A Characterization of Lexicographic Max-Ordering Solutions

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Abstract

In this paper we give the definition of a solution concept in multicriteria combinatorial optimization. We show how Pareto, max-ordering and lexicographically optimal solutions can be incorporated in this framework. Furthermore we state some properties of lexicographic max-ordering solutions, which combine features of these three kinds of optimal solutions. Two of these properties, which are desirable from a decison maker's point of view, are satisfied if and only if the solution concept is that of lexicographic max-ordering.

Keywords: Multiple Criteria, Decision Making, Combinatorial Optimization, Pareto Optimality, Lexicographic Order, Max-Ordering

AMS subject classification: 90C29, 90B27, 90B50

1 Introduction

Before discussing optimal solutions we begin with the description of the framework of combinatorial optimization in which we will define solution concepts. A combinatorial optimization problem is usually defined by a finite ground set $E = \{e_1, \ldots, e_m\}$ and a set of feasible solutions $\mathcal{F}$ which consists in a subset of the set of all subsets of E, denoted by $\mathcal{P}(E)$. Furthermore there are Q objective functions $f_q : \mathcal{F} \to \mathbb{N}$. The problem is then to find some feasible solution "minimizing" the Q objectives.

Usually the objectives are defined by means of weight vectors $w(e) \in \mathbb{R}^Q$ for the elements e of E. The most common objective functions are the sum of the weights, i.e. $f_q(F) = \sum_{e \in F} w_q(e)$, or their maximum, i.e. $f_q(F) = \max_{e \in F} w_q(e)$.

In the context of multiple criteria optimization it is clear that the common definition of minimizing an objective when only one criterion is considered is no longer valid. Below we list several definitions of "optimal" solutions of multiple criteria combinatorial optimization problems, which appear most often in the literature. The interested reader is referred to [3] for even more.

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Pareto optimality A feasible solution $F \in \mathcal{F}$ is called Pareto optimal if there is no other solution which is not worse with respect to all objective functions and strictly better with respect to at least one, i.e.

$$\nexists F' \in \mathcal{F} : f_q(F') \leq f_q(F), \quad q = 1, \ldots, Q \text{ and } f(F) \neq f(F').$$

It should be obvious that under the assumption that the problem is formulated correctly no solution other than a Pareto optimal one should be considered as "optimal" in any sense. However, the set of Pareto optimal solutions may be prohibitively large from a decision maker's point of view. We refer to [6] and [5] where examples are given in which all feasible solutions are Pareto optimal and have different objective value vectors.

Max-ordering optimality Here a feasible solution is considered optimal if the worst objective value is as small as possible, i.e.

$$\max_{q \in \{1,\ldots,Q\}} f_q(F) \leq \max_{q \in \{1,\ldots,Q\}} f_q(F') \quad \forall F' \in \mathcal{F}.$$

The advantage of this definition is that the optimal solution value is uniquely defined. However not all max-ordering optimal solutions are also Pareto optimal, although at least one is, as we will see later. Another weakness is that, once the objectives are evaluated for a feasible solution, only one (the worst) determines whether the solution is considered as optimal. Therefore a decison maker may ask: Why use several objectives to start with?

Lexicographic optimality The definition of optimality which seems to be closest to single objective optimization is that of lexicographic optimality. A feasible solution F is lexicographically optimal if

$$f(F) \leq_{lex} f(F') \quad \forall F' \in \mathcal{F}.$$

Recall that for $x, y \in \mathbb{R}^Q$ we say that $x <_{lex} y$ if there exists an index q^* such that $x_q = y_q$ for $q = 1, \ldots, q^* - 1$ and $x_{q^*} < y_{q^*}$. Obviously due to the comparison of solutions according to the lexicographic order this definition implies a ranking of the objectives (f_1 is more important than f_2 and so on). A decision maker may not be willing, or even be unable, to do that, because he is indifferent with respect to the single objectives. But again the optimal solution value, a $Q-$dimensional vector in this case, is unique.

