A reduction algorithm for integer multiple objective linear programs

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We consider a multiple objective linear program $(MOLP)\max\{Cx|Ax = b, x \in \mathbb{N}_0^n\}$ where $C = (c_{ij})$ is the $p \times n$-matrix of p different objective functions $z_i(x) = c_{i1}x_1 + ... + c_{in}x_n, i = 1, ..., p$ and A is the $m \times n$-matrix of a system of m linear equations $a_{k1}x_1 + ... + a_{kn}x_n = b_k, k = 1, ..., m$ which form the set of constraints of the problem. All coefficients are assumed to be natural numbers or zero. The set M of admissable solutions is given by $M = \{x|Ax = b, x \in \mathbb{N}_0^n\}$. An efficient solution $\overline{x}$ is an admissible solution such that there exists no other admissable solution x‘ with $C\overline{x} < Cx$‘. The efficient solutions play the role of optimal solutions for the MOLP and it is our aim to determine the set of all efficient solutions.

From the p different objective function we generate a new parametric objective function f which has the property to perserve the canonical order on $(\mathbb{N}_0^n; \leq_{lex})$. Therefore we can use f to find all efficient solutions in a lexikographic order. An efficient solution which is already found will be eliminated by some constraints to get the next efficient solution.

This is a theoretical approach to solve the problem because of the high complexity of the procedures involved. But we have already shown in [] that for small bicriteria integer linear programs it is applicable. This approach generalizes the results in [] to the multicriteria case by introducing an objective function f which is strictly monotone for two different orderings.

1 The scaling

We determine for every objective function $z_i(x)$ the minimal value a_i and the maximal value b_i. We denote the difference by $d_i = b_i - a_i + 1$. If for some objective function $z_i(x)$ we have $d_i = 1$ then we may drop this objective function because it is unnecessary as a criterion for the decision. Without a loss of generality we assume that $d_i = 1$ then we may drop this objective function because it is unnecessary as a criterion for the decision. Without a loss of generality we assume that $d_i > 1$ and introduce a new parametrix objective function $f : \mathbb{N}_0^n \to \mathbb{N}_0^n$

$$f(x) = \sum_{i=1}^{p} \left(\prod_{j=1}^{i} d_j \right)^{-1} z_i(x)$$

We write $z_i(x) = c_i(x)$ where $c_i = (c_{i1}, ..., c_{in})`$ and have $f(x) = f(x) = \sum_{i=1}^{p} \left(\prod_{j=1}^{i} d_j \right)^{-1}$
$c_i(x) = \frac{1}{d_1} c_1 x + \frac{1}{d_1 d_2} c_2 x + ... + \frac{1}{d_1 ... d_p} c_p x$.We consider the canonical order $\leq$ on $\mathbb{N}_0^P$ which is defined componentwise.$(a_1, ..., a_p) \leq (b_1, ..., b_p)$ if and only if $a_i \leq b_i$ for every $i = 1, ..., p$. The criterion space of the MOLP is given by

$$Z = \left\{ z \in \mathbb{N}_0^P | z = Cx, x \in \mathbb{N} \right\}$$

under the above hypothesis. It is obvious that $(Z; \leq)$ is a suborder of $\left(\mathbb{N}_0^P; \leq \right)$. Furthermore the function $g : Z \to \mathbb{R}_0$ $g(z) = g(c_1 x, ... c_n x) = f(x)$ is strictly monotone on Z as

$$\left(\prod_{j=1}^{i} d_j \right)^{-1} > 0.$$

On the other hand let us consider the lexicographic order $<_{lex}$ on $\mathbb{N}_0^P$ which is defined in the following way. $(a_1, ..., a_p) <_{lex} (b_1, ..., b_p)$ if there exists $m \in \mathbb{R}_0$ with $0 \leq m < p$ such that $a_k = a_k$ for $k = 1, ..., m-1$ and $\mathrm{a}_m < b_{m.}$ The linear order $\left(\mathbb{N}_0^P; < lex \right)$ has the linear suborder $(Z; <_{lex})$. Furthermore we may consider $(\mathbb{R}; <)$ as a lexicographic order as well. Our aim is to show that $g : Z \to \mathbb{R}_0$ also preserves the lexicographic order. We will use the following

