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As it was conjectured in [D F] and proved in [D 1], finite-dimensional algebras of infinite type (i.e. having infinitely many indecomposable representations) split into two classes. For the first one, called tame, indecomposable representations of any fixed dimension form a finite set of at most 1-parameter families, while for the second one, called wild, there exist arbitrarily large families of non-isomorphic indecomposable representations. Moreover, in some sense, knowing representations of one wild algebra, one would know those of any other algebras.

A lot of examples showed that the same should hold for Cohen-Macaulay modules over Cohen-Macaulay algebras of Krull dimension 1. In this paper we give a proof of it based on the same method of "matrix problems" or so called representations of bocses (cf. §1). But we had to consider a new situation, namely that of "open subcategories" (§2) and first reprove the results of [D 1] for it. This new shape seems to be unavoidable in the case of Cohen-Macaulay modules but it should be also of use for other questions in representation theory. In §3 we propose a method to reduce the calculation of Cohen-Macaulay modules to some open subcategory and use the results of §2 to prove the tame-wild dichotomy.

1 Preliminaries

As the notions of bocses and their representations are not well-known, remind the main definitions (cf. [Roi], [D 1]). All considered categories will be linear over some basic field K which will always be supposed algebraically closed. Respectively, all functors are K-linear (bifunctors bilinear). We write Hom, $\otimes$ instead of Hom_K, $\otimes_K$. A **module** over a category A is a functor $M : A \to Vect$ (the category of K-vector spaces); an A-B-**bimodule** (where A, B are categories) is a bifunctor $V : A^{op} \times B \to Vect$; if $A = B$, we call V an A-bimodule. For $v \in V(X,Y)$, $a \in A(X',X)$, $b \in B(Y,Y')$ we write bva instead of $V(a,b)(v)$. A **bocs** is a pair $\mathbf{a} = (A, V)$ where A is some category and V an A-coalgebra, i.e. an A-bimodule V supplied with a comultiplication $\mu : V \to V \otimes_A V$ and a counit $\varepsilon : V \to A$ satisfying the usual conditions.

A **representation** of **a** over some algebra R is defined as a functor $M : A \longrightarrow$ *pr*–R, the category of finitely generated projective R-modules. If N is another representation, define

$$Hom_{\mathbf{a}}(M,N) = Hom_{A-A}(V,(M,N))$$

where (M,N) is an A-bimodule defined by the rules:

$$(M,N)(X,Y) = Hom_R(M(X),N(Y)) \quad \text{for} X,Y \in \mathbf{ob}A;$$

$$afb = N(a)fM(b) \quad \text{for} f \in (M,N)(X,Y),$$
$$a : Y \longrightarrow Y', \quad b : X' \longrightarrow X \ \text{ in } A.$$

The product of $\varphi \in Hom_{\mathbf{a}}(M,N)$ and $\psi \in Hom_{\mathbf{a}}(L,M)$ is defined as the composition

$$V \xrightarrow{\mu} V \otimes_A V \xrightarrow{\varphi\otimes\psi} (M,N) \otimes_A (L,M) \xrightarrow{m} (L,N)$$

where m is the multiplication of R-homomorphisms. Thus the **category of representations** $Rep(\mathbf{a}, R)$ is defined. We write $Rep(\mathbf{a})$ instead of $Rep(\mathbf{a}, K)$.

Any algebra R can be considered as a bocs ("principal bocs") if we put $A = V = R$. Of course, representations of such bocses are just representations of R. Remark that if $M \in Rep(\mathbf{a}, R)$ and $L \in Rep(R, R')$, then their tensor product $M(L) = M \otimes_R L$ lies in $Rep(\mathbf{a}, R')$; so M can be viewed as "a family of representations of **a** paramatrized by R".