Extremal Pareto optimality A feasible solution F is called extremal Pareto optimal if it is the solution of a scalarized problem, i.e. if there exists a scalarizing vector $(\lambda_1, \ldots, \lambda_Q)$, all λ_q strictly positive and $\sum_{q=1}^{Q} \lambda_q = 1$ such that F is an optimal solution of the single criterion minimization problem with objective function $\sum_{i=1}^{Q} \lambda_q f_q$, i.e.

$$\sum_{q=1}^{Q} \lambda_q f_q(F) \leq \sum_{q=1}^{Q} \lambda_q f_q(F') \quad \forall F' \in \mathcal{F}.$$

Extremal Pareto solutions are always Pareto solutions. Their objective function vectors are extreme points of the convex hull of $\{f(F)|F \in \mathcal{F}\}$. In continuous multicriteria optimization these solutions are also called properly efficient solutions, see [4]. Even the set of extremal Pareto solutions may be very large, although in most cases its cardinality is much smaller than that of the Pareto set. Choosing a scalarizing vector λ is equivalent to assigning relative preferences to the Q objective functions, something decision makers may want to avoid.

The rest of the paper is organized as follows. In Section 2 we define lexicographic max-ordering solutions. In Section 3 we introduce the notion of solution concepts and prove some properties of lexicographic max-ordering solutions. Section 4 is devoted to the main result, the characterization of lexicographic max-ordering solutions by means of two properties. Conclusions are summarized in Section 5.

2 Lexicographic Max-ordering Solutions

In this paper we want to propose a definition of "optimality" which combines the features of Pareto, max-ordering and lexicographic optimality, namely consideration of all objective functions, unique optimal solution value, and lexicographic ordering. In combining these features we will not only be able to preserve the advantages of these three optimality definitions but we can at the same time get rid of the main disadvantages such as prohibitively large cardinality of the set of optimal solutions and neglect of objective functions. First of all we need the following definition.

Definition 1 *For any element $x \in \mathbb{R}^Q$ we define $\Theta(x) = (\Theta_1(x), \ldots, \Theta_Q(x))$ to be the vector containing the components of x in nonincreasing order: $\Theta_1(x) \geq \ldots \geq \Theta_Q(x)$, $\{x_1, \ldots, x_Q\} = \{\Theta_1(x), \ldots, \Theta_Q(x)\}$.*

Definition 1 is essential for the definition of lexicographic max-ordering solutions, Lex-MO solutions for short, of a multiple criteria combinatorial optimization problem.

Definition 2 *A feasible solution $F \in \mathcal{F}$ is a Lex-MO solution if its objective function vector is lexicographically minimal with respect to $\Theta(f(F))$. Therefore*

$$\Theta(f(F)) \leq_{lex} \Theta(f(F')) \quad \forall F' \in \mathcal{F}.$$

Lexicographic max-ordering solutions have been considered for several special problems in the literature, under other names. We refer to [8] for location problems, Lex-MO solutions are called lexicographic centers, and [7] for multicriteria linear programming, Lex-MO solutions are called nucleolar solutions. In [1] the relations between multicriteria linear programming and game theory are investigated with reference to the concept of the nucleolus in game theory. The

nucleolus has similarities to that of Lex-MO solutions presented in our paper. In none of these papers the relevance for general multiple criteria optimization has been considered. This was done for the first time in [2].
We now provide a simple example, taken from [2], illustrating the relations of Lex-MO solutions and most of the definitions of optimality presented in Section 1.

Example 1 *Consider a problem the set of feasible solutions of which is $\mathcal{F} = \{a, b, c, d, e\}$. Let us assume that the objective function values and Θ vectors are as presented in Table 1.*

F	$f(F)$	$\Theta(f(F))$
a	(1,3,8,2,4)	(8,4,3,2,1)
b	(4,3,8,1,1)	(8,4,3,1,1)
c	(7,5,4,6,1)	(7,6,5,4,1)
d	(3,7,4,6,5)	(7,6,5,4,3)
e	(4,7,5,6,5)	(7,6,5,5,4)

Table 1: Objectives and Θ Values in Example 1

Note that a, b, c, and d are Pareto solutions. The lexicographically optimal solution is obviously a. The set of max-ordering solutions consists of c, d, and e, wheras c is the unique Lex-MO solution. We will see later that it is always the case that Lex-MO solutions are Pareto as well as max-ordering optimal.

3 Solution Concepts

Before we prove the main results of this paper we proceed by introducing the general notation of a solution concept for multiple criteria combinatorial optimization problems. We also define what is the set of optimal solution values and what is to be understood as an optimal solution with respect to a solution concept.

