi = \operatorname{Spec} f$ is the induced map of spectra, then clearly $\varphi^{-1}V(I) = V(f(I))$ and $\varphi^{-1}D(m) = D(f(m))$ for all $I \subseteq M$ and $m \in M$. Thus, $\operatorname{Spec} f$ is continuous and preserves distinguished open subsets.

Proposition 3.1.8. *Let M be a monoid, and let $S \subseteq M$ be a submonoid.*

a) The map $\operatorname{Spec} \overline{M} \to \operatorname{Spec} M$ induced by the quotient map $M \to \overline{M}$ is a homeomorphism.

b) If M is integral, then the map $\operatorname{Spec} M^{\mathrm{sat}} \to \operatorname{Spec} M$ induced by the inclusion $M \to M^{\mathrm{sat}}$ is a homeomorphism.

c) The map $\operatorname{Spec} M[-S] \to \operatorname{Spec} M$ induced by the natural morphism $M \to M[-S]$ is a homeomorphism onto $\{\mathfrak{p} \in \operatorname{Spec} M \mid \mathfrak{p} \cap S = \emptyset\}$. This image is open if S is finitely generated.

Proof. For a prime ideal $\mathfrak{p} \in \operatorname{Spec} M$ we easily verify that its image $\overline{\mathfrak{p}}$ in $\overline{M}$ is a prime ideal in $\operatorname{Spec} M$. This defines a continuous map $\operatorname{Spec} M \to \operatorname{Spec} \overline{M}$ which is inverse to the given map, proving part a). For part b) the inverse map can also be written down directly: it maps $\mathfrak{p}$ to $\mathfrak{p}^{\mathrm{sat}} = \{m \in M^{\mathrm{gp}} \mid \exists k \in \mathbb{N}_+ : km \in \mathfrak{p}\}$ showing that the natural map is bijective. To see that the topology on $\operatorname{Spec} M^{\mathrm{sat}}$ is induced by that on $\operatorname{Spec} M$ it is sufficient to observe that $D(m) = D(km)$ for all $k \in \mathbb{N}_+$. For part c) note that $D(m - s) = D(m)$ for all $m \in M$ and $s \in S$, so the topology on $\operatorname{Spec} M[-S]$ is induced by that on $\operatorname{Spec} M$. The proof of the injectivity and the description of the image of $\operatorname{Spec} M[-S] \to \operatorname{Spec} M$ is completely analogous to that of the corresponding statement in commutative algebra. Finally, if S is finitely generated, say by $s_1, \ldots, s_k$, then the image is equal to $D(s_1 + \ldots + s_k)$ which is open. □

Corollary 3.1.9. *Let M be a fine monoid. Then every maximal chain of prime ideals of M has length $\dim(M) = \operatorname{rk}(M^{\mathrm{gp}}/M^*)$.*

Proof. By Proposition 3.1.8 b) the posets of prime ideals of M and M^{sat} coincide. Since M^* has finite index in $(M^{\mathrm{sat}})^*$, we may thus assume that M is saturated. By Lemma 3.1.2 and Proposition 3.1.8 a) we also may assume M to be sharp. By Lemma 3.1.3 this implies that M is a sharp toric monoid. In this case every maximal chain of prime ideals of M corresponds to a maximal chain of faces of the strongly convex full-dimensional cone $\mathbb{R}_{\geq 0} M \subseteq M^{\mathrm{gp}}_{\mathbb{R}}$ by Proposition 3.1.6. But these all have length $\operatorname{rk} M^{\mathrm{gp}} = \dim_{\mathbb{R}}(M^{\mathrm{gp}}_{\mathbb{R}})$. □

Recall that for two points x, y of a topological space X we say that x *specializes* to y, or y *generalizes* to x, if $y \in \overline{\{x\}}$. We denote this by $x \rightsquigarrow y$. The preorder determined by $\rightsquigarrow$ is an order if and only if X is Kolmogorov (T_0). In this case the closed points of X are exactly those which are maximal with respect to specialization. Closed subsets of X are closed under specialization, and dually open subsets are closed under generalization. In any finite topological space, this preorder actually determines the topology, that is a subset is open if and only if it is closed under generalization (topological spaces with this property are called *Alexandrov spaces*). In such a space X every point x has a smallest neighborhood, the set $V_{\rightsquigarrow x} = \{y \in X \mid y \rightsquigarrow x\}$. This is relevant for us, since $\operatorname{Spec} M$ is finite for every fine monoid M. To see this, note that if $m_1, \ldots, m_k$ are generators of M, and $F \subseteq M$ is a face, then the set $\{m_i \mid m_i \in F\}$ generates F as a submonoid. In $\operatorname{Spec} M$ we have $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$. So if $\mathfrak{m}$

denotes the unique maximal ideal of Spec M, then $V_{\rightsquigarrow \mathfrak{m}} = \operatorname{Spec} M$ is the only open set containing it. In particular, Spec M is quasi-compact.

As in algebraic geometry, we want to consider the topological space Spec M with the additional structure of a structure sheaf. Afterwards, we want to be able to glue these "affine" patches to larger objects, some sort of monoidal schemes. Similarly as the category of schemes is a full subcategory of the category of locally ringed spaces, the category of monoidal schemes should be a full subcategory of the category $(\mathbf{LMS})$ of *locally monoidal spaces*. The objects of $(\mathbf{LMS})$ are pairs $(X, \mathcal{M}_X)$ consisting of a topological space X and a sheaf of monoids $\mathcal{M}_X$ on X. A morphism $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a pair consisting of a continuous map $f\colon X \to Y$ and a morphism of sheaves of monoids $f^\flat\colon f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ such that the induced maps on stalks $f^\flat_x\colon \mathcal{M}_{Y,f(x)} \to \mathcal{M}_{X,x}$ are all local, that is that $(f^\flat_x)^{-1}\mathcal{M}^*_{X,x} = \mathcal{M}^*_{Y,f(x)}$. The composite of two morphisms is defined analogously as in the case of locally ringed spaces. As suggested by the notation, we will usually omit the structure sheaves and the maps between structure sheaves when referring to locally monoidal spaces and morphisms between them. The objects we will be most interested in are actually living in the full subcategory of $(\mathbf{LMS})$ consisting of *sharp monoidal spaces*, that is spaces X such that $\Gamma(U, \mathcal{M}_X)$ is sharp for all open subsets $U \subseteq X$. We denote this category by $(\mathbf{SMS})$. For a locally monoidal space X the space $\overline{X} = (X, \overline{\mathcal{M}}_X)$, with $\overline{\mathcal{M}}_X$ the sheafification of $U \mapsto \overline{\Gamma(U, \mathcal{M}_X)}$, is a sharp monoidal space, and it has a natural morphism to X. It follows immediately from the universal property of the sharpening of monoids that this morphism is universal among all morphisms from sharp monoidal spaces to X. In other words, the sharpening functor $\overline{(-)}\colon (\mathbf{LMS}) \to (\mathbf{SMS})$ is right adjoint to the inclusion.

Remark 3.1.10. Unlike in the case of ringed spaces, a topological space with a sheaf of monoids does not have to satisfy any additional requirements to be a locally monoidal space. The reason for this is that all monoids are automatically local in the sense that they have a unique maximal ideal. However, not all morphisms of monoidal spaces are morphisms of locally monoidal spaces.

Now let M be a monoid. In [Dei05], Deitmar introduces a sheaf of monoids $\mathcal{M}$ on Spec M by defining $\Gamma(U, \mathcal{M})$ to be the monoid of elements $(m_\mathfrak{p}) \in \prod_{\mathfrak{p} \in U} M_\mathfrak{p}$ that are locally differences of elements of M. More precisely, for every $\mathfrak{q} \in U$ there has to exist an open neighborhood $\mathfrak{q} \in V \subseteq U$ and $m, s \in M$ such that $V \subseteq D(s)$ and $m_\mathfrak{p} = m - s$ for all $\mathfrak{p} \in V$. We denote the resulting locally monoidal space by $\operatorname{Spec}_D M$.

Proposition 3.1.11 ([Dei05, Prop. 2.1])**.** *Let M be a monoid.*

a) For every point $\mathfrak{p} \in \operatorname{Spec}_D M$ we have $\mathcal{M}_{\operatorname{Spec}_D M, \mathfrak{p}} \cong M_\mathfrak{p}$.

b) For every $m \in M$ we have $\Gamma(D(m), \mathcal{M}_{\operatorname{Spec}_D M}) \cong M_m$

In particular, $\Gamma(\operatorname{Spec}_D M, \mathcal{M}_{\operatorname{Spec}_D M}) = M$.

Proof. The natural morphism $M \to \Gamma(\operatorname{Spec}_D M, \mathcal{M}_{\operatorname{Spec}_D M}) \to \mathcal{M}_{\operatorname{Spec}_D M, \mathfrak{p}}$ sends $M \setminus \mathfrak{p}$ to units and hence induces a morphism $M_\mathfrak{p} \to \mathcal{M}_{\operatorname{Spec}_D M, \mathfrak{p}}$ which is easily seen to be an isomorphism. For part b) we use Proposition 3.1.8 c) to see that $D(m)$ has a unique maximal point $\mathfrak{q}$, which is the inverse image of the maximal ideal of M_m under $M \to M_m$. Thus, $\Gamma(\operatorname{Spec}_D M, \mathcal{M}_{\operatorname{Spec}_D M}) \cong \mathcal{M}_{\operatorname{Spec}_D M, \mathfrak{q}} \cong M_{\mathfrak{q}}$. The latter is naturally isomorphic to M_m (the isomorphism is induced by $M \to M_m$). For the "in particular" statement we use b) with $m = 0$. □

The preceding proposition showed that $\operatorname{Spec}_D M$ is a locally monoidal space whose monoid of global sections is equal to M. As one would expect from algebraic geometry, it is universal with this property.

Proposition 3.1.12 ([Dei05, Prop. 2.5]). *Let X be a locally monoidal space. Then there is a natural bijection*

$$\operatorname{Mor}_{(\mathbf{LMS})}(X, \operatorname{Spec}_D M) \cong \operatorname{Hom}(M, \Gamma(X, \mathcal{M}_X))$$

In particular, if M and N are monoids, then the morphisms $\operatorname{Spec}_D N \to \operatorname{Spec}_D M$ are in one-to-one correspondences with morphisms $M \to N$ of monoids.

Proof. To define the map from left to right, observe that a morphism $f \colon X \to \operatorname{Spec}_D M$ induces a homomorphism

$$M \to \Gamma(\operatorname{Spec}_D M, \mathcal{M}_{\operatorname{Spec}_D M}) \xrightarrow{f^\flat} \Gamma(X, \mathcal{M}_X)$$

which we denote by φ. We will show how to reconstruct f from φ, thus providing a construction for an inverse map. Let $x \in X$, and let $\mathfrak{p} = f(x)$. Now consider the commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\cong} & \Gamma(\operatorname{Spec}_D M, \mathcal{M}_{\operatorname{Spec}_D M}) & \xrightarrow{f^\flat} & \Gamma(X, \mathcal{M}_X) \\ \downarrow & & \downarrow & & \downarrow \\ M_{\mathfrak{p}} & \xrightarrow{\cong} & \mathcal{M}_{\operatorname{Spec}_D M, \mathfrak{p}} & \xrightarrow{f^\flat_x} & \mathcal{M}_{X,x}. \end{array}$$

The composite of the upper horizontal maps is φ. As $f^\flat_x$ is local, we see that $\mathfrak{p}$ is the inverse image of the maximal ideal under the composite $M \xrightarrow{\varphi} \Gamma(X, \mathcal{M}_X) \to \mathcal{M}_{X,x}$. Thus, the underlying map of f is uniquely determined by φ. Because the horizontal maps in the left square of the diagram are isomorphisms, the diagram also shows how to reconstruct $f^\flat_x$. Since $f^\flat$ is uniquely determined by the collection $(f^\flat_x)_{x \in X}$, we see that f can be reconstructed from φ. It is easily checked that this construction produces a morphism $X \to \operatorname{Spec}_D M$ for every morphism $\varphi \colon M \to \Gamma(X, \mathcal{M}_X)$. The "in particular" statement follows immediately. □

We have now seen that the locally monoidal spaces of the form $\operatorname{Spec}_D M$ for a monoid M are analogous to affine schemes in algebraic geometry in many ways. One could now carry on and use them as local models for a theory of monoidal schemes, and in fact this is what Deitmar did in [Dei05], where he developed his approach to $\mathbb{F}_1$-geometry. However, as mentioned earlier we are more interested in sharp monoidal spaces, and hence want to assign a sharp monoidal space to a monoid M. The natural way to do this is to take the sharpening of $\operatorname{Spec}_D M$. So from here on, we will use $\operatorname{Spec} M$ to denote the sharp monoidal space $\overline{(\operatorname{Spec}_D M)}$. By the preceding results and the universal property of the sharpening we obtain the following corollary.

Corollary 3.1.13. *Let M be a monoid.*

a) For every prime ideal $\mathfrak{p} \in \operatorname{Spec} M$ we have $\mathcal{M}_{\operatorname{Spec} M,\mathfrak{p}} \cong \overline{M_\mathfrak{p}}$.

b) For every $f \in M$ we have $\Gamma(D(f), \mathcal{M}_{\operatorname{Spec} M}) \cong \overline{M_f}$.

c) For every sharp monoidal space X there is a natural bijection

$$\operatorname{Mor}_{(\mathbf{SMS})}(X, \operatorname{Spec} M) \cong \operatorname{Hom}(M, \Gamma(X, \mathcal{M}_X)).$$

Definition 3.1.14. A sharp monoidal space isomorphic to $\operatorname{Spec} M$ for some monoid M is called an *affine Kato fan*. A *Kato fan* is a sharp monoidal space F which can be covered by affine Kato fans. If all of these are isomorphic to $\operatorname{Spec} M$ for a finitely generated monoid we say that F is *locally of finite type*. If furthermore, only finitely many suffice, we say X is of *finite type*. If F can be covered by spectra of integral monoids we say that it is *integral*. If the monoids can even be chosen to be saturated F said to be *saturated*. We call F *fine* if it is of finite type and integral. We will denote the category of Kato fans by $(\mathbf{Fan})$ and its full subcategory of fine and saturated Kato fans by $(\mathbf{fs\text{-}Fan})$.

Remark 3.1.15. Note that unlike for schemes, a Kato fan F for which $\Gamma(U, \mathcal{M}_F)$ is integral for all open subsets $U \subseteq F$ does not need to be connected. The reason for this different behavior is that products of integral monoids are integral again, while products of integral rings are certainly not.

Example 3.1.16.

a) The affine Kato fan $\operatorname{Spec} \mathbb{N}$ consists of the two points $\emptyset$ and $\mathbb{N}_+$. The stalk of $\mathcal{M}_{\operatorname{Spec} \mathbb{N}}$ at $\emptyset$ is equal to 0, so the open subfan $\{\emptyset\}$ is isomorphic to $\operatorname{Spec} 0$. Thus, when gluing copies of $\operatorname{Spec} \mathbb{N}$ along $\{\emptyset\}$ the gluing data is always trivial. Therefore, for every $n \in \mathbb{N}$ we obtain a fine and saturated Kato fan by gluing n copies of $\operatorname{Spec} \mathbb{N}$ along $\{\emptyset\}$. It is depicted in Figure 3.1 on the left.

b) The affine Kato fan $\operatorname{Spec} \mathbb{N}^2$ consists of the four points $\emptyset$, $\mathbb{N}^2 \setminus \mathbb{N}e_1$, $\mathbb{N}^2 \setminus \mathbb{N}e_2$, and $\mathbb{N}^2 \setminus \{0\}$. The two open subsets $D(e_i)$ are isomorphic to $\operatorname{Spec} \mathbb{N}$. Because $\mathbb{N}$ has no nontrivial automorphism, when gluing several copies of $\operatorname{Spec} \mathbb{N}^2$ along the $D(e_i)$ the gluing data is always trivial. Figure 3.1 contains a picture of a Kato fan obtained by gluing three copies of $\operatorname{Spec} \mathbb{N}^2$ in a way analogous to the construction of $\mathbb{P}^2$.

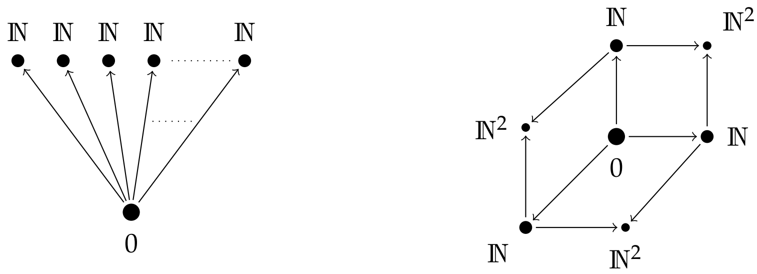


Figure 3.1: Visualizations of Kato fans (as in [Uli13]) arrows indicate specialization, monoids at points are the local monoids.

Here are some properties of Kato fans.

Proposition 3.1.17. *Let F be a Kato fan.*

a) *The underlying topological space of F is sober, that is every closed irreducible subset has a unique generic point.*

b) *The irreducible components of F are equal to its connected components. Their generic points are those points $x \in F$ with $\mathcal{M}_{F,x} = 0$.*

c) *If $F \neq \emptyset$, and G is a connected Kato fan, then every morphism $F \to G$ is dominant, that is has dense image in G.*

d) *The fan F is locally of finite type if and only if all local monoids $\mathcal{M}_{F,x}$ are finitely generated. It is of finite type if and only if it is of locally finite type and quasi-compact.*

e) *The fan F is integral (saturated) if and only if all local monoids are integral (saturated).*

Proof. For part a) let $A \subseteq F$ be a closed irreducible subset, and let $U = \operatorname{Spec} M$ be an open affine subset of F with $U \cap A \neq \emptyset$. Then the generic points of A are in bijection with the generic points of $U \cap A$, and thus we may assume that F is affine, equal to $\operatorname{Spec} M$. Suppose $\mathfrak{p}$ and $\mathfrak{q}$ are generic points of A. Then $\mathfrak{p} \in \overline{\{\mathfrak{q}\}} = V(\mathfrak{q})$ and $\mathfrak{q} \in \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. It follows that $\mathfrak{p} = \mathfrak{q}$, so there is at most one generic point. For the existence let $\mathfrak{p} = \bigcap_{\mathfrak{q} \in A} \mathfrak{q}$. Then we clearly have $A = V(\mathfrak{p})$ and it is left to show that $\mathfrak{p}$ is prime. Suppose $m, n \in M$ with $m + n \in \mathfrak{p}$. Then $A = V(\mathfrak{p}) \subseteq V(m+n) = V(m) \cup V(n)$, and therefore $A \subseteq V(m)$ or $A \subseteq V(n)$. Without loss of generality assume the former. Let $\mathfrak{m}$ be the preimage of the maximal ideal of $M[-m]$ in M. Then $\mathfrak{m} \notin V(\mathfrak{p})$ and hence there exists $p \in \mathfrak{p} \setminus \mathfrak{m}$. This p goes to a unit in $M[-m]$ so that there is $q \in M$ and $k \in \mathbb{N}_+$ such that $km = p + q \in \mathfrak{p}$. Since $\mathfrak{p}$ is an intersection of prime ideals, this implies $m \in \mathfrak{p}$, and hence that $\mathfrak{p}$ is prime.

For part b) first note that affine Kato fans are irreducible, and therefore F is locally irreducible. So it suffices to show that a connected Kato fan F is irreducible, the generic point being the unique $x \in X$ with $\mathcal{M}_{F,x} = 0$. Let $U = \operatorname{Spec} M$ be an open affine subset of F. It is irreducible, and its generic point x corresponds to the prime ideal $\emptyset$ of M, so $\mathcal{M}_{F,x} = \overline{M^{\mathrm{gp}}} = 0$. We claim that $F = \overline{\{x\}}$. Suppose that $\partial\overline{\{x\}}$ is nonempty, and let V be an affine open neighborhood of one of its elements. Then $U \cap V$ is a nonempty open subset of both U and V. Thus, the generic points of U and V coincide, showing that $V \subseteq \overline{\{x\}}$, a contradiction. We see that $F \setminus \overline{\{x\}}$ is closed, and hence empty by the connectedness of F.

Now let G be a connected Kato fan, $f\colon F \to G$ a morphism, and $x \in F$ a point with $\mathcal{M}_{F,x} = 0$. Since $f_x^\flat$ is local, we have $\mathcal{M}_{G,f(x)} = 0$. By what we just saw this means that the generic point of G is in the image of f, and hence that $f(F)$ is dense in G, proving c).

For part d) and e) it is sufficient to note that finitely generated, integral, and saturated are properties which are preserved by localization, and that every monoid is equal to its localization at its maximal ideal. □

Recall that the *(Krull) dimension* $\dim(X)$ of a topological space X is the supremum over all lengths of chains of irreducible closed subsets of X. If X is sober, then chains of irreducible closed subsets correspond to chains in the specialization poset so that $\dim(X)$ is the height (maximal length of a chain) of the poset $(X, \rightsquigarrow)$. The *codimension* of an irreducible closed subset $Y \subseteq X$ is the maximal length of a chain of irreducible closed subsets of X which contain Y. We denote it by $\operatorname{codim}(Y, X)$. For $x \in X$ we will use the notation $\operatorname{codim}(x, X) = \operatorname{codim}(\overline{\{x\}}, X)$.

Proposition 3.1.18. *Let F be a Kato fan, let $x \in F$, and let $\iota\colon V_{\rightsquigarrow x} \to F$ be the inclusion. Then there is a natural isomorphism*

$$(V_{\rightsquigarrow x}, \iota^{-1}\mathcal{M}_F) \cong \operatorname{Spec} \mathcal{M}_{F,x}.$$

In particular, we have $\operatorname{codim}(x, F) = \dim(\mathcal{M}_{F,x})$.

Proof. Let $U = \operatorname{Spec} M$ be an affine open neighborhood of x, and let $\mathfrak{p}$ be the prime ideal of M corresponding to x. Then by Proposition 3.1.13 there is a natural isomorphism $\mathcal{M}_{F,x} \cong \overline{M_{\mathfrak{p}}}$, giving rise to a natural morphism

$$\operatorname{Spec} \mathcal{M}_{F,x} \to \operatorname{Spec} \overline{M_{\mathfrak{p}}} \to \operatorname{Spec} M \to F.$$

By Proposition 3.1.8, this morphism is a homeomorphism onto $V_{\rightsquigarrow x}$. To finish the proof it suffices to show that

$$\overline{M_{\mathfrak{q}^c}} = \mathcal{M}_{\operatorname{Spec} M, \mathfrak{q}^c} \to \mathcal{M}_{\operatorname{Spec} \overline{M_{\mathfrak{p}}}, \mathfrak{q}} = \overline{(\overline{M_{\mathfrak{p}}})_{\mathfrak{q}}} \tag{3.1.1}$$

is an isomorphism for all $\mathfrak{q} \in \operatorname{Spec} \overline{M_{\mathfrak{p}}}$, where we denote the image of $\mathfrak{q}$ in $\operatorname{Spec} M$ by $\mathfrak{q}^c$. Let $G = M \setminus \mathfrak{p}$, and $H = M \setminus \mathfrak{q}^c$. Then

$$\overline{M_{\mathfrak{q}^c}} = \overline{M[-H]} \qquad \text{and} \qquad \overline{(\overline{M_{\mathfrak{p}}})_{\mathfrak{q}}} = \overline{\overline{M[-G]}[-H]}.$$

Using the universal properties of sharpenings and localizations, we see that the canonical morphisms from M into these monoids have the same universal property:

they are universal among all morphisms from M to sharp monoids that map H to 0. Thus, the two monoids are canonically isomorphic, the isomorphism being that of (3.1.1). Since the codimension of x in F is the height of the poset $(V, \rightsquigarrow)$, and specialization is equivalent to inclusion on $\operatorname{Spec} \mathcal{M}_{F,x}$, the "in particular" statement follows. □

Fine and saturated Kato fans are closely related to cone complexes. In fact, the categories (**fs-Fan**) and (**CC**) are equivalent. Before showing this, let us make the following definition:

Definition 3.1.19. Let F and G be Kato fans. Then the set of *G-valued points of F*, denoted by $F(G)$ is the set $\mathrm{Mor}_{(\mathbf{Fan})}(G, F)$ of morphisms from F to G. For a monoid M the $(\operatorname{Spec} M)$-valued points are also called *M-valued points* and we abbreviate $F(\operatorname{Spec} M)$ to $F(M)$.

Construction 3.1.20 ([Uli13, Prop. 3.8])**.** Let F be a fine and saturated Kato fan. We construct a *cone complex* $\Sigma_F = \{\sigma_x \mid x \in F\}$ *associated to* F. The underlying set will be $|\Sigma_F| = F(\mathbb{R}_{\geq 0})$. There is a natural map $r\colon |\Sigma_F| \to F$ mapping a morphism $\operatorname{Spec}(\mathbb{R}_{\geq 0}) \to F$ to the image of $\mathbb{R}_{>0}$. If $x \in F$ is a point of F, then $V_{\rightsquigarrow x} = \operatorname{Spec} \mathcal{M}_{F,x}$ is open in F, and $\sigma_x := r^{-1} V_{\rightsquigarrow x}$ is naturally identified with $\operatorname{Hom}(\mathcal{M}_{F,x}, \mathbb{R}_{\geq 0})$. Taking the topology of pointwise convergence on σ_x, and considering $\mathcal{M}_{F,x}^{\mathrm{gp}}$ as a lattice of continuous real-valued functions on σ_x, we obtain a cone in the sense of Subsection 2.1.1. To see this, note that σ_x injects into $\operatorname{Hom}(\mathcal{M}_{F,x}^{\mathrm{gp}}, \mathbb{R})$, that is the dual space of $(\mathcal{M}_{F,x}^{\mathrm{gp}})_{\mathbb{R}}$, and that an element of this dual space is contained in σ_x if and only if it is nonnegative on the cone $\mathbb{R}_{\geq 0}\mathcal{M}_{F,x}$. Therefore, we have $\sigma_x = (\mathbb{R}_{\geq 0}\mathcal{M}_{F,x})^{\vee}$. The faces of σ_x are in one-to-one correspondence with the faces of $\mathbb{R}_{\geq 0}\mathcal{M}_{F,x}$ [CLS11, Prop. 1.2.10], which are in one-to-one correspondence with the prime ideals of $\mathcal{M}_{F,x}$ by Proposition 3.1.6. These in turn correspond to points of $V_{\rightsquigarrow x}$. Let y be such a point, then there exists $m \in \mathcal{M}_{F,x}$ such that $V_{\rightsquigarrow y} = \operatorname{Spec} \overline{M[-m]}$. We conclude that the inclusion $\sigma_y \to \sigma_x$ is the face map

$$\operatorname{Hom}(\overline{\mathcal{M}_{F,y}[-m]}, \mathbb{R}_{\geq 0}) \to \operatorname{Hom}(\mathcal{M}_{F,x}, \mathbb{R}_{\geq 0})$$

onto the face of σ_x where m vanishes. It follows that the coarsest topology on $|\Sigma_F|$ such that all inclusions $\sigma_x \to |\Sigma_F|$ for $x \in F$ are continuous, together with the collection of cones $\{\sigma_x\}_{x \in F}$, defines a cone complex Σ_F. Note that $r^{-1}\{x\}$ is equal to the relative interior of σ_x.

If $f\colon F \to G$ is a morphism, then the canonical map $F(\mathbb{R}_{\geq 0}) \to G(\mathbb{R}_{\geq 0})$ clearly defines a morphism of cone complexes $\Sigma_f\colon \Sigma_F \to \Sigma_G$. Obviously, this defines a functor

$$\Sigma\colon (\mathbf{fs\text{-}Fan}) \to (\mathbf{CC}),$$

where (**CC**) denotes the category of cone complexes. Note that for $x \in F$ the fact that $f_x^\flat$ is local implies that $\sigma_{f(x)}$ is the smallest cone of Σ_G containing $\Sigma_f(\sigma_x)$. ◊

Example 3.1.21. Let F be the fine and saturated Kato fan consisting of two affines $U_{1/2} = \operatorname{Spec} \mathbb{N}^2$ glued along the open subsets $D(e_1) = \operatorname{Spec} \mathbb{N}$ of U_1 and U_2, where e_1 and e_2 are the generators of $\mathbb{N}^2$. Let $G = \operatorname{Spec} \mathbb{N}^2$. Furthermore, let $f\colon F \to G$ be the morphism given by

$$\begin{aligned} \mathbb{N}^2 &\to \mathbb{N}^2, \\ e_1 &\mapsto e_1 + e_2 \\ e_2 &\mapsto e_2 \end{aligned}$$

on U_1, and by

$$\begin{aligned} \mathbb{N}^2 &\to \mathbb{N}^2, \\ e_1 &\mapsto e_2 \\ e_2 &\mapsto e_1 + e_2 \end{aligned}$$

on U_2. We observe that this is really well-defined; the function e_1 vanishes on $D(e_1)$, so the restriction of $e_1 + e_2$ to $D(e_1)$ is equal to the restriction of e_2. Let x denote the codimension-1 point of $U_1 \cap U_2$, and let y_i be the other codimension-1 point of $U_i \setminus V_{\rightsquigarrow x}$. To compute $f(y_1)$ we have to use the first chart. The point y_1 is given by the ideal $\langle e_1 \rangle$ generated by e_1. Because $\lambda \cdot (e_1 + e_2) + \mu \cdot e_2$ is contained in $\langle e_1 \rangle$ if and only $\lambda \neq 0$, we conclude that $f(y_1) = \langle e_1 \rangle$. Similarly, we see that $f(y_2) = \langle e_2 \rangle$ and that $f(x) = \langle e_1, e_2 \rangle$. From this it already follows that the closed points of F are also mapped to $\langle e_1, e_2 \rangle$. This is illustrated in the upper part of Figure 3.2. The lower part shows the induced morphism Σ_f of cone complexes. Since F consists of two copies of $\operatorname{Spec} \mathbb{N}^2$ glued along $\operatorname{Spec} \mathbb{N}$, it is clear that Σ_F is consists of two copies of $\mathbb{R}^2_{\geq 0}$ glued along one of their rays. To compute Σ_f, we have to dualize the description of f from above. We see that a point $(a, b) \in \mathbb{R}^2_{\geq 0} = U_1(\mathbb{R}_{\geq 0})$ is mapped to $(a + b, b)$, and a point $(a, b) \in \mathbb{R}^2_{\geq 0} = U_2(\mathbb{R}_{\geq 0})$ is mapped to $(b, a + b)$. In particular Σ_f coincides with the morphism depicted in Figure 3.2.

Proposition 3.1.22 ([Uli13, Prop. 3.8])**.** *The functor Σ from Construction 3.1.20 is an equivalence of categories.*

Proof. We construct an inverse functor as follows. Let Σ be a cone complex. Let F_Σ be the set of cones of Σ, equipped with the Alexandrov topology associated to the inclusion, that is the topology generated by $\{\tau \in \Sigma \mid \tau \leq \sigma\}$ for $\sigma \in \Sigma$. We also define a sheaf on monoids $\mathcal{M}_{F_\Sigma}$ on F_Σ by setting

$$\Gamma(U, \mathcal{M}_{F_\Sigma}) = \left\{ (m_\sigma) \in \prod_{\sigma \in U} M_+^\sigma \;\middle|\; m_\tau = m_\sigma|_\tau \text{ whenever } \tau \subseteq \sigma \right\}$$

for an open subset U. Clearly, $V_{\rightsquigarrow\sigma} \cong \operatorname{Spec} M_+^\sigma$ for all $\sigma \in \Sigma$ so that the resulting sharp monoidal space is a fine and saturated Kato fan. It follows immediately from the definition of morphisms of cone complexes that this construction is functorial.

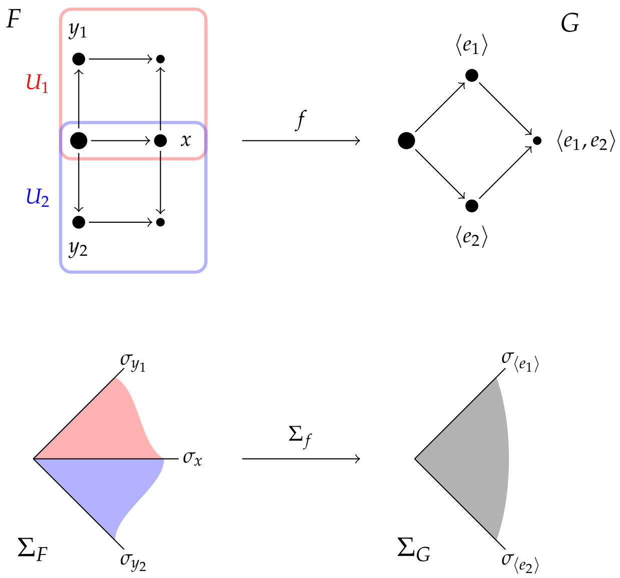


Figure 3.2: The morphism f and the induced morphism Σ_f of cone complexes.