As a rule, the category A will be finitely generated over K, i.e. with finite object set and a finite set of morphisms (generators) whose products span all spaces of morphisms $A(X,Y)$. A **dimension** of a representation of $\mathbf{a}$ is defined as a function $\underline{d} : \mathbf{ob}A \longrightarrow \mathbf{N}$. In cases when there is a notion of rank for finitely generated projective R-modules, we can associate to $M \in Rep(\mathbf{a}, R)$ its dimension $\underline{dim}M : \mathbf{ob}A \longrightarrow \mathbf{N}$, namely, $(\underline{dim}M)(X) = \text{rank } M(X)$ and denote by $Rep_{\underline{d}}(\mathbf{a}, R)$ the set of representations having dimension $\underline{d}$. For instance, this is the case if $R = K$ (hence $rank = dim$), so $Rep_{\underline{d}}(\mathbf{a})$ is defined. If S is a system of generators for A, then each representation $M \in Rep(\mathbf{a})$ determines (and is determined by) linear mappings $M(a) : M(X) \longrightarrow M(Y)$, $a \in S, a : X \longrightarrow Y$. Hence, treating all linear mappings $M(a)$ as matrices, we can consider $Rep_{\underline{d}}(\mathbf{a})$ as an algebraic variety lying in affine space $\mathbf{A}^{\|d\|}$, carrying the Zariski topology, where

$$\|\underline{d}\| = \sum_{\substack{a \in S, \\ a: X \to Y}} \underline{d}(X)\underline{d}(Y).$$

All considered bocses are supposed **normal** — which means that for any $X \in \mathbf{ob}A$ an element $\omega_X \in V(X,X)$ exists such that $\varepsilon(\omega_X) = 1_X$, $\mu(\omega_X) = \omega_X \otimes \omega_X$. In this case the bimodule structure on V is completely determined if we know the **kernel** of the bocs $\mathbf{a}, \bar{V} = Ker\varepsilon$ and for each $a \in A(X,Y)$ its **differential** $\partial a = a\omega_X - \omega_Y a \in \bar{V}$. Moreover, the coalgebra structure is determined if we know the **differentials** $\partial v = \mu(v) - v \otimes \omega_X - \omega_Y \otimes v \in \bar{V} \otimes_A V$ for all $v \in \bar{V}(X,Y)$.

In main applications **free bocses** arise, i.e. such that A is a free category (that of paths $K\Gamma$ of an oriented graph Γ) and the kernel $\bar{V}$ is a free A-bimodule. A free bocs is completely determined if we know the set S_0 of free generators of A, the set S_1 of free generators of $\bar{V}$ and their differentials. The set $S = S_0 \cup S_1$ is called a **set of free generators** of the bocs $\mathbf{a}$.

For technical purposes, **semi-free bocses** are needed. A **semi-free category** is, by definition, a category of the form $K\Gamma[g_a(a)^{-1}]$ where a ranges through the set of loops (i.e. elements of S_0 such that $a : X \longrightarrow X$) and $g_a(t) \in K[t]$ is a non-zero polynomial (depending on a). If $g_a \neq const$, call the loop a **marked**. A bocs is called **semi-free** if A is a semi-free category, $\bar{V}$ a free A-bimodule and $\partial a = 0$ for all marked loops. In this case call S a set of **semi-free generators** of $\mathbf{a}$.

If $\mathbf{a}$ is free, then, of course, $Rep_{\underline{d}}(\mathbf{a}) \simeq \mathbf{A}^{\|\underline{d}\|}$; if $\mathbf{a}$ is semi-free, then $Rep_{\underline{d}}(\mathbf{a})$ is an open subset in $\mathbf{A}^{\|\underline{d}\|}$.

A semi-free category is called **triangular** if there exists a system S of semi-free generators and a function $h : S \longrightarrow \mathbf{N}$ such that for any $a \in S$ ∂a belongs to the subbocs generated by $b \in S$ with $h(b) < h(a)$.

A representation $M \in Rep(\mathbf{a}, R)$ is called **strict** if it satisfies the following two conditions:

1. If $L \in Rep(R, R')$ is indecomposable, then $M(L) \in Rep(\mathbf{a}, R')$ is also indecomposable.
2. If $L, L' \in Rep(R, R')$ are non-isomorphic, then $M(L) \not\simeq M(L')$, too.