Definition 3
1. *A solution concept is defined by a triple $(p, \theta, \preceq)$, where p is a nonnegative integer, θ is a mapping from $\mathbb{R}^Q$ to $\mathbb{R}^p$, and $\preceq$ is a partial order in $\mathbb{R}^p$.*
2. *The set of all optimal solution values is the set of all minimal elements of $\{\theta(f(F)) | F \in \mathcal{F}\}$, denoted by $\mathcal{V}_{opt}(p, \theta, \preceq)$.*
3. *A solution $F \in \mathcal{F}$ is an optimal solution of the combinatorial optimization problem with respect to solution concept $(p, \theta, \preceq)$ if*

$$\theta(f(F)) \in \mathcal{V}_{opt}(p, \theta, \preceq).$$

The set of all optimal solutions is denoted by $\mathcal{F}_{opt}(p, \theta, \preceq)$.

In other words we can say that F is an optimal solution with respect to $(p, \theta, \preceq)$ if it does not exist another feasible solution F' such that $\theta(f(F')) \preceq \theta(f(F))$ and $\theta(f(F')) \neq \theta(f(F))$. Then $\mathcal{V}_{opt}(p, \theta, \preceq) = \{\theta(f(F)) | F \in \mathcal{F}_{opt}(p, \theta, \preceq)\}$. Recall that $\preceq$ can be a partial order only.

In this paper we only consider solution concepts which satisfy the following **normalization property**. For the special case that $Q = 1$ it is natural to assume that the usual single objective minimization with the canonical order of $\mathbb{R}$ should be used. Therefore if $(p, \theta, \preceq)$ is any solution concept, we require that $Q = 1$ implies $(p, \theta, \preceq) = (1, \mathrm{id}, \leq_{\mathbb{R}})$, which actually means ordinary minimization of one criterion.

In Example 2 we see that the various definitions of optimality presented in Section 1 can easily be incorporated in the definition of a solution concept. We do this by giving in Table 2 the appropriate p, θ, and $\preceq$ definitions for all optimality definitions, including Lex-MO optimality.

Example 2 *The relation of the optimality definitions of Section 1 and the notion of solution concepts is summarized in Table 2.*

Optimality	p	θ	$\preceq$
Pareto	Q	$\theta(x) = x$	$\leq_{\mathbb{R}^Q}$
Max-ordering	1	$\theta(x) = \max_{q \in \{1,\ldots,Q\}} x_q$	$\leq_{\mathbb{R}}$
Lexicographic	Q	$\theta(x) = x$	$\leq_{lex}$
Extremal Pareto	1	$\theta(x) =< \lambda, x >$	$\leq_{\mathbb{R}}$
Lex-MO	Q	$\theta(x) = \Theta(x)$	$\leq_{lex}$

Table 2: Optimality Definitions and Solution Concepts

In Table 2 $\leq_{\mathbb{R}^Q}$ *denotes the componentwise order in* $\mathbb{R}^Q$. *We will later use* id *and* max *to denote the mappings* $\theta(x) = x$ *and* $\theta(x) = \max_{q \in \{1,\ldots,Q\}} x_q$. *Of course* $< \lambda, x >$ *denotes the scalar product in* $\mathbb{R}^Q$.
Note that concerning extremal Pareto optimality we get a different θ *for each* λ. *Therefore the set of extremal pareto solutions is actually* $\cup \mathcal{F}_{opt}(1, < \lambda, x >, \leq_{\mathbb{R}})$, *where the union is taken over all scalarizing vectors* λ.

Let us now prove some properties of Lex-MO solutions in terms of solution concepts. We start with the one already mentioned in Example 1: Lex-MO solutions are always Pareto optimal and max-ordering optimal.

Proposition 1
1. $\mathcal{F}_{opt}(Q, \Theta, \leq_{lex}) \subseteq \mathcal{F}_{opt}(Q, id, \leq_{\mathbb{R}^Q})$
2. $\mathcal{F}_{opt}(Q, \Theta, \leq_{lex}) \subseteq \mathcal{F}_{opt}(1, \max, \leq_{\mathbb{R}})$

The proof of this result has already been given in [2]. We illustrate it with the data of Example 1.

Hence, due to Proposition 1 the intersection of the set of Pareto and max-ordering optimal solutions is never empty.