It is left to show that the resulting functor is inverse to Σ. First note that $F \to F_{\Sigma_F}$, $x \to \sigma_x$ is a homeomorphism by construction, and with the natural identifications $\mathcal{M}_{F_{\Sigma_F}, \sigma_x} = M_+^{\sigma_x} = \mathcal{M}_{F,x}$ it even becomes an isomorphism of Kato fans which is functorial by construction. Conversely, if Σ is a cone complex, then for $\sigma \in \Sigma$ we have a canonical isomorphism

$$\sigma \to \mathrm{Hom}(M_+^\sigma, \mathbb{R}_{\geq 0}) = \mathrm{Hom}(\mathcal{M}_{F_\Sigma, \sigma}, \mathbb{R}_{\geq 0}) = \sigma_\sigma$$

of cones. These isomorphisms glue to an isomorphism $\Sigma \to \Sigma_{F_\Sigma}$, which is clearly functorial. □

Let F be a connected Kato fan and let $Z \subseteq F$ be an irreducible closed subset. One might expect that there is an analogue of the induced reduced structure known for schemes, that is a naturally defined sheaf of monoids on Z such that the inclusion $Z \to F$ is a closed embedding in an appropriate sense. Unfortunately, this cannot work in general. If the inclusion $Z \to F$ was a morphism of Kato fans, it would be dominant by Proposition 3.1.17 c), thus forcing $Z = F$. However, there still is a canonically defined sheaf of monoids on Z making it a Kato fan again. For fine and saturated Kato fans this structure on Z corresponds to the star construction for cone complexes introduced in section 2.1.1.

Construction 3.1.23. Let F be a Kato fan, let $x \in F$, and let $\iota \colon \overline{\{x\}} \to F$ be the inclusion. We will construct a Kato fan $F(x)$ with underlying set $\overline{\{x\}}$. Let $\mathcal{M}_{F(x)}$ be the sheaf of sharp monoids on $F(x) := \overline{\{x\}}$ with

$$\Gamma(U, \mathcal{M}_{F(x)}) = \{f \in \Gamma(U, \iota^{-1}\mathcal{M}_F) \mid f_x = 0\}$$

for $U \subseteq F(x)$ open. We claim that $(F(x), \mathcal{M}_{F(x)})$ is a Kato fan. To see this, we may assume that F is affine, say $F = \operatorname{Spec} M$, and x corresponds to a face G of M. Then we have $\Gamma(F(x), \mathcal{M}_{F(x)}) = G$. We need to show that the induced morphism $F(x) \to \operatorname{Spec} G$ is an isomorphism. First note that for a point $y \in F(x)$ corresponding to a face H of M we have $\mathcal{M}_{F,y} = \overline{M[-H]}$ by Corollary 3.1.13 a). It follows directly from the definition of $\mathcal{M}_F$ that G is the preimage of $\mathcal{M}_{F(x),y} \subseteq \mathcal{M}_{F,y}$ in M, and hence that $\mathcal{M}_{F(x),y} = \overline{G[-H]}$. Since H also is a face of G, we see that y is mapped to the point of $\operatorname{Spec} G$ corresponding to H, and that the morphism is an isomorphism on stalks. We also see that the corresponding map of faces maps a face H of M which is contained in G to the face H of G. Because faces of G bijectively correspond to faces of M contained in G, we see that the map is a bijection. Finally, the image of $D(m) \cap F(x)$ is equal to $D(m)$ if $m \in G$, and empty else, so the map also is a homeomorphism. ♢

Proposition 3.1.24. *Let F be a fine and saturated Kato fan, and let $x \in F$. Then*

$$\mathrm{S}_{\Sigma_F}(\sigma_x) = \Sigma_{F(x)},$$

where $\mathrm{S}_{\Sigma_F}(\sigma_x)$ denotes the star of Σ_F at σ_x (cf. Subsection 2.1.1).

Proof. The cones of $\Sigma_{F(x)}$ are in one-to-one correspondence with the elements of $F(x)$, which in turn are in one-to-one correspondence to the cones of $\mathrm{S}(\sigma_x)$. Of course, this correspondence is order-preserving with respect to inclusion. So it suffices to show that σ_y^F/σ_x^F is naturally isomorphic to $\sigma_y^{F(x)}$ for every $y \in F(x)$, where we write σ_z^G for the cone of Σ_G corresponding to a point z of a Kato fan G. By the construction of Σ_F, the elements of $\mathcal{M}_{F,y}$ are the nonnegative integral linear functions on σ_y^F, so the elements of $\mathcal{M}_{F(x),y}$ are the nonnegative integral linear functions on σ_y^F that vanish on σ_x^F. Every such function induces a nonnegative integral linear function on σ_y^F/σ_x^F, yielding a morphism $\mathcal{M}_{F(x),y} \to M_+^{\sigma_y^F/\sigma_x^F}$. Because every element of $M_+^{\sigma_y^F/\sigma_x^F}$ is induced by its pull-back to σ_y, which automatically vanishes on σ_x^F, we see that this morphism is an isomorphism. Therefore, we have

$$\sigma_y^{F(x)} = \operatorname{Hom}(\mathcal{M}_{F(x),y}, \mathbb{R}_{\geq 0}) = \operatorname{Hom}(M_+^{\sigma_y^F/\sigma_x^F}, \mathbb{R}_{\geq 0}) = \sigma_y^F/\sigma_x^F$$

as required. □

Remark 3.1.25. We have already seen that the inclusion $F(x) \to F$ cannot be made a morphism of Kato fans. However, there are slight modifications $F(x)_\circ$ and $F_\circ$ of $F(x)$ and F, respectively, which have the same underlying topological spaces and are still sharp monoidal spaces, such that the inclusion is naturally a morphism $F(x)_\circ \to F_\circ$

of sharp monoidal spaces. The construction requires absorbing elements. Recall that an element $\infty \in M$ of a monoid M is called an absorbing element if $\infty + M = \{\infty\}$. A monoid is called pointed if it has an absorbing element, which then has to be unique. Every monoid M has a natural embedding into a pointed monoid $M_\circ$: just take $M_\circ = M \cup \{\infty\}$ and define $\infty + m = \infty$ for all $m \in M_\circ$. Every face F of M is also a face of $M_\circ$ and there is a natural morphism $M_\circ \to F_\circ$ which is the identity on $F_\circ$ and maps $M_\circ \setminus F$ to ∞. Of course, this also works for sheaves of monoids so that for every sharp monoidal space X there is a sharp monoidal space $X_\circ$ with the same underlying topological space as X and whose structure sheaf is the sheafification of the presheaf which satisfies

$$\Gamma(U, \mathcal{M}_{X_\circ}) = \Gamma(U, \mathcal{M}_X)_\circ$$

for all open subsets $U \subset X$.

In the situation where F is a connected Kato fan and $x \in F$, the local monoid $\mathcal{M}_{F(x),y}$ is a face of $\mathcal{M}_{F,y}$ for all $y \in F(x)$ by definition. Thus, there are natural morphisms

$$\mathcal{M}_{F_\circ,y} = (\mathcal{M}_{F,y})_\circ \to (\mathcal{M}_{F(x),y})_\circ = \mathcal{M}_{F(x)_\circ,y}$$

for all $y \in F(x)$ which define a morphism $F(x)_\circ \to F_\circ$ of sharp monoidal spaces. Clearly, $F_\circ$ is not a Kato fan anymore. But it is close: adding one additional point η to $F_\circ$, making η the generic point, and extending the structure sheaf to $F_\circ \cup \{\eta\}$ such that the stalk at η is 0 actually yields a Kato fan. To see this it suffices to check that for every monoid M we have $\operatorname{Spec}(M_\circ) = \operatorname{Spec} M \cup \{\emptyset\}$ as sets, where we identify a prime ideal $\mathfrak{p}$ of M with the prime ideal $\mathfrak{p} \cup \{\infty\}$ of $M_\circ$, and

$$(\operatorname{Spec} M)_\circ = (\operatorname{Spec} M, \mathcal{M}_{\operatorname{Spec} M_\circ}|_{\operatorname{Spec} M}).$$

We refer to [CC10] and [CHWW15] for a more detailed treatment of locally monoidal spaces which locally look like the spectrum of a pointed monoid.

Before introducing weak embeddings for Kato fans, let us prove a result that makes it very easy to specify sections of sheaves on Kato fans. Let X be a topological space, and let $\mathcal{S}$ be a sheaf of sets on X. By the sheaf property, every section $s \in \Gamma(X, \mathcal{S})$ is uniquely determined by its family of germs $(s_x)_{x \in X}$, that is by the corresponding section of $Y = \prod_{x \in X} \mathcal{S}_x \to X$. Recall that Y, together with the topology generated by $\{(s_u)_{u \in U} \mid s \in \Gamma(U, \mathcal{S})\}$ is called the espace étalé of X, and the sections of Y belonging to $\mathcal{S}$ are exactly the continuous ones. The sheaf on X of all sections of Y, into which $\mathcal{S}$ canonically embeds, is therefore called the sheaf of discontinuous sections. If X is the underlying space of a Kato fan, continuity can be checked easily, namely via cospecialization. We recall that the *cospecialization* map $\mathcal{S}_y \to \mathcal{S}_x$ for two points $x, y \in X$ with $x \rightsquigarrow y$ sends the germ s_y of a section $s \in \Gamma(U, \mathcal{S})$ defined in a neighborhood U of y to the germ s_x.

Lemma 3.1.26. *Let F be a Kato fan, and let $\mathcal{S}$ be a sheaf of sets on F. Then a discontinuous section $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{S}_x$ on an open subset $U \subseteq F$ is continuous, that is belongs to $\Gamma(U, \mathcal{S})$ if and only if s_x cospecializes to s_y whenever $y \rightsquigarrow x$.*

Proof. Clearly, every section of $\mathcal{S}$ respects cospecialization. For the converse let $U \subseteq F$ open, let $s = (s_x)_{x \in U} \in \prod_{x \in U} \mathcal{S}_x$ be a family respecting cospecialization, and let $V = \operatorname{Spec} M$ be an affine open subset of U. Then V has a largest point with respect to specialization, say z, and $\Gamma(V, \mathcal{S}) = \mathcal{S}_z$. Therefore, s_z defines an element of $\Gamma(V, \mathcal{S})$. Sections of $\Gamma(V, \mathcal{S})$ respect cospecialization, so the corresponding discontinuous section is $(s_x)_{x \in V} = (s_x)_{x \in U}|_V$. We see that s is in fact continuous, that is an element of $\mathcal{S}$. □

Remark 3.1.27. Note that the proof of the preceding proposition did only need the purely topological property that every point of F has a neighborhood basis of open sets having a largest element with respect to specialization. In particular, the result holds for any Alexandrov space.

3.1.3 Weakly Embedded Kato Fans

We have seen in Proposition 3.1.22 that fine and saturated Kato fans are nothing but algebro-geometric descriptions of cone complexes. We would like to extend this analogy to include weak embeddings, that is we want to define weak embeddings for Kato fans in a way such that fine and saturated weakly embedded Kato fans correspond to weakly embedded cone complex. Let F be a fine and saturated Kato fan. A weak embedding for Σ_F is comprised of a lattice N^F and a map $|\Sigma_F| \to N^F_{\mathbb{R}}$, which is integral linear on all cones of Σ_F. Taking coordinates on N^F, we see that this data is equivalent to a morphism $M^F = \operatorname{Hom}(N^F, \mathbb{Z}) \to \operatorname{Div}(\Sigma_F)$. The following more algebraic description of $\operatorname{Div}(\Sigma_F)$ leads naturally to a definition of weak embeddings for Kato fans.

Proposition 3.1.28. *Let F be a fine and saturated Kato fan. Then we have*

$$\operatorname{Div}(\Sigma_F) = \Gamma(F, \mathcal{M}_F^{\mathrm{gp}}).$$

Proof. A divisor on Σ_F is determined by its restrictions to the cones of Σ_F. Such a collection $(\varphi_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_F} M^\sigma$ defines an element of $\operatorname{Div}(\Sigma_F)$ if and only if $\varphi_\sigma|_\tau = \varphi_\tau$ whenever $\tau \le \sigma$. By construction, for $x \in F$ the lattice M^{σ_x} is equal to $\mathcal{M}_{F,x}^{\mathrm{gp}}$. Thus, the elements of $\operatorname{Div}(\Sigma_F)$ are precisely the elements $(\varphi_x)_x \in \prod_{x \in F} \mathcal{M}_{F,x}^{\mathrm{gp}}$ such that φ_y cospecializes to φ_x whenever $x \rightsquigarrow y$. By Lemma 3.1.26 these are precisely the elements of $\Gamma(F, \mathcal{M}_F^{\mathrm{gp}})$. □

Definition 3.1.29. A *weakly embedded sharp monoidal space* is a pair $(X, M^X \to \mathcal{M}_X^{\mathrm{gp}})$ consisting of a sharp monoidal space X and a morphism from an abelian group M^X to $\Gamma(X, \mathcal{M}_X^{\mathrm{gp}})$, or equivalently a morphism $M^X \to \mathcal{M}_X^{\mathrm{gp}}$, where we consider M^X as a constant sheaf on X. The elements of M^X are called the *rational functions on X*. A morphism between two weakly embedded sharp monoidal spaces X and Y is a pair (f, f^*) consisting of a map $f\colon X \to Y$ of sharp monoidal spaces, and morphism $f^*\colon M^Y \to M^X$ of abelian groups such that the obvious diagram commutes. We denote the resulting category of weakly embedded sharp monoidal spaces by (**we-SMS**). We will mostly deal with its full subcategories (**we-Fan**), (**fwe-Fan**), and (**fswe-Fan**) of *weakly embedded Kato fans, fine weakly embedded Kato fans, and fine and saturated weakly embedded Kato fans*, having as objects the weakly embedded sharp monoidal spaces $(X, M^X \to \mathcal{M}_X^{\mathrm{gp}})$ for which X is a Kato fan, fine Kato fan, and fine and saturated Kato fan, respectively. In the fine case we also require M^X to be finitely generated, and in the saturated case we furthermore require it to be torsion-free.

Before showing the equivalence of the categories (**fswe-Fan**) and (**we-CC**) we introduce some more notation.

Definition 3.1.30. Let M be a monoid, A an abelian group, and let $f\colon A \to M^{\mathrm{gp}}$ be a morphism of abelian groups. Then we denote by $\operatorname{Spec}(A \xrightarrow{f} M^{\mathrm{gp}})$ the weakly embedded Kato fan with underlying fan $\operatorname{Spec} M$ and weak embedding

$$f\colon A \to M^{\mathrm{gp}} = \Gamma(\operatorname{Spec} M, \mathcal{M}_{\operatorname{Spec} M}^{\mathrm{gp}}).$$

Let X be a weakly embedded Kato fan, and let $U \subseteq X$ be an open subset. Then U is itself a Kato fan, and the restriction $M^X \to \Gamma(X, \mathcal{M}_X^{\mathrm{gp}}) \to \Gamma(U, \mathcal{M}_U^{\mathrm{gp}})$ naturally makes U a weakly embedded Kato fan. For example, if $X = \operatorname{Spec}(\mathbb{Z} \to \mathbb{N}^{\mathrm{gp}})$, and $U = \{\emptyset\}$, then the resulting weakly embedded Kato fan is $\operatorname{Spec}(\mathbb{Z} \to 0^{\mathrm{gp}})$.

Definition 3.1.31. Let F and G be weakly embedded Kato fans, then we denote by $F(G)$ the set of *G-valued points of F*, which equals $\operatorname{Mor}_{(\mathbf{we\text{-}Fan})}(G, F)$. If G does not have a weak embedding, we will use the same notation for the G valued points of the underlying Kato fan of F, or equivalently, the G-valued points after equipping G with the trivial weak embedding $\mathrm{id}\colon \mathcal{M}_G^{\mathrm{gp}} \to \mathcal{M}_G^{\mathrm{gp}}$. If $f\colon A \to M^{\mathrm{gp}}$ is a morphism from an abelian group into the groupification of a monoid M, we abbreviate $F(\operatorname{Spec}(A \xrightarrow{f} M^{\mathrm{gp}}))$ as $F(A \xrightarrow{f} M^{\mathrm{gp}})$.

Lemma 3.1.32. *Let F be a Kato fan with a weak embedding $M^F \xrightarrow{\varphi} \Gamma(X, \mathcal{M}_F)$, let M be a monoid, and let A be an abelian group. Then*

$$F(M^{\mathrm{gp}} \xrightarrow{\mathrm{id}} M^{\mathrm{gp}}) = F(M) \qquad \text{and} \qquad F(A \to 0^{\mathrm{gp}}) = \operatorname{Hom}(M^F, A) \times \pi_0(F),$$

where $\pi_0(F)$ denotes the set of connected components of F.

Proof. The set $F(M^{\mathrm{gp}} \to M^{\mathrm{gp}})$ consists of all pairs of a morphism $f\colon \operatorname{Spec} M \to F$ of Kato fans, and $f^*\colon M^F \to M^{\mathrm{gp}}$ of abelian groups such that $(f^\flat)^{\mathrm{gp}} \circ \varphi = \mathrm{id}_{M^{\mathrm{gp}}} \circ f^*$. We see that f^* is uniquely determined by the morphism of Kato fans, and hence $F(M^{\mathrm{gp}} \to M^{\mathrm{gp}}) = F(M)$.

Similarly a map from $\operatorname{Spec}(A \to 0^{\mathrm{gp}})$ to F is comprised by a morphism of Kato fans $f\colon \operatorname{Spec} 0 \to F$, and a morphism of abelian groups $f^*\colon M^F \to A$, but this time the condition on (f, f^*) to be a morphism of weakly embedded Kato fans is $0 \circ f^* = (f^\flat)^{\mathrm{gp}} \circ \varphi$, which is always trivially true. Because a morphism $M \to 0$ of sharp monoids can only be local if $M = 0$, the morphisms of $\operatorname{Spec} 0$ to F are in one-to-one correspondence with the components of F (by part b) of Proposition 3.1.17). Therefore, we have a natural identification

$$F(A \to 0^{\mathrm{gp}}) = \pi_0(F) \times \mathrm{Hom}(M^F, A). \qquad \square$$

Construction 3.1.33. Let F be a fine and saturated weakly embedded Kato fan. As pointed out at the beginning of this section, defining a weak embedding for Σ_F is equivalent to specifying a lattice M^Σ and a morphism $M^\Sigma \to \mathrm{Div}(\Sigma_F)$. By Proposition 3.1.28 we have $\mathrm{Div}(\Sigma_F) = \Gamma(F, \mathcal{M}_F^{\mathrm{gp}})$, so we can take $M^\Sigma = M^F$ and the morphism $M^F \to \mathrm{Div}(\Sigma_F)$ given by the weak embedding of F. We thus obtain a weakly embedded cone complex which we again denote by Σ_F. The construction is clearly functorial so that we get a functor

$$\Sigma\colon (\textbf{fswe-Fan}) \to (\textbf{we-CC}),$$

where $(\textbf{we-CC})$ denotes the category of weakly embedded cone complexes. $\Diamond$

As a direct consequence of Propositions 3.1.22 and 3.1.33 we obtain

Corollary 3.1.34. *The functor Σ of Construction 3.1.33 is an equivalence of categories.*

Remark 3.1.35. Let us give an alternative construction of the weak embedding for the cone complex Σ_F associated to a fine and saturated weakly embedded Kato fan F. By Lemma 3.1.32 the inclusion $\mathrm{Spec}(\mathbb{R} \to 0^{\mathrm{gp}}) \to \mathrm{Spec}(\mathbb{R} \to \mathbb{R}_{\geq 0}^{\mathrm{gp}})$ induces a map

$$\varphi\colon |\Sigma_F| = F(\mathbb{R} \to \mathbb{R}_{\geq 0}^{\mathrm{gp}}) \to F(\mathbb{R} \to 0^{\mathrm{gp}}) = \pi_0(F) \times N_{\mathbb{R}}^F \to N_{\mathbb{R}}^F,$$

where we denote $N^F = \mathrm{Hom}(M^F, \mathbb{Z})$. To see that φ is a weak embedding, that is that it is linear on the cones of Σ_F, let $x \in F$, let $w \in \sigma_x = \mathrm{Hom}(\mathcal{M}_{F,x}, \mathbb{R}_{\geq 0})$, and let $m \in M^F$. The value $\varphi^*(m)(w)$ of the pull-back of m at w is by definition equal to $\langle m, \varphi(w)\rangle$. Looking back at the proof of Lemma 3.1.32 we see that this pairing is given by the image of m under the composite

$$M^F \to \Gamma(F, \mathcal{M}_F^{\mathrm{gp}}) \to \mathcal{M}_{F,x}^{\mathrm{gp}} \to \mathbb{R},$$

where the last arrow is induced by w. So denoting the image of m in $\mathcal{M}_{F,x}^{\mathrm{gp}}$ by m', we have $\varphi^*(m)(w) = \langle m', w\rangle$ for all $w \in \sigma_x$. Since $\mathcal{M}_{F,x}^{\mathrm{gp}} = M^{\sigma_x}$ by construction, this shows that $\varphi^*(m)$ is linear on all cones of Σ_F for all $m \in M^F$. Choosing coordinates on N^F we see that this implies that φ is linear on the cones of Σ_F as well. This also shows that φ coincides with the weak embedding defined in Construction 3.1.33.

Construction 3.1.36. Let F be an integral weakly embedded Kato fan, and let $x \in F$. We define M_x^F as the kernel of the morphism $M^F \to \mathcal{M}_{F,x}^{\mathrm{gp}}$. Then for every $y \in F(x)$ the morphism $M^F \to \mathcal{M}_{F,y}^{\mathrm{gp}}$ maps M_x^F into $\mathcal{M}_{F(x),y}^{\mathrm{gp}}$. Thus, there is an induced map $M_x^F \to \mathcal{M}_{F(x)}^{\mathrm{gp}}$, making $F(x)$ into a weakly embedded Kato fan. We will denote it by $F(x)$ again. Note that by construction we have $M^{F(x)} = M_x^F$. $\Diamond$

Remark 3.1.37. Note that we had to assume that F is integral in Construction 3.1.36 to assure that $\mathcal{M}^{\mathrm{gp}}_{F(x),y}$ is a subgroup of $\mathcal{M}^{\mathrm{gp}}_{F,y}$ for all $y \in F(x)$. This would also hold under the weaker assumption that the local monoids are *s-cancellative* as defined in [Pir15]: a monoid M is s-cancellative if for all $a, b, c \in M$ and $\mathfrak{p} \in \operatorname{Spec} M$ such that $a + c = b + c$ and $a, b \notin \mathfrak{p}$ there exists $c' \notin \mathfrak{p}$ with $a + c' = b + c'$.

Corollary 3.1.38. *Let F be a fine and saturated Kato fan, and let $x \in F$. Then*

$$\Sigma_{F(x)} = \mathrm{S}_{\Sigma_F}(\sigma_x)$$

as weakly embedded cone complexes.

Proof. That the equality holds between the underlying cone complexes has been shown in Proposition 3.1.24. What is left to show is that the weak embeddings coincide. Recall from section 2.1.1 that $M^{\mathrm{S}(\sigma_x)}$ is the sublattice of $M^{\Sigma_F} = M^F$ consisting of all functions vanishing on σ_x. By the construction of Σ_F, these are precisely the elements of M^F_x. Following the proof of Proposition 3.1.24 we see that every element of $M^{F(x)} = M^{\mathrm{S}(\sigma_x)}$ defines the same function on $\mathrm{S}(\sigma_x)$ and $\Sigma_{F(x)}$ □

3.2 Intersection Theory for Kato Fans

Now that we know monoids and Kato fans sufficiently well, we want to show that they serve the purpose for which we introduced them: to use their analogy to schemes in order to give an algebraic definition of their Chow groups, and to obtain a duality between these Chow groups and the groups of Minkowski weights on cone complexes. Of course, we will not be content with the mere definition of Chow groups, but also want to study their functorial behavior and how to intersect with divisors. Our treatment is inspired by the first two chapters of the classic book [Ful98], where the theory is developed for schemes.

3.2.1 Chow Groups of Fine Kato Fans

As in algebraic geometry, the Chow group of a fine weakly embedded Kato fan F will be a graded abelian group, consisting of classes of formal $\mathbb{Z}$-linear combinations of irreducible closed subsets of F. We will, however, identify these "subfans" with their generic points. The Chow groups will also not be graded by dimension, as they usually are in algebraic geometry. The reason for this is that dimension is not well-behaved on Kato fans. For example, $\operatorname{Spec} 0$ is open in $\operatorname{Spec} \mathbb{N}$, yet it has dimension 0 while $\operatorname{Spec} \mathbb{N}$ is 1-dimensional. On an irreducible algebraic variety X, the grading of $Z(X)$ by negative codimension differs from the grading by dimension only by a shift of $\dim(X)$. As the codimension of closed subsets of F is well behaved, this leads to the following definition.

Definition 3.2.1. Let F be a fine weakly embedded Kato fan, and let $k \in \mathbb{N}$. Then we define the group $Z_k(F)$ of *k-cycles on F* as the free abelian group on the elements of F with codimension $-k$ (that is on those $x \in F$ with $\operatorname{codim}(x, F) = -k$). We will refer to the graded abelian group $Z_*(F) = \bigoplus_{k\in\mathbb{Z}} Z_k(F)$ as the *cycle group* of F. We denote the generator of $Z_*(F)$ corresponding to $x \in F$ by $[x]$. If G is an irreducible closed subset of F with generic point x, we will sometimes write $[G]$ instead of $[x]$. Elements of $Z_{-1}(F)$ will also be called *Weil divisors*. The *support* of a cycle $\alpha = a_1[x_1] + \ldots + a_n[x_n]$ with $a_i \neq 0$ for all i is $|\alpha| = \bigcup_{i=1}^n F(x_i)$.

It follows immediately from the definition that $Z_k(F) = 0$ whenever $k > 0$ or $k < -\dim(F)$. Note also that for every $x \in F$ the cycle group $Z(F(x))$ is naturally a subgroup of $Z(F)$. We have to be careful with the grading though: the inclusion map $Z(F(x)) \to Z(F)$ is graded of degree $-\operatorname{codim}(x, F)$.

To define rational equivalence for cycles on Kato fans we need the concept of the order of an element $m \in M_F$ at a codimension 1 point x of a fine weakly embedded Kato fan F. By Proposition 3.1.18 the monoid $\mathcal{M}_{F,x}$ is 1-dimensional. By Proposition 3.1.8 the same is true for the sharp and saturated monoid $\overline{\mathcal{M}^{\text{sat}}_{F,x}}$ which therefore is isomorphic to $\mathbb{N}$ (because $\mathbb{R}_{\geq 0}\overline{\mathcal{M}^{\text{sat}}_{F,x}}$ is a ray by Proposition 3.1.6). Since there is a unique isomorphism $\overline{\mathcal{M}^{\text{sat}}_{F,x}} \to \mathbb{N}$ we obtain a natural map

$$\operatorname{ord}^F_x \colon \mathcal{M}^{\text{gp}}_{F,x} \to \mathbb{Z},$$

which we abbreviate to ord_x if there is no risk of confusion. For $m \in M^F$ we write $\operatorname{ord}_x(m)$ for the image of m under the composite $M^F \to \mathcal{M}^{\text{gp}}_{F,x} \xrightarrow{\operatorname{ord}_x} \mathbb{Z}$ and call it the *order of m at x*.

For $m \in M^F$ we define

$$[\operatorname{div}^F(m)] = \sum_x \operatorname{ord}_x(m)[x] \in Z_{-1}(F),$$

where the sum is over all codimension-1 points of F. This sum is well-defined because F is finite. Again, when F is clear from the context we abbreviate $[\operatorname{div}(m)] = [\operatorname{div}^F(m)]$. Now we can define rational equivalence just as we would expect it from algebraic geometry.

Definition 3.2.2. Let $k \in \mathbb{Z}$. We define $R_k(F)$ as the subgroup of $Z_k(F)$ generated by $\operatorname{div}^{F(x)}(m)$ for $x \in F$ of codimension $-k-1$ and $m \in M^F_x$. The *k-th Chow group* of F is defined as $A_k(F) = Z_k(F)/R_k(F)$. The *total Chow group* is the graded abelian group $A_*(F) = \oplus_{k\in\mathbb{Z}} A_k(F)$. We say that two cycles α and β on F are *rationally equivalent* if their images in $A_*(F)$ coincide.

Note that $A_k(F) = 0$ for $k > 0$. Furthermore, $A_0(F)$ is the freely generated by the classes of the components of F.

Remark 3.2.3. We are most interested in the case when F is fine and saturated. Some of our results sill hold when we replace saturated by torsion-free, i.e. require that all local monoids are torsion-free. In the presence of torsion, everything seems to fail badly, yet interestingly.

Example 3.2.4. Let F be the Kato fan of Example 3.1.21, and let us use the notation of that example. To specify an element of $\Gamma(F, \mathcal{M}_F^{\mathrm{gp}})$ we need to give two elements $m_i \in \Gamma(U_i, \mathcal{M}_F^{\mathrm{gp}}) = \mathbb{Z}^2$ that coincide on $U_1 \cap U_2$. Let $m_1 = e_1 + e_2$ and $m_2 = e_2$. Since e_1 vanishes on $D(e_1)$, this does indeed define an element $m \in \Gamma(F, \mathcal{M}_F^{\mathrm{gp}})$. We consider F as a weakly embedded cone complex with the weak embedding

$$M^F := \mathbb{Z} \to \Gamma(F, \mathcal{M}_F^{\mathrm{gp}}), \ \ 1 \to m.$$

Let us compute $\mathrm{div}^F(1)$. By definition, 1 goes to $e_1 + e_2$ on U_1. At both of the two codimension-1 points x and y_1 of U_1, one of the two generators e_1, e_2 vanishes in the local monoid, whereas the other one generates the maximal ideal. Therefore, we have $\mathrm{ord}_x(1) = \mathrm{ord}_{y_1}(1) = 1$. On U_2, the rational function 1 goes to e_2, which vanishes at y_2. Thus, we have

$$[\mathrm{div}^F(1)] = [y_1] + [x].$$

Proposition 3.2.5. *Let F be a fine and saturated weakly embedded Kato fan, let $x \in F$, let $y \in F(x)$ with* $\mathrm{codim}(y, F(x)) = 1$, *and let $m \in \mathcal{M}_{F(x),y} \subseteq \mathcal{M}_{F,y}$. Then*

$$\mathrm{ord}_y(m) = \langle m, u \rangle,$$

where we interpret m as a linear function on the ray $\sigma_y/\sigma_x \in \mathrm{S}_{\Sigma_F}(\sigma_x)$, whose primitive generator we denote by u. In particular, if we interpret $m' \in M_x^F$ as an element of $M_{\sigma_x}^{\Sigma_F}$, then we have

$$\mathrm{ord}_y(m') = \langle m', u_{\sigma_y/\sigma_x} \rangle,$$

where u_{σ_y/σ_x} is the lattice normal vector introduced in section 2.2.1.

Proof. Since $\Sigma_{F(x)} = \mathrm{S}_{\Sigma_F}(\sigma_x)$ by Proposition 3.1.24 we may assume that F is connected with generic point x. Recall that $\mathrm{ord}_x \colon \mathcal{M}_{F,y}^{\mathrm{gp}} \to \mathbb{Z}$ is the unique isomorphism mapping $\mathcal{M}_{F,y}$ onto $\mathbb{N}$. By Construction 3.1.20 the group $\mathcal{M}_{F,y}^{\mathrm{gp}}$ is naturally isomorphic to M^{σ_y}, where the isomorphism identifies $\mathcal{M}_{F,y}$ with $M_+^{\sigma_y}$. The morphism

$$M^{\sigma_y} \to \mathbb{Z}, \ \ l \to \langle l, u \rangle$$

is an isomorphism and maps $M_+^{\sigma_y}$ onto $\mathbb{N}$. Hence, the diagram

$$\begin{array}{ccc} \mathcal{M}_{F,y}^{\mathrm{gp}} & \xrightarrow{\mathrm{ord}_y} & \\ \downarrow \cong & & \mathbb{Z} \\ M^{\sigma_y} & \xrightarrow{\langle -, u_{\sigma_y} \rangle} & \end{array}$$

is commutative, and the main statement of the lemma follows. The "in particular" statement follows directly from the fact that the lattice normal vector u_{σ_y/σ_x} is the image of u in $N^{\Sigma_F}/N_{\sigma_x}^{\Sigma_F}$. □

Recall that one motivation for our treatment of Kato fans was to interpret the groups of Minkowski weights on a weakly embedded cone complex Σ_F as duals of the Chow groups of a fine and saturated weakly embedded Kato fan F. Our next result shows that we have made the correct definition.