One can say that if such M exists, the representation theory of $\mathbf{a}$ is at least as complicated as that of R.

If a set $F = \{M_i \mid M_i \in Rep(\mathbf{a}, R_i)\}$ is given (each M_i can be a representation over its own R_i), we call F **strict** provided each M_i is strict and if $i \neq j$, then $M_i(L) \not\simeq M_j(L')$ for any $L \in Rep(R_i, R), L' \in Rep(R_j, R)$.

We need also **"bimodule categories"** defined as follows. Let U be an R_1-R_2-bimodule where R_1, R_2 are some algebras. For each algebra R let $P_i = P_i(R)$ be the category of finitely generated projective $R_i \otimes R^{op}$-modules. Consider a P_1-P_2-bimodule U_R such that $U_R(P_1, P_2) = Hom_{R_1 \otimes R^{op}}(P_1, U \otimes_{R_2} P_2)$.

Take the elements of all $U_R(P_1, P_2)$ as objects of a new category $U(R)$ and as morphisms from $u \in U_R(P_1, P_2)$ to $u' \in U_R(P_1', P_2')$ take all pairs (f_1, f_2) with $f_i \in Hom_{R_i \otimes R^{op}}(P_i, P_i')$ such that $u' f_1 = f_2 u$.

If $L \in Rep(R, R')$, then $P_i \otimes_R L \in P_i(R')$, so L defines a natural mapping

$$\otimes L : U_R(P_1, P_2) \longrightarrow U_{R'}(P_1 \otimes_R L, P_2 \otimes_R L).$$

Hence, one can reproduce for bimodule categories the above notion of strictness.

Note that this definition is formally distinct from that of [D1] though they provide equivalent categories.

Usually the algebras R_i are finite-dimensional and in this case the following theorem is valid [D1]:

Theorem 1

If R_1, R_2 are finite-dimensional algebras and U is a finite-dimensional $R_1 - R_2$-bimodule, then there exists a free triangular bocs $\mathbf{a} = \mathbf{a}_U$ and for each algebra R an equivalence of categories $T_R : Rep(\mathbf{a}, R) \longrightarrow U(R)$ commuting with tensor products, i.e.

$$T_{R'}(M \otimes_R L) \simeq T_R(M) \otimes_R L \quad \text{for any} \quad L \in Rep(R, R').$$

2 Tame and wild open subcategories

Let $\mathbf{a}$ be a finitely generated bocs and $\mathbf{X} \subset Rep(\mathbf{a})$ a full subcategory. Call $\mathbf{X}$ an **open subcategory** if it satisfies the following conditions:

1. If $M \in \mathbf{X}$ and $N \simeq M$, then $N \in \mathbf{X}$;
2. $M \oplus N \in \mathbf{X}$ if and only if $M \in \mathbf{X}$ and $N \in \mathbf{X}$;
3. for each dimension $\underline{d}$ the subset $\mathbf{X}_{\underline{d}} = \mathbf{X} \cap Rep_{\underline{d}}(\mathbf{a})$ is open in $Rep_{\underline{d}}(\mathbf{a})$.

For any algebra R put $\mathbf{X}(R) = \{M \in Rep(\mathbf{a}, R) \mid M(L) \in \mathbf{X} \text{ for any} L \in Rep(R)\}$. It is clear that if $M \in \mathbf{X}(R)$ and $L \in Rep(R, R')$, then $M(L) \in \mathbf{X}(R')$.

Call $\mathbf{X}$ **wild** if for any finitely generated algebra R there exists a strict representation $M \in \mathbf{X}(R)$. Non-formally this means that to know the representations of $\mathbf{X}$ we have to know the representations for all finitely generated algebras.

It is well-known (and easy to check) that to prove wildness it is sufficient to find a strict representation $M \in \mathbf{X}(K < x, y >)$ (free non-commutative algebra with 2 generators), as the latter has a strict representation over any other one. A little more complicated but also known (cf. [GP] or [D2]) is that here we can replace $K < x, y >$ by the polynomial ring $K[x, y]$ or even the power series ring $K[| x, y |]$.