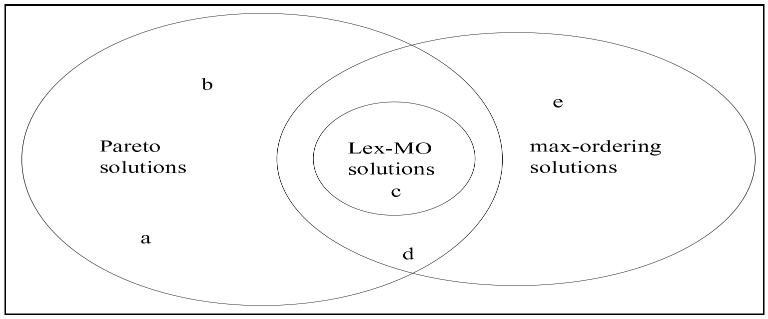

Figure 1: Pareto, Lex-MO and Max-ordering Solutions

We now repeat, without proof, two other properties which have been shown in [2]. They show another main advantage of Lex-MO solutions: They can be used to parametrize the set of Pareto optimal solutions.

Proposition 2 *Let $\lambda_q > 0, \quad q = 1, \ldots, Q$ be Q strictly positive multipliers and define Q new objective functions by $f'_q := \lambda_q f_q$ and let $f' = (f'_1, \ldots, f'_Q)$. Furthermore let F^* be an element of $\mathcal{F}_{opt}(Q, \Theta, \leq_{lex})$ with respect to f'. Then $F^* \in \mathcal{F}_{opt}(Q, id, \leq_{I\!R^Q})$ with respect to f.*

Proposition 3 *If F^* is an element of $\mathcal{F}_{opt}(Q, id, \leq)$ with respect to f then there exist strictly positive multipliers $\lambda_q > 0, \quad q = 1, \ldots, Q$ such that F^* is in $\mathcal{F}_{opt}(Q, \Theta, \leq_{lex})$ with respect to f'.*

4 Characterization of Lex-MO Solutions

This Section is devoted to the main result of the paper. We will discuss two other properties of Lex-MO solutions which characterize this solution concept. The first is related to a reduced optimization problem. Let us suppose that, for whatever reason, for some objective functions the values that are taken for some optimal solution (with respect to a given solution concept, of course) are known. Then it should suffice to consider for the minimization only the remaining objectives, with the additional constraints, that for the known objectives the known values are taken. If the optimal solutions of this reduced minimization problem are exactly those of the original problem which have the given values for the specified objectives, we say that the solution concept satisfies the **reduction property**. Formally we define the **reduced problem** as follows.

Definition 4 *Let $(y_1, \ldots, y_Q) \in I\!R^Q$ be such that there exists at least one $F \in \mathcal{F}_{opt}(p, \theta, \preceq)$ with $f_q(F) = y_q, \quad q = 1, \ldots, Q$. Furthermore let $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, Q\}$ be an index set. Then the reduced problem for $\{i_1, \ldots, i_k\}$, denoted by $RP(f^k)$, where $f^k = (f_{i_1}, \ldots, f_{i_k})$ is defined by*

$$\min_{F \in \mathcal{F}} \theta(f^k(F))$$

$$s.t. \;\; f_q(F) = y_q \quad q \in \{1, \ldots, Q\} \setminus \{i_1, \ldots, i_k\}.$$

If we refer to optimal solutions or solution values with respect to a reduced problem we will use $RP(f^k)$ as superscript to clarify notation. Using the definition of the reduced problem, we define the reduction property.

Definition 5 *A solution concept $(p, \theta, \preceq)$ satisfies the reduction property if for all index sets $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, Q\}$ and for all $(y_1, \ldots, y_Q) \in \mathbb{R}$ such that a feasible solution $F \in \mathcal{F}_{opt}(p, \theta, \preceq)$ exists with $f_q(F) = y_q, \quad q = 1, \ldots, Q$ it holds that*

$$\mathcal{F}_{opt}^{RP(f^k)}(p, \theta, \preceq) = \{F \in \mathcal{F}_{opt}(p, \theta, \preceq) | f_q(F) = y_q \quad \forall q \notin \{i_1, \ldots, i_k\}\}.$$

The second property we need may seem as natural as the reduction property from a decision maker's point of view. Apart from considering only Pareto optimal solutions as really good decisions one may also be interested in minimizing the worst objective function, i.e. considerig only max-ordering solutions. Recall that due to Proposition 1 such solutions always exist, namely at least the Lex-MO solutions.