Proposition 3.2.6. *Let F be a fine and saturated weakly embedded Kato fan. Then for $k \in \mathbb{Z}$ we have*

$$\mathrm{Hom}(A_k(F), \mathbb{Z}) \cong \mathrm{M}_{-k}(\Sigma_F).$$

Proof. By Definition 3.2.1 the morphism

$$\mathbb{Z}^{(\Sigma_F)_{(k)}} \to A_{-k}(F), \ \ e_{\sigma_x} \mapsto [x], \tag{3.2.1}$$

where e_σ denotes the generator of $\mathbb{Z}^{(\Sigma_F)_{(k)}}$ corresponding to a k-dimensional cone, is surjective, the kernel being generated by

$$\sum_{x\colon y \rightsquigarrow x} \mathrm{ord}_x^{F(y)}(m) e_{\sigma_x},$$

where $y \in F$ has codimension $k-1$ and $m \in M_y^F$. By Lemma 3.2.5, these relations are the same as

$$\sum_{\sigma\colon \tau \leq \sigma} \langle m, u_{\sigma/\tau} \rangle e_\sigma,$$

where $\tau \in \Sigma_F$ is $k-1$ dimensional, and $m \in M_\tau^{\Sigma_F}$. Therefore, dualizing the map of (3.2.1) identifies $\mathrm{Hom}(A_{-k}(F), \mathbb{Z})$ with the group of weights $(c_\sigma) \in \mathbb{Z}^{(\Sigma_F)_{(k)}}$ satisfying

$$0 = \sum_{\sigma\colon \tau \leq \sigma} \langle m, u_{\sigma/\tau} \rangle c_\sigma = \left\langle m, \sum_{\sigma\colon \tau \leq \sigma} c_\sigma u_{\sigma/\tau} \right\rangle$$

for all $(k-1)$-dimensional $\tau \in \Sigma_F$ and $m \in M_\tau^{\Sigma_F}$. Since $M_\tau^{\Sigma_F}$ is the dual lattice of $N_\tau^{\Sigma_F}$, this is equivalent to the balancing condition. □

Now that we have established that the groups of Minkowski weights on Σ_F are dual to the Chow groups of F we want to investigate the connection between $A_*(F)$ and $A_*(\overline{\Sigma}_F)$. It seems unlikely that $A_*(\overline{\Sigma}_F)$, which (neglecting the boundary for simplicity) is a quotient of a direct limit of the groups of Minkowski weights of proper subdivisions of Σ_F, should still have a close relation to $A_*(F)$. However, under a mild assumption, this is indeed the case.

Proposition 3.2.7. *Let F be a fine and saturated weakly embedded Kato fan, and let $n = \dim \Sigma_F$. Assume that assigning 1 to all n-dimensional cones of Σ_F defines a Minkowski weight. This is the case for example if Σ_F is the cone complex associated to a complete toroidal variety. For $\sigma \in \Sigma_F$ denote the cycle class of the pull-back of this weight to $\mathrm{S}_{\Sigma_F}(\sigma)$ by $[\mathrm{S}_{\Sigma_F}(\sigma)] \in A_*(\overline{\Sigma}_F)$. Then the morphisms*

$$\varphi\colon Z_k(F) \to A_{k+n}(\overline{\Sigma}_F), \ \ [x] \mapsto [\mathrm{S}_{\Sigma_F}(\sigma_x)]$$

induce a morphism $A_*(F) \to A_*(\overline{\Sigma}_F)$.

Proof. Let $x \in F$ and $m \in M^F_x$. We need to show that $\varphi([\mathrm{div}^{F(x)}(m)]) = 0$. For $y \in F(x)$ we have $[\mathrm{S}_{\Sigma_F}(\sigma_y)] = [\mathrm{S}_{\mathrm{S}_{\Sigma_F}(\sigma_x)}(\sigma_y/\sigma_x)]$ and $\mathrm{S}_{\Sigma_F}(\sigma_x) = \Sigma_{F(x)}$, so we may assume that F is connected and x is its generic point. Then

$$\varphi([\mathrm{div}(m)]) = \sum_y \mathrm{ord}_y(m)[\mathrm{S}_{\Sigma_F}(\sigma_y)],$$

where the sum runs over all $y \in F$ with $\mathrm{codim}(y, F) = 1$. By Lemma 3.2.5 we have $\mathrm{ord}_y(m) = \langle m, u_{\sigma_y}\rangle$, where u_{σ_y} denotes the primitive generator of the ray σ_y. Therefore, $\varphi([\mathrm{div}(m)]) = \mathrm{div}(m) \cdot [\Sigma_F]$, where we interpret $\mathrm{div}(m)$ as a principal divisor on Σ_F, and "$\cdot$" is the product of Construction 2.2.12. But since $\mathrm{div}(m)$ is principal, we have $\mathrm{div}(m) \cup [\Sigma_F] = 0$ and hence

$$\varphi([\mathrm{div}(m)]) = \mathrm{div}(m) \cdot [\Sigma_F] = \mathrm{div}(m) \cdot [\Sigma_F] - \mathrm{div}(m) \cup [\Sigma_F] = 0$$

in $A_*(\Sigma_F)$ by Proposition 2.2.23. □

Remark 3.2.8. The requirement for Proposition 3.2.7 that the constant weight 1 is a Minkowski weight on Σ_F is not really necessary to obtain a morphism $A_*(F) \to A_*(\overline{\Sigma}_F)$. We could also start with any Minkowski weight $c \in \mathrm{M}_n(\Sigma_F)$ and define $[\mathrm{S}_{\Sigma_F}(\sigma)]$ as the cycle class associated to the restriction of c to $\mathrm{S}_{\Sigma_F}(\sigma)$. The statement of the proposition would then still hold, but the morphism obtained like this would not be canonical.

Proposition 3.2.9. *Let X be a toroidal embedding, and let F be a weakly embedded Kato fan with $\Sigma_F \cong \Sigma(X)$. Furthermore, let $n = \dim X$, and let $k \in \mathbb{Z}$. Then the morphism*

$$\varphi\colon Z_k(F) \to Z_{n+k}(X), \quad [x] \mapsto [V(\sigma_x)]$$

induces a morphism $A_k(F) \to A_{n+k}(X)$ which even is an isomorphism if X is toric.

Proof. Let $x \in F$ and $m \in M^F_x$. The image of $[\mathrm{div}^{F(x)}(m)]$ in $A_*(X)$ is equal to

$$\sum_y \mathrm{ord}_y^{F(x)}(m)[V(\sigma_y)],$$

where the sum is over all $y \in F(x)$ with $\mathrm{codim}(y, F(x)) = 1$. By Lemma 3.2.5 this is equal to

$$\sum_y \langle m, u_{\sigma_y/\sigma_x}\rangle [V(\sigma_y)].$$

Recall that $M^F = M^{\Sigma(X)} = M^X$ is the lattice of invertible functions on X_0 (modulo units of the ground field), and that $m \in M^F_x = M^{\Sigma(X)}_{\sigma_x}$ implies that the support of the divisor of m on X does not contain $\mathrm{O}(\sigma_x)$, that is that m defines a rational function on $\mathrm{V}(\sigma_x)$. Arguing as in the proof of Proposition 2.3.1, we see that

$$\varphi([\mathrm{div}^{F(x)}(m)]) = \sum_y \langle m, u_{\sigma_y/\sigma_x}\rangle [\mathrm{V}(\sigma_y)] = [\mathrm{div}^{\mathrm{V}(\sigma_x)}(m)] = 0$$

in $A_*(X)$.

Now assume X is toric. Then the fact that φ induces an isomorphism follows from Fulton and Sturmfels' description of the Chow groups of toric varieties by generators and relations [FS97, Prop. 1.1]; the Chow group $A_{n+k}(X)$ is generated by the classes of $[V(\sigma_x)] = \varphi([x])$ for $[x] \in Z_k(F)$, and the relations between these classes are given by

$$\sum_{x\colon \operatorname{codim}(x,F(y))=1} \langle m, u_{\sigma_x/\sigma_y}\rangle [V(\sigma_x)] = \varphi(\operatorname{div}(m))$$

for $y \in F$ with $\operatorname{codim}(y, F) = -k-1$ and $m \in M^{\Sigma}_{\sigma_y} = M^F_y$. □

Remark 3.2.10. Let X be a complete toroidal embedding, and let F be a fine and saturated weakly embedded Kato fan with $\Sigma_F \cong \Sigma(X)$. Then by Proposition 3.2.7 there is a natural morphism $A_*(F) \to A_*(\overline{\Sigma}(X))$, and by Proposition 3.2.9 there is a natural morphism $A_*(F) \to A_*(X)$. It is immediate the former is equal to the composite

$$A_*(F) \to A_*(X) \xrightarrow{\text{Trop}} A_*(\overline{\Sigma}(X)).$$

Chow groups of toric varieties are well-studied (we refer to [CLS11] for a thorough treatment). Thus, the preceding proposition produces plenty of examples.

Example 3.2.11 ([CLS11, Example 4.1.4]). Let Σ be the embedded cone complex consisting of the single cone $\operatorname{cone}\{(d,-1),(0,1)\} \subseteq \mathbb{R}^2$, and let F be a fine and saturated Kato fan with $\Sigma_F \cong \Sigma$. Then F has two codimension-1 points x and y corresponding to the rays spanned by $(d,-1)$ and $(0,1)$, respectively. Therefore, $A_{-1}(F)$ is equal to the quotient of the free abelian group generated by $[x]$ and $[y]$ by its subgroup generated by

$$[\operatorname{div}((1,0))] = d[x] \quad \text{, and}$$
$$[\operatorname{div}((0,1))] = -[x] + [y].$$

Hence, $A_{-1}(F) \cong \mathbb{Z}/d\mathbb{Z}$. We see that the Chow groups of a weakly embedded Kato fan may have torsion, as opposed to the groups of Minkowski weights of its associated cone complex, which are always torsion free.

3.2.2 Push-Forwards for Proper Subdivisions

In algebraic geometry, every proper morphism $f\colon X \to Y$ induces a push-forward morphism $f_*\colon A_*(X) \to A_*(Y)$. In our effort to copy the algebro-geometric intersection theory, we would like to have an analogous construction for weakly embedded Kato fans. For a morphism $f\colon F \to G$ of fine and saturated Kato fans, a push-forward $f_*\colon A_*(F) \to A_*(G)$ would be dual to a pull-back map $\mathrm{M}_*(\Sigma_G) \to \mathrm{M}_*(\Sigma_F)$ of Minkowski weights. So far, we only know the existence of such a map in very few instances. Most importantly, we know that it works if Σ_F is a proper subdivision of Σ_G. The following definition translates the concept of subdivisions to the language of weakly embedded Kato fans.

Definition 3.2.12. A morphism $f\colon F \to G$ of fine weakly embedded Kato fans is called a subdivision if

a) for all $x \in F$ the morphism $(f_x^\flat)^{\mathrm{gp}}\colon \mathcal{M}^{\mathrm{gp}}_{G,f(x)} \to \mathcal{M}^{\mathrm{gp}}_{F,x}$ is surjective,

b) $F(\mathbb{N}) \to G(\mathbb{N})$ is injective, and

c) $M^G \to M^F$ is surjective.

We say that f is a proper subdivision if additionally $F(\mathbb{N}) \to G(\mathbb{N})$ is surjective.

Remark 3.2.13.

a) In case the weak embeddings are trivial, our definition of proper subdivisions coincides with the definition of integral subdivisions in [GR14, Def. 4.5.22].

b) If $f\colon F \to G$ is a subdivision, and $x \in F$, then the rank of $\mathcal{M}^{\mathrm{gp}}_{F,x}$ is less or equal the rank of $\mathcal{M}^{\mathrm{gp}}_{G,f(x)}$. Thus, $\operatorname{codim}(x, F) \leq \operatorname{codim}(f(x), G)$ by Corollary 3.1.9.

Example 3.2.14. Let M be a fine sharp monoid, then the natural morphism

$$\operatorname{Spec} \overline{M^{\mathrm{sat}}} \to \operatorname{Spec} M$$

satisfies conditions a) and b) of Definition 3.2.12. If A is an abelian group, $A \to M^{\mathrm{gp}}$ is a morphism, and A' is the quotient of A by its torsion subgroup, then there is an induced morphism $A' \to (\overline{M^{\mathrm{sat}}})^{\mathrm{gp}}$. Clearly, the natural morphism

$$\operatorname{Spec}(A' \to (\overline{M^{\mathrm{sat}}})^{\mathrm{gp}}) \to \operatorname{Spec}(A \to M^{\mathrm{gp}})$$

is a subdivision. It is also universal among all morphisms from fine and saturated weakly embedded Kato fans to $\operatorname{Spec}(A \to M^{\mathrm{gp}})$. This universal property can be used to show that the construction glues, that is that the inclusion $(\textbf{fswe-Fan}) \to (\textbf{fwe-Fan})$ has a right adjoint $(-)^{\mathrm{sat}}$. For every fine weakly embedded Kato fan F and open subset $U \subseteq F$ the preimage of U in F^{sat} is equal to U^{sat}, and $M^{F^{\mathrm{sat}}}$ is equal to the quotient of M^F by its torsion subgroup. In particular, $F^{\mathrm{sat}} \to F$ is a proper subdivision. It is called the *saturation of F*.

Proposition 3.2.15. *Let $f\colon F \to G$ be a morphism of fine and saturated weakly embedded Kato fans. Then f is a (proper) subdivision if and only if Σ_f is a (proper) subdivision of weakly embedded cone complexes as defined in Subsection 2.1.1.*

Proof. First note that we may ignore the weak embeddings because the morphisms $M^{\Sigma_G} \to M^{\Sigma_F}$ and $M^G \to M^F$ are identical by construction, and $M^G \to M^F$ is surjective if and only if N^F is identified with a saturated sublattice of N^G.

Now assume that f is a subdivision. First we show that

$$\Sigma_F = F(\mathbb{R}_{\geq 0}) \to G(\mathbb{R}_{\geq 0}) = \Sigma_G$$

is injective. Suppose $v, w \in \Sigma_F$ are distinct points with the same image $\Sigma_f(v) = \Sigma_f(w)$. For each $x \in X$, the restriction $\sigma_x \to \sigma_{f(x)}$ of Σ_f is injective on the integral points

$\sigma_x \cap N^{\sigma_x} \subseteq F(\mathbb{N})$ by assumption. This clearly implies that Σ_f is injective on all of σ_x. So there are distinct $x, x' \in F$ with $v \in \operatorname{relint}(\sigma_x)$ and $w \in \operatorname{relint}(\sigma_{x'})$. Then the relative interiors of the two subcones $\Sigma_f(\sigma_x)$ and $\Sigma_f(\sigma_y)$ of Σ_G have nonempty intersections. Therefore, this intersection contains an integral point. But then an integral multiple of this point is contained in

$$\Sigma_f(\operatorname{relint}(\sigma_x) \cap F(\mathbb{N})) \cap \Sigma_f(\operatorname{relint}(\sigma_{x'}) \cap F(\mathbb{N})),$$

a contradiction. Because Σ_f is linear on the cones of Σ_F, this shows that Σ_f identifies Σ_F with a subspace of Σ_G.

To see that Σ_f is a subdivision we still need to show that all cones of Σ_F are subcones of Σ_G, that is that the integral functions of σ_x for $x \in F$ are precisely the restrictions of the integral functions of $\sigma_{f(x)}$. But by Construction 3.1.20, the lattices of integral functions on σ_x and $\sigma_{f(x)}$ are equal to $\mathcal{M}^{\mathrm{gp}}_{F,x}$ and $\mathcal{M}^{\mathrm{gp}}_{G,f(x)}$, respectively, so this follows directly from the surjectivity of $(f_x^\flat)^{\mathrm{gp}}$.

Conversely, if Σ_F is a subdivision of Σ_G, then Σ_f is injective. In particular, it is injective on integral points, that is $F(\mathbb{N}) \to G(\mathbb{N})$ is injective. And as we saw above the surjectivity of $(f_x^\flat)^{\mathrm{gp}}$ follows from σ_x being a subcone of $\sigma_{f(x)}$ for all $x \in F$.

To finish the proof we show that if f is a subdivision, Σ_f is surjective if and only if $F(\mathbb{N}) \to G(\mathbb{N})$ is surjective. Surely, if Σ_F is a proper subdivision of Σ_G, then Σ_F and Σ_G have the same set of integral points, showing one direction. For the other direction assume that $F(\mathbb{N}) \to G(\mathbb{N})$ is surjective, and let $y \in G$. The intersections of σ_y with the image of Σ_f is equal to

$$\bigcup_{x : f(x) \rightsquigarrow y} \Sigma_f(\sigma_x).$$

In particular, it is a union of finitely many cones. So if σ_y wasn't contained in the image of Σ_f, there would be integral points in σ_y which are not in the image either, a contradiction. Thus, Σ_f is surjective. □

Definition 3.2.16. Let F and G be fine weakly embedded Kato fans, and let $f \colon F \to G$ be a proper subdivision. We define the *push-forward* $f_* \colon Z_*(F) \to Z_*(G)$ as the graded-morphism

$$f_*[x] = \begin{cases} [f(x)] & \text{if } \operatorname{codim}(x, F) = \operatorname{codim}(y, G), \\ 0 & \text{else.} \end{cases}$$

Note that if x is a point of a fine weakly embedded Kato fan F, and $\iota \colon F(x) \to F$ is the inclusion, then with the same definition we obtain a push-forward $\iota_* \colon Z_*(F(x)) \to Z_*(F)$. That this morphism induces a morphism $A_*(F(x)) \to A_*(F)$ follows directly from the definition of the Chow groups.

Proposition 3.2.17. *Let $f \colon F \to G$ be a proper subdivision of fine and saturated weakly embedded Kato fans. Then $f_* \colon Z_*(F) \to Z_*(G)$ induces a graded morphism $A_*(F) \to A_*(G)$, which we again denote by f_*.*

Proof. Let $x \in F$ and $m \in M^X_x$. We need to show that $f_*[\mathrm{div}(m)] = 0$ in $A_*(G)$. Let $y = f(x)$. Because f is a subdivision, we have $\mathrm{codim}(x, F) \leq \mathrm{codim}(y, G)$. First assume that $\mathrm{codim}(x, F) \leq \mathrm{codim}(y, G) - 2$. Then for every $z \in F(x)$ with $\mathrm{codim}(z, F(x)) = 1$ we have

$$\mathrm{codim}(f(z), G) \geq \mathrm{codim}(y, G) > \mathrm{codim}(x, F) + 1 = \mathrm{codim}(z, F).$$

Therefore, we have $f_*[\mathrm{div}(m)] = 0$ in $Z_*(G)$. Now assume $\mathrm{codim}(x, F) = \mathrm{codim}(y, F) - 1$. So for $z \in F(x)$ with $\mathrm{codim}(z, F(x)) = 1$ we have $f_*[z] \neq 0$ if and only if $f(z) = y$. Looking at the cone complex picture, σ_x being a codimension 1 subcone of σ_y implies that there are exactly 2 cones σ_z and $\sigma_{z'}$ with $\mathrm{codim}(z, F(x)) = \mathrm{codim}(z', F(x)) = 1$ which are contained in σ_y. Note that for $w \in F(x)$ the conditions $\sigma_w \subseteq \sigma_y$ and $f(w) = y$ are equivalent. It also follows that the two lattice normal vectors u_{σ_z/σ_x} and $u_{\sigma_{z'}/\sigma_x}$ are negatives of each other and thus that

$$f_*[\mathrm{div}(m)] = \Big(\mathrm{ord}_z(m) + \mathrm{ord}_{z'}(m)\Big)[y] = \Big(\langle m, u_{\sigma_z/\sigma_x}\rangle + \langle m, u_{\sigma_{z'}/\sigma_x}\rangle\Big)[y] = 0,$$

where the second equality follows from Lemma 3.2.5.

Now suppose $\mathrm{codim}(x, F) = \mathrm{codim}(y, G)$. Then $\dim(\sigma_x) = \dim(\sigma_y)$, and hence the induced morphism

$$\Sigma_{F(x)} = \mathrm{S}_{\Sigma_F}(\sigma_x) \to \mathrm{S}_{\Sigma_G}(\sigma_y) = \Sigma_{G(y)}$$

of weakly embedded cone complexes is a proper subdivision again. By Proposition 3.2.15 this implies that the induced morphism $g\colon F(x) \to G(y)$ is a proper subdivision. Let $i\colon F(x) \to F$ and $j\colon G(y) \to G$ be the inclusions. Then $f_* \circ i_* = j_* \circ g_*$, and hence it suffices to show that $g_*[\mathrm{div}(m)] = 0$ in $A_*(G(y))$, that is we may assume F and G are connected, with generic points x and y, respectively. By Proposition 3.2.15 the morphism Σ_f is a proper subdivision, and hence every ray of Σ_G contains precisely on ray of Σ_F. This implies that every $z \in G$ with $\mathrm{codim}(z, G) = 1$ has exactly one preimage $z' \in F$, which has codimension 1 again. Furthermore, the surjectivity of $(f^\flat_{z'})^{\mathrm{gp}}$ implies that $\mathrm{ord}^F_{z'}(f^* m') = \mathrm{ord}^G_z(m')$ for every $m' \in M^G$. This shows that for any $m' \in M^G$ with $f^* m' = m$ we have $f_*[\mathrm{div}(m)] = [\mathrm{div}(m')] = 0$ in $A_*(G)$. □

Example 3.2.18. Let $f\colon F \to G$ be the morphism of Kato fans from Example 3.1.21. We consider F as a weakly embedded Kato fan with the weak embedding of Example 3.2.4. Furthermore, we consider the weak embedding

$$M^G := \mathbb{Z} \to (\mathbb{N}^2)^{\mathrm{gp}} = \mathbb{Z}^2, \;\; 1 \mapsto e_1$$

on G. Then we convince ourselves that $f^* = \mathrm{id}_{\mathbb{Z}}$ makes f into a morphism of weakly embedded Kato fans. Using Proposition 3.2.15 and the description of Σ_f given in Example 3.1.21, we see that f is a proper subdivision. In particular, the push-forward f_* is defined. Recall that the images of y_1 and x under f are $\langle e_1\rangle$ and $\langle e_1, e_2\rangle$, respectively. So it follows from Example 3.2.4 that $f_*[\mathrm{div}^F(1)] = [\langle e_1\rangle]$, which is clearly equal to $[\mathrm{div}^G(1)]$.

Remark 3.2.19. In algebraic geometry, push-forwards are defined for all proper morphisms between algebraic schemes. It is an interesting problem to determine the correct notion of properness for morphisms between weakly embedded Kato fans and to define proper push-forwards for them. Let us remark on one possible definition of properness. Analogous to the valuative criterion for properness of morphisms of schemes, or Deitmar schemes [Gil16, Def. 4.5.2], we could call a morphism $f\colon F \to G$ of fine weakly embedded Kato fans proper (or valuatively proper) if the diagram

$$\begin{array}{ccc} F(\mathbb{Z} \to \mathbb{N}^{\mathrm{gp}}) & \longrightarrow & F(\mathbb{Z} \to 0^{\mathrm{gp}}) \\ \downarrow & & \downarrow \\ G(\mathbb{Z} \to \mathbb{N}^{\mathrm{gp}}) & \longrightarrow & G(\mathbb{Z} \to 0^{\mathrm{gp}}) \end{array}$$

is a Cartesian diagram of sets. This is certainly true for proper subdivisions. It is also true for the morphism

$$F = \operatorname{Spec}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{N}^{\mathrm{gp}}) \to \operatorname{Spec}(\mathbb{Z} \xrightarrow{\cdot 4} \mathbb{N}^{\mathrm{gp}}) = G$$

given by the identity on $\operatorname{Spec}\mathbb{N}$ and multiplication by 2 on $\mathbb{Z}$. It sends the closed point x of $\operatorname{Spec}\mathbb{N}$ to itself, and the induced morphism

$$\operatorname{Spec}(0 \to 0^{\mathrm{gp}}) = F(x) \to G(x) = \operatorname{Spec}(0 \to 0^{\mathrm{gp}})$$

is trivial and valuatively proper again, as one would expect. Because the push-forward $A_*(F) \to A_*(G)$ should be compatible with the push-forward $A_*(F(x)) \to A_*(G(x))$, it should thus map $[x]$ to itself. But then $2[x] = [\operatorname{div}(1)] = 0 \in A_*(F)$ maps to $2[x]$, which is nonzero in $A_*(G)$. We see that valuative properness is not sufficient to be able to reasonably define a push-forward.

Let $f\colon F \to G$ be a proper subdivision of fine and saturated weakly embedded Kato fans. Then it follows immediately from the definitions that the morphism $\mathrm{M}_{-k}(\Sigma_G) \to \mathrm{M}_{-k}(\Sigma_F)$ dual to the push-forward $f_*\colon A_k(F) \to A_k(G)$ is nothing but the subdivision of Minkowski weights.

Corollary 3.2.20. *Let F be a fine and saturated weakly embedded Kato fan, and let $k \in \mathbb{Z}$. Then*

$$\varinjlim_G \operatorname{Hom}(A_k(G), \mathbb{Z}) \cong Z_{-k}(\Sigma_F),$$

where the limit is over all proper subdivisions G of F.

Proof. By Proposition 3.2.6, we have $\operatorname{Hom}(A_k(G), \mathbb{Z}) \cong \mathrm{M}_{-k}(\Sigma_G)$ for every proper subdivision G of F. Since the proper subdivisions of F are in one-to-one correspondence with the proper subdivisions of Σ_F by Proposition 3.2.15, we see that

$$Z_{-k}(\Sigma_F) = \varinjlim_{\Sigma'} \mathrm{M}_{-k}(\Sigma') \cong \varinjlim_G \operatorname{Hom}(A_k(G), \mathbb{Z}),$$

where the first limit is over all proper subdivisions Σ' of Σ, and the second limit is over all proper subdivisions G of F. □

3.2.3 Pull-backs for Locally Exact Morphisms

Now we want to define pull-backs $f^*: A_*(G) \to A_*(F)$ for $f: F \to G$ belonging to an appropriate class of morphisms. In algebraic geometry, the appropriate class of morphisms to define these pull-backs are flat morphisms. For weakly embedded cone complexes it will be locally exact morphisms.

Definition 3.2.21. A morphism $g: M \to N$ of integral monoids is called *exact* if $(g^{\text{gp}})^{-1}N = M$. A morphism $f: F \to G$ of integral weakly embedded Kato fans is called *locally exact* if $f_x^\flat$ is exact for all $x \in F$.

Example 3.2.22. Consider the morphism $g: \mathbb{N}^2 \to \mathbb{N}^2$ which maps the generators e_1 and e_2 of $\mathbb{N}^2$ to $e_1 + e_2$ and e_2, respectively. Then $g^{\text{gp}}(e_1 - e_2) = e_1 \in \mathbb{N}^2$, but $e_1 - e_2 \notin \mathbb{N}^2$. It follows that g is not exact.

Here are some properties of exact morphisms.

Proposition 3.2.23 ([Ogu06, Section 4.1]). *Let $g: M \to N$ be a morphism of integral monoids, and let $f: F \to G$ be a morphism of integral weakly embedded Kato fans.*

a) If g is exact, then $\overline{g}: \overline{M} \to \overline{N}$ it is injective.

b) If g is exact, then the induced morphism Spec $N \to$ Spec M *is surjective. The converse is true if M is fine and saturated.*

c) If f is locally exact, then generalizations lift along f, that is whenever $x \in F$, and $y \rightsquigarrow f(x)$, then there exists $x' \in F$ with specializing to x with $f(x') = y$. The converse is true if G is fine and saturated.

Proposition 3.2.24 (cf. [Gil16, Thm. 4.7.16]). *Let $f: F \to G$ be a morphism of fine and saturated weakly embedded Kato fans. Then the following are equivalent:*

a) f is locally exact.

b) $\Sigma_f: \Sigma_F \to \Sigma_G$ maps cones of Σ_F onto cones of Σ_G.

Proof. First assume f is locally exact, and let $x \in F$. Recall that the cone $\sigma_{f(x)}$ is the smallest cone of Σ_G that contains $\Sigma_f(\sigma_x)$. In particular, if $\sigma_{f(x)}$ is a ray, then it is equal to $\Sigma_f(\sigma_x)$. In any dimension $\sigma_{f(x)}$ is spanned by its rays which correspond to codimension 1 points in $V_{\rightsquigarrow f(x)}$. As generalizations lift along f, the induced map $V_{\rightsquigarrow x} \to V_{\rightsquigarrow f(x)}$ is surjective. Hence, every ray of $\sigma_{f(x)}$ is contained in the image of σ_x, showing that $\Sigma_f(\sigma_x) = \sigma_{f(x)}$.

Conversely, assume that Σ_f maps cones onto cones. Let $x \in F$ and let $y \in G$ with $y \rightsquigarrow f(x)$. Then σ_y is a face of $\sigma_{f(x)} = \Sigma_f(\sigma_x)$, and hence $\Sigma_f^{-1}(\sigma_y) \cap \sigma_x$ is a face of σ_x mapping onto σ_y. Let $x' \in V \rightsquigarrow x$ such that this face is equal to $\sigma_{x'}$, then $f(x') = y$. Therefore, generalizations lift along f, and hence f is locally exact by 3.2.23 c). □

Example 3.2.25. Using Proposition 3.2.24 we see that the morphism f of Example 3.1.21 is not locally exact. We can also check this directly; if z denotes the maximal point of U_1, then $f_z^\flat$ is equal to the morphism considered in Example 3.2.22 for which we have seen that it is not exact.

Lemma 3.2.26. *Let $f: F \to G$ be a locally exact morphism of fine weakly embedded Kato fans, let $y \in G$, and let $x \in G$ with $f(x) = y$. Then $\operatorname{codim}(x, F) \geq \operatorname{codim}(y, G)$. Equality holds if and only if x is also minimal with respect to specialization among the elements of $f^{-1}\{y\}$. In this case, the induced morphism $V_{\rightsquigarrow x} \to V_{\rightsquigarrow y}$ is a homeomorphism, and $(f_x^\flat)^{\mathrm{gp}}$ is injective with finite cokernel.*

Proof. By considering the induced morphism $V_{\rightsquigarrow x} \to V_{\rightsquigarrow y}$ we may assume that F and G are affine with closed points x and y, say $F = \operatorname{Spec} N$ and $G = \operatorname{Spec} M$, and that f is induced by an exact morphism $g: M \to N$. Generalizations lift along f by Proposition 3.2.23 c), and hence every maximal chain of prime ideals of M can be lifted to a chain of prime ideals in N. Thus,

$$\operatorname{codim}(x, F) = \dim N \geq \dim M = \operatorname{codim}(y, G).$$

Since this is true for all points of F, the equality $\operatorname{codim}(x, F) = \operatorname{codim}(y, G)$ implies that x is minimal (w.r.t. specialization) among the preimages of y. Conversely, assume that x is minimal. Replacing M and N by $\overline{M}^{\mathrm{sat}}$ and $\overline{N}^{\mathrm{sat}}$ we may assume that F and G are saturated. Suppose $\operatorname{codim}(x, F) > \operatorname{codim}(y, F)$. The map of cones $\sigma_x \to \sigma_y$ is surjective by Proposition 3.2.24. Let $w \in \operatorname{relint}(\sigma_y)$. Then the preimage $\Sigma_f^{-1}\{w\}$ is equal to the intersection of σ_x with an affine linear subspace of $N_{\mathbb{R}}^{\sigma_x}$ of dimension greater 0, namely a suitable translate of the kernel of $N_{\mathbb{R}}^{\sigma_x} \to N_{\mathbb{R}}^{\sigma_y}$. As σ_x is sharp, it cannot contain an affine linear space of dimension greater 0, and therefore there exists an element v in the boundary of σ_x mapping to w. Let $\sigma_{x'}$ be the unique face of σ_x with $v \in \operatorname{relint}(\sigma_{x'})$. Then $x' \rightsquigarrow x$ and $f(x') = y$ even though $x' \neq x$, a contradiction. Thus, we have $\operatorname{codim}(x, F) = \operatorname{codim}(y, G)$. In this case, the induced map $\sigma_x \to \sigma_y$ is bijective, and hence $F \to G$ is a homeomorphism. Furthermore, $(f_x^\flat)^{\mathrm{gp}}$ has finite cokernel because it is an injective morphism of lattices of equal rank. □

Definition 3.2.27. Let $f: F \to G$ be a locally exact morphism of fine weakly embedded Kato fans. We define $f^*: Z_*(F) \to Z_*(G)$ by

$$f^*[y] = \sum_{x \in f^{-1}\{y\}^{\min}} [\mathcal{M}_{F,x}^{\mathrm{gp}} : \mathcal{M}_{G,y}^{\mathrm{gp}}][x],$$

where $f^{-1}\{y\}^{\min}$ denotes the set of minimal elements of $f^{-1}\{y\}$ with respect to specialization. The lattice indices occurring in the definition are well-defined by Lemma 3.2.26, and by the same lemma f^* respects the grading.