Call a **rational algebra** any algebra of the form $K[x, f(x)^{-1}]$ for a non-zero polynomial $f(x)$, i.e. the affine algebra of a smooth rational affine curve.

Theorem 2

Let $\mathbf{a} = (A, V)$ be a finitely generated semi-free bocs, $\mathbf{X} \subset Rep(\mathbf{a})$ an open subcategory. Then the following conditions are equivalent:

1. $\mathbf{X}$ is non-wild;

2. for each dimension $\underline{d}$ there exists a subvariety $X_{\underline{d}} \subset \mathbf{X}_{\underline{d}}$ such that

$$dim X_{\underline{d}} \leq \mid \underline{d} \mid = \sum_{T \in \mathbf{obA}} \underline{d}(T)$$

and any representation from $\mathbf{X}_{\underline{d}}$ is isomorphic to one belonging to $X_{\underline{d}}$;

3. for each dimension $\underline{d}$ there exists a subvariety $Y_{\underline{d}} \subset \mathbf{X}_{\underline{d}}$ such that $dim Y_{\underline{d}} \leq 1$ and any indecomposable representation from $\mathbf{X}_{\underline{d}}$ is isomorphic to one belonging to $Y_{\underline{d}}$;

4. there exists a strict set $\{M_i \mid i \in I, M_i \in \mathbf{X}(R_i)\}$ with rational algebras R_i such that for each dimension $\underline{d}$ all indecomposable representations from $\mathbf{X}_{\underline{d}}$ except a finite number (up to isomorphism) are isomorphic to $M_i(L)$ for some $i \in I_{\underline{d}}$ and some $L \in Rep(R_i)$ where $I_{\underline{d}}$ is a finite subset of I (depending on $\underline{d}$).

(If these conditions are satisfied, call **X tame**).

Proof

(4) $\Longrightarrow$ (3) as any indecomposable n-dimensional representation L of a rational algebra $K[x, f(x)^{-1}]$ maps x to a Jordan cell $J(\lambda)$ with eigenvalue λ such that $f(\lambda) \neq 0$. Hence representations $M_i(L)$ for such L produce a 1-dimensional subvariety of $\mathbf{X}_{\underline{d}}$ and as $\underline{d}$ is fixed, n is also fixed.

(3) $\Longrightarrow$ (2) is quite evident as $\mid \underline{d} \mid$ is an upperbound for the maximal number of indecomposable direct summands of any representation of dimension $\underline{d}$.

(2) $\Longrightarrow$ (1) if $M \in \mathbf{X}_{\underline{d}}(K < x, y >)$ is strict, then $M(L)$ for $L \in Rep_n$ $(K < x, y >)$ form in $\mathbf{X}_{n\underline{d}}$ a subset of dimension at least n^2 consisting of pairwise non-isomorphic representations and $n^2 > \mid n\underline{d} \mid$ if $n > \mid \underline{d} \mid$.

At last, (1) $\Longrightarrow$ (4) can be proved just by repeating the proof of the above Theorem 1 given in [D1] if we make the following simple observation. Let $a \in A(X, Y)$. Denote $\mathbf{X}(a) = \{M(a) \mid M \in \mathbf{X}\}$. Then the only possibilities for $\mathbf{X}(a)$ are:

- if $X \neq Y$, either all linear mappings, or those $F : L \longrightarrow L'$ with $rkF = dim L$, or those with $rkF = dim L'$ or isomorphisms only;
- if $X = Y$ there exists a finite subset $E(a) \subset K$ such that $\mathbf{X}(a) = \{F : L \longrightarrow L \mid F \text{ has no eigenvalue from } E(a)\}$.

Of course, the proof of [D 1], based on algorithms of reduction of matrices, is rather complicated. Unfortunately, till now the only known way to obtain the equivalences $(1) \iff (2) \iff (3)$ is to prove that $(1) \implies (4)$.