Definition 6 *A solution concept $(p, \theta, \preceq)$ satisfies the* **regularity property** *if*

$$\mathcal{F}_{opt}(p, \theta, \preceq) \subseteq \mathcal{F}_{opt}(1, \max, \leq_{\mathbb{R}}).$$

To prove our main result we first show that $(Q, \Theta, \leq_{lex})$ does indeed satisfy the reduction and regularity property.

Proposition 4 $(Q, \Theta, \leq_{lex})$ *satisfies reduction and regularity property.*

Proof:
That $(Q, \Theta, \leq_{lex})$ satisfies the regularity property follows directly from Proposition 1. To prove that it also satisfies the reduction property we have to show that

$$\mathcal{F}_{opt}^{RP(f^k)} = \{F \in \mathcal{F}_{opt} | f_q(F) = y_q \quad \forall q \notin \{i_1, \ldots, i_k\}\}$$

for every choice of $\{i_1, \ldots, i_k\}$ and $(y_1, \ldots, y_Q)$ as in Definition 4. We will denote the latter set in the equation by $\mathcal{F}'$ for brevity.
First let $\bar{F}$ be an element of $\mathcal{F}_{opt}^{RP(f^k)}$. By definition of $\mathcal{F}'$ it follows that

$$\Theta(f(F)) \leq_{lex} \Theta(f(\bar{F})) \quad \forall F \in \mathcal{F}' \tag{1}$$

But for all elements $F \in \mathcal{F}'$ by the definition of the reduced problem it holds that

$$f_q(F) = f_q(\bar{F}) = y_q \quad \forall q \in \{1, \ldots, Q\} \setminus \{i_1, \ldots, i_k\}. \tag{2}$$

From (1) and (2) we conclude that

$$\Theta(f^k(F)) \leq_{lex} \Theta(f^k(\bar{F}))$$

which due to the choice of $\bar{F}$ must be satisfied with equality. Therefore it holds that $\Theta(f(F)) = \Theta(f(\bar{F}))$ which clearly implies $\bar{F} \in \mathcal{F}'$.
Now assume that we have some $F \in \mathcal{F}'$. Such a solution is by definition of $RP(f^k)$ feasible for the reduced problem. Now suppose that there exists a solution $\bar{F} \in \mathcal{F}_{opt}^{RP(f^k)}$ such that $\Theta(f^k(\bar{F})) <_{lex} \Theta(f^k(F))$. This would imply that also $\Theta(\bar{F}) <_{lex} \Theta(F)$, contradicting the choice of F. Hence it must hold that

$$\Theta(f(\bar{F})) \leq_{lex} \Theta(f(F)). \tag{3}$$

(1) and (3) imply that F belongs to $\mathcal{F}_{opt}^{RP(f^k)}$.
Both arguments together complete the proof.

□

In Example 3 we will see that other solution concepts do not always satisfy these properties. We now prove the main result of the paper, namely that reduction and regularity are characteristic properties of Lex-MO solutions.

Theorem 1 *A solution concept $(p, \theta, \preceq)$ satisfies reduction and regularity property if and only if $(p, \theta, \preceq) = (Q, \Theta, \leq_{lex})$.*

Proof:
That $(Q, \Theta, \leq_{lex})$ satisfies both properties has been shown in Proposition 4. Therefore let us assume that $(p, \theta, \preceq)$ is a solution concept satisfying both reduction and regularity property.
We prove the result by induction on the number q of objectives. For $q = 1$ the result follows immediately from the normalization property, since all solution concepts coincide in that case.
Let us now assume that the result holds for $q = 1, \ldots, Q-1$, i.e. $(p, \theta, \preceq) = (q, \Theta, \leq_{lex})$. and consider the case $q = Q$. We have to show that $\mathcal{F}_{opt}(p, \theta, \preceq) = \mathcal{F}_{opt}(Q, \Theta, \leq_{lex})$.
Let $\bar{F}$ be an element of $\mathcal{F}_{opt}(p, \theta, \preceq)$. Furthermore denote by y the optimal value of the max-ordering solution, i.e. $y \in \mathcal{V}_{opt}(1, \max, \leq_{\mathbb{R}})$. Since $(p, \theta, \preceq)$ satisfies the regularity property there must exist an index q^* such that $f_{q^*}(\bar{F}) = y$ (and $f_q(\bar{F}) \leq y \quad \forall q \neq q^*$). Therefore $\bar{F}$ belongs also to $\{F \in \mathcal{F}_{opt}(p, \theta, \preceq) | f_{q^*}(F) = y\}$.
Now we consider the reduced problem with $\{i_1, \ldots, i_k\} = \{1, \ldots, Q\} \setminus \{q^*\}$ and the value of f_{q^*} fixed at y. Then