Proposition 3.2.28. *Let $f: F \to G$ be a locally exact morphism of fine and saturated weakly embedded Kato fans, let $y \in G$, and let $m \in M_y^G$. Then*

$$f^*[\operatorname{div}^{G(y)}(m)] = \sum_{x \in f^{-1}\{y\}^{\min}} [\mathcal{M}_{F,x}^{\mathrm{gp}} : \mathcal{M}_{G,y}^{\mathrm{gp}}] \cdot [\operatorname{div}^{F(x)}(f^* m)]. \tag{3.2.2}$$

In particular, there is an induced morphism $A_*(G) \to A_*(F)$, which we again denote by f^*.

Proof. Let $x \in F$ with $\operatorname{codim}(x, F) = \operatorname{codim}(y, G) + 1$. First assume that $f(x) \notin \overline{\{y\}}$. Then clearly the multiplicity of $[x]$ in both sides of (3.2.2) is 0. Next, assume that $f(x) = y$. Then the multiplicity of $[x]$ in the left hand side of (3.2.2) is obviously 0. Let $x' \in f^{-1}\{y\}^{\min}$ be a point specializing to x. The image of f^*m in $\mathcal{M}^{\mathrm{gp}}_{F(x'),x}$ is the image of m under the composite

$$M^G_y \to \mathcal{M}^{\mathrm{gp}}_{G(y),y} \to \mathcal{M}^{\mathrm{gp}}_{F(x'),x}.$$

By definition we have $\mathcal{M}_{G(y),y} = 0$, and hence the multiplicity of $[x]$ in the cycle $[\operatorname{div}^{F(x')}(f^*m)]$ is 0. Thus, the multiplicity of $[x]$ is also 0 in the right hand side of (3.2.2).

Finally, assume that $y \rightsquigarrow f(x)$ and $\operatorname{codim}(f(x), G) = \operatorname{codim}(x, G)$. By Lemma 3.2.26 this implies that $x \in f^{-1}\{f(x)\}^{\min}$, so the multiplicity of $[x]$ in $f^*[\operatorname{div}(m)]$ is $[\mathcal{M}^{\mathrm{gp}}_{F,x} : \mathcal{M}^{\mathrm{gp}}_{G,f(x)}] \operatorname{ord}^{G(y)}_{f(x)}(m)$. Furthermore, the lemma implies that the induced morphism $V_{\rightsquigarrow x} \to V_{\rightsquigarrow f(x)}$ is a homeomorphism, so there exists a unique $x' \in f^{-1}\{y\}^{\min} \cap V_{\rightsquigarrow x}$. Therefore, the multiplicity of $[x]$ in the right hand side of (3.2.2) is equal to $[\mathcal{M}^{\mathrm{gp}}_{F,x'} : \mathcal{M}^{\mathrm{gp}}_{G,y}] \operatorname{ord}^{F(x')}_x(f^*m)$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{M}^{\mathrm{gp}}_{G(y),f(x)} & \to & \mathcal{M}^{\mathrm{gp}}_{G,f(x)} & \to & \mathcal{M}^{\mathrm{gp}}_{G,y} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{M}^{\mathrm{gp}}_{F(x'),x} & \to & \mathcal{M}^{\mathrm{gp}}_{F,x} & \to & \mathcal{M}^{\mathrm{gp}}_{F,x'} & \to & 0.
\end{array}$$

Because all vertical arrows are injective, an application of the snake lemma yields

$$[\mathcal{M}^{\mathrm{gp}}_{F(x'),x} : \mathcal{M}^{\mathrm{gp}}_{G(y),f(x)}] \cdot [\mathcal{M}^{\mathrm{gp}}_{F,x'} : \mathcal{M}^{\mathrm{gp}}_{G,y}] = [\mathcal{M}^{\mathrm{gp}}_{F,x} : \mathcal{M}^{\mathrm{gp}}_{G,f(x)}].$$

The image of f^*m in $\mathcal{M}^{\mathrm{gp}}_{F(x'),x} \cong \mathbb{Z}$ is precisely the image of m under the composite $M^G_y \to \mathcal{M}^{\mathrm{gp}}_{G(y),f(x)} \to \mathcal{M}^{\mathrm{gp}}_{F(x'),x}$. From this it follows directly that

$$\operatorname{ord}^{F(x')}_x(f^*m) = [\mathcal{M}^{\mathrm{gp}}_{F(x'),x} : \mathcal{M}^{\mathrm{gp}}_{G(y),f(x)}] \operatorname{ord}^{G(y)}_{f(x)}(m),$$

which shows that the multiplicities of $[x]$ in the two sides of (3.2.2) are in fact equal. □

Remark 3.2.29. In the proof of the preceding proposition we did not need that the local monoids of F and G are saturated, but only that they are torsion-free. Thus, if $F \to G$ is a morphism of fine and torsion-free (that is the local monoids are all torsion-free) weakly embedded Kato fans, then the pull-back still induces a morphism $A_*(G) \to A_*(F)$.

Proposition 3.2.30. *Let $f\colon F \to G$ be a locally exact morphism of fine and saturated weakly embedded Kato fans. Then the morphism $(f^*)^\vee\colon \mathrm{M}_*(\Sigma_F) \to \mathrm{M}_*(\Sigma_G)$ dual to the pull-back $f^*\colon A_*(G) \to A_*(F)$ coincides with the push-forward $(\Sigma_f)_*$ of Minkowski weights introduced in Construction 2.2.10.*

Proof. Let $c \in \mathrm{M}_k(\Sigma_F)$ be a Minkowski weight, and let $\sigma \in \Sigma_G$ be a k-dimensional cone. Then, the weight of $(f^*)^\vee(c)$ at σ is equal to

$$\sum_\tau [M^\tau : M^\sigma] c(\sigma),$$

where the sum is over all k-dimensional $\tau \in \Sigma_F$ with $\Sigma_f(\tau) = \sigma$. The index of M^σ in M^τ is equal to the index of $\Sigma_f(N^\tau)$ in N^σ, hence the weights of $(f^*)^\vee(c)$ and $(\Sigma_f)_*(c)$ at σ coincide. Because σ was arbitrary, this finishes the proof. □

3.2.4 Cartier Divisors on Weakly Embedded Kato Fans

We will now introduce Cartier divisors on weakly embedded Kato fans. For a fine and saturated weakly embedded Kato fan F we want the definition to coincide with that for Σ_F. This leads to the following definition.

Definition 3.2.31. Let F be a fine weakly embedded Kato fan. Then the group of *Cartier divisors* on F is given by $\mathrm{Div}(F) = \Gamma(F, \mathcal{M}_F^{\mathrm{gp}})$. A Cartier divisor $D \in \mathrm{Div}(F)$ is said to be *cp* if it is locally given by an element of M^F. More precisely, it has to be a global section of the (sheaf-theoretic) image of $M^F \to \mathcal{M}_F^{\mathrm{gp}}$. We denote the subgroup of $\mathrm{Div}(F)$ consisting of cp-divisors by $\mathrm{CP}(F)$. For $m \in M^F$ we denote the image of m in $\mathrm{CP}(F)$ by $\mathrm{div}(m)$ and call it the *principal divisor* associated to m. The quotient $\mathrm{CP}(F)/M^F$ of cp-divisors modulo principal divisors will be denoted by $\mathrm{ClCP}(F)$ and we will call its elements cp-divisor classes.

Remark 3.2.32. In [FW14] the authors defined Cartier divisors and line bundles on irreducible and fine Deitmar schemes and showed that for such a monoidal scheme the group of Cartier divisors is naturally isomorphic to the Picard group. This interpretation is lost when we sharpen the spaces to obtain Kato fans. Nevertheless, a typical cp-divisor will be denoted by $\mathcal{L}$, just if it was a given by a line bundle.

Every morphism $f\colon F \to G$ of fine weakly embedded Kato fans induces a morphism $f^{-1}\mathcal{M}_F^{\mathrm{gp}} = (f^{-1}\mathcal{M}_F)^{\mathrm{gp}} \to \mathcal{M}_G^{\mathrm{gp}}$ and hence a *pull-back* morphism $f^*\colon \mathrm{Div}(F) \to \mathrm{Div}(G)$. This morphism clearly maps principal divisors to principal divisors and cp-divisors to cp-divisors, so it induces morphisms $\mathrm{CP}(F) \to \mathrm{CP}(G)$ and $\mathrm{ClCP}(F) \to \mathrm{ClCP}(G)$, which we again denote by f^*.

Corollary 3.2.33. *Let F be a fine and saturated weakly embedded Kato fan. Then* $\mathrm{Div}(F) = \mathrm{Div}(\Sigma_F)$, $\mathrm{CP}(F) = \mathrm{CP}(\Sigma_F)$, *and* $\mathrm{ClCP}(F) = \mathrm{ClCP}(\Sigma_F)$.

Proof. The first equality is the statement of Proposition 3.1.28. The rest follows from the fact that $M^{\Sigma_F} = M^F$ and the commutativity of the diagram

$$\begin{array}{ccc} M^F & \longrightarrow & \mathcal{M}^{\text{gp}}_{F,x} \\ \Big\downarrow = & & \Big\downarrow = \\ M^{\Sigma_F} & \longrightarrow & M^{\sigma_x} \end{array}$$

for all $x \in F$. □

Exactly as in algebraic geometry, every Cartier divisor D on a fine weakly embedded Kato fan F has an *associated Weil divisor*

$$[D] = \sum_{\operatorname{codim}(x,F)=1} \operatorname{ord}_x(D_x),$$

where $D_x \in \mathcal{M}^{\text{gp}}_{F,x}$ denotes the germ of $D \in \Gamma(X, \mathcal{M}^{\text{gp}}_F)$ at x. This clearly defines a morphism $\operatorname{Div}(F) \to Z_{-1}(F)$. If $m \in M^F$, then $[\operatorname{div}(f)]$ agrees with the previous definition. In particular, taking associated Weil divisors induces a map $\operatorname{ClCP}(F) \to A_{-1}(F)$. We will denote the image of $\mathcal{L} \in \operatorname{ClCP}(F)$ by $c_1(\mathcal{L}) \cap [F]$.

As one expects from algebraic geometry, a Cartier divisor D on a general fine weakly embedded Kato fan is not determined by its associated Weil divisor $[D]$. In algebraic geometry, a sufficient criterion for this to be true is normality. The property of monoids analogous to the ring-theoretic normality is being saturated. And in fact, the following proposition shows that on a fine and saturated weakly embedded Kato fan, the map $D \mapsto [D]$ is injective. Similarly, not every Weil divisor comes from a Cartier divisor in general. In algebraic geometry, the most important class of schemes where the notions of Weil and Cartier divisors coincide are regular schemes. The monoidal analogue of the ring-theoretic regularity is being *strictly simplicial*. Here, we say that a fine monoid M is strictly simplicial if it is free, that is isomorphic to $\mathbb{N}^r$ for some $r \in \mathbb{N}$.

Proposition 3.2.34. *Let F be a fine and saturated weakly embedded Kato fan. Then*

$$\operatorname{Div}(F) \to Z_{-1}(F), \quad D \mapsto [D]$$

is injective. If F is strictly simplicial, then it is also surjective.

Proof. Let $D \in \operatorname{Div}(F)$, and let $x \in F$ be a codimension-1 point. By Proposition 3.1.28 this corresponds to a divisor $\varphi \in \operatorname{Div}(\Sigma_F)$. The point x corresponds to a ray $\sigma_x \in \Sigma_F$, and the value of φ at the primitive generator of σ_x is equal to $\operatorname{ord}_x(D_x)$ by Lemma 3.2.5. Since φ is linear on all cones of Σ_F, it is determined by its restrictions to the rays of Σ_F. But the values of φ at the primitive generators of the rays of σ_F are precisely the multiplicities of the corresponding codimension-1 points in $[D]$. Therefore, $D \mapsto [D]$ is injective.

Now assume that F is strictly simplicial, and let $x \in F$ be a codimension-1 point. The strict simplicity of F is equivalent to Σ_F being strictly simplicial, and hence there exists a Cartier divisor $\varphi \in \operatorname{Div}(\Sigma_F)$ such that φ is one on the primitive generator of σ_x, and vanishes on all other rays of Σ_F. Let $D \in \operatorname{Div}(F)$ be the corresponding

Cartier divisor on F. Then by what we saw above, we have $[D] = [x]$. Because $Z_{-1}(F)$ is generated by the cycles $[x]$ for codimension-1 points x, this implies the surjectivity. □

Remark 3.2.35. We say that a fine and saturated Kato fan F is *simplicial* if Σ_F is a simplicial cone complex. With exactly the same proof as for Proposition 3.2.34 we obtain the result that $\mathrm{Div}(F)_{\mathbb{Q}} \to Z_{-1}(F)_{\mathbb{Q}}$ is an isomorphism if F is simplicial. In other words, every Weil divisor on a simplicial Kato fan is $\mathbb{Q}$-Cartier.

Let $D \in \mathrm{Div}(F)$. The *support* of D is defined as $|D| = \{x \in F \mid D_x \neq 0\}$. If F is fine and saturated, then $|D| = |[D]|$. Similarly as in algebraic geometry, this follows from the fact that for a fine and saturated monoid M we have

$$M = \bigcap_{\mathfrak{p}} M_{\mathfrak{p}},$$

where the intersection is over all prime ideals of M of codimension 1 [Ogu06, Cor. 2.3.11].

Construction 3.2.36 (Pull-backs to the boundary)**.** Let F be a fine weakly embedded Kato fan, let $x \in F$, and let $D \in \mathrm{Div}(F)$. Furthermore, let $\iota\colon F(x) \to F$ be the inclusion (which is a continuous map, but not a morphism of Kato fans). If $x \notin |D|$, then it is immediate from the definitions that D induces a Cartier divisor ι^*D on $F(x)$. For $m \in M_x^F = M^{F(x)}$ we have $\iota^* \mathrm{div}^F(m) = \mathrm{div}^{F(x)}(m)$. Similarly, we see that ι^*D is cp if D is cp.

Now assume that $D \in \mathrm{CP}(F)$, but no longer that $x \notin |D|$. By definition, there exists $f \in M^F$ such that $D_x = \mathrm{div}(f)_x$, and hence $x \notin |D - \mathrm{div}(f)|$. Thus, the divisor $\iota^*(D - \mathrm{div}(f))$ is defined. If $g \in M^F$ is another element with $D_x = \mathrm{div}(g)_x$, then

$$\iota^*(D - \mathrm{div}(f)) - \iota^*(D - \mathrm{div}(g)) = \iota^* \mathrm{div}(g - f) = \mathrm{div}^{F(x)}(g - f)$$

is principal on $F(x)$. Hence, we obtain a well-defined *pull-back* map $\iota^*\colon \mathrm{CP}(F) \to \mathrm{ClCP}(F(x))$. The principal divisors on F are in the kernel of ι^*, so there is an induced pull-back map $\mathrm{ClCP}(F) \to \mathrm{ClCP}(F(x))$ which we again denote by ι^*. ♢

Definition 3.2.37. Let F be a fine weakly embedded Kato fan, let $D \in \mathrm{Div}(F)$, and let $x \in F$. Furthermore, let $\iota\colon F(x) \to F$ be the inclusion. If $x \notin |D|$, we define the *intersection product* $D \cdot [F(x)]$ as the Weil divisor associated to ι^*D. If $D \in \mathrm{CP}(x)$ and x is possibly in the support of $|D|$, we define $D \cdot [F(x)]$ as the Weil divisor class associated to the cp-divisor class ι^*D. For $\mathcal{L} \in \mathrm{ClCP}(F)$, we define the *"∩"-product* $c_1(\mathcal{L}) \cap [F(x)]$ as $c_1(\iota^*\mathcal{L}) \cap [F(x)]$. Extending by linearity, we obtain bilinear maps

$$\mathrm{CP}(F) \times Z_k(F) \to A_{k-1}(F),\ (D, \alpha) \mapsto D \cdot \alpha \quad \text{, and}$$
$$\mathrm{ClCP}(F) \times Z_k(F) \to A_{k-1}(F),\ (\mathcal{L}, \alpha) \mapsto c_1(\mathcal{L}) \cap \alpha.$$

Remark 3.2.38. Let F be a fine weakly embedded Kato fan, let $x \in F$, and let $\iota\colon F(x) \to F$ be the inclusion. If we denote the inclusion $\Sigma_{F(x)} = \mathrm{S}_{\Sigma_F}(\sigma_x) \to \overline{\Sigma}_F$ by Σ_ι, then it is clear from the constructions that the diagram

$$\begin{array}{ccc} \mathrm{CP}(F) & \xrightarrow{\iota^*} & \mathrm{ClCP}(F(x)) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{CP}(\Sigma_F) & \xrightarrow{\Sigma_\iota^*} & \mathrm{ClCP}(\mathrm{S}_{\Sigma_F}(\sigma_x)), \end{array}$$

where Σ_ι^* is the pull-back from Construction 2.2.8, and the vertical maps are the isomorphisms from Corollary 3.2.33, is commutative.

Example 3.2.39. Let F be as in Example 3.1.21, with the weak embedding as in Example 3.2.4. With the notation as before, let $D_x \in \mathrm{Div}(F)$ be the unique divisor with $[D_x] = [x]$ which exists by Proposition 3.2.34. It is given by e_2 on both U_1 and U_2, and therefore it is not cp. However, it is locally cp at $[x]$, so we can still define $D_x \cdot [x]$. First, we subtract $\mathrm{div}(1)$ from D_x and obtain the divisor given by $-e_1$ on U_1 and by 0 on U_2. Since $-e_1$ cospecializes to 0 in $\mathcal{M}_{F,x}^{\mathrm{gp}}$, this does indeed induce a divisor on $F(x)$. Its associated Weil divisor is clearly given by $-[z]$, where z denotes the maximal point of U_1. Thus, we have $D_x \cdot [x] = -[z]$.

Lemma 3.2.40. *Let $f\colon F \to G$ be a morphism of fine weakly embedded Kato fans, and let $x \in F$ with $f(x) = y$. Let $g\colon F(x) \to G(y)$ be the induced morphism, and let $\iota\colon F(x) \to F$ and $\kappa\colon G(y) \to G$ be the inclusion maps. Furthermore, let $\mathcal{L} \in \mathrm{ClCP}(G)$. Then*

$$\iota^*(f^*\mathcal{L}) = g^*(\kappa^*\mathcal{L}).$$

Proof. Let $D \in \mathrm{CP}(G)$ be a representative of $\mathcal{L}$ with $D_y = 0$. Then by definition $f^*\mathcal{L}$ is represented by f^*D and we have $(f^*D)_y = 0$. Thus, $\iota^*(f^*\mathcal{L})$ is represented by $\iota^*(f^*D)$. The germ of this divisor at $z \in F(x)$ is, after identifying $\mathcal{M}_{F(x),z}^{\mathrm{gp}}$ with a subgroup of $\mathcal{M}_{F,z}^{\mathrm{gp}}$, equal to $(f^*D)_z = (f_z^\flat)^{\mathrm{gp}}(D_{f(z)})$. On the other hand, $D_{f(z)}$ also is the local equation of κ^*D at $f(z)$. The statement follows form the fact that $g_z^\flat$ is the restriction of $f_z^\flat$ to $\mathcal{M}_{G(y),f(z)}$. □

Proposition 3.2.41. *Let F be a fine and saturated weakly embedded Kato fan, and let $D, D' \in \mathrm{CP}(F)$. Then*

$$D \cdot [D'] = D' \cdot [D]$$

in $A_{-2}(F)$.

Proof. For each $x \in X$ let $f^x, g^x \in M^F$ be local equations at x for D and D', respectively. Furthermore, let $\iota_x\colon F(x) \to F$ denote the inclusion. Note that by definition $\iota_x^* D$ is represented by $\iota_x^*(D - \mathrm{div}(f^x))$ whereas $\iota_x^* D'$ is represented by $\iota_x^*(D' - \mathrm{div}(g^x))$. Plugging in definitions we see that $D \cdot [D']$ is represented by

$$\sum_{x:\mathrm{codim}(x,F)=1} \mathrm{ord}_x^F(g^x) \sum_{y:\mathrm{codim}(y,F(x))=1} \mathrm{ord}_y^{F(x)}(f^y - f^x)[y] =$$

$$\sum_{y:\mathrm{codim}(y,F)=2} \left(\sum_{\substack{x:\mathrm{codim}(x,F)=1 \\ x \rightsquigarrow y}} \mathrm{ord}_x^F(g^x)\, \mathrm{ord}_y^{F(x)}(f^y - f^x) \right) [y].$$

By symmetry, a representative for $D' \cdot [D]$ can be found by exchanging f and g in the expression above. We claim that the difference of these two expressions is given by

$$\sum_{x:\mathrm{codim}(x,F)=1} [\mathrm{div}^{F(x)}\,(\mathrm{ord}_x(f^x)g^x - \mathrm{ord}_x(g^x)f^x)]. \tag{3.2.3}$$

Since this is 0 in $A_*(F)$ this would finish the proof. Note that $\mathrm{ord}_x(f^x)g^x$ and $\mathrm{ord}_x(g^x)f^x$ have the same order at x and therefore $\mathrm{ord}_x(f^x)g^x - \mathrm{ord}_y(g^x)f^x$ vanishes at x, i.e. is contained in M^F_x, because F is saturated. Now let $y \in F$ be a point of height 2. Then $\mathcal{M}_{F,y}$ is a 2-dimensional sharp toric monoid and thus has exactly two 1-dimensional faces, corresponding to two points x and x' of height 1 specializing to y. The multiplicity of $[y]$ in the difference of the representatives of $D \cdot [D']$ and $D' \cdot [D]$ is thus

$$\begin{aligned}\mathrm{ord}^F_x(g^x)\,\mathrm{ord}^{F(x)}_y(f^y - f^x) + \mathrm{ord}^F_{x'}(g^{x'})\,\mathrm{ord}^{F(x')}_y(f^y - f^{x'}) - \\ \mathrm{ord}^F_x(f^x)\,\mathrm{ord}^{F(x)}_y(g^y - g^x) - \mathrm{ord}^F_{x'}(f^{x'})\,\mathrm{ord}^{F(x')}_y(g^y - g^{x'}).\end{aligned}$$

Let us only look at the contribution of x to this expression, which consists of the first and the third summand. We have

$$\begin{aligned}&\mathrm{ord}^F_x(g^x)\,\mathrm{ord}^{F(x)}_y(f^y - f^x) - \mathrm{ord}^F_x(f^x)\,\mathrm{ord}^{F(x)}_y(g^y - g^x) = \\ &\quad= \mathrm{ord}^{F(x)}_y\left(\mathrm{ord}^F_x(g^x)(f^y - f^x) - \mathrm{ord}^F_x(f^x)(g^y - g^x)\right) = \\ &\quad= \mathrm{ord}^{F(x)}_y\left(\mathrm{ord}^F_x(f^x)g^x - \mathrm{ord}^F_x(g^x)f^x\right) + \mathrm{ord}^{F(x)}_y\left(\mathrm{ord}^F_x(g^x)f^y - \mathrm{ord}^F_x(f^x)g^y\right).\end{aligned}$$

The first summand after the last equality is precisely the contribution of x to the multiplicity of $[y]$ in (3.2.3). As the same calculation works for x and x' interchanged, we see that the coefficient of $[y]$ in the difference of the representatives of $D \cdot [D']$ and $D' \cdot [D]$ is equal to the coefficient of $[y]$ in (3.2.3) plus

$$\mathrm{ord}^{F(x)}_y\left(\mathrm{ord}^F_x(g^x)f^y - \mathrm{ord}^F_x(f^x)g^y\right) + \mathrm{ord}^{F(x')}_y\left(\mathrm{ord}^F_{x'}(g^{x'})f^y - \mathrm{ord}^F_{x'}(f^{x'})g^y\right). \tag{3.2.4}$$

To show that this is 0 we relate $\mathrm{ord}^{F(x)}_y(m)$ and $\mathrm{ord}^F_{x'}(m)$ for $m \in \mathcal{M}^{\mathrm{gp}}_{F(x),y} \subseteq \mathcal{M}^{\mathrm{gp}}_{F,y}$, where by $\mathrm{ord}^F_{x'}(m)$ we mean the order of the image of m under the cospecialization morphism $\mathcal{M}^{\mathrm{gp}}_{F,y} \to \mathcal{M}^{\mathrm{gp}}_{F,x'}$. Let K be the kernel of this morphism. It is the groupification of the face of $\mathcal{M}_{F,y}$ corresponding to x' and therefore is isomorphic to $\mathbb{Z}$. Let u be the generator of K which is contained in $\mathcal{M}_{F,y}$, and let v be the unique generator of $\mathcal{M}_{F(x),y}$. Then the index $i = [\mathcal{M}^{\mathrm{gp}}_{F,y} : \mathbb{Z}v + \mathbb{Z}u]$ is finite, and equal to the index $[\mathcal{M}_{F,x'} : \mathbb{Z}v']$, where v' is the cospecialization of v. This in turn is equal to $\mathrm{ord}^F_{x'}(v')$ so that we get

$$\mathrm{ord}^F_{x'}(m) = \mathrm{ord}^F_{x'}(\mathrm{ord}^{F(x)}_y(m) \cdot v) = \mathrm{ord}^F_{x'}(v) \cdot \mathrm{ord}^{F(x)}_y(m) = i \cdot \mathrm{ord}^{F(x)}_y(m).$$

Interchanging the roles of x and x', and thus of u and v, we see that we also have $\operatorname{ord}_x^F(m) = i \cdot \operatorname{ord}_y^{F(x')}(m)$ for all $m \in \mathcal{M}_{F(x'),y}$. Applying this to (3.2.4) we obtain

$$i^{-1}\big(\operatorname{ord}_x^F(g^x)\operatorname{ord}_{x'}^F(f^y) - \operatorname{ord}_x^F(f^x)\operatorname{ord}_{x'}^F(g^y)\big) + \\ + i^{-1}\big(\operatorname{ord}_{x'}^F(g^{x'})\operatorname{ord}_x^F(f^y) - \operatorname{ord}_{x'}^F(f^{x'})\operatorname{ord}_x^F(g^y)\big).$$

To finish the proof, note that because y generalizes to x and x', a local equation of a divisor at y is also a local equation at x and x'. It follows we can replace f^x and $f^{x'}$ by f^y, and g^x and $g^{x'}$ by g^y, and see that the expression is in fact equal to 0. □

Corollary 3.2.42. *Let F be a fine and saturated weakly embedded Kato fan, let $\mathcal{L}, \mathcal{L}' \in$ ClCP(F), and let $\alpha \in Z_k(F)$.*

a) If α is rationally equivalent to 0, then the same is true for $c_1(\mathcal{L}) \cap \alpha$. Therefore, there is and induced map
$$c_1(\mathcal{L}) \cap -\colon A_k(F) \to A_{k-1}(F).$$

b) We have
$$c_1(\mathcal{L}) \cap (c_1(\mathcal{L}') \cap \alpha) = c_1(\mathcal{L}') \cap (c_1(\mathcal{L}) \cap \alpha).$$

Proof. For part a) we may assume that $\alpha = [\operatorname{div}^{F(x)}(m)]$ for some $x \in F$ and $m \in M_x^F$. If $D \in \operatorname{CP}(F(x))$ is a representative of the pull-back of $\mathcal{L}$ to $F(x)$, then $c_1(\mathcal{L}) \cap \alpha = D \cdot [\operatorname{div}^{F(x)}(m)]$. By Proposition 3.2.41 this is equal to $\operatorname{div}^{F(x)}(m) \cdot [D]$, which is clearly rationally equivalent to 0.

For part b) we may assume that $\alpha = [F(x)]$ for some $x \in F$. Then after pulling back $\mathcal{L}$ and $\mathcal{L}'$ to $F(x)$, we may assume that F is connected with generic point x, and that $\alpha = [F]$. Choosing representatives of $\mathcal{L}$ and $\mathcal{L}'$ we further reduce to the statement of Proposition 3.2.41. □

Proposition 3.2.43. *Let $f\colon F \to G$ be a proper subdivision of fine and saturated weakly embedded Kato fans. Furthermore, let $\alpha \in A_*(F)$, and let $\mathcal{L} \in$ ClCP(F). Then the* projection formula *holds, that is*
$$c_1(\mathcal{L}) \cap f_*\alpha = f_*(c_1(f^*\mathcal{L}) \cap \alpha).$$

Proof. In Proposition 3.2.17 we proved this on a level of cycles when $\alpha = [x]$ for some $x \in F$, and $\mathcal{L}$ represented by $\operatorname{div}(m)$ for some $m \in M_{f(x)}^G$. Since the argument was local and $\mathcal{L}$ is locally principal, the exact same reasoning proves the proposition. □

Proposition 3.2.44. *Let $f\colon F \to G$ be a locally exact morphism of fine and saturated weakly embedded Kato fans. Furthermore, let $\alpha \in Z_*(F)$ and $D \in$ Div(F) such that no component of α is contained in $|D|$. Then no component of $f^*\alpha$ is contained in $|f^*D|$, and we have*
$$f^*(D \cdot \alpha) = f^*D \cdot f^*\alpha.$$

In particular, we have
$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$$
for all $\beta \in A_*(F)$ and $\mathcal{L} \in$ ClCP(F).

Proof. Suppose a component of $f^*\alpha$ is contained in $|f^*D| = f^{-1}|D|$. Then the image of its generic point is contained in $|D|$. But this image is the generic point of a component of α, a contradiction. To prove the equality we may assume that $\alpha = [y]$ for some $y \in G$. In this case we have proven the assertion for principal divisors in Proposition 3.2.28. The proof carries over immediately to the more general situation here.

For the "in particular" statement we may assume that $\beta = [y]$ for some $y \in G$. As we may also assume that $\mathcal{L}$ is represented by a divisor $D \in \mathrm{Div}(F)$ with $y \notin |D|$, we are done. □

Proposition 3.2.45. *Let F be a fine and saturated weakly embedded Kato fan, and let $\mathcal{L} \in \mathrm{ClCP}(F) = \mathrm{ClCP}(\Sigma_F)$. Then for $k \in \mathbb{Z}$, the dual of*

$$A_k(F) \to A_{k-1}(F), \ \ \alpha \mapsto c_1(\mathcal{L}) \cap \alpha$$

is equal to the morphism

$$\mathrm{M}_{-k+1}(\Sigma_F) \to \mathrm{M}_{-k}(\Sigma_F), \ \ c \mapsto \mathcal{L} \cup c$$

introduced in Construction 2.2.14.

Proof. Let $c \in \mathrm{M}_{-k+1}(\Sigma_F)$, and let $x \in F$ with $\mathrm{codim}(x, F) = -k$. Furthermore, let $D \in \mathrm{CP}(F)$ be a representative of $\mathcal{L}$, and let $m_y \in M^F$ be a local equation for D at $y \in F$. Then the pull-back of c under $c_1(\mathcal{L}) \cap -$ has weight

$$\sum_y \mathrm{ord}_y(m_y - m_x)c(\sigma_y) \tag{3.2.5}$$

at σ_x, where the sum is over all $y \in F(x)$ with $\mathrm{codim}(y, F(x)) = 1$. When identifying D with a divisor on Σ_F, the restriction of D to σ_y is given by the pull-back of $m_y \in M^{\Sigma_F}$ via the weak embedding. Applying Lemma 3.2.5 we can rewrite (3.2.5) as

$$\sum_y \langle m_y - m_x, u_{\sigma_y/\sigma_y} \rangle c(\sigma_y),$$

the sum again running over all $y \in F(x)$ with $\mathrm{codim}(y, F(x)) = 1$. But those $y \in F(x)$ correspond precisely to the $(-k+1)$-dimensional cones of Σ_F containing σ_x. A comparison with the weight defined in Construction 2.2.14 yields that we indeed obtain $\mathcal{L} \cup c$. □

Proposition 3.2.46. *With the requirements and notation of Proposition 3.2.7, the morphism φ is compatible with intersections with cp-divisors, that is*

$$\varphi(c_1(\mathcal{L}) \cap \alpha) = \mathcal{L} \cup \varphi(\alpha)$$

for all $\mathcal{L} \in \mathrm{ClCP}(F)$ and $\alpha \in A_*(F)$.