Proposition 1.1

Let $\Delta \overline{c}_i d_i \in \mathbb{N}$ with $0 < \Delta c_i < d_i$ and $i = 1, ..., p$. If $a_i = \Delta c_i + \frac{a_{i+1}}{d_{i+1}}$for $i = 1, ..., p-1$ and $a_p = \Delta c_p$ for $i = p$ then we have $\frac{a_i}{d_i} < 1$ for $i = 1, ..., p$.

Proof. For $i = p$ we have $a_p < d_p$ and hence $\frac{a_p}{d_p} < 1$. Assume that we have proved $\frac{a_{i+1}}{d_{i+1}} < 1$ for some $i = 1, ..., p-1$.We have

$$a_i = \Delta c_i + \frac{a_i + 1}{d_i + 1} \leq d_i - 1 + \frac{d_i + 1}{d_i + 1} < d_i - 1 + 1 = d_i.$$

Hence we have $\frac{a_i}{d_i} < 1$.

Theorem 1.2

The function $g : Z \to \mathbb{R}_0$ defined by $(z) = g(c_1 x, ..., c_p x) = f(x)$ preserves the lexicographic order.

Proof. Let $z^1 <_{lex} z^2$ and hence $(c_1 x^1, ..., c_n x^1) L_{lex} (c_1 x^2, ..., c_p x^2)$. We have

$$\left(\prod_{j=1}^{i}\right)^{-1} c_i x^1 = \left(\prod_{j=1}^{i} d_j^{-1}\right) c_i x^2 \text{ for } i = 1, ..., m-1$$

and

$$\left(\prod_{j=1}^{m} d_j\right)^{-1} c_m \cdot x^1 < \left(\prod_{j=1}^{m} d_j\right)^{-1} c_m x^2.$$

It remains to show that

$$\left| \sum_{i=m+1}^{p} \left(\prod_{j=1}^{m} d_j\right)^{-1} c_i \left(x^2 - x^1\right) \right| < \left(\prod_{j=1}^{m} d_j\right)^{-1} c_m \left(x^2 - x^1\right)$$

or after a division that

$$\sum_{i=m+1}^{p} \left(\prod_{j=m+1}^{i} d_j\right)^{-1} c_i \left|x^2 - x^1\right| < c_m \left(x^2 - x^1\right)$$

(We also notice that $1 \leq c_m \left(x^2 - x^1\right)$ because we have only integers). For our convenience we put $\triangle \overline{c}_i := \left|c_i \left(x^2 - x^1\right)\right|$

$$\sum_{i=m+1}^{p} \left(\prod_{j=m+1}^{i} d_j\right)^{-1} \triangle \, \overline{c}_i =$$

$$\frac{\triangle c_{m+1}}{d_{m+1}} + \frac{\triangle c_{m+2}}{d_{m+1} \cdot d_{m+2}} + ... + \frac{\triangle c_{p-1}}{d_{m+1} ... \cdot d_{p-1}} + \frac{\triangle c_p}{d_{m+1} ... \cdot d_p} = \frac{1}{d_{m+1}}$$

$$(\triangle c_{m+1} + \frac{1}{d_{m+2}}(c_{m+2} + ... + \frac{1}{d_{p-1}}(\triangle c_{p-1} + \frac{1}{d_p} \triangle c.$$