$$\begin{aligned}
\{F \in \mathcal{F}_{opt}(p, \theta, \preceq) | f_{q^*}(F) = y\} &= \mathcal{F}_{opt}^{RP(f^k)}(p, \theta, \preceq) \\
&= \mathcal{F}_{opt}^{RP(f^k)}(q, \Theta, \leq_{lex}) \\
&= \{F \in \mathcal{F}_{opt}(Q, \Theta, \leq_{lex}) | f_{q^*} = y\} \\
&\subseteq \mathcal{F}_{opt}(Q, \Theta, \leq_{lex}).
\end{aligned}$$

The first equation follows from the reduction property for $(p, \theta, \preceq)$. The second follows from the induction hypothesis since $RP(f^k)$ is a multiple criteria problem

with $Q-1$ objective functions and feasible set $\mathcal{F}^{RP(f^k)} = \{F in \mathcal{F} | f_{q^*}(F) = y\}$. Finally the third is implied by the reduction property for $(Q, \Theta, \leq lex)$. The inclusion is trivial. Hence $\bar{F} \in \mathcal{F}_{opt}(Q, \Theta, \leq_{lex})$.
Proving the converse inclusion analogously completes the proof. We let F' be an element of $\mathcal{F}_{opt}(Q, \Theta, \leq_{lex})$. By regularity we see that there are q^* and y such that $F' \in \{F \in \mathcal{F}_{opt}(Q, \Theta, \leq_{lex}) | f_{q^*}(F) = y\}$. In the same way as above we conclude that $F' \in \mathcal{F}_{opt}(p, \theta, \preceq)$.

□

We are closing this section with an example giving an overview on which of the solution concepts mentioned in this paper satisfy the regularity and the reduction property.

Example 3 *In Table 3 we list the solution concepts mentioned in this paper (see Example 2), indicating which do or do not satisfy the reduction and regularity property.*

Solution concept	Reduction	Regularity
$(Q, \text{id}, \leq_{\mathbb{R}^Q})$	no	no
$(1, \max, \leq_{\mathbb{R}})$	no	yes
$(Q, \text{id}, \leq_{lex})$	yes	no
$(1, < \lambda, x >, \leq_{\mathbb{R}})$	yes	no
$(Q, \Theta, \leq_{lex})$	yes	yes

Table 3: Solution Concepts and Properties

Concerning regularity $(1, \max, \leq_{\mathbb{R}})$ *satisfies this property by definition. For the other solution concepts the entries in Table 3 are verified by Proposition 4 and Example 1. For reduction it is easy to construct examples where either the reduced problem has pareto optimal solutions which are not pareto optimal for the original problem or some optimal solutions of the original problem are not optimal for* $RP(f^k)$. *It is also not hard to show that* $(Q, id, \leq_{lex})$ *and* $(1, < \lambda, x >, \leq_{\mathbb{R}})$ *satisfy the reduction property.*

5 Conclusions

In this paper we have discussed lexicographic max-ordering solutions of multiple criteria combinatorial optimization problems. We have seen how this and other definitions of optimality fit into the notion of solution concepts we have defined. Reduction and regularity properties have been used to characterize Lex-MO solutions. It is important to notice that decision makers probably consider both properties as important features of optimal solutions of optimization problems.

Let us shortly consider the case of a discrete decision problem. In this case the (finite) set of alternatives a decision maker may choose from is given explicitly, rather than implicitly as is usually the case in combinatorial optimization problems. Then the values of all objectives can be evaluated for all the alternatives. Provided all criteria are really equally important, i.e. there are no preferences or rankings at all, the decision maker can choose a Lex-MO optimal alternative and be sure that:

- he has chosen a Pareto optimal solution, i.e. he cannot improve one criterion without worsening another one,
- a solution that has the smallest value of the worst objective, and
- that all Lex-MO optimal alternatives are equivalent in the sense that their Θ vectors are the same.

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