Proof. As both sides are linear in α, we may assume that $\alpha = [x]$ for some $x \in F$. We recall that $\mathcal{L} \cup [\mathrm{S}_{\Sigma_F}(\sigma_x)]$ is defined by pulling back $\mathcal{L}$ to S_{Σ_F} and then using Construction 2.2.14. Similarly, $c_1(\mathcal{L}) \cap [x]$ is defined by pulling back $\mathcal{L}$ to $F(x)$ and then taking the associated Weil-divisor class. Since the pull-back of $\mathcal{L}$ to $F(x)$ corresponds to the divisor class on $\Sigma_{F(x)} = \mathrm{S}_{\Sigma_F}(\sigma_x)$ obtained by pulling back $\mathcal{L} \in \mathrm{Div}(\Sigma_F)$, we see that we may assume that F is connected with generic point x. Let $D \in \mathrm{Div}(F)$ be a representative of $\mathcal{L}$. Then we have

$$\varphi(c_1(\mathcal{L}) \cap [F]) = \sum_y \mathrm{ord}_y(D_y)[\mathrm{S}_{\Sigma_F}(\sigma_y)] =$$
$$= \sum_y \langle D_y, u_{\sigma_y}\rangle [\mathrm{S}_{\Sigma_F}(\sigma_y)] = D \cdot [\Sigma_F] = \mathcal{L} \cup \varphi([F])$$

where the sums are over all $y \in F$ with $\mathrm{codim}(y, F) = 1$, and the third equality follows directly from Construction 2.2.12. □

Remark 3.2.47. Even though the morphism $A_*(F) \to A_*(\overline{\Sigma}_F)$ of Proposition 3.2.7 respects intersections with cp-divisors, the functorial behavior of $A_*(F)$ and $A_*(\overline{\Sigma}_F)$ is still very different. While every morphism $f\colon F \to G$ of fine and saturated weakly embedded Kato fans induces a push-forward $(\Sigma_f)_*\colon A_*(\overline{\Sigma}_F) \to A_*(\overline{\Sigma}_G)$ (by Proposition 2.2.21), a push-forward $f_*\colon A_*(F) \to A_*(G)$ does not exist in general (cf. Remark 3.2.19). Conversely, if f is locally exact there is a pull-back $f^*\colon A_*(G) \to A_*(F)$, yet it is unclear how to define a pull-back $f^*\colon A_*(\overline{\Sigma}_G) \to A_*(\overline{\Sigma}_F)$.

3.3 Tropicalization for Logarithmic Schemes

After having studied basic concepts of intersection theory on Kato fans, we want to relate them to the intersection theory of algebraic varieties. To be able to assign a Kato fan to a variety, it needs to come with some additional structure. In the toroidal case, where this works, this data was given by a specified open subset. A very natural generalization of the toroidal setup is given by logarithmic geometry.

3.3.1 Logarithmic Schemes

This section is meant as a crash course to logarithmic geometry. Excellent references for further details are [Kat89, Kat94, Ogu06].

Definition 3.3.1. A *prelogarithmic structure* on a scheme X is given by a sheaf of monoids $\mathcal{M}$ on X, together with a morphism $\alpha\colon \mathcal{M} \to \mathcal{O}_X$. It is called a *logarithmic structure* if additionally the induced morphism $\alpha^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$ is an isomorphism. We frequently abbreviate "logarithmic" to "log". A morphism of two prelog structures $\alpha\colon \mathcal{M} \to \mathcal{O}_X$ and $\beta\colon \mathcal{M}' \to \mathcal{O}_X$ is given by a morphism $\varphi\colon \mathcal{M} \to \mathcal{M}'$ such that $\beta \circ \varphi = \alpha$. This defines a category $(\mathbf{PreLog}_X)$ of prelog structures on X. The category $(\mathbf{Log}_X)$ of log structures on X is its full subcategory whose objects are the log structures.

Remark 3.3.2. Log structures on a scheme X are often defined on the étale site $X_{\text{ét}}$. We will restrict ourselves to log structures defined on the Zariski site X_{Zar}.

Example 3.3.3.

a) Let X be a scheme. Then the inclusion $\mathcal{O}_X^* \to \mathcal{O}_X$ is a log structure. It is called the *trivial log structure* on X. It is the initial element of $(\mathbf{Log}_X)$. Similarly, the identity $\mathcal{O}_X \to \mathcal{O}_X$ is the terminal element of $(\mathbf{Log}_X)$.

b) Let X_0 be an open subset of a scheme X. For an open subset $U \subseteq X$, define

$$\Gamma(U, \mathcal{M}) = \{f \in \Gamma(U, \mathcal{O}_X) \mid f|_{U \cap X_0} \in \Gamma(U \cap X_0, \mathcal{O}_X^*)\}.$$

Then $\mathcal{M}$ is a subsheaf of $\mathcal{O}_X$ and the inclusion $\mathcal{M} \to \mathcal{O}_X$ defines a log structure. It is called the divisorial log structure associated to $X \setminus X_0$. As suggested by the notation, in this way every toroidal embedding comes with a canonical log structure. But we also obtain log structures which do not come from any toroidal embedding. For example, if $X = \mathbb{A}^2$ and $X \setminus X_0 = V(x^2 - y^3)$, then the boundary's singularity is too bad to be toroidal.

c) Let k be a field, and let $M = \mathbb{N} \oplus k^*$. Then the morphism

$$M \to k, \quad (n, a) \mapsto 0^n \cdot a,$$

where we use the convention that $0^0 = 1$, defines a log structure on $\operatorname{Spec} k$. The scheme $\operatorname{Spec} k$ equipped with this log structure is called the *standard log point* (over k).

The inclusion $(\mathbf{Log}_X) \to (\mathbf{PreLog}_X)$ has a left adjoint, which can be constructed as follows: given a prelog structure $\alpha \colon \mathcal{M} \to \mathcal{O}_X$, we define a sheaf $\mathcal{M}^{\log}$ as the sheafification of the presheaf assigning to an open subset U of X the pushout of the diagram

$$\begin{array}{ccc} \alpha^{-1}\big(\Gamma(U, \mathcal{O}_X^*)\big) & \longrightarrow & \Gamma(U, \mathcal{O}_X^*) \\ \big\downarrow & & \\ \Gamma(U, \mathcal{M}) & & \end{array}$$

in the category of monoids. The natural morphism $\alpha^{\log} \colon \mathcal{M}^{\log} \to \mathcal{O}_X$ defines a log structure on X, the *logification of* α. There also is a natural morphism $\mathcal{M} \to \mathcal{M}^{\log}$ which is universal among all morphisms from $\mathcal{M}$ to a log structure on X.

Remark 3.3.4. Taking a pushout (i.e. amalgamated sum) of sheaves of monoids on a topological space X is local, that is for every point $x \in X$, the stalk of the pushout at x is equal to the pushout of the stalks at x. Therefore, to understand pushouts of sheaves of monoids, it suffices to understand pushouts of monoids. Let P, Q_1 and Q_2 be monoids, and let $f_i \colon P \to Q_i$ be two morphisms. Then the amalgamated sum $Q_1 \otimes_P Q_2$ is equal to $Q_1 \oplus Q_2 / \sim$, where $\sim$ is the congruence generated by

$(f_1(p), 0) \sim (0, f_2(p))$ for $p \in P$. If one of the three monoids is a group, then $\sim$ can be described explicitly: in that case we have $(q_1, q_2) \sim (q_1', q_2')$ if and only there exist $p, p' \in P$ such that $q_1 + f_1(p) = q_1' + f_1(p')$ and $q_2 + f_2(p') = q_2' + f_2(p)$ [Ogu06, Prop. 1.1.4].

Definition 3.3.5. Let X be a scheme. A *chart* for a logarithmic structure $\mathcal{M} \to \mathcal{O}_X$ on X consists of a monoid P, together with a morphism $\alpha\colon P \to \mathcal{O}_X$ (where we consider P as a constant sheaf) and an isomorphism $\varphi\colon P^{\log} \to \mathcal{M}$.

Remark 3.3.6. Note that the data of a chart can be more compactly presented by a morphism $P \to \mathcal{M}$. Then α can be recovered as the composite $P \to \mathcal{M} \to \mathcal{O}_X$, and the morphism $\varphi\colon P^{\log} \to \mathcal{M}$ is induced by $P \to \mathcal{M}$ (using the universal property of $P^{\log}$). Of course, if $P \to \mathcal{M}$ should really be a chart, we have to require that φ is an isomorphism.

Definition 3.3.7. A log structure is said to be *fine* if it locally admits a chart $P \to \mathcal{M}|_U$ with P a fine monoid. It is said to be *fine and saturated* if P can be chosen to be saturated as well.

Example 3.3.8. Consider the standard log point $(\operatorname{Spec} k, \mathcal{M})$ over a field k. Let $\alpha\colon \mathbb{N} \to k$ be the morphism specified by $1 \mapsto 0$. Then α defines a chart. Indeed, the preimage of k^* under α is trivial, and hence the amalgamated sum $\mathbb{N} \otimes_{\alpha^{-1}k^*} k$ is equal to the direct sum $\mathbb{N} \oplus k^* = \Gamma(\operatorname{Spec} k, \mathcal{M})$. The morphism $\alpha^{\log}$ sends (n, u) to $\alpha(n) \cdot u = 0^n \cdot u$, thus $\alpha^{\log}$ is equal to the log structure $\mathcal{M} \to \mathcal{O}_{\operatorname{Spec} k}$.

Definition 3.3.9. A scheme equipped with a log structure is called a *logarithmic scheme*. If X is a log scheme, we will denote its log structure by $\alpha_X\colon \mathcal{M}_X \to \mathcal{O}_X$. The underlying scheme of X will be denoted by $\underline{X}$. The sheaf $\overline{\mathcal{M}}_X$, that is the sheafification of the presheaf $U \mapsto \overline{\Gamma(U, \mathcal{M}_X)}$ (recall that the overline denotes the sharpening), is called the *characteristic sheaf* of X. We will denote the sharp monoidal space $(X, \overline{\mathcal{M}}_X)$ by $\overline{X}$. The locus

$$\{x \in X \mid \overline{\mathcal{M}}_{X,x} = 0\}$$

where the log structure is trivial will be denoted by X^*.

A morphism $f\colon X \to Y$ of log schemes consists of a morphism $\underline{f}\colon \underline{X} \to \underline{Y}$ of the underlying schemes, together with a morphism $f^\flat\colon f^{-1}\mathcal{M}_X \to \mathcal{M}_Y$ such that the diagram

$$\begin{array}{ccc} f^{-1}\mathcal{M}_Y & \xrightarrow{f^\flat} & \mathcal{M}_X \\ \downarrow{\scriptstyle f^{-1}\alpha_Y} & & \downarrow{\scriptstyle \alpha_X} \\ f^{-1}\mathcal{O}_Y & \xrightarrow{f^\sharp} & \mathcal{O}_X \end{array}$$

commutes. We denote the induced morphism $\overline{X} \to \overline{Y}$ by $\overline{f}$.

Example 3.3.10. Let X and Y be logarithmic schemes. Assume the log structure on Y is trivial, and let $g\colon \underline{X} \to \underline{Y}$ be a morphism of the underlying schemes. Then there is exactly one morphism $f\colon X \to Y$ of log schemes such that $\underline{f} = g$. Indeed, the image of $g^\sharp \circ g^{-1}\alpha_Y$ is contained in $\mathcal{O}_X^*$ so that we must have $f^\flat = \alpha_X|_{\mathcal{M}_X^*}^{-1} \circ g^\sharp \circ g^{-1}\alpha_Y$. Similarly if the log structure on Y is arbitrary, but α_X is the identity, then the only way to extend g to a morphism f of log schemes is by setting $f^\flat = g^\sharp \circ g^{-1}\alpha_Y$.

Definition 3.3.11. Let X be a log scheme and let $f\colon Y \to \underline{X}$ be a morphism of schemes. Then the inverse image $f^*\mathcal{M}_X \to \mathcal{O}_Y$ of the log structure of X is defined as the logification of the composite

$$f^{-1}\mathcal{M}_X \xrightarrow{f^{-1}\alpha_X} f^{-1}\mathcal{O}_X \xrightarrow{f^\sharp} \mathcal{O}_Y.$$

It is immediate that after equipping Y with this log structure, the morphism f lifts to a morphism $Y \to X$ of log schemes. It is also clear that if Y already had a log structure, and f is the underlying morphism of schemes of a morphism g of log schemes, then g factors uniquely through $(Y, f^*\mathcal{M}_X)$.

Definition 3.3.12. Let X be a fine and saturated log scheme such that $\underline{X}$ is locally Noetherian. For for a point $x \in X$ define $I(x, \mathcal{M}_X) \trianglelefteq \mathcal{O}_{X,x}$ as the ideal generated by the image of the maximal ideal of $\mathcal{M}_{X,x}$. Then X is called log-regular if for all $x \in X$

a) $\mathcal{O}_{X,x}/I(x, \mathcal{M}_X)$ is a regular local ring, and

b) $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}/I(x, \mathcal{M}_x) + \operatorname{rk}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X,x})$.

Note that in b) the inequality "$\leq$" always holds [Kat94, Lemma 2.3].

If $\underline{X}$ is of finite type over an algebraically closed field, then X is log-regular if and only if $X^* \subseteq X$ is a strict toroidal embedding and $\mathcal{M}_X$ is the divisorial log structure associated to $X \setminus X^*$.

Definition 3.3.13. We say that a log scheme X is integral if $\underline{X}$ is an integral scheme, and the log structure $\alpha_X\colon \mathcal{M}_X \to \mathcal{O}_X$ factors through $\mathcal{O}_X^\bullet := \mathcal{O}_X \setminus \{0\}$, that is α_X maps no section of $\mathcal{M}_X$ to 0.

3.3.2 Characteristic Fans

In order to tropicalize subschemes of a logarithmic scheme X we need a notion analogous to the associated cone complex of a toroidal embedding. In the logarithmic setting this will be a Kato fan F together with a *strict* morphism $\chi\colon \overline{X} \to F$ of sharp monoidal spaces. Here, strict means that the morphism $\chi^\flat\colon \chi^{-1}\mathcal{M}_F \to \overline{\mathcal{M}}_X$ is an isomorphism. Note that if $P \to \mathcal{M}_X$ is a chart, then the induced morphism $\overline{X} \to \operatorname{Spec} \overline{P}$ is strict. We will see shortly that if the morphisms $\overline{U} \to \operatorname{Spec} \overline{P}_U$ for local charts $P_U \to \mathcal{M}_U$ glue to a strict morphism $\chi_X\colon \overline{X} \to F_X$, then χ_X has a universal property. This has been done in [Uli13] for fine and saturated log schemes. Even though our primary interest is in the fine and saturated case, we will do it in a more general framework to pave the way to work with not necessarily fine log schemes, occurring for example when working over a non-discrete valuation ring. A detailed account of the construction of χ_X will also help us to bring weak embeddings into the picture, which are necessary for intersection-theoretic considerations on F_X.

Definition 3.3.14. A monoid P is called *quasi-finite* if $\operatorname{Spec} P$ is finite. A logarithmic scheme X is *quasi-finite* if its underlying topological space is locally Noetherian and locally there exists a chart $P \to \mathcal{M}_X$ with P quasi-finite.

Definition 3.3.15. A log scheme X is called *small* if its underlying topological space is Noetherian, $\Gamma(X, \overline{\mathcal{M}}_X)$ is quasi-finite, and the canonical morphism $\underline{X} \to \operatorname{Spec} \Gamma(X, \overline{\mathcal{M}}_X)$ is strict, has the closed point of $\operatorname{Spec} \Gamma(X, \overline{\mathcal{M}}_X)$ in its image, and the intersection C of the closures of the irreducible components of its fibers is nonempty. We call C the *center* of X. An open subset U of a log scheme X is called small if it is small as a logarithmic scheme.

The idea for the construction of a Kato fan associated to a log scheme X is to cover X with small open subsets and then glue the affine Kato fans $\operatorname{Spec} \Gamma(U, \overline{\mathcal{M}}_X)$. The following sufficient condition for a log scheme to be small will be very helpful.

Proposition 3.3.16. *Let X be a log scheme with Noetherian underlying topological space, and let $P \to \mathcal{M}_X$ be a quasi-finite chart which is local it $x \in X$. Furthermore, assume that the intersection of the closures of the components of the fibers of the induced morphism $\underline{X} \to \operatorname{Spec} \overline{P}$ contains x. Then X is small with x in its center, and $\Gamma(X, \overline{\mathcal{M}}_X) \cong \overline{P}$.*

Proof. Since P is a chart on U, we know that $\chi : \underline{X} \to \operatorname{Spec} \overline{P}$ is strict [Ogu06, Prop. 2.1.4], and because the chart is local at x, we know that x is mapped to the closed point. Furthermore, χ factors through $\underline{X} \to \operatorname{Spec} \Gamma(X, \overline{\mathcal{M}}_X)$. Thus, it suffices to show that the natural map $\overline{P} \to \Gamma(X, \overline{\mathcal{M}}_X)$ is an isomorphism. The composite

$$\overline{P} \to \Gamma(X, \overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X,x}$$

is an isomorphism because the chart is local at x [Ogu06, Def. 2.2.8]. Hence we only need to show that $\Gamma(X, \overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X,x}$ is injective. So assume $s, t \in \Gamma(X, \overline{\mathcal{M}}_X)$ have the same stalk at x. Let $y \in X$ be any point, and let ξ be the generic point of an irreducible component of $\chi^{-1}\{\chi(y)\}$ containing y. By assumption, ξ specializes to x, thus $s_\xi = t_\xi$. The strictness of χ implies that the arrows from left to right in the commutative diagram

$$\begin{array}{ccc} & \xrightarrow{\chi_y^\flat} & \overline{\mathcal{M}}_{X,y} \\ \mathcal{M}_{\operatorname{Spec} P,\chi(y)} & & \downarrow \\ & \xrightarrow{\chi_\xi^\flat} & \overline{\mathcal{M}}_{X,\xi} \end{array}$$

are isomorphisms. It follows that the cospecialization map $\overline{\mathcal{M}}_{X,y} \to \overline{\mathcal{M}}_{X,\xi}$ is an isomorphism as well, and hence that $s_y = t_y$. □

Example 3.3.17. Let $X = \mathbb{A}^2_k$ be affine 2-space over a field k, and consider the log structure given by the chart $\mathbb{N}^2 \to k[x,y]$ which maps the free generators e_1 and e_2 of $\mathbb{N}^2$ to x and $x - y^2$, respectively. Both x and $x - y^2$ vanish at $(0,0)$, so the chart is local at the origin. The points of $\operatorname{Spec} \mathbb{N}^2$ are $\emptyset$, $e_1 + \mathbb{N}^2$, $e_2 + \mathbb{N}^2$, and $\mathbb{N}^2 \setminus \{0\}$. The preimage of $\mathbb{N}^2 \setminus \{0\}$ under $X \to \operatorname{Spec} \mathbb{N}^2$ is the set of points where both x and $x - y^2$ vanish, that is it is equal to $\{(0,0)\}$. The preimage of $e_1 + \mathbb{N}^2$ is equal to the points where x vanishes, but $x - y^2$ is invertible, so it is equal to $V(x) \setminus \{(0,0)\}$. Similarly, the preimage of $e_2 + \mathbb{N}^2$ is equal to $V(x - y^2) \setminus \{(0,0)\}$. Finally, the preimage of $\emptyset$ is $\mathbb{A}^2 \setminus V(x(x - y^2))$. We see that the prerequisites of Proposition 3.3.16 are met, and hence that X is small, has $(0,0)$ in its center, and $\Gamma(X, \overline{\mathcal{M}}_X) = \mathbb{N}^2$. In fact, we immediately see that the center is equal to $\{(0,0)\}$.

Corollary 3.3.18. *Let X be a quasi-finite log scheme, and let $x \in X$. Then there exists a small open subset of X having x in its center.*

Proof. Let $P \to \mathcal{M}_U$ be a quasi-finite chart in a Noetherian open neighborhood U of x. After possible shrinking U and replacing P by one of its localizations we may assume that it is local at x [Ogu06, Remark 2.2.9]. Let Y be the union of those closures of irreducible components of the fibers of $\underline{U} \to \operatorname{Spec} \overline{P}$ which do not contain x. Replacing U by $U \setminus Y$ we may assume that Y is empty. Applying Proposition 3.3.16 finishes the proof. □

For a small log scheme X the morphism $\overline{X} \to \operatorname{Spec} \Gamma(X, \overline{\mathcal{M}}_X)$ is universal among the morphisms from $\overline{X}$ to Kato fans, as the following proposition shows.

Proposition 3.3.19. *Let X be a small log scheme, and let $\psi \colon \overline{X} \to F$ be a morphism to a Kato fan F. Then ψ factors uniquely through $\chi \colon \overline{X} \to \operatorname{Spec} \Gamma(X, \overline{\mathcal{M}}_X)$. Furthermore, if ψ is strict then so is the induced morphism $\operatorname{Spec} \Gamma(X, \overline{\mathcal{M}}_X) \to F$.*

Proof. The statement is clear if F is affine, thus it suffices to prove that ψ factors through an affine subfan of F. We do so by showing that ψ is constant on the irreducible components of the fibers of χ. Let S be such a component, let ξ be its generic point, and let $s \in S$ arbitrary. Let $V = \operatorname{Spec} P$ be an affine open neighborhood of $\psi(s)$, and let $\mathfrak{p}$ and $\mathfrak{q}$ be the ideals of P corresponding to $\psi(s)$ and $\psi(\xi)$, respectively. Then we have a commutative diagram

$$\begin{array}{ccc} \overline{P}_{\mathfrak{p}} & \xrightarrow{\psi_s^\flat} & \overline{\mathcal{M}}_{X,s} \\ \downarrow & & \downarrow \\ \overline{P}_{\mathfrak{q}} & \xrightarrow{\psi_\xi^\flat} & \overline{\mathcal{M}}_{X,\xi}, \end{array}$$

where the vertical arrows are cospecialization maps. Since ψ is a morphism of locally monoidal spaces, the horizontal arrows are local, and by the same argument as given in the proof of Proposition 3.3.16, the vertical map on the right is an isomorphism. It follows that the cospecialization map $\overline{P}_{\mathfrak{p}} \to \overline{P}_{\mathfrak{q}}$ is local as well, which implies that $\mathfrak{p} = \mathfrak{q}$. Let x be in the center of X. Then x is in the closures of all irreducible components of all fibers of χ. Hence, every point in the image of ψ specializes to $\psi(x)$. In particular, the image of ψ is contained in any affine open neighborhood of $\psi(x)$.

Now assume that ψ is strict, and let $\varphi\colon \operatorname{Spec}\Gamma(X, \overline{\mathcal{M}}_X) \to F$ be the induced morphism. Then the composite map

$$\mathcal{M}_{F,\psi(x)} \xrightarrow{\varphi^\flat_{\chi(x)}} \mathcal{M}_{\operatorname{Spec}\Gamma(X,\overline{\mathcal{M}}_X),\chi(x)} \xrightarrow{\chi^\flat_x} \overline{\mathcal{M}}_{X,x}$$

is equal to $\psi^\flat_x$, which is an isomorphism by the strictness of ψ. Because $\chi^\flat_x$ is an isomorphism as well, the same is true for $\varphi^\flat_{\chi(x)}$. But $\chi(x)$ is the closed point of $\operatorname{Spec}\Gamma(X, \overline{\mathcal{M}}_X)$, so φ has to be strict. □

For a general quasi-finite log scheme, a universal fan in the sense of Proposition 3.3.19 cannot be expected to be affine. But we can expect the existence of a universal fan which is glued from the universal fans of small open subsets. To understand how these fans glue, consider the restriction morphism $r : \Gamma(X, \overline{\mathcal{M}}_X) \to \Gamma(U, \overline{\mathcal{M}}_X)$ in case U is a small open subset of a small log scheme X. Let x be an element in the center of U. Since $\overline{X} \to \operatorname{Spec}\Gamma(X, \overline{\mathcal{M}}_X)$ is strict, the morphism $\Gamma(X, \overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X,x}$ is a *sharpened localization*, that is an identification of $\overline{\mathcal{M}}_{X,x}$ with $\overline{\Gamma(X, \overline{\mathcal{M}}_X)_{\mathfrak{p}}}$ for some prime ideal $\mathfrak{p} \in \operatorname{Spec}\Gamma(X, \overline{\mathcal{M}}_X)$. On the other hand, as x is in the center of U, the morphism $\Gamma(U, \overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X,x}$ is an isomorphism. Therefore, r is a sharpened localization as well.

Lemma 3.3.20. *Let $f, g\colon X \to Y$ be two morphisms of locally monoidal spaces such that their underlying continuous maps are open, let Z be the coequalizer of f and g in the category of locally monoidal spaces, and let $h\colon Y \to Z$ be the universal morphism. Then h is open as well. If furthermore the morphisms*

$$f^\flat_x\colon \mathcal{M}_{Y,f(x)} \to \mathcal{M}_{X,x} \quad \textit{and}$$
$$g^\flat_x\colon \mathcal{M}_{Y,g(x)} \to \mathcal{M}_{X,x}$$

are injective for all $x \in X$, then the same is true for the morphisms

$$h^\flat_y\colon \mathcal{M}_{Z,h(y)} \to \mathcal{M}_{Y,y}$$

for all $y \in Y$.

Proof. The underlying topological space of Z is the coequalizer of f and g in the category of sets equipped with the quotient topology. More precisely, Z is equal to $Y/\sim$, where $\sim$ is the equivalence relation generated by $f(x) \sim g(x)$ for $x \in X$. Consider the action of the free monoid $P = \langle a, b\rangle$ on two elements a and b on the set $\mathcal{T}_Y$ of open subsets of Y given by $a.V = g(f^{-1}V)$ and $b.V = f(g^{-1}V)$ for $V \in \mathcal{T}_Y$. Then for every open subset $U \subseteq Y$ we have

$$h^{-1}h(U) = \bigcup_{p\in P} p.U,$$

which is open in X. This shows the openness of h. Now let $y \in Y$, and assume that $s', t' \in \mathcal{M}_{Z,h(y)}$ have the same image under $h^\flat_y$. As the structure sheaf of Z is the equalizer of

$$h_*\mathcal{M}_Y \rightrightarrows (h \circ f)_*\mathcal{M}_X = (h \circ g)_*\mathcal{M}_X$$

there exists an open neighborhood $W \subseteq Z$ of $h(y)$ and sections $s, t \in \Gamma(h^{-1}W, \mathcal{M}_Y)$ which represent s' and t', respectively. By assumption we have $s_y = t_y$ in $\mathcal{M}_{Y,y}$. Let

$$U = \{u \in h^{-1}W \mid t_u = s_u\}.$$

For every $v \in a.U$ there exists $x \in f^{-1}U$ with $g(x) = v$. Then in $\mathcal{M}_{X,x}$ we the equality

$$g^\flat_x(s_v) = g^\flat(s)_x = f^\flat(s)_x = f^\flat_x(s_{f(x)}) = f^\flat_x(t_{f(x)}) = f^\flat(t)_x = g^\flat(t)_x = g^\flat_x(t_v).$$

So injectivity of $g^\flat_x$ implies that $v \in U$ and hence that $a.U \subseteq U$. Similarly, we see that $b.U \subseteq U$. Together, this shows that $U = h^{-1}h(U)$. The equality $s|_{h^{-1}h(U)} = t|_{h^{-1}h(U)}$ proves that $s' = t'$, and thus that $h^\flat_y$ is injective. □

Proposition and Definition 3.3.21 (cf. [Uli13]). *Let X be a quasi-finite log scheme, and let $X = \bigcup_{i\in I} U_i$ and $U_i \cap U_j = \bigcup_{k \in K_{i,j}} V_k$ be covers by small open subsets. Then the following are equivalent:*

a) *There exists a strict morphism $\chi_X\colon \overline{X} \to F_X$ to a Kato fan F_X through which all other such morphisms factor uniquely.*

b) *There exists a strict morphism $\overline{X} \to G$ to a Kato fan G.*

c) *The coequalizer of the diagram*

$$\coprod_{i,j\in I} \coprod_{k \in K_{i,j}} \operatorname{Spec}\Gamma(V_k, \overline{\mathcal{M}}_X) \rightrightarrows \coprod_{i \in I} \operatorname{Spec}\Gamma(U_i, \overline{\mathcal{M}}_X)$$

with the two obvious morphisms exists in the category of Kato fans with strict morphisms.

In case these equivalent statements are true, we say that X is without monodromy. *The morphism χ_X from a) is unique up to canonical isomorphisms, and F_X is equal to the coequalizer of the diagram in c) taken in the category of locally monoidal spaces. We call F_X the characteristic fan of X and sometimes refer to χ_X as the characteristic morphism of X.*

Proof. The implication a) ⇒ b) is obvious. For the implication b) ⇒ c) let F be the coequalizer of the diagram in the category of locally monoidal spaces. For every small open subset U of X the composite $\overline{U} \to \overline{X} \to G$ is strict, so by Proposition 3.3.19 there exists a unique strict morphism $\alpha_U\colon \operatorname{Spec}\Gamma(U, \overline{\mathcal{M}}_X) \to G$ through which it factors. These morphisms obviously respect inclusions of small open sets, hence induce a morphism $\alpha\colon F \to G$. By the strictness and hence injectivity of the $\alpha_i := \alpha_{U_i}$, the canonical maps $\beta_i : \operatorname{Spec}\Gamma(U_i, \overline{\mathcal{M}}_X) \to F$ are injective as well. Since the two maps in the coequalizer diagram are open, the quotient map $\coprod_{i\in I}\Gamma(U_i,\overline{\mathcal{M}}_X)\to F$, and hence the morphisms β_i are open by Lemma 3.3.20. Therefore, β_i maps $\operatorname{Spec}\Gamma(U_i,\overline{\mathcal{M}}_X)$ homeomorphically onto its image, which is an open subset of F. Furthermore, the strictness of α_i implies that the composite morphism

$$\mathcal{M}_{G,\alpha_i(x)} \xrightarrow{\alpha^\flat_{\beta_i(x)}} \mathcal{M}_{F,\beta_i(x)} \xrightarrow{(\beta_i)^\flat_x} \mathcal{M}_{\operatorname{Spec}\Gamma(U_i,\overline{\mathcal{M}}_X),x}$$

is an isomorphism for all $x\in\operatorname{Spec}\Gamma(U_i,\overline{\mathcal{M}}_X)$, and hence that $\beta_i^{-1}\mathcal{M}_F\to\mathcal{M}_{\operatorname{Spec}\Gamma(U_i,\overline{\mathcal{M}}_X)}$ is surjective on stalks. But again by Lemma 3.3.20, it is injective as well. This proves that the β_i provide affine charts for the locally monoidal space F, which is hence a Kato fan. Of course, this also means that all β_i are strict. That F also is the coequalizer in the category of Kato fans with strict morphisms follows immediately.

Now assume c) and denote the coequalizer of the diagram by F. Then for every $i\in I$ we have a morphism $\overline{U}_i\to\operatorname{Spec}\Gamma(U_i,\overline{\mathcal{M}}_X)\to F$. By the definition of F these morphisms glue and yield a strict morphism $\chi\colon\overline{X}\to F$. Let $\psi\colon\overline{X}\to G$ be another strict morphism into a Kato fan G. The proof of the implication b) $\Rightarrow$ c) shows that there is a strict morphism $\alpha\colon F\to G$ which is uniquely defined by the commutative diagrams

$$\begin{array}{ccc}
\operatorname{Spec}\Gamma(U_i,\overline{\mathcal{M}}_X) & \longleftarrow & \overline{U}_i \\
\downarrow & & \downarrow \\
F & \xrightarrow{\alpha} & G
\end{array}$$

for $i\in I$. By the definition of χ the commutativity of these diagrams is equivalent to that of

$$\begin{array}{ccc}
 & \overset{\chi}{\nearrow} & F \\
\overline{X} & & \downarrow{\scriptstyle\alpha} \\
 & \underset{\psi}{\searrow} & G.
\end{array}$$

This shows that ψ factors through F via α, and, simultaneously, that α is unique with this property. □

The construction of characteristic fans is functorial. In fact, we have the following global version of Proposition 3.3.19.