Using the proposition 1.1 we get

$$= \frac{1}{d_{m+1}}(\triangle c_{m+1} + \frac{1}{d_{m+2}}(\triangle c_{m+2} + ... + \frac{1}{d_{p-1}}(\triangle c_{p-1} + \frac{a_p}{d_p}.$$

$$= \frac{1}{d_{m+1}}(\triangle c_{m+1} + \frac{1}{d_{m+2}}(\triangle c_{m+2} + ... + \frac{a_{p-1}}{d_{p-1}}) = ...$$

$$= \frac{a_{m+1}}{d_{m+1}} < 1 \leq c_m \left(x^2 - x^1\right)$$

Corollary 1.3

The function f has the properties

(1.3.1) $Cx^1 < Cx^2$ implies $f(x^1) < f(x^2)$(1.3.2) $Cx^1 <_{lex} Cx^2$ implies $f(x^1) < f(x^2)$

Assume that the admissable solution $x^0 \in M$ is not efficient. Then there exists an admissable solution x' with $Cx^0 < Cx'$. By (1.3.1) x^0 is not an optimal solution of $\max\{f(x)|Ax = b, x \in \mathbb{N}_0^n\}$.

Corrollary 1.4

If x^0 is an optimal solution of $\max\{f(x)|Ax = b, x \in \mathbb{N}_0^n\}$ then x^0 is an efficient solution of $\max\{Cx|Ax = b, x \in \mathbb{N}_0^n\}$

2 The adaption of constraints

Let x^0 be the effecient solution which is found as an optimal solution of the linear program $\max\{f(x)|Ax = b, x \in \mathbb{N}_0^n\}$ Let $f_0 = f(x^0)$ the optimal value of the objective function f. Then we eliminate this efficient solution by the constraint $f(x) < f_0$.

We call a solution $x^1 \in M$ dominated by a solution $x^2 \in M$ if $Cx^1 < Cx^2$. We eliminate all solutions $x \in M$ which are dominated by x^0 with the constraint $y'(Cx - (x^0) > 0$ then x^0 determinates no $x \in X$ for $y \in \mathbb{N}^n, (y \neq 0)$. By adding these constraints the set of admissable solutions changes.

Lemma 2.1

Let x be the set of all admissable solutions. If for every $x \in X$ there is a vector $y \in \mathbb{N}^n$ with $y'(Cx - Cx^0) > 0$ then x^0 dominates no $x \in X$.

Proof. If we have $y'(Cx - Cx^0) = \sum_{i=1}^{p}((Cx)_i - (Cx^0)i)y_i > 0$ then there exists at least one index i such that $(Cx)_i - (Cx^0)_i > 0$. It means that at least in one component i the value of the new solution x in the objective function is greater than the value of x^0. Hence x will not be dominated by x^0.

Let $z^1 = Cx^1, z^2 = Cx^2, ..., z^j = Cx^j$ be different efficient solutions for the problem $\max\{Cx|Ax = b, x \in \mathbb{N}_0$. Let $L = \{x^1, ..., x^j\}$ be the set of the efficient solutions which were found till now.

Lemma 2.2

Let $(Cx)_i > 0$ for $i = 1, ..., p$ and $x \in X$. For $x^j \in L$ there exists $y^j \in \mathbb{N}^n$ with $y^{j'}(Cx - Cx^j) > 0$ if and only if $(Cx)_i - (Cx^j)_i y_i^j > 0$ for $i = 1, ..., p$ and $\sum_{i=1}^{p} y_i^j \geq 1$.

Proof. If $y^{j'}(Cx - Cx^0) > 0$ holds then we have $(Cx - Cx^0)i_0 > 0$ for at least one component i_0. We choose $y_{i0}^j = 1$ and all other components $y_i^j = 0$. As $Cx > 0$ we have $((Cx)_i - (Cx^j)_i)y_i^j > 0$ for every $i = 1, ..., p$ and $\sum_{i=1}^{p} y_i^j \geq 1$. On the other hand from $\sum_{i=1}^{p} y_i^j \geq 1$ it follows that there is a vector $y_k^j = m \geq 1$ for some k. For this k we have

$((Cx)_k - (Cx^j)k) \cdot y_i^j > C$. Now we choose $y_k^j = 1$ and every other component $y_i^j = 0$ and we have $y^{j'}(Cx - Cx^j) > 0$.