Proposition 3.3.22. *Let X be a quasi-finite log scheme without monodromy, and let $\psi\colon\overline{X}\to G$ be a morphism to a Kato fan G. Then there exists a unique morphism $\varphi\colon F_X\to G$ of Kato fans such that $\psi=\varphi\circ\chi_X$. Furthermore, if ψ is strict, then so is φ.*

Proof. First note that if X is small, then the statement is the same as that of Proposition 3.3.19. Now let X be arbitrary and suppose that $\varphi,\varphi'\colon F_X\to G$ satisfy $\varphi\circ\chi_X=\psi=\varphi'\circ\chi_X$. Let $U\subseteq X$ be a small open subset. Then there is a commutative diagram

$$\begin{array}{ccccc} \overline{U} & \longrightarrow & \overline{X} & & \\ \downarrow & & \downarrow & \searrow^{\psi} & \\ F_U & \longrightarrow & F_X & \xrightarrow[\varphi,\,\varphi']{} & G. \end{array}$$

As U is small we conclude that the restrictions of φ and φ' to F_U are equal. By Proposition 3.3.21 there is an open cover of F_X by affines of the form F_U for $U \subseteq X$ small and open. Therefore, $\varphi = \varphi'$, showing the uniqueness. For the existence we use Proposition 3.3.19 to obtain morphisms $\varphi_U \colon F_U \to G$ with $\varphi_U \circ \chi_U = \psi|_U$ for small open subsets $U \subseteq X$. By the uniqueness part these morphism are compatible with inclusions $F_V \to F_U$ whenever $V \subseteq U \subseteq X$ are small and open. This is exactly what we need for the φ_U to glue to a morphism $\varphi \colon F_X \to G$ by Proposition 3.3.21 c). By construction, it satisfies $\varphi \circ \chi_X = \psi$. If ψ is strict, then so are all φ_U for $U \subseteq X$ small and open by Proposition 3.3.19. Because strictness is a local property, this implies that φ is strict. □

Example 3.3.23.

a) Let X be the nodal curve consisting of two irreducible components C_1 and C_2 which are isomorphic to $\mathbb{P}^1$ and intersect each other in two points P and Q. It is depicted on the left of Figure 3.3. For $i = 1, 2$ let φ_i be a regular function on $X \setminus \{Q\}$ that vanishes on C_i and gives a local equation for P on C_{3-i}. Equip X with the log structure given by the charts

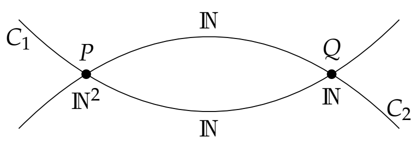
$$\begin{aligned} \mathbb{N}^2 &\to \Gamma(X \setminus \{Q\}, \mathcal{O}_X), \;\; e_i \mapsto \varphi_i \quad \text{and} \\ \mathbb{N} &\to \Gamma(X \setminus \{P\}, \mathcal{O}_X), \;\; 1 \mapsto 0, \end{aligned}$$

glued by identifying φ_i with 0 on $X \setminus C_{3-i}$. Then $U_1 = X \setminus \{Q\}$, $U_2 = X \setminus \{P\}$, and $V_i = X \setminus C_i$ are small subsets of X with $F_{U_1} = \operatorname{Spec} \mathbb{N}^2$, $F_{U_2} = \operatorname{Spec} \mathbb{N}$ and $F_{V_i} = \operatorname{Spec} \mathbb{N}$. The natural morphisms $F_{V_i} \to F_{U_1}$ for $i = 1, 2$ are dual to the sharpened localizations corresponding to the faces $\mathbb{N}e_i$ of $\mathbb{N}^2$. On the other hand, the natural morphisms $F_{V_i} \to F_{U_2}$ are the identity for $i = 1, 2$. It follows that the two codimension-1 points of F_{U_1} get identified in any commutative diagram

$$F_{V_1} \coprod F_{V_2} \rightrightarrows F_{U_1} \coprod F_{U_2} \to F.$$

In particular, $F_{U_1} \coprod F_{U_2} \to F$ cannot be a strict morphism of Kato fans. We conclude that X has monodromy.

b) Now let X be the nodal curve consisting of three irreducible components C_i, $1 \leq i \leq 3$, which are isomorphic to $\mathbb{P}^1$ and whose pairwise intersection is a point, say $C_i \cap C_j = \{P_{ij}\}$. It is depicted on the right of Figure 3.3. For $i = 1, 2$ let φ_i be a regular function on $X \setminus C_{3-i}$ which is 0 on C_i and gives a local equation



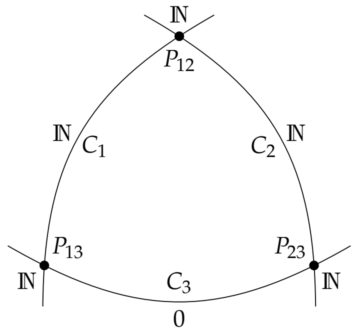


Figure 3.3: Two logarithmic curves, one with monodromy and one without. The depicted monoids represent the characteristic sheaves.

for P_{i3} on C_3. We consider the log structure on X given by the charts

$$\begin{array}{l} \mathbb{N} \to \Gamma(X \setminus C_1), \ \ 1 \to \varphi_2, \\ \mathbb{N} \to \Gamma(X \setminus C_2), \ \ 1 \to \varphi_1 \\ \mathbb{N} \to \Gamma(X \setminus C_3), \ \ 1 \to 0, \end{array} \quad \text{, and}$$

glued by identifying φ_i with 0 on $C_i \setminus \{P_{i,3-i}, P_{i3}\}$ for $i = 1, 2$. The open subsets $U_i = X \setminus C_i$ are small for $1 \leq i \leq 3$ with $F_{U_i} = \operatorname{Spec} \mathbb{N}$. The open subsets $V_{ij} = U_i \cap U_j$ are also small for $i \neq j$ with $F_{V_{ij}} = \operatorname{Spec} 0$ if $3 \notin \{i, j\}$ and $F_{V_{ij}} = \operatorname{Spec} \mathbb{N}$ else. The natural morphisms $F_{V_{ij}} \to F_{U_i}$ are either the identity or the inclusion of $\operatorname{Spec} 0$ into $\operatorname{Spec} \mathbb{N}$, so the diagram

$$F_{V_{12}} \coprod F_{V_{13}} \coprod F_{V_{23}} \rightrightarrows F_{U_1} \coprod F_{U_2} \coprod F_{U_3}$$

has $\operatorname{Spec} \mathbb{N}$ as its coequalizer in the category of Kato fans with strict morphisms. Thus, X does not have monodromy and $F_X = \operatorname{Spec} \mathbb{N}$. Note that even though $F_{U_1} = \operatorname{Spec} \mathbb{N}$ and $F_{U_2} = \operatorname{Spec} \mathbb{N}$ we have $F_{U_1 \cap U_2} = F_{V_{12}} = \operatorname{Spec} 0$, that is taking characteristic fans does not preserve fiber products.

By what we have seen so far, we can canonically assign a Kato fan to any quasi-finite log scheme X without monodromy. This Kato fan has a completely combinatorial description, especially if X is fine and saturated in which case we may describe it with a cone complex. Yet to be able to assign a balanced tropical cycle to X, we still lack a weak embedding of F_X. As in the toroidal case, this weak embedding should be related to the sheaf of invertible functions on X.

Lemma 3.3.24. *Let*

$$V \overset{f}{\underset{g}{\rightrightarrows}} U \xrightarrow{h} X$$

be a coequalizer diagram of topological spaces such that h is a local homeomorphism. Furthermore, let $\mathcal{F}$ be a sheaf of sets on X, and denote by $\mathcal{F}_U$ and $\mathcal{F}_V$ its pull-backs to U and V, *respectively. Then*

$$\Gamma(X,\mathcal{F}) \longrightarrow \Gamma(U,\mathcal{F}_U) \rightrightarrows \Gamma(V,\mathcal{F}_V)$$

is an equalizer diagram.

Proof. Denote the equalizer of $\Gamma(U,\mathcal{F}_U) \rightrightarrows \Gamma(V,\mathcal{F}_V)$ by E. It is the subset of $\Gamma(U,\mathcal{F}_U)$ consisting of all sections s with $f^*s = g^*s$. It follows directly from the equality $h \circ f = h \circ g$ that $\Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F}_U)$ factors through E. Furthermore, the induced map $\Gamma(X,\mathcal{F}) \to E$ is injective by the surjectivity of h. To show that it is surjective as well, let $s \in E$. For every $x \in U$ the image of s_x under the natural isomorphism $(\mathcal{F}_U)_x \cong \mathcal{F}_{h(x)}$ defines a germ of $\mathcal{F}$ at $h(x)$. If we show that any other point $x' \in U$ with $h(x') = h(x)$ defines the same germ at $h(x)$, we can use this to obtain a well-defined section of the espace étalé of $\mathcal{F}$, which then also has to be continuous since h is a local homeomorphism. The thus defined element of $\Gamma(X,\mathcal{F})$ will clearly be a preimage of s. To prove that this element is well-defined we may assume that there exists $y \in V$ with $f(y) = x$ and $g(y) = x'$ (because the underlying set of X is the set-theoretic coequalizer of f and g). In that case, we have a commutative diagram

$$\begin{array}{ccccc} & & (\mathcal{F}_U)_x & & \\ & \overset{h_x^*}{\nearrow} & & \overset{f_y^*}{\searrow} & \\ \mathcal{F}_{h(x)} = \mathcal{F}_{h(x')} & & & & (\mathcal{F}_V)_y \\ & \underset{h_{x'}^*}{\searrow} & & \underset{g_y^*}{\nearrow} & \\ & & (\mathcal{F}_U)_{x'} & & \end{array}$$

in which all arrows are isomorphisms. Furthermore, since s is in E, the germs s_x and $s_{x'}$ are mapped to the same element in $(\mathcal{F}_V)_y$. Thus, both s_x and $s_{x'}$ have the same preimage in $\mathcal{F}_{h(x)}$, which finishes the proof. □

Proposition 3.3.25. *Let X be a quasi-finite log scheme without monodromy, and let $\mathcal{F}$ be a sheaf of sets on F_X. Then the natural map*

$$\Gamma(F_X,\mathcal{F}) \to \Gamma(X,\chi_X^{-1}\mathcal{F})$$

is an isomorphism.

Proof. First assume X is small. Let $x \in X$ be a point in the center. Then $\chi_X(x)$ is the unique closed point of F_X and thus $\Gamma(F_X,\mathcal{F}) \to \mathcal{F}_{\chi(x)}$ is an isomorphism. Since there also is a natural isomorphism $(\chi_X^{-1}\mathcal{F})_x \cong \mathcal{F}_{\chi_X(x)}$ we obtain a morphism

$$\Gamma(X,\chi_X^{-1}\mathcal{F}) \to (\chi_X^{-1}\mathcal{F})_x \to \mathcal{F}_{\chi_X(x)} \to \Gamma(F_X,\mathcal{F})$$

which is obviously left-inverse to $\Gamma(F_X,\mathcal{F}) \to \Gamma(X,\chi_X^{-1}\mathcal{F})$. So to finish the proof in case X is small it suffices to show that $\Gamma(X,\chi_X^{-1}\mathcal{F}) \to (\chi_X^{-1}\mathcal{F})_x$ is injective. Let $s,t \in \Gamma(X,\chi_X^{-1}\mathcal{F})$ with $s_x = t_x$, and let $U = \{y \in X \mid s_y = t_y\}$. As the cospecialization maps $(\chi_X^{-1}\mathcal{F})_y \to (\chi_X^{-1}\mathcal{F})_\xi$ are isomorphisms whenever $\xi \rightsquigarrow y$, and ξ and y are in the same fiber of χ_X, it follows that a component of a fiber of χ_X is either fully contained in U or disjoint from it. It now follows from U being open and from x belonging to the center of X that $U = X$ and hence that $s = t$.

For the general case let $X = \bigcup_{i\in I} U_i$ and $U_i \cap U_j = \bigcup_{k\in K_{ij}} V_k$ be covers by small open subsets. By Proposition 3.3.21 the characteristic fan F_X is the coequalizer of

$$\coprod_{i,j\in I}\coprod_{k\in K_{ij}} F_{V_k} \rightrightarrows \coprod_{i\in I} F_{U_i}.$$

and we may consider the F_{U_i} and the F_{V_k} as open subsets of F_X. In particular, we may apply Lemma 3.3.24 and see that

$$\Gamma(F_X,\mathcal{F}) \to \prod_{i\in I}\Gamma(F_{U_i},\mathcal{F}) \rightrightarrows \prod_{i,j\in I}\prod_{k\in K_{ij}}\Gamma(F_{V_k},\mathcal{F})$$

is an equalizer diagram. The proposition now follows from the small case and the sheaf property of $\chi_X^{-1}\mathcal{F}$. □

Definition 3.3.26. Let X be a quasi-finite log scheme without monodromy. We denote by M^X the quotient $\Gamma(X,\mathcal{M}_X^{\mathrm{gp}})/\Gamma(X,\mathcal{O}_X^*)$. Together with the canonical morphism $M^X \to \Gamma(X,\overline{\mathcal{M}}_X^{\mathrm{gp}})$, this makes $\overline{X}$ into a weakly embedded sharp monoidal space. Because pull-backs commute with groupifications, we have a canonical isomorphism $\Gamma(X,\overline{\mathcal{M}}_X^{\mathrm{gp}}) \cong \Gamma(F_X,\mathcal{M}_{F_X}^{\mathrm{gp}})$, making F_X a weakly embedded Kato fan with $M^{F_X} = M^X$, and χ_X a morphism of weakly embedded sharp monoidal spaces. We will henceforward understand $\overline{X}$, F_X, and χ_X as given with this structure.

Example 3.3.27. Let $X_0 \subseteq X$ be a complete (strict) toroidal embedding defined over an algebraically closed field k, considered as a log scheme with the divisorial log structure. Then for every cone $\sigma \in \Sigma(X)$ the set $X(\sigma)$ is small with $\Gamma(X(\sigma),\overline{\mathcal{M}}_X) = M_+^\sigma$. So by Proposition 3.3.21 and by the construction of $\Sigma(X)$, the cone complexes Σ_{F_X} and $\Sigma(X)$ are canonically isomorphic. Note that the characteristic morphism maps $\mathrm{O}(\sigma)$ to the point $x \in F_X$ with $\sigma = \sigma_x$. By [Kat94, Thm. 11.6] we have

$$\Gamma(X,\mathcal{M}_X^{\mathrm{gp}}) = \Gamma(X_0,\mathcal{O}_X^*),$$

showing that M^X in the sense of the Definition 3.3.26 coincides with M^X as defined in Section 2.1.3, that is that $\Sigma(X)$ and Σ_{F_X} are also isomorphic as weakly embedded cone complexes.

When considering the characteristic morphism χ_X as a morphism of weakly embedded sharp monoidal spaces, it has a universal property analogous to that of Proposition 3.3.22.

Corollary 3.3.28. *Let X be a quasi-finite log scheme without monodromy, and let $\psi\colon \overline{X} \to G$ be a morphism of weakly embedded sharp monoidal spaces to a weakly embedded Kato fan G. Then there exists a unique morphism $\varphi\colon F_X \to G$ of weakly embedded Kato fans such that $\psi = \varphi \circ \chi_X$. Furthermore, if ψ is strict, then so is φ.*

Proof. We know from Proposition 3.3.22 that there exists a unique morphism $\varphi \colon F_X \to G$ of Kato fans such that $\psi = \varphi \circ \chi_X$, and that φ is strict if and only if ψ is strict. Since $M^X = M^{F_X}$, there is a unique way to lift φ to a morphism of weakly embedded Kato fans such that the equality $\psi = \varphi \circ \chi_X$ still holds in the presence of weak embeddings. □

As we will see later, a strict morphism $\overline{X} \to F$ to a weakly embedded Kato fan F is exactly what we need to tropicalize subvarieties of a logarithmic scheme X. Even though χ_X is universal among those morphisms, it will be convenient to allow different ones as well. This motivates the following definition.

Definition 3.3.29. A *tropical scheme* is a pair $(X, \chi \colon \overline{X} \to F)$ consisting of a fine log scheme X such that $\underline{X}$ is Noetherian, and a strict morphism χ of weakly embedded sharp monoidal spaces to a fine weakly embedded Kato fan F. We usually omit χ and F from the notation and refer to them as $\chi = \chi_X$ and $F = F_X$. A morphism $X \to Y$ of tropical schemes is a pair (f, F_f) consisting of a morphism $f \colon X \to Y$ of log schemes and a morphism $F_f \colon F_X \to F_Y$ of weakly embedded Kato fans such that $F_f \circ \chi_X = \chi_Y \circ \overline{f}$. We usually omit F_f from the notation. We denote the thus defined category of tropical schemes by (**TropSch**). The forgetful functor (**TropSch**) $\to$ (**fwe-Fan**) is denoted by F, as implicit in the notation.

Remark 3.3.30. Note that our notation is abusive; if X is a tropical scheme, then its characteristic morphism χ_X does not necessarily coincide with the characteristic morphism of its underlying log scheme. Let us stress that by χ_X we always mean the morphism from $\overline{X}$ to a Kato fan which comes with the data. If X is a fine log scheme without monodromy, then we will consider it as a tropical scheme via its characteristic morphism. In this case the two notions of characteristic morphism (one for X as a log scheme, one for X as a tropical scheme) coincide.

For the definition of tropicalizations in the toroidal setting in Chapter 2, toroidal modifications played a central role. To be able to define tropicalizations for subvarieties of a tropical scheme X, we need to assign a modification $Y \to X$ to every subdivision G of F_X. The following result ensures that there is a universal choice for Y.

Proposition 3.3.31. *Let X be a tropical scheme, let G be a fine weakly embedded Kato fan, and let $\varphi \colon G \to F_X$ be a subdivision. Then there exists a morphism of tropical schemes $f \colon Y \to X$ such that $F_f = \varphi$ and for every tropical scheme Z the diagram of sets*

$$\begin{array}{ccc} \mathrm{Mor}_{(\mathbf{TropSch})}(Z, Y) & \longrightarrow & \mathrm{Mor}_{(\mathbf{fwe\text{-}Fan})}(F_Z, G) \\ \big\downarrow f_* & & \big\downarrow \varphi_* \\ \mathrm{Mor}_{(\mathbf{TropSch})}(Z, X) & \longrightarrow & \mathrm{Mor}_{(\mathbf{fwe\text{-}Fan})}(F_Z, F_X) \end{array}$$

is cartesian. It is unique up to isomorphism, and we denote $Y = X \times_{F_X} G$. If X is an integral tropical scheme (i.e. the underlying log scheme is integral), then there also is a morphism *of integral tropical schemes $Y' \to X$ with almost the same universal property (Z has to be integral as well). This morphism maps $(Y')^*$ isomorphically onto X^* and is proper if φ is a proper subdivision. We denote $Y' = X \overset{\bullet}{\times}_{F_X} G$.*

Proof. The existence of f follows easily from [GR14, Prop. 10.6.14 (iii)], where it is proven without the weak embeddings. So assume that X is integral. By [GR14, Remark 10.6.17] the preimage $f^{-1}(X^*)$ is equal to $(X \times_{F_X} G)^*$ and is mapped isomorphically onto X^*. Let Y' be the reduced closed subscheme of $X \times_{F_X} G$ whose underlying set is the closure of $(X \times_{F_X} G)^*$, made into a tropical scheme by pulling back the log structure from $X \times_{F_X} G$ and taking the composite $\overline{Y'} \to \overline{X \times_{F_X} G} \to G$ as characteristic morphism. Since $\underline{Y'}$ is integral and $(Y')^* = (X \times_{F_X} G)^* \neq \emptyset$, this tropical scheme is integral. If $g \colon Z \to X$ is a morphism of integral tropical schemes, and $\psi \colon F_Z \to G$ is a morphism of weakly embedded Kato fans such that $\varphi \circ \psi = F_g$, then by the universal property of $X \times_{F_X} G$ there exists exactly one factorization $g = f \circ h$ with $\psi = F_h$. From $g(Z^*) \subseteq X^*$ we conclude that h maps Z into Y' and thus that h factors through Y'. Therefore, $Y' \to X$ is universal. The additional statements follow directly from [GR14, Prop. 10.6.31 (ii)]. □

3.3.3 Pulling back from the Characteristic Fan

Under certain conditions the intersection theory of a tropical scheme X, or rather its underlying scheme $\underline{X}$, is closely related to that of its characteristic fan. More precisely, we will construct a morphism $\chi_X^* \colon A_*(F_X) \to A_*(X)$ which behaves similarly as a flat pull-back in algebraic geometry. Even though there is a well-developed intersection theory on general Noetherian schemes, we will assume for simplicity that X (or rather $\underline{X}$) is of finite type over a fixed field.

For the rest of this chapter, all schemes will be of finite type over a fixed perfect field k.

The precise condition for the pull-back to exist is that of trop-regularity.

Definition 3.3.32. A tropical scheme X is called *trop-regular* if it is integral, F_X is strictly simplicial, and for every $x \in F_X$ the preimage $\chi_X^{-1}\{x\}$ has pure codimension $\dim \mathcal{M}_{F_X,x} = \operatorname{rk} \mathcal{M}_{F_X,x}^{\mathrm{gp}}$. If F_X is only simplicial, we will say that X is *almost trop-regular*.

Example 3.3.33.

a) Assume that k is algebraically closed and X is a toroidal embedding. We have already seen in Example 3.3.27 that $\Sigma_{F_X} = \Sigma(X)$ and that for every $x \in F_X$ we have $\chi_X^{-1}\{x\} = \mathrm{O}(\sigma_x)$. We see that the condition on fiber dimensions is always fulfilled for toroidal embeddings. If, furthermore, X is smooth, then $\Sigma(X)$ and hence F_X is strictly simplicial. Therefore, smooth toroidal embeddings are trop-regular.

b) Let X be the log scheme from Example 3.3.17. We have already seen in the example that $F_X = \operatorname{Spec} \mathbb{N}^2$ and that the fibers of χ_X have the expected dimensions. Thus, X is trop-regular. But it cannot be toroidal because the boundary strata $V(x)$ and $V(x - y^2)$ do not meet transversally.

Let X be a trop-regular tropical scheme. The idea for the construction of the pull-back $\chi_X^*\colon A_*(F_X) \to A_*(X)$ is to map the class $[x] \in A_*(F_X)$ of a point $x \in F_X$ to a weighted sum of the classes of the components of $\chi_X^{-1}\{x\}$. The weight of a component C of $\chi_X^{-1}\{x\}$ should be defined similarly as the multiplicity $e_C X$ of X along C appearing in classical intersection theory [Ful98, Example 4.3.4], but in a way respecting the log structure. Let η be the generic point of C, and let U be a small open subset of X having η in its center. Then the characteristic fan F_U is equal to $V_{\rightsquigarrow x}$ and the restriction $(U, \overline{U} \to F_U)$ is trop-regular again. Since our definition of the logarithmic multiplicity will be local, we may assume that $X = U$. Let $n_1, \ldots, n_r$ be the free generators of $\mathcal{M}_{F_X,x}$. The preimage $\chi_X^{-1}\{x\}$ consists precisely of those points of X where none of the $\chi_X^\flat(n_i) \in \Gamma(X, \overline{\mathcal{M}}_X)$ is zero. Shrinking X further, we may assume that $\chi_X^\flat(n_i)$ lifts to some $s_i \in \Gamma(X, \mathcal{M}_X)$ for all i. Then $\chi_X^{-1}\{x\}$ is equal to the locus where none of the s_i is invertible. By the definition of log structures, this is precisely the locus where all of the images $t_i = \alpha_X(s_i) \in \Gamma(X, \mathcal{O}_X)$ vanish. This shows that the germs $(t_1)_\eta, \ldots, (t_r)_\eta$ form a system of parameters in $\mathcal{O}_{X,\eta}$ (i.e. $r = \dim \mathcal{O}_{X,\eta}$ and the maximal ideal of $\mathcal{O}_{X,\eta}$ is minimal over the ideal they generate). Note that the ideal they generate is equal to $I(\eta, \mathcal{M}_X)$. In particular, it does not depend on the choices we made.

Definition 3.3.34. Let X be a trop-regular tropical scheme, and let C be a component of a fiber of χ_X with generic point η. We define the *logarithmic multiplicity of X along C*, denoted by $e_C^{\log} X$, as the multiplicity $e(I(\eta, \mathcal{M}_X))$, that is as the unique natural number such that

$$\ell(\mathcal{O}_{X,\eta}/I(\eta, \mathcal{M}_X)^t) = e_C^{\log} X/r! \cdot t^r + \text{ lower terms}$$

for $t \gg 0$, where r is the codimension of C, and $\ell(R)$ denotes the length of an Artinian ring R seen as a module over itself.

Example 3.3.35.

a) Let X be a log-regular and trop-regular tropical scheme. For example, X could be a smooth toroidal embedding over an algebraically closed field. If η and C are as above, then $\mathcal{O}_{X,\eta}/I(\eta, \mathcal{M}_X)$ is 0-dimensional and hence a field. Therefore, $I(\eta, \mathcal{M}_X)$ is the maximal ideal of $\mathcal{O}_{X,\eta}$. Furthermore, $\overline{\mathcal{M}}_{X,x}$ is free of rank $\operatorname{rk} \overline{\mathcal{M}}_{x,x}^{\text{gp}} = \dim \mathcal{O}_{X,x}$, and hence $I(\eta, \mathcal{M}_X)$ is generated by $\dim O_{X,x}$ elements. In other words, $I(\eta, \mathcal{M}_X)$ is the maximal ideal of a regular local ring. Thus, $e_C^{\log} X = 1$.

b) Let X be the trop-regular log scheme from Example 3.3.17 and Example 3.3.33 b). The origin of $\underline{X} = \mathbb{A}_k^2$ is the unique component of the preimage of the maximal point of F_X, and $I((0,0), \mathcal{M}_X)$ is generated by x and $x - y^2$. Therefore, we have $e_{\{(0,0)\}}^{\log} X = 2$.

Remark 3.3.36. Let X be any tropical scheme, and let C be a component of a fiber of χ_X, say with generic point η. Then the same arguments as for the trop-regular case show that the maximal ideal of $\mathcal{O}_{X,\eta}$ is minimal over $I(\eta, \mathcal{M}_X)$. So the multiplicity $e(I(\eta, \mathcal{M}_X))$ is well-defined in this more general context. However, these numbers are not suitable to define more general logarithmic multiplicities as they fail to have the expected properties. For example, they are not necessarily 1 for log-regular schemes, as we can see in the toric example $X = \operatorname{Spec} k[P]$ with $P = \langle a, b, c \mid a + b = 2c \rangle$ the toric monoid from Example 3.1.5. The cone $\mathbb{R}_{\geq 0}P$ is a simplicial cone of multiplicity 2, and hence the multiplicity of $I(x, \mathcal{M}_X)$, where x is the unique point in the center of X, is 2 [BG09, Thm. 6.54].

Definition 3.3.37. Let X be a trop-regular tropical scheme. We define the pull-back for cycles along χ_X by

$$\chi_X^*\colon Z_*(F_X) \to Z_*(X), \quad [x] \mapsto \sum_C (e_C^{\log} X)[\overline{C}],$$

where the sum is taken over the components C of $\chi_X^{-1}\{x\}$. Note that all of these components have codimension $\operatorname{codim}(x, F)$, that is χ_X^* is graded of degree $\dim(X)$.

Before we show that χ_X^* respects rational equivalence, let us define a pull-back for Cartier divisors as well. We observe that on an integral scheme X we have $(\mathcal{O}_X^\bullet)^{\mathrm{gp}} = \mathcal{K}_X^*$, that is the groupification of the sheaf of nonzero regular functions is precisely the sheaf of invertible rational functions. In particular, the sheaf $\overline{\mathcal{O}_X^\bullet}^{\mathrm{gp}} = (\mathcal{O}_X^\bullet)^{\mathrm{gp}}/\mathcal{O}_X^*$ is equal to the sheaf of Cartier divisors.

Definition 3.3.38. Let X be an integral tropical scheme. We define the pull-back along χ_X for Cartier divisors as the composite morphism

$$\chi_X^*\colon \operatorname{Div}(F_X) = \Gamma(F_X, \mathcal{M}_{F_X}^{\mathrm{gp}}) \to \Gamma(X, \overline{\mathcal{M}}_X^{\mathrm{gp}}) \to \Gamma(X, \overline{\mathcal{O}_X^\bullet}^{\mathrm{gp}}) = \operatorname{Div}(X).$$

As χ_X is a morphism of weakly embedded sharp monoidal spaces, this morphism is compatible with the composite

$$M^{F_X} \xrightarrow{\chi_X^*} M^X = \Gamma(X, \mathcal{M}_X^{\mathrm{gp}})/\Gamma(X, \mathcal{O}_X^*) \xrightarrow{\alpha_X^{\mathrm{gp}}} \Gamma(X, \mathcal{K}^*)/\Gamma(X, \mathcal{O}_X^*),$$

that is pull-backs of principal divisors are principal again. Therefore, there is an induced morphism

$$\operatorname{ClCP}(F_X) \to \operatorname{Pic}(X)$$

which we again denote by χ_X^*.

If X is trop-regular and $x \in F_X$ has codimension 1, then the Weil divisor $[x]$ is Cartier by Proposition 3.2.34. To avoid confusion, we use the notation D_x for the unique Cartier divisor with $[D_x] = [x]$. The strictness of χ_X implies that the support of $\chi_X^* D_x$ is precisely the preimage of $|D_x| = F(x)$. It is also clear from the definitions that $[\chi_X^* D_x] = \chi_X^*[x]$. The generalization of this simple equation to higher codimensions will be a very important tool.

Lemma 3.3.39. *Let X be a trop-regular tropical scheme, and let $x_1, \dots, x_r$ be distinct codimension-1 points of F_X. Then we have*

$$\chi_X^*(D_{x_1} \cdots D_{x_r} \cdot [F_X]) = \chi_X^* D_{x_1} \cdots \chi_X^* D_{x_r} \cdot [X].$$

Proof. The generic points of the components of $F(x_1) \cap \dots \cap F(x_r)$ all have codimension r because every simplicial cone of which $\sigma_{x_1}, \dots, \sigma_{x_r}$ are rays has a r-dimensional face whose rays are precisely $\sigma_{x_1}, \dots, \sigma_{x_r}$. This, together with he trop-regularity of X implies that all components of

$$|\chi_X^* D_{x_1}| \cap \dots \cap |\chi_X^* D_{x_r}| = \chi_X^{-1}(F(x_1) \cap \dots \cap F(x_r))$$

have codimension r, that is that the intersection is proper. In particular, the product $\chi_X^* D_{x_1} \cdots \chi_X^* D_{x_r} \cdot [X]$ is well-defined as a cycle on X. By [Ful98, Example 7.1.10] the multiplicity of a component C in this intersection product is precisely the Samuel multiplicity of the ideal of $\mathcal{O}_{X,\eta}$ generated by representatives of $(\chi_X^* D_{x_i})_\eta \in \overline{\mathcal{O}}_{X,\eta}$ for $1 \leq i \leq r$, where η is the generic point of C. As this ideal is equal to $I(\eta, \mathcal{M}_X)$, this multiplicity is equal to $e_C^{\log} X$. Therefore, if S is the set of of generic points of the components of $F(x_1) \cap \dots \cap F(x_r)$, we have

$$\chi_X^* \left(\sum_{\xi \in S} [\xi] \right) = \chi_X^* D_{x_1} \cdots \chi_X^* D_{x_r} \cdot [X].$$

On the other hand, we easily see that $D_{x_1} \cdots D_{x_r} \cdot [F_X]$ is a well-defined cycle on F_X, and that it is equal to $\sum_{\xi \in S} [\xi]$. □

The preceding lemma shows us how to define pull-backs of cycles from the Kato fan for almost trop-regular tropical schemes.

Definition 3.3.40. Let X be an almost trop-regular tropical scheme. Let $x \in F_X$, and let $x_1, \dots, x_r$ be the codimension-1 points of $V_{\rightsquigarrow x}$. Each of the cycles $[x_i]$ is $\mathbb{Q}$-Cartier (cf. Remark 3.2.35) and we denote the corresponding $\mathbb{Q}$-Cartier divisor by $D_{x_i} \in \mathrm{Div}(F_X)$. We define $\chi_X^*[x] \in Z_*(X)$ as the unique cycle supported on $\chi_X^{-1}\{x\}$ such that

$$\chi_X^*[x]|_{\chi_X^{-1} V_{\rightsquigarrow x}} = \left(\mathrm{mult}(\sigma_x) \chi_X^* D_{x_1} \cdots \chi_X^* D_{x_r} \cdot [X] \right) \Big|_{\chi_X^{-1} V_{\rightsquigarrow x}},$$

where $\mathrm{mult}(\sigma_x)$ is the lattice index $[N^{\sigma_x} : N^{\sigma_{x_1}} + \dots + N^{\sigma_{x_r}}]$. That this is in fact in $Z_*(X)$ follows inductively; let σ_r be the face of σ_x which corresponds to the unique codimension-r point of $V_{\rightsquigarrow x}$ that contains $x_1, \dots, x_r$. Then the linear function on σ_r corresponding to

$$\frac{\mathrm{mult}(\sigma_r)}{\mathrm{mult}(\sigma_{r-1})} D_{x_r}$$

is easily seen to be integral linear.

Extending by linearity, we obtain a morphism

$$\chi_X^* \colon Z_*(F_X) \to Z_*(X).$$

By Lemma 3.3.39 this morphism coincides with that of Definition 3.3.37 in case X is trop-regular.

Remark 3.3.41. The factor $\operatorname{mult}(\sigma_x)$ in Definition 3.3.40 appears due to the fact that the multiplicity of $[x]$ in the intersection product $D_{x_1} \cdots D_{x_r} \cdot [F_X]$ is equal to $\operatorname{mult}(\sigma_x)^{-1}$. This can be seen inductively, using that if y has codimension one less than x, and z is the unique codimension-1 point in $V_{\rightsquigarrow x} \setminus V_{\rightsquigarrow y}$, then the multiplicity of $[x]$ in $D_z \cdot [y]$ is equal to

$$\frac{\operatorname{mult}(\sigma_y)}{\operatorname{mult}(\sigma_x)}.$$

Proposition 3.3.42. *Let X be an almost trop-regular tropical scheme, let $\alpha \in Z_*(F_X)$, and let $D \in \operatorname{Div}(F_X)$ such that no component of α is contained in $|D|$. Then no component of $\chi_X^* \alpha$ is contained in $|\chi_X^* D|$, and*

$$\chi_X^*(D \cdot \alpha) = \chi_X^* D \cdot \chi_X^* \alpha.$$

In particular, χ_X^*, induces a morphism

$$A_*(F) \to A_*(X)$$

which we again denote by χ_X^*, and for every $\mathcal{L} \in \operatorname{ClCP}(F_X)$ and $\beta \in A_*(F_X)$ we have

$$\chi_X^*(c_1(\mathcal{L}) \cap \beta) = c_1(\chi_X^* \mathcal{L}) \cap \chi_X^* \beta.$$

Proof. The preimages of distinct points of F_X have distinct components, and both sides of the first equation are linear in α, so we may assume that $\alpha = [x]$ for some $x \in F_X$. Let ξ be the generic point of a component of $\chi_X^{-1}\{x\}$. Then the germ of $\chi_X^* D$ at ξ is equal to the image of D_x under the composite

$$\mathcal{M}^{\mathrm{gp}}_{F_X,x} \xrightarrow{(\chi_X^\flat)^{\mathrm{gp}}} \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\xi} \to \overline{\mathcal{O}}^{\mathrm{gp}}_{X,x}.$$

By assumption, we have $D_x = 0$, and hence $(\chi_X^* D)_\xi = 1$, meaning that ξ is not contained in the support of D.

Now let η be the generic point of a component of $\chi_X^* D \cdot \chi_X^*[x]$. Then $y := \chi_X(\eta)$ is a proper specialization of x because $\chi_X^* D$ is nontrivial at η. For dimension reasons this implies that y is a codimension-1 point of $F(x)$, and that η is the generic point of a component of $\chi_X^{-1}\{y\}$. We see that both $\chi_X^*(D \cdot [x])$ and $\chi_X^* D \cdot \chi_X^*[x]$ are supported on the components of the fibers of the codimension-1 points of $F(x)$. Let ζ be the generic point of such a component, and let $z = \chi_X(\zeta)$. To show that the multiplicity of $\overline{\{\zeta\}}$ is equal on both sides of the equation, it suffices to show that the restrictions of $\chi_X^*(D \cdot [x])$ and $\chi_X^* D \cdot \chi_X^*[x]$ to $\chi_X^{-1} V_{\rightsquigarrow z}$ are equal. Therefore, we may assume that F_X is affine with closed point z. Let $z', x_1, \ldots, x_r$ be the codimension-1 points of F_X, where z' is the unique one not specializing to x. Since x is the only point in F_X which generalizes to all x_i, we have

$$\chi_X^*[x] = \operatorname{mult}(\sigma_x) \chi_X^* D_{x_1} \cdots \chi_X^* D_{x_r} \cdot [X]$$

by Lemma 3.3.39, and similarly we have

$$\chi_X^*[z] = \operatorname{mult}(\sigma_z)\chi_X^* D_{z'} \cdot \chi_X^* D_{x_1} \cdots \chi_X^* D_{x_r} \cdot [X].$$

Furthermore, the fact that $x \notin |D|$ implies that $D = \operatorname{ord}_{z'}(D) D_{z'}$, and hence that $\chi_X^* D = \operatorname{ord}_{z'}(D)\chi_X^* D_{z'}$. This yields

$$\begin{aligned}\chi_X^* D \cdot \chi_X^*[x] &= \operatorname{ord}_{z'}(D)\operatorname{mult}(\sigma_x)\chi_X^* D_{z'} \cdot \chi_X^* D_{x_1} \cdots \chi_X^* D_{x_r} \cdot [X] = \\ &= \operatorname{ord}_{z'}(D)\frac{\operatorname{mult}(\sigma_x)}{\operatorname{mult}(\sigma_z)}\chi_X^*[z] = \operatorname{ord}_{z'}(D)\chi_X^*(D_{z'} \cdot [x]) = \chi_X^*(D \cdot [x]),\end{aligned}$$

where the second to last equality follows from the fact that $D_{z'} \cdot [x] = \frac{\operatorname{mult}(\sigma_z)}{\operatorname{mult}(\sigma_x)}$, as already noticed in Remark 3.3.41.

For the "in particular" statement note that for $x \in F_X$ and $m \in M_x^{F_X}$ we have

$$\chi_X^*[\operatorname{div}^{F_X(x)}(m)] = \chi_X^*(\operatorname{div}(m) \cdot [x]) = \chi_X^*(\operatorname{div}(m)) \cdot \chi_X^*[x].$$

Because $\chi_X^* \operatorname{div}(m)$ is principal, it follows that $\chi_X^*[\operatorname{div}^{F_X(x)}(m)] = 0$ in $A_*(X)$, and hence that χ_X^* induces a morphism $A_*(F_X) \to A_*(X)$. For the last equality let $\mathcal{L} \in \operatorname{ClCP}(F_X)$ and $\beta \in A_*(X)$. As both sides are linear in β, we may assume that β is the class of $[x]$ for some $x \in F_X$. Furthermore, we may choose a representative $D \in \operatorname{Div}(F_X)$ of $\mathcal{L}$ such that $x \notin |D|$. Then we have

$$\begin{aligned}\chi_X^*(c_1(\mathcal{L}) \cap \beta) &= \chi_X^*(D \cdot [x]) = \chi_X^* D \cdot \chi_X^*[x] = \\ &= c_1(\mathcal{O}_X(\chi_X^* D)) \cap \chi_X^*[x] = c_1(\chi_X^*\mathcal{L}) \cap \chi_X^*[x].\end{aligned}$$

□

Proposition 3.3.43. *Let $f\colon X \to Y$ be a morphism between almost trop-regular tropical schemes such that $\underline{f}\colon \underline{X} \to \underline{Y}$ is flat and $F_f\colon F_X \to F_Y$ is locally exact. Then the diagram*

$$\begin{array}{ccc} Z_*(F_Y) & \xrightarrow{F_f^*} & Z_*(F_X) \\ \downarrow{\scriptstyle \chi_Y^*} & & \downarrow{\scriptstyle \chi_X^*} \\ Z_*(Y) & \xrightarrow{f^*} & Z_*(X) \end{array}$$

is commutative.

Proof. It suffices to show that $f^*(\chi_Y^*[y]) = \chi_X^*(F_f^*[y])$ for all $y \in F_Y$. The open set

$$U = (\chi_Y \circ f)^{-1} V_{\rightsquigarrow y} = (F_f \circ \chi_X)^{-1} V_{\rightsquigarrow y}.$$

meets all components of both $f^*(\chi_Y^*[x])$ and $\chi_X^*(F_f^*[y])$. Thus, it suffices to show that their restrictions to U coincide, that is we may assume that F_Y is affine with closed point y. Let $y_1, \ldots, y_r$ be the codimension-1 points of F_Y. Denoting $\operatorname{mult}(\sigma_y)$ by m, we have

$$f^*(\chi_Y^*[y]) = f^*\Big(m \cdot \chi_Y^* D_{y_1} \cdots \chi_Y^* D_{y_r} \cdot [Y]\Big) = m \cdot \chi_X^* F_f^* D_{y_1} \cdots \chi_X^* F_f^* D_{y_r} \cdot [X] =$$
$$= m \cdot \chi_X^* \Big(F_f^* D_{y_1} \cdots F_f^* D_{y_r} \cdot F_f^*[F_Y]\Big) = \chi_X^* F_f^* \Big(m \cdot D_{y_1} \cdots D_{y_r} \cdot [F_Y]\Big) = \chi_X^* F_f^*[y]$$

where the third equality follows from Proposition 3.3.42, and the fourth one from Proposition 3.2.44. □

As in the toroidal case, a key step in the tropicalization procedure for subvarieties of tropical schemes will be the choice of a sufficiently fine proper subdivision of the characteristic fan. The following result will be the key ingredient to show that the tropicalization is independent of the choice of the subdivision.

Proposition 3.3.44. *Let X be an almost trop-regular tropical scheme, let $g\colon G \to F_X$ be a simplicial proper subdivision, let $Y = X \overset{\bullet}{\times}_{F_X} G$, and let $f\colon Y \to X$ be the natural morphism. Then the diagram*

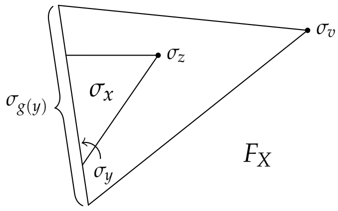
$$\begin{array}{ccc} Z_*(G) & \xrightarrow{\;g_*\;} & Z_*(F_X) \\ \big\downarrow \chi_Y^* & & \big\downarrow \chi_X^* \\ Z_*(Y) & \xrightarrow{\;f_*\;} & Z_*(X) \end{array}$$

is commutative.

Proof. It suffices to show that $f_*\chi_Y^*[x] = \chi_X^* g_*[x]$ for all $x \in G$. We prove this by induction on $\operatorname{codim}(x, G)$. If $\operatorname{codim}(x, G) = 0$, then x is the generic point of G, and both sides are equal to $[X]$ because f is birational by Proposition 3.3.31. Now let $n \in \mathbb{N}_+$ and assume that the equality holds for all $y \in G$ of codimension strictly less that n. Let $x \in G$ be a codimension-n point. Considering the induced morphism $(\chi_X \circ f)^{-1} V_{\rightsquigarrow g(x)} \to \chi_X^{-1} V_{\rightsquigarrow g(x)}$ we may assume that F_X is affine with maximal point $g(x)$. The images of the components of $\chi_Y^{-1}\{x\}$ under f are contained in $\chi_X^{-1}\{g(x)\}$, so if $\operatorname{codim}(g(x), F_X) > n$ they have codimension strictly larger than n. It follows that in this case we have $f_*\chi_Y^*[x] = 0 = \chi_X^* g_*[x]$. We may thus assume that $\operatorname{codim}(g(x), F_X) = n$. We first consider the case that a facet σ_y of σ_x is contained in a facet of $\sigma_{g(x)}$ (which must then be equal to $\sigma_{g(y)}$). Let z be the codimension-1 point of $V_{\rightsquigarrow x} \setminus V_{\rightsquigarrow y}$, and let v be the codimension-1 point of $F_X \setminus V_{\rightsquigarrow g(y)}$. Furthermore, let D be the $\mathbb{Q}$-Cartier divisor on F_X with $|D| = F(v)$ and $D \cdot [g(y)] = [g(x)]$. By induction hypothesis we know that $f_*\chi_Y^*[y] = \chi_X^*[g(y)]$. This yields

$$f_*\chi_Y^*(g^*D \cdot [y]) = f_*\Big(f^*\chi_X^* D \cdot \chi_Y^*[y]\Big) = \chi_X^* D \cdot \chi_X^*[g(y)] = \chi_X^*(D \cdot [g(y)]) = \chi_X^*[g(x)]$$

where the first and third equality follow from Proposition 3.3.42, and the second one follows from the projection formula. The function on $\sigma_{g(x)}$ corresponding to D vanishes on $\sigma_{g(y)}$, and the function on the ray $\sigma_{g(x)}/\sigma_{g(y)}$ it induces is 1 on the ray's primitive generator by Lemma 3.2.5. Let D' be the unique $\mathbb{Q}$-Cartier divisor on F_Y with $|D'| = F(z)$ and $D' \cdot [y] = [x]$. Similarly as for D, its associated function on Σ_{F_Y} vanishes on σ_y and its value at the primitive generator of the ray σ_x/σ_y is 1. Since $\sigma_{g(x)}/\sigma_{g(y)}$ and σ_x/σ_y are naturally isomorphic, it follows that

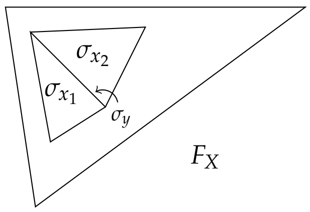


$$g^*D = D' + \sum_w a_i D_w$$

for some $a_i \in \mathbb{Q}$, where the sum is taken over all codimension-1 points $w \in F_Y$ not equal to z for which $g(w) \notin V_{\rightsquigarrow g(y)}$. For every such w the two cones σ_w and σ_y do not span a cone of Σ_G, and therefore $D_w \cdot [y] = 0$. It follows that $g^*D \cdot [y] = D' \cdot [y] = [x]$, and hence that

$$f_*\chi_Y^*[x] = \chi_X^*[g(x)] = \chi_X^* g_*[x].$$

If no facet of σ_x is contained in a facet of $\sigma_{g(x)}$, then there is a sequence $x = x_0, \ldots, x_n$ of codimension-n points of G such that a facet of σ_{x_n} is contained in a facet of $\sigma_{g(x)} = \sigma_{g(x_n)}$, and σ_{x_i} and $\sigma_{x_{i+1}}$ share a facet for every $0 \leq i < n$. So it suffices to show that $f_*\chi_Y^*[x_1] = f_*\chi_Y^*[x_2]$ whenever $x_1, x_2 \in G$ are two codimension-n points for which there exists a codimension-$(n-1)$ point $y \in V_{\rightsquigarrow x_1} \cap V_{\rightsquigarrow x_2}$. There exits a unique linear function on $\sigma_{g(x)}$ which vanishes on σ_y and whose induced function on the ray σ_{x_1}/σ_y is 1 at the ray's primitive generator. Since $N^{\sigma_{x_1}}/N^{\sigma_y}$ and $N^{\sigma_{x_2}}/N^{\sigma_y}$ are naturally isomorphic, but σ_{x_2} is on the other side of σ_y, this function has value -1 at the primitive generator of σ_{x_2}/σ_y. Let $D \in \mathrm{Div}(F_X)_{\mathbb{Q}}$ be the $\mathbb{Q}$-Cartier divisor on F_X associated to the function. Then by construction we have $g^*D \cdot [y] = [x_1] - [x_2]$. Using this equality we obtain



$$f_*\chi_Y^*[x_1] - f_*\chi_Y^*[x_2] = f_*\chi_Y^*(g^*D \cdot [y]) = f_*\left(f^*\chi_X^*D \cdot \chi_Y^*[y]\right) = \chi_X^*D \cdot f_*\chi_Y^*[y] = 0,$$

where the second equality follows from Proposition 3.3.42 and the third one follows from the projection formula. □

3.3.4 Tropicalization

In this section we define the tropicalization of a subvariety Z of a tropical scheme X. Note that we still work with the standing assumption that X is of finite type over a fixed prefect field k.

As in the toroidal case, we obtain the tropicalization of Z by defining suitable weights on Ulirsch's set-theoretic tropicalization, whose construction we will briefly recall: Let R be a rank-1 valuation ring with valuation $v\colon R \to \overline{\mathbb{R}}_{\geq 0}$. Consider $\operatorname{Spec} R$ with the terminal log structure $\mathrm{id}\colon \mathcal{O}_{\operatorname{Spec} R} \to \mathcal{O}_{\operatorname{Spec} R}$. Then the identity $R \to R$ is a chart. The set of prime ideals of the multiplicative monoid R is finite – it consists of $\emptyset$ and the two prime ideals of the ring R – so $\operatorname{Spec} R$ is quasi-finite. The chart is also local at the maximal point $\mathfrak{m}$ of $\operatorname{Spec} R$, and the fibers of $\operatorname{Spec} R \to \operatorname{Spec} \overline{R}$ are $\emptyset$, $\{0\}$ and $\{\mathfrak{m}\}$. Thus, X is small, and its characteristic fan is equal to $\operatorname{Spec} \overline{R}$ by Proposition 3.3.16. As we have seen in Example 3.3.10, any R-valued point of $\underline{X}$ can be uniquely lifted to a morphism of log schemes $\operatorname{Spec} R \to X$. Taking characteristic fans we obtain a morphism $\operatorname{Spec} \overline{R} \to F_X$. The valuation ν induces a morphism $\overline{R} \to \overline{\mathbb{R}}_{\geq 0}$, and hence a morphism $\operatorname{Spec} \overline{\mathbb{R}}_{\geq 0} \to \operatorname{Spec} \overline{R}$. In total, we obtain a tropicalization map as the composite

$$\mathrm{trop}_X^R\colon \underline{X}(R) = X(\operatorname{Spec} R) \to F_X(\overline{R}) \to F_X(\overline{\mathbb{R}}_{\geq 0}).$$

Whenever R is dominated by a rank-1 valuation ring R', the diagram

$$\begin{array}{ccc} X(R) & \xrightarrow{\mathrm{trop}_X^R} & \\ \big\downarrow & & F_X(\overline{\mathbb{R}}_{\geq 0}) \\ X(R') & \xrightarrow{\mathrm{trop}_X^{R'}} & \end{array}$$

is commutative so that there is a well-defined map

$$\mathrm{trop}_X\colon X^{\sqsupset} \to F_X(\overline{\mathbb{R}}_{\geq 0}).$$

Definition 3.3.45 ([Uli15a, Section 6]). Let X be a tropical scheme, and let Z be a closed subvariety of X. Then we define the *set-theoretic tropicalization* $\mathrm{trop}_X(Z)$ as the image of the composite

$$Z^{\sqsupset} \to X^{\sqsupset} \xrightarrow{\mathrm{trop}_X} F_X(\overline{\mathbb{R}}_{\geq 0}).$$

Note that we have

$$F_X(\overline{\mathbb{R}}_{\geq 0}) = \coprod_{x \in F_X} (F_X(x))(\overline{\mathbb{R}}_{\geq 0}) = \coprod_{x \in F_X} \Sigma_{F_X(x)^{\mathrm{sat}}},$$

so it makes sense to talk about subcones of $F_X(\overline{\mathbb{R}}_{\geq 0})$. We will see in Proposition 3.3.55 that $\mathrm{trop}_X(Z)$ is a union of subcones of $F_X(\overline{\mathbb{R}}_{\geq 0})$ of dimension less or equal to $\dim(Z)$.

To define weights on the tropicalization of Z we will perform three reduction steps.

1. Reduce to the case where $Z = X$.
2. Reduce to the case where X is integral (cf. Definition 3.3.13).
3. Reduce to the case where X is trop-regular.

The first reduction is easy: we pull back the log structure of X to Z and consider Z as a tropical scheme with characteristic morphism

$$\overline{Z} \to \overline{X} \xrightarrow{\chi_X} F_X.$$

For the second reduction we make the following definition.

Definition 3.3.46. Let X be a log scheme such that $\underline{X}$ is integral. We define $X^\bullet$ as the log scheme with underlying scheme $\underline{X}$ and log structure

$$\mathcal{M}_{X^\bullet} = \mathcal{M}_X^\bullet := \alpha_X^{-1}\mathcal{O}_X^\bullet \to \mathcal{O}_X,$$

where $\alpha_X \colon \mathcal{M}_X \to \mathcal{O}_X$ denotes the log structure of X, and $\mathcal{O}_X^\bullet = \mathcal{O}_X \setminus \{0\}$. It is clearly integral.

For this definition to be helpful we need to show that a tropical structure on a log scheme X with $\underline{X}$ integral induces a tropical structure on $X^\bullet$. More precisely, we need to show that $X^\bullet$ is fine again, and that the characteristic morphism of X induces a strict morphism from $\overline{X^\bullet}$ to a fine weakly embedded Kato fan. This is the content of the following two results.

Lemma 3.3.47. *Let X be a log scheme such that $\underline{X}$ is integral, and let $\beta \colon P \to \mathcal{M}_X$ be a chart. Then the induced morphism $\beta^\bullet \colon P^\bullet := \beta^{-1}\Gamma(X, \mathcal{M}_X^\bullet) \to \mathcal{M}_X^\bullet$ is a chart for $\mathcal{M}_X^\bullet$. In particular, if X is fine (resp. fine and saturated resp. quasi-finite) then so is $X^\bullet$.*

Proof. Being a chart is a local property. For $x \in X$ we consider the commutative diagram

$$\begin{array}{ccccc} \beta^{-1}\mathcal{M}_{X,x}^* & \longrightarrow & \mathcal{M}_{X,x}^* & \searrow & \\ \downarrow & & \downarrow & & \\ P^\bullet & \longrightarrow & (P^\bullet)^{\log} & \longrightarrow & \mathcal{M}_{X,x}^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ P & \longrightarrow & P^{\log} & \xrightarrow{\cong} & \mathcal{M}_{X,x}. \end{array}$$

As the two squares on the left are pushout diagrams, every element of $(P^\bullet)^{\log}$ can be written as $p + u$ for some $p \in P^\bullet$ and $u \in \mathcal{M}_{X,x}^*$. Suppose that two elements $p + u$ and $p' + u'$ of $(P^\bullet)^{\log}$ have the same image in $\mathcal{M}_{X,x}^\bullet$. Then they also have the same image in $\mathcal{M}_{X,x}$ and hence in $P^{\log}$. By the explicit description of the logification $P^{\log}$ given in Remark 3.3.4 there exist $s, s' \in \beta^{-1}\mathcal{M}_{X,x}^*$ such that $p + s = p' + s'$ and $u + \beta(s') = u' + \beta(s)$. Applying that same description to the logification $(P^\bullet)^{\log}$ we conclude that $p + u = p' + u'$ in $(P^\bullet)^{\log}$, showing the injectivity of

$$(P^\bullet)^{\log} \to \mathcal{M}_{X,x}^\bullet.$$

For surjectivity let $m \in \mathcal{M}^\bullet_{X,x}$. Because β is a chart, there exist $p \in P$ and $u \in \mathcal{M}^*_{X,x}$ such that $m = \beta(p) + u$. The fact that $m \in \mathcal{M}^\bullet_{X,x}$ means that

$$0 \neq \alpha_X(m) = \alpha_X(\beta(p)) \cdot \alpha_X(u)$$

in $\mathcal{O}_{X,x}$, and hence that $\alpha_X(\beta(p)) \neq 0$ in $\mathcal{O}_{X,x}$. By definition, this implies that $\beta(p) \in \mathcal{M}^\bullet_X$ and $p \in P^\bullet$. As a consequence, we see that m is in the image of $(P^\bullet)^{\log}$.

For the "in particular" statement it suffices to note that $P^\bullet$ is a face of P and faces of fine (resp. fine and saturated resp. quasi-finite) monoids are fine (resp. fine and saturated resp. quasi-finite) again. □

Proposition 3.3.48. *Let X be a tropical scheme such that $\underline{X}$ is integral, and let ξ be its generic point. Then there is a natural strict morphism $\overline{X^\bullet} \to F_X(\chi_X(\xi))$ induced by the characteristic morphism of X.*

Proof. We first consider χ_X as a morphism of sharp monoidal spaces, and then show that everything is compatible with the weak embeddings. Let $y := \chi_X(\xi)$ and let $\iota\colon F_X(y) \to F_X$ be the inclusion. The underlying map of χ_X factors through $F_X(y)$, so we obtain a strict morphism $\overline{X} \to (F_X(y), \iota^{-1}\mathcal{M}_{F_X})$. To prove that this induces a strict morphism $\overline{X} \to F_X(y)$, it suffices to show that the inverse image of the subsheaf $\mathcal{M}_{F_X(y)}$ of $\iota^{-1}\mathcal{M}_{F_X}$ is mapped isomorphically onto the subsheaf $\overline{\mathcal{M}^\bullet_X}$ of $\overline{\mathcal{M}}_X$. This can be checked on stalks. Let $x \in X$. Then there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_{F_X,\chi_X(x)} & \xrightarrow{\cong} & \overline{\mathcal{M}}_{X,x} & \longrightarrow & \overline{\mathcal{O}}_{X,x} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_{F_X,y} & \xrightarrow{\cong} & \overline{\mathcal{M}}_{X,\xi} & \longrightarrow & \overline{\mathcal{O}}_{X,\xi}, \end{array}$$

where the vertical arrows are cospecialization maps. By definition, the stalk $\overline{\mathcal{M}^\bullet_{X,x}}$ is the preimage of $\overline{\mathcal{O}^\bullet_{X,x}}$ under the upper horizontal map of the right square. Since $\underline{X}$ is integral, $\overline{\mathcal{O}^\bullet_{X,x}}$ is the preimage of $\overline{\mathcal{O}^\bullet_{X,\xi}} = \{1\}$ under the cospecialization map. So the commutativity of the right square yields that $\overline{\mathcal{M}^\bullet_{X,x}}$ is the preimage of $\{0\} \subseteq \overline{\mathcal{M}}_{X,\xi}$ under the cospecialization map. In particular, the preimage of $\overline{\mathcal{M}^\bullet_{X,x}}$ in $\mathcal{M}_{F_X,\chi_X(x)}$ is equal to the preimage of $\{0\} \subseteq \mathcal{M}_{F_X,y}$ under the cospecialization map on the very left. But this preimage is equal to $\mathcal{M}_{F_X(y),\chi_X(x)}$ by definition.

Now let us consider the weak embeddings. Recall that $M^{F_X(y)} = M^{F_X}_y$ is the kernel of the canonical morphism $M^{F_X} \to \mathcal{M}_{F_X,y}$. We need to show that the image of $M^{F_X}_y$ under

$$\chi_X^*\colon M^{F_X} \to M^X = \Gamma(X, \mathcal{M}^{\mathrm{gp}}_X)/\Gamma(X, \mathcal{O}^*_X)$$

is contained in the subgroup $M^{X^\bullet} = \Gamma(X, (\mathcal{M}^\bullet_X)^{\mathrm{gp}})/\Gamma(X, \mathcal{O}^*_X)$ of M^X. By the strictness of χ_X we know that for every $m \in M^{F_X}_y$ the stalk $\chi^*_X(m)_\xi \in \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\xi}$ is 0. Therefore, it suffices to show that the sequence

$$0 \to \Gamma(X, (\mathcal{M}^\bullet_X)^{\mathrm{gp}}) \to \Gamma(X, \mathcal{M}^{\mathrm{gp}}_X) \to \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\xi}$$

is exact. It is clear from the definitions that $(\mathcal{M}_X^\bullet)^{\mathrm{gp}}$ is a subsheaf of $\mathcal{M}_X^{\mathrm{gp}}$, so it suffices to check that for every $x \in X$ the sequence

$$0 \to (\mathcal{M}_{X,x}^\bullet)^{\mathrm{gp}} \to \mathcal{M}_{X,x}^{\mathrm{gp}} \to \overline{\mathcal{M}}_{X,\xi}^{\mathrm{gp}}$$

is exact. Let $P \to \mathcal{M}_U$ be a chart in an open neighborhood U of x which is exact at x. By Lemma 3.3.47 there is an induced chart $P^\bullet \to \mathcal{M}_U^\bullet$, where $P^\bullet$ is the preimage of $\mathcal{O}_{X,\xi}^*$ under the composite

$$P \to \mathcal{M}_{X,\xi} \to \mathcal{O}_{X,\xi}.$$

By the construction of the logification, the chart induces a commutative diagram

$$\begin{array}{ccccccc} P^\bullet \otimes_{P^*} \mathcal{O}_{X,x}^* & \cong & \mathcal{M}_{X,x}^\bullet & \hookrightarrow & \mathcal{M}_{X,x} & \cong & P \otimes_{P^*} \mathcal{O}_{X,x}^* \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P^\bullet \otimes_{P^\bullet} \mathcal{O}_{X,\xi}^* & \cong & \mathcal{O}_{X,\xi}^* & \hookrightarrow & \mathcal{M}_{X,\xi} & \cong & P \otimes_{P^\bullet} \mathcal{O}_{X,\xi}^*. \end{array}$$

Taking groupifications in this diagram, and using that groupification commutes with pushouts (it has a right adjoint), we obtain the commutative diagram

$$\begin{array}{ccccccc} (P^\bullet)^{\mathrm{gp}} \oplus \mathcal{O}_{X,x}^* \big/ P^* & \cong & (\mathcal{M}_{X,x}^\bullet)^{\mathrm{gp}} & \hookrightarrow & \mathcal{M}_{X,x}^{\mathrm{gp}} & \cong & P^{\mathrm{gp}} \oplus \mathcal{O}_{X,x}^* \big/ P^* \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (P^\bullet)^{\mathrm{gp}} \oplus \mathcal{O}_{X,\xi}^* \big/ (P^\bullet)^{\mathrm{gp}} & \cong & \mathcal{O}_{X,\xi}^* & \hookrightarrow & \mathcal{M}_{X,\xi}^{\mathrm{gp}} & \cong & P^{\mathrm{gp}} \oplus \mathcal{O}_{X,\xi}^* \big/ (P^\bullet)^{\mathrm{gp}} \end{array}$$

from which the assertion follows immediately. □

For the third reduction step, we would expect from the toroidal case that if we choose a strictly simplicial proper subdivision G of F_X such that $\mathrm{trop}_X(X)$ is a union of cones of Σ_G, then the modification $X \overset{\bullet}{\times}_{F_X} G$ is trop-regular. Unfortunately, the following example shows that our situation is more complicated.

Example 3.3.49. Let $X = \mathbb{A}_k^2$ be the tropical scheme with log structure defined by the chart

$$\begin{aligned} \mathbb{N}^2 &\to k[x_1, x_2], \\ e_1 &\mapsto x_1(x_1 + x_2 + 1) \\ e_2 &\mapsto x_2(x_1 + x_2 + 1), \end{aligned}$$

where e_1 and e_2 are the generators of $\mathbb{N}^2$, and with the induced morphism $\overline{X} \to \operatorname{Spec} \mathbb{N}^2$ as characteristic morphism. Then the fiber of the maximal point of $\operatorname{Spec} \mathbb{N}^2$ equals

$$V(x_1 + x_2 + 1) \cup \{(0,0)\},$$

which does not have pure codimension 2 so that X is not trop-regular. However, for $\omega \in \overline{\mathbb{R}}^2_{\geq 0}$ the valuation

$$\nu_\omega \colon k[x,y] \to \overline{\mathbb{R}}_{\geq 0}, \ \sum_\alpha c_\alpha \mathbf{x}^\alpha \mapsto \min\{\langle \omega, \alpha \rangle \mid c_\alpha \neq 0\}$$

defines an element in $X^{\sqsupset}$ whose image under trop_X is

$$\big(\nu_\omega(x_1) + \nu_\omega(x_1 + x_2 + 1), \nu_\omega(x_2) + \nu_\omega(x_1 + x_2 + 1)\big) = \big(\nu_\omega(x_1), \nu_\omega(x_2)\big) = \omega.$$

Thus, $\overline{\mathbb{R}}^2_{\geq 0} = F_X(\overline{\mathbb{R}}_{\geq 0}) \subseteq \mathrm{trop}_X(X)$ which does not tell us how to modify X to make it trop-regular. The problem here is that the characteristic fan of X does not coincide with the characteristic fan of the underlying log scheme. The latter consists of two copies of $\operatorname{Spec} \mathbb{N}^2$, one for each component where the characteristic sheaf is maximal, glued along their rays. The tropicalization with respect to this larger fan contains a ray in the cone corresponding to $V(x_1 + x_2 + 1)$ that tells us how to subdivide.