Theorem 3.3

Let X be the set of all admissable solutions which fulfill the following constraints $Ax = b, f(x) < f(x^j), y^{j'}(Cx - Cx^j) > 0$ for every $x^j \in L$ with $x \in \mathbb{N}_0^n, y^j \in \mathbb{N}_0^P$. If $x \neq 0$ then the linear program $\max\{f(x)|x \in X\}$ generates a new efficient solution. If $x = \emptyset$ then all efficient solutions have been already found in the list L.

Proof. The new solution x' has the property that for every efficient solution x^j of our list L we have $f(x^j) > f(x')$. We have to show that x' is efficient. If x' is not efficient then either x' is dominated by an element of the admissable set x of the actual calculation or by an already eliminated element.
Case1.$z' = Cx'$ is dominated by $z = Cx$ of the actual admissable set X. Then we have $Cx > Cx'$ and as f is strictly monotone $f(x) > f(x')$, a contradiction.
Case 2. z' is dominated by the already eliminated point z^j. But this contradicts the constraint $y^{j'}(Cx - Cx^j) > 0$. Hence x' is an efficient solution.

3 The reduction algorithm.

We use the notations of the preceding sections.

Step1. Calculation of the objective function f for $i = 1, ..., p$ do

$b_i = \max\{c_i x | Ax = b, x \in \mathbb{N}_0^n\}$

$a_i = \min\{c_i x | Ax = b, x \in \mathbb{N}_0^n\}$

$d_i = b_i - a_i + 1$

$$f(x) = \sum_{i=1}^{p} \left(\prod_{j=1}^{i} d_j \right)^{-1} c_i x$$

Step 2.Initial solution (z^1, x^1)

(z^1, x^1) is calculated by $\max\{f(x)|Ax = b, x \in \mathbb{N}_0^n\}$

$L := \{(z^1, x^1)\}$

$x^1 := \{x|Ax = b, x \in \mathbb{N}_0^n$, there is $y^1 \in \mathbb{N}_0^p$ with $(Cx)_j - z_j^1 y_j^1 \geq 1\}$

$i := 1$

Step3: Searching loop

for $x^i \neq \emptyset$ do (z^{i+1}, x^{i+1}) is calculated by $\max\{f(x)|x \in x^i\}$

$L := Lu\{(z^{i+1}, x^{i+1})\}$

$x^{i+1} = \{x \in x^i f(x) < f(x^{i+1})$, there is $y^{i+1} \in \mathbb{N}_0^p$ with $\sum_{j=1}^{p} y_j^{i+1} \geq 1$ and $(Cx)_j - z^{i+1} y_j^{i+1} > 0$ for every $j = 1, ..., p\}$

$i := i + 1$

Step 4.Output

for $j = 1, ..., i$ do print (z^j, x^j)

Theorem 3.1

The reduction algorithm finds every efficient solution of the integer multiple objective linear program
$\max\{Cx|Ax = b, x \in \mathbb{N}_0^n\ \}$

Proof. Assume there is an efficient solution x^0 with $z^0 = Cx^0$ of $\max\{Cx|Ax = b, x \in \mathbb{N}_0^n\ \}$. Then there exists $z^k, z^{k+1} \in L$ such that $z^k >_{lex} z >_{lex} z^i$ and such that for $z' \in L$ we have either $z^i >_{lex} z$
k or $z^{k+1} >_{lex} z'$.We consider the $(k+1)^{st}$ iteration of the searching loop in step 3. In this state z belongs to admissable set x as f(x^k) $> f(x)$ and z is efficient by hypothesis. The algorithm found z^{k+1} as $f(x^{k+1}) \geq f(x)$ holds in contradiction to $z >_{lex} z^{k+1}$ and hence $f(x) > f(x^{k+1})$.

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