Proposition 3.3.50. *Let X be an integral tropical scheme. Then there exists a subdivision $G \to F_X$ such that $X \overset{\bullet}{\times}_{F_X} G$ is trop-regular.*

Proof. First note that after a base change to F_X^{sat} we may assume that F_X is fine and saturated. By de Jong's theorem [dJ96, Thm. 3.1] there exists an alteration, that is a proper, dominant, and generically finite morphism $\psi \colon X' \to \underline{X}$ such that X' is regular, of finite type over a finite extension field of k, and $\psi^{-1}(X \setminus X^*)$ is a simple normal crossings divisor. Equipping X' with the divisorial log structure it becomes a log-regular log scheme. The morphism ψ can be made into a morphism of log schemes in a unique way. Indeed, if $m \in \Gamma(U, \mathcal{M}_X)$ for some open subset $U \subseteq X$, then $\alpha_X(m)$ is invertible on X^*, and hence $\psi^\sharp(\alpha(m))$ is invertible on $\psi^{-1}X^*$. Therefore, it defines an element of $\Gamma(\psi^{-1}U, \mathcal{M}_{X'})$. The morphism ψ of log schemes can be uniquely extended to a morphism of tropical schemes by Corollary 3.3.28. Similarly as in Construction 2.2.10 we obtain a strictly simplicial proper subdivision $G \to F_X$, a proper subdivision $G' \to F_{X'}$, and a locally exact morphism $\varphi \colon G' \to G$ such that

$$\begin{array}{ccc} G' & \longrightarrow & F_{X'} \\ \big\downarrow & & \big\downarrow \\ G & \longrightarrow & F_X \end{array}$$

commutes. By the universal property of base changes there is an induced commutative diagram

$$\begin{array}{ccc} X' \overset{\bullet}{\times}_{F_{X'}} G' & \longrightarrow & X' \\ \big\downarrow{\scriptstyle \psi'} & & \big\downarrow{\scriptstyle \psi} \\ X \overset{\bullet}{\times}_{F_X} G & \longrightarrow & X \end{array}$$

of tropical schemes. As the horizontal arrows in this diagram are proper and birational by Proposition 3.3.31, the left vertical arrow, let us denote it by ψ', is an alteration again. Note also that $X' \times_{F_X} G'$ is log-regular again [GR14, Cor. 10.6.18 (iii)].

We observe that to finish the proof it suffices to show the following statement: if there is a morphism $f\colon X' \to X$ of integral tropical schemes such that F_X is strictly simplicial, X' is log-regular, f is an alteration, and F_f is locally exact, then X is trop-regular. Let $x \in F_X$, and let C be a component of $\chi_X^{-1}\{x\}$. Then by Lemma [Kat94, Lemma 2.3] we have $\operatorname{codim}(C, X) \leq \operatorname{codim}(x, F_X)$. By 3.2.26 all points of $F_f^{-1}\{x\}$, and hence all components of $\chi_{X'}^{-1}(F_f^{-1}\{x\})$ have codimension at least $\operatorname{codim}(x, F_X)$. Therefore, all components of $f(\chi_{X'}^{-1}(F_f^{-1}\{x\}))$ have codimension at least $\operatorname{codim}(x, F_X)$ as well. But we have

$$C \subseteq f(f^{-1}C) \subseteq f(\chi_{X'}^{-1}(F_f^{-1}\{x\})),$$

thus proving that $\operatorname{codim}(C, X) = \operatorname{codim}(x, F(x))$. □

Example 3.3.51. Let X be the integral tropical scheme of Example 3.3.49. We want to find a proper subdivision G of F_X such that $X \times_{F_X} G$ is trop-regular. Applying the procedure of the proof of Proposition 3.3.50, we are almost done if we know a resolution of singularities of the pair $(X, X \setminus X^*)$. Since $X = \mathbb{A}^2$, and $X \setminus X_0$ is the union of $V(x_1)$, $V(x_2)$, and $V(1 + x_1 + x_2)$, this is trivial: $X \setminus X^*$ already is a simple normal crossings divisor in the smooth variety X. Let $X' = \mathbb{A}^2$ with the divisorial log structure, and let $f\colon X' \to X$ be the natural morphism. It is clearly isomorphic to $\mathbb{P}^2$ with the divisorial structure of the toric boundary. Thus, the cone complex $\Sigma_{F_{X'}}$ consists of three copies of $\mathbb{R}^2_{\geq 0}$, one for every intersection of the three lines, glued in the standard way. Let us consider the cone corresponding to the intersection point P in $V(x_1) \cap V(1 + x_1 + x_2)$. The local coordinates of F_X around $\chi_X(P)$ are, by definition, given by $x_1(1 + x_1 + x_2)$ and $x_2(1 + x_1 + x_2)$ modulo units. On the other hand, the local coordinates of $F_{X'}$ around $\chi_{X'}(P)$ are given by x_1 and $1 + x_1 + x_2$. Since $x_2(1 + x_1 + x_2)$ only differs by a unit from $1 + x_1 + x_2$ around P, we see that the restriction of Σ_{F_f} to $\sigma_{\chi_{X'}(P)}$ is equal to

$$\mathbb{R}^2_{\geq 0} = \sigma_{\chi_{X'}(P)} \to \sigma_{\chi_X(P)} = \mathbb{R}^2_{\geq 0}, \ \ (a, b) \mapsto (a + b, b).$$

Therefore, its image is $\{(a, b) \in \mathbb{R}^2_{\geq 0} \mid a \geq b\}$. Similarly, we see that with the same coordinates on the target, the cone $\sigma_{\chi_{X'}(P')}$, where $\{P'\} = V(x_2) \cap V(1 + x_1 + x_2)$, has image $\{(a, b) \in \mathbb{R}^2_{\geq 0} \mid a \leq b\}$, and the third cone has image $\mathbb{R}^2_{\geq 0}$. Looking back at the proof of Proposition 3.3.50, we conclude that subdividing $\Sigma_{F_X} = \mathbb{R}^2_{\geq 0}$ in the middle will produce a trop-regular modification.

Now we have all the tools to perform the reduction steps at our disposal. Starting with a subvariety Z of a tropical scheme X, the reduction steps produce a trop-regular tropical scheme Z' whose Kato fan is a proper subdivision of $F_X(x)$ for some $x \in X$. Using the characteristic morphism of Z' we will obtain a morphism from $A_{-\dim(Z)}(F_{Z'})$ to $A_0(X)$, and this is what we will call the tropicalization of Z. In summary, the tropicalization of Z will be a morphism from a Chow group of a refinement of $F_X(x)$ for some $x \in F_X$ to some abelian group. This motivates the following definition.

Definition 3.3.52. Let F be a fine weakly embedded Kato fan, and let H be an Abelian group. For $d \in \mathbb{Z}$ we define

$$C_d^\circ(F; H) := \varinjlim_G \operatorname{Hom}(A_{-d}(F), H),$$

where the limit is over all fine and saturated proper subdivisions G of F. We use this to define the *group of tropical cycles on F with coefficients in H* by

$$C_d(F; H) := \bigoplus_{x \in F} C_d^\circ(F(x); H).$$

Finally, we define a graded group $C_*(F; H) := \bigoplus_{d \in \mathbb{Z}} C_d(F; H)$. Let $C \in C_d(F; H)$ be represented by a morphism $A_{-d}(G) \to H$ for some subdivision G of $F(x)$ for some $x \in F$. We define the support $|C| \subseteq F(\overline{\mathbb{R}}_{\geq 0})$ of C as the union of all codimension-d cones $\sigma_z \in \Sigma_G$ such that the image of $[z]$ in H is nonzero. This does clearly not depend on the chosen representative.

Obviously, the definition of $C_*(F; H)$ is (covariantly) functorial in H. But it is also (covariantly) functorial in F: if $f \colon F \to G$ is a morphism of fine weakly embedded Kato fans, and $x \in F$, then there is an induced morphism $f(x) \colon F(x) \to G(f(x))$. As we know from Construction 2.2.10, there exist sufficiently many fine and saturated proper subdivisions F' and G' of $F(x)$ and $G(f(x))$ such that $f(x)$ lifts to a locally exact morphism $F' \to G'$. We can use the pull-backs $A_*(G') \to A_*(F')$ existing for such pairs of subdivisions to define a push-forward $C_*^\circ(F(x); H) \to C_*^\circ(G(f(x)); H)$. Taking a direct sum of those we obtain a push-forward $f_* \colon C_*(F; H) \to C_*(G; H)$.

Construction 3.3.53 (Tropicalization)**.** Let X be a tropical scheme, and let Z be a d-dimensional subvariety of $\underline{X}$. We make Z into a tropical scheme by pulling back the log structure of X and taking the composite $\overline{Z} \to \overline{X} \to F_X$ as characteristic morphism. Let ξ be the generic point of Z. By Lemma 3.3.47 and Proposition 3.3.48 the log scheme $Z^\bullet$ with the induced morphism $\overline{Z^\bullet} \to F_X(\chi_X(\xi))$ is an integral tropical scheme. Applying Proposition 3.3.50 we find a strictly simplicial subdivision $G \to F_X(\chi_X(\xi))$ such that $Z' := Z \overset{\bullet}{\times}_{F_X(\chi_X(\xi))} G$ is trop-regular. The morphism $\chi_{Z'}^* \colon A_{-d}(G) \to A_0(Z')$ from Proposition 3.3.42 then defines an element of $C_d(F_X; A_0(Z'))$. The pushforward induced by the proper morphism $Z' \to Z \to X$ transforms this into an element of $C_d(F_X; A_0(X))$. This element does not depend on the choice of G by Proposition 3.3.44. We denote it by $\operatorname{Trop}_X(Z)$. Extending by linearity we obtain a graded morphism

$$\operatorname{Trop}_X \colon Z_*(X) \to C_*(F_X; A_0(X)).$$

We denote $\operatorname{Trop}_X(X) = \operatorname{Trop}_X[X]$. ◊

Remark 3.3.54.

a) Since pull-backs from the Kato fan work for almost trop-regular tropical schemes just as well as for trop-regular tropical schemes, it suffices to take an appropriate simplicial proper subdivision of $F_X(\chi_X(\xi))$ instead of a strictly simplicial one.

b) Given a cocycle $c \in A^l(Z)$ we can tropicalize the pair (Z, c) as follows: choose a subdivision G of $F_X(\chi_X(\xi))$ as before, but now consider the map

$$\chi_{Z'}^*\colon A_{l-d}(G) \to A_l(Z')$$

and take its composite with the morphism $A_l(Z') \to A_0(Z')$ defined by c. Taking the composite with the push-forward defined by $Z' \to X$ then yields a well-defined element of $C_{d-l}(F_X; A_0(X))$.

Proposition 3.3.55. *Let X be a tropical scheme, and let $Z \subseteq X$ be a subvariety. Then* $\mathrm{trop}_X(Z)$ *is a union of subcones of* $F_X(\overline{\mathbb{R}}_{\geq 0})$ *of dimension less or equal to* $\dim(Z)$*. Furthermore, we have* $|\mathrm{Trop}_X(Z)| \subseteq \mathrm{trop}_X(Z)$*.*

Proof. We begin with showing that $\mathrm{trop}_X(Z)$ is a union of cones. After performing the three reduction steps for Z we may assume that $Z = X$ and X is trop-regular. As in the proof of Proposition 3.3.50 there exists a morphism $X' \to X$ of tropical schemes such that X' is log-regular, defined over a finite extension field of k, and the underlying morphism of schemes $\underline{X}' \to \underline{X}$ is an alteration. Note that the induced morphism $(X')^{\beth} \to X^{\beth}$ is surjective by the surjectivity of $(X')^{\mathrm{an}} \to X^{\mathrm{an}}$ [Ber90, Prop. 3.4.6 (7)] and the valuative criterion for properness. By the commutativity of the diagram

$$\begin{array}{ccc} (X')^{\beth} & \longrightarrow & F_{X'}(\overline{\mathbb{R}}_{\geq 0}) \\ \downarrow & & \downarrow \\ X^{\beth} & \longrightarrow & F_X(\overline{\mathbb{R}}_{\geq 0}), \end{array}$$

the set $\mathrm{trop}_X(X)$ is the image of $\mathrm{trop}_{X'}(X')$, thus reducing to the case where X is log-regular. By Corollary 3.3.28 we may further assume that χ_X coincides with the characteristic morphism of the underlying log scheme. But in this case Σ_{F_X} is at most $\dim(X)$-dimensional and trop_X is surjective [Uli13, Cor. 1.3] (Note that the cited corollary uses that k is perfect).

For the second part of the statement we also perform the reduction steps and assume that $Z = X$ is trop-regular. After taking an appropriate proper subdivision of F_X we may even assume that $\mathrm{trop}_X(X)$ is a union of cones of Σ_{F_X}. Therefore, it suffices to prove an appropriate version of the Tevelev lemma [Tev07, Lemma 2.2]: if X is trop-regular, and $x \in F_X$ then $\chi_X^*[x] \neq 0$ implies that $\mathrm{relint}(\sigma_x) \cap \mathrm{trop}_X(X) \neq \emptyset$. So assume that $\chi_X^*[x] \neq 0$. Then there exists $y \in \chi_X^{-1}\{x\}$ and a discrete valuation ring R dominating the local ring $\mathcal{O}_{X,y}$. By construction, the tropicalization of the element of $X^{\sqsupset}$ represented by the canonical morphism $\operatorname{Spec} R \to X$ is the element of $V_{\rightsquigarrow x}(\overline{\mathbb{N}})$ defined by the composite

$$\mathcal{M}_{F_X,x} \to \overline{\mathcal{M}}_{X,y} \to \overline{\mathcal{O}}_{X,y} \to R/R^* \cong \overline{\mathbb{N}}.$$

The image of this morphism is contained in $\mathbb{N}$, and as a composite of local morphisms it is local. We thus obtain an integral point of $\operatorname{relint}(\sigma_x)$. $\square$

If Z is a subvariety of a complete toroidal embedding X over an algebraically closed field k, we have now defined its tropicalization $\operatorname{Trop}_X(Z)$ in two different ways. First, as an element of $Z_*(\overline{\Sigma}(X))$ in Definition 2.3.8, and second as an element of $C_*(F_X; A_0(X))$. Pushing forward along the degree morphism $A_0(X) \to \mathbb{Z}$ the latter object gives rise to an element of $C_*(F_X; \mathbb{Z})$, and since $\Sigma_{F_X} = \Sigma(F)$ (cf. Example 3.3.27) we have $C_*(F; \mathbb{Z}) = Z_*(\overline{\Sigma}(X))$ by Proposition 3.2.6.

Proposition 3.3.56. *Let X be a complete toroidal embedding. Then the diagram*

$$\begin{array}{ccc} & & C_*(F_X; A_0(X)) \\ & \nearrow & \downarrow \deg_* \\ Z_*(X) & & \\ & \searrow & \\ & & Z_*(\overline{\Sigma}(X)), \end{array}$$

where the upper horizontal map is the tropicalization map as defined in Construction 3.3.53, and the lower horizontal map is the tropicalization map as defined in Definition 2.3.8, is commutative.

Proof. Let Z be a subvariety of X, and let $\sigma \in \Sigma(X)$ be the unique cone such that the generic point ξ of Z is contained in $\mathrm{O}(\sigma)$, that is $\sigma = \sigma_{\chi_X(\xi)}$. Consider Z and $\mathrm{V}(\sigma)$ as tropical schemes with the log structure pulled back from X. Then the log structure of Z also is the inverse image of the log structure of $\mathrm{V}(\sigma)$, and hence the log structure of $Z^\bullet$ is the inverse image of the log structure of $\mathrm{V}(\sigma)^\bullet$. Note that the log structure of $\mathrm{V}(\sigma)^\bullet$ is equal to the divisorial one given by $\mathrm{V}(\sigma) \setminus \mathrm{O}(\sigma)$, and $\Sigma_{F_X(\chi_X(\xi))} = \mathrm{S}_{\Sigma(X)}(\sigma)$ by Proposition 3.1.24. This reduces to the case where $Z \cap X^*$ is nonempty, that is where $\sigma = 0$. Let G be a strictly simplicial proper subdivision of F_X such that $\operatorname{trop}_X(Z)$ is a union of cones of Σ_G. Then $X \overset{\bullet}{\times}_{F_X} G$ is equal to what we denoted by $X \times_{\Sigma(X)} \Sigma_G$ in Chapter 2. Furthermore, using the universal property of the base change, we easily see that the proper transform of Z along $X \overset{\bullet}{\times}_{F_X} G \to X$, equipped with the induced tropical structure, is equal to $Z \overset{\bullet}{\times}_{F_X} G$. This allows us to reduce to the case where Z intersects all boundary strata properly. Let $d = \dim(Z)$, and let $x \in F_X$ be a codimension-d point. Moreover, let $x_1, \dots, x_d$ be the codimension-1 points of F_X specializing to x. Then the restriction of cycle $[Z] \cdot [\mathrm{V}(\sigma_x)] = [Z] \cdot \chi_X^*[x]$ (which is well-defined because the intersection is proper) to $X(\sigma_x) = \chi_X^{-1} V_{\rightsquigarrow x}$ is equal to the restriction of

$$\chi_X^* D_{x_1} \cdots \chi_X^* D_{x_d} \cdot [Z] = \chi_Z^* D_{x_1} \cdots \chi_Z^* D_{x_d} \cdot [Z]$$

to $X(\sigma_x) \cap Z$. This in turn is equal to the restriction of $\chi_Z^*[x]$ to $X(\sigma_x) \cap Z$. As all components of $V(\sigma_x) \cap Z$ and $\chi_Z^{-1}\{x\}$ have their generic point in $X(\sigma_x)$, this shows that

$$\deg(\chi_Z^*[x]) = \deg([V(\sigma_x)] \cdot [Z]).$$

The left hand side of this equation is the degree of the weight of the "new" $\mathrm{Trop}_X(Z)$ at σ_x, whereas the right hand side is the weight of the "old" $\mathrm{Trop}_X(Z)$ at σ_x, finishing the proof. □

Not surprisingly, tropicalizations are compatible with push-forwards along proper morphisms.

Proposition 3.3.57. *Let $f\colon X \to Y$ be a morphism of tropical schemes such that $\underline{f}$ is proper. Then the diagram*

$$\begin{array}{ccc} Z_*(X) & \xrightarrow{\quad f_* \quad} & Z_*(Y) \\ \downarrow{\scriptstyle \mathrm{Trop}_X} & & \downarrow{\scriptstyle \mathrm{Trop}_Y} \\ C_*(F_X;\, A_0(X)) & \xrightarrow{F_f^* \circ f_*} & C_*(F_Y;\, A_0(Y)) \end{array}$$

is commutative.

Proof. The proof is very similar to that of Theorem 2.3.20, so we only sketch it. Let Z be a closed subvariety of X. If $\dim(f(Z)) < \dim Z$, we use Proposition 3.3.55 to see that $\mathrm{Trop}_X(Z)$ is mapped to 0 for dimension reasons. Otherwise, let $d := \dim(f(Z)) = \dim Z$. We perform the reduction steps for Z and $f(Z)$ simultaneously; pulling back the respective log structures of X and Y to Z and $f(Z)$, and replacing Z and $f(Z)$ by $Z^\bullet$ and $f(Z)^\bullet$ is straightforward. To find compatible subdivision of F_X and F_Y we proceed as in the proof of Proposition 2.3.10. We may then assume that $Z = X$ and $f(Z) = Y$ are integral, that X and Y are almost trop-regular, and that F_f is locally exact. Then the tropicalizations $\mathrm{Trop}_X([X])$ and $\mathrm{Trop}_Y(f_*[X])$ are represented by the morphisms

$$\begin{aligned} \chi_X^*\colon A_{-d}(F_X) &\to A_0(X) \qquad \text{, and} \\ [K(X) : K(Y)]\chi_Y^*\colon A_{-d}(F_Y) &\to A_0(Y), \end{aligned}$$

respectively, and we need to show that the diagram

$$\begin{array}{ccc} A_{-d}(F_X) & \xrightarrow{\quad \chi_X^* \quad} & A_0(X) \\ {\scriptstyle F_f^*}\uparrow & & \downarrow{\scriptstyle f_*} \\ A_{-d}(F_Y) & \xrightarrow{[K(X):K(Y)]\chi_Y^*} & A_0(Y) \end{array}$$

commutes. If $y \in F_Y$ is a codimension-d point, then to show that $[y]$ is mapped to the same image in $A_0(Y)$ by the two maps in the diagram, we may restrict to the case in which $F_Y = V_{\rightsquigarrow y}$. In this case $[y]$ is the intersection of d $\mathbb{Q}$-Cartier divisors on F_Y and we can use Proposition 3.2.44, Proposition 3.3.42, the projection formula, and a straightforward calculation of lattice indices to finish the proof. □

Let us conclude this thesis with a brief discussion of possible extensions of the theory developed in this chapter. We first note that even though we have proved the result that for an almost trop-regular scheme X the morphism χ_X^* respects intersections with divisors (Proposition 3.3.44), we have not included a result about how Trop_X respects intersections with divisors. The reason for this is that if X^* is empty, then Cartier divisors on F_X do not define Cartier divisors on X in general. Therefore, it is not clear what to intersect with on the algebraic side.

The definition of tropical schemes requires the underlying log schemes to not have monodromy. In the presence of monodromy the characteristic fans should be replaced by some sort of Kato stack [Uli16, Section 2]. It would be interesting to define Chow groups for an appropriate class of such stacks and to relate them to the Chow groups of log schemes along the lines of our treatment of the monodromy-free case.

Finally, we have not studied how tropicalizations behave in families. More precisely, given a tropical scheme over a discrete valuation ring, it would be interesting to relate the tropicalizations of the special and the generic fiber. It is safe to assume that in sufficiently nice situations, e.g. in the log smooth case, the tropicalization of the generic fiber is some sort of recession fan of the tropicalization of the special fiber. There is also strong evidence to assume that intersections with divisors have related effects on the two fibers [OP13, Thm. 4.4.2].

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Notation

We have organized the notation used in this thesis by topic. Each entry comes with the number of the subsection where the notation first appears. If we were abusive and used the same notation for several meanings, several subsections may be given.

Moduli Spaces

$M_{0,I}$	moduli space of I-marked smooth rational algebraic curves	1.1.4
$\overline{M}_{0,I}$	moduli space of I-marked rational stable curves	2.4
$M_{0,I}^{\text{trop}}$	moduli space of I-marked rational tropical curves	1.1.2
$\text{LSM}_I^\circ(r,d)$	moduli space of I-marked generic logarithmic stable maps to $\mathbb{P}^r$ of degree d	1.1.4
$\text{LSM}_I^\circ(X,\Delta)$	moduli space of I-marked generic logarithmic stable map to X contact order Δ	1.1.4
$\text{LSM}_I(X,\Delta)$	moduli space of I-marked logarithmic stable maps to X with contact order Δ	1.3.1
$\text{TSM}_I^\circ(r,d)$	moduli space of I-marked tropical stable maps to $\mathbb{R}^{r+1}/\mathbb{R}\mathbf{1}$ of degree d	1.1.2
$\text{TSM}_I^\circ(N_\mathbb{R},\Delta)$	moduli space of I-marked tropical stable maps to $N_\mathbb{R}$ of degree Δ	1.1.2
ev_i	evaluation map for the i-th mark	1.1.4
ev_i^{t}	tropical evaluation map for the i-th mark	1.1.2
pl	Plücker embedding for $M_{0,n}$	1.1.4
pl^{t}	tropical Plücker embedding for $M_{0,n}^{\text{trop}}$	1.1.2
ft	forgetful map forgetting the map	1.1.4
ft^{t}	tropical forgetful map forgetting the map	1.1.2

Cones and Cone Complexes

M^σ	lattice of integral linear functions on σ	2.1.1
M_+^σ	monoid of nonnegative integral linear functions on σ	2.1.1
N^σ	dual lattice of M^σ	2.1.1
N_+^σ	integral points of σ (considered as subset of $N_\mathbb{R}^\sigma$)	2.1.1

$\lvert\Sigma\rvert$	underlying set of Σ	2.1.1
N^{Σ}	integral points of the codomain of the weak embedding	2.1.1
M^{Σ}	dual lattice of N^{Σ}	2.1.1
φ_{Σ}	weak embedding of Σ	2.1.1
N^{Σ}_{σ}	integral points of $\varphi_{\Sigma}(\sigma)$	2.1.1
$\mathrm{S}_{\Sigma}(\tau)$	star of Σ at τ	2.1.1
$\overline{\sigma}$	extended cone of σ	2.1.2
$\overline{\Sigma}$	extended cone complex of Σ	2.1.2
$\mathrm{Div}(\Sigma)$	Cartier divisors, i.e. piecewise integral linear functions, on Σ	2.2.1
$\mathrm{CP}(\Sigma)$	combinatorially principal divisors	2.2.1
$\mathrm{ClCP}(\Sigma)$	the quotient of $\mathrm{CP}(\Sigma)$ by M^{Σ}	2.2.1
$M_k(\Sigma)$	k-dimensional Minkowski weighs on Σ	2.2.1
$Z_k(\Sigma)$	k-cycles on Σ, i.e. Minkowski weights on some proper subdivision	2.2.1
$Z_k(\overline{\Sigma})$	direct sum of the groups $Z_k(\mathrm{S}_{\Sigma}(\Sigma))$ for $\sigma \in \Sigma$	2.2.1
$A_k(\overline{\Sigma})$	k-th Chow group of Σ, that is the quotient of $Z_k(\overline{\Sigma})$ by cycles rationally equivalent to 0	2.2.4
$\cup$	tropical cup product	2.2.3
"."	tropical intersection product	2.2.3, 2.2.4
f^*	pull-back of cp-divisors induced by a morphism of weakly embedded extended cone complexes	2.2.1
f_*	push-forward of cycle (classes) induced by a morphism of weakly embedded extended cone complexes	2.2.2, 2.2.4

Toroidal Embeddings

X_0	open stratum of X	2.1.3
$X(Y)$	combinatorial open subset of X associated to a stratum Y	2.1.3
$M^Y(X)$	boundary divisors on $X(Y)$	2.1.3
$M^Y_+(X)$	effective boundary divisors on $X(Y)$	2.1.3
$N^Y(X)$	dual lattice of $M^Y(X)$	2.1.3
σ^Y_X	cone of nonnegative linear functions on $M^Y_+(X)$	2.1.3
$\Sigma(X)$	cone complex associated to X	2.1.3
φ_X	weak embedding of $\Sigma(X)$	2.1.3
N^X	integral points of the codomain of φ_X	2.1.3
M^X	dual lattice of N^X; invertible functions on X_0 modulo k^*	2.1.3
$X(\sigma)$	combinatorial open subset of X associated to $\sigma \in \Sigma(X)$	2.1.3

$\mathrm{O}(\sigma)$	stratum associated to a cone $\sigma \in \Sigma(X)$	2.1.3
$\mathrm{V}(\sigma)$	closure of $\mathrm{O}(\sigma)$	2.1.3
$\mathrm{Trop}(f)$	induced map $\overline{\Sigma}(X) \to \overline{\Sigma}(Y)$ of a toroidal morphism $f\colon X \to Y$	2.1.3
$X \times_\Sigma \Sigma'$	toroidal modification induced by a subdivision $\Sigma' \to \Sigma$	2.1.3
Trop_X	tropicalization morphism for divisors, cocycles, or cycles cycle classes	2.3.1, 2.3.2, 2.3.4, 2.3.5
Trop_ι	tropicalization for subvarieties of X_0 w.r.t. a morphism $\iota\colon X_0 \to T$ to an algebraic torus	2.3.6
$\Sigma_\iota(X)$	the cone complex $\Sigma(X)$ with the weak embedding induced by a morphism $\iota\colon X_0 \to T$ to a torus	2.3.6
$\varphi_{X,\iota}$	the weak embedding of $\Sigma_\iota(X)$	2.3.6

Monoids and Kato fans

$M[-S]$	localization of M at S	3.1.1
$M_\mathfrak{p}$	localization of M at the prime ideal $\mathfrak{p}$	3.1.1
M^{gp}	groupification of M	3.1.1
M^*	group of units of M	3.1.1
M^{sat}	saturation of the monoid M	3.1.1
$\overline{M}$	the sharpening of M, that is M modulo M^*	3.1.1
$\mathrm{Spec}\, M$	affine spectrum of M	3.1.1, 3.1.2
$\mathcal{M}_F$	structure sheaf of the Kato fan F	3.1.2
$f^\flat$	the morphism $f^{-1}\mathcal{M}_G \to \mathcal{M}_F$ included in the data of a morphism f of Kato fans	3.1.2
F^{sat}	saturation of the Kato fan F	3.2.2
$V_{\rightsquigarrow x}$	set of points specializing to x	3.1.2
$F(M)$	set of morphism $\mathrm{Spec}\, M \to F$	3.1.2
Σ_F	cone complex associated to F	3.1.2, 3.1.3
σ_x	cone of Σ_F corresponding to $x \in F$	3.1.2
$F(x)$	induced Kato fan with underlying set $\overline{\{x\}}$	3.1.2, 3.1.3
M^F	group of rational functions of F	3.1.3
M^F_x	group of rational functions of $F(x)$	3.1.3
$\mathrm{ord}^F_x(m)$	order of m at x	3.2.1

$\mathrm{div}^F(m)$	principal divisor of a rational function m	3.2.1
$Z_k(F)$	group of k-cycles on F	3.2.1
$A_k(F)$	k-th Chow group of F	3.2.1
f_*	push-forward of cycle (classes) along a subdivision f	3.2.2
f^*	pull-back of cycle (classes) along a locally exact morphism f, or of divisors along arbitrary morphisms	3.2.3, 3.2.4
$\mathrm{Div}(F)$	Cartier divisors on F	3.2.4
$\mathrm{CP}(F)$	cp-divisors on F	3.2.4
$\mathrm{ClCP}(F)$	cp-divisors modulo principal divisors	3.2.4
$\lvert D\rvert$	support of a divisor D	3.2.4
$D \cdot \alpha$	intersection product of divisor and cycle	3.2.4
$c_1(\mathcal{L}) \cap \alpha$	cap-product of divisor class and cycle class	3.2.4

Logarithmic and Tropical Schemes

$\mathcal{M}^{\log}$	logification of a sheaf of monoids $\mathcal{M}$	3.3.1
$\alpha_X \colon \mathcal{M}_X \to \mathcal{O}_X$	log structure on X	3.3.1
$\overline{\mathcal{M}}_X$	characteristic sheaf of X, equal to $\mathcal{M}_X/\mathcal{O}_X^*$	3.3.1
$\overline{X}$	the sharp monoidal space $(X, \overline{\mathcal{M}}_X)$; later also the weakly embedded sharp monoidal space $(X, \overline{\mathcal{M}}_X, M^X \to \overline{\mathcal{M}}_X^{\mathrm{gp}})$	3.3.1
$\underline{X}$	the underlying scheme of the log scheme X	3.3.1
$f^\flat$	the morphism $f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ included in the data of a morphism $X \to Y$ of log schemes	3.3.1
$I(x, \mathcal{M}_X)$	the ideal in $\mathcal{O}_{X,x}$ generated by the maximal ideal of $\mathcal{M}_{X,x}$	3.3.1
F_X	the characteristic fan of the logarithmic or tropical scheme X	3.3.2
χ_X	the characteristic morphism of the logarithmic or tropical scheme X	3.3.2
$X \times_{F_X} G$	the universal tropical scheme with characteristic fan G over the tropical scheme X	3.3.2
$X \overset{\bullet}{\times}_{F_X} G$	the universal integral tropical scheme with characteristic fan G over the integral tropical scheme X	3.3.2
$e_C^{\log} X$	the logarithmic multiplicity of X along C	3.3.3
χ_X^*	the pull-back of cycles, cycle classes, or divisors from F_X to X	3.3.3
$\mathcal{O}_X^\bullet$	the sheaf of nonzero regular functions on an integral scheme X	3.3.1, 3.3.4
$\mathcal{M}_X^\bullet$	the preimage of $\mathcal{O}_X^\bullet$ under α_X	3.3.4
$C_d(F; H)$	the group of tropical d-cycles with coefficients in H on F	3.3.4

trop_X	the tropicalization map $X^{\daleth} \to F_X(\overline{\mathbb{R}}_{\geq 0})$	3.3.4
Trop_X	the tropicalization morphism for cycles on the tropical scheme X	3.3.4

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