3 Cohen-Macaulay Algebras

In this paragraph we consider algebras Λ over K satisfying the following conditions:

(A1) The centre Z of Λ is a complete local noetherian Cohen-Macaulay ring of Krull dimension 1 with residue field K;

(A2) Λ is a (finitely generated) Cohen-Macaulay module over Z;

(A3) Λ is semi-prime, i.e. has no nilpotent ideals.

We call such algebras ***CM*-Algebras**. Denote by $CM(\Lambda)$ the category of Λ-modules which are maximal Cohen-Macaulay modules over Z, i.e., in our case, finitely generated and torsion free. Call them **CM-Λ-modules**.

If Λ is a CM-algebra, its full quotient ring Q is a semi-simple artinian ring and there exists a (not necessarily unique) **maximal overring** $\bar{\Lambda}$, i.e. a CM-algebra such that $\Lambda \subset \bar{\Lambda} \subset Q$ and there are no CM-algebras $\Lambda' \neq \bar{\Lambda}$ with $\bar{\Lambda} \subset \Lambda' \subset Q$ (cf. D3]). It follows from [Rog] that $\bar{\Lambda}$ is always hereditary, i.e. any $CM - \bar{\Lambda}$-module is projective over $\bar{\Lambda}$.

If R is any K-algebra, denote by $CM(\Lambda, R)$ the category of R-Λ-bimodules M satisfying the following conditions:

(M1) M is finitely generated as bimodule;

(M2) ${}_ZM$ is torsion free;

(M3) M_R is flat;

(M4) $M(L) = M \otimes_R L$ is a CM-Λ-module for any $L \in Rep(R)$.

If R/m is finite-dimensional over K for any maximal left ideal $m \subset R$, then (M4) is equivalent to

(M4′) for any non-zero divisor $\lambda \in Z$ the R-module $M/\lambda M$ is also flat.

Surely, if $M \in CM(\Lambda, R)$ and $L \in Rep(R, R')$, then $M(L) \in CM(\Lambda, R')$. So we are able to define strict modules $M \in CM(\Lambda, R)$ and strict sets of such modules just as in §1. If R is a finitely generated commutative K-algebra of Krull dimension d, call any bimodule $M \in CM(\Lambda, R)$ a **d-parameter family** of CM-Λ-modules (with base R).

Call Λ **CM – wild** if for every finitely generated algebra R there exists a strict module $M \in CM(\Lambda, R)$. Again we have to check the existence of M only for $R = K\langle x, y\rangle$, or $R = K[x, y]$, or $R = K[| x, y |]$.

If a Λ-module M is torsion free (over Z) it can be embedded into the Q-module $Q \otimes_\Lambda M$, so if Λ' is an overring of Λ, i.e. a CM-algebra such that $\Lambda \subset \Lambda' \subset Q$, we can consider the Λ'-module $\Lambda' M$, which is the image of $\Lambda' \otimes_\Lambda M$ in $Q \otimes_\Lambda M$. If M was a CM-module, then so is $\Lambda' M$. In this case $Q \otimes_\Lambda M$ is finitely generated over Q,thus $Q \otimes_\Lambda M \simeq r_1 Q_1 \oplus \cdots \oplus r_t Q_t$ where $Q_1, \cdots, Q_t$ are all pairwise non-isomorphic simple Q-modules. Call the vector $\underline{r}(M) = (r_1, \cdots, r_t)$ the **(vector) rank** of M and denote $CM_{\underline{r}}(\Lambda)$ the set of all CM-Λ-modules of rank $\underline{r}$.

Theorem 3

For a CM-algebra Λ the following conditions are equivalent:

1. Λ is not CM-wild;
2. for any rank $\underline{r} = (r_1, \cdots, r_t)$ there exists a d-parameter family M of CM-Λ-modules with $d \leq |\ \underline{r}\ | = \sum_{i=1}^{t} r_i$ such that any CM-Λ-module of rank $\underline{r}$ is isomorphic to some $M(L)$;
3. for any rank $\underline{r}$ there exists a 1-parameter family M of CM-Λ-modules such that any indecomposable CM-Λ-module of rank $\underline{r}$ is isomorphic to some $M(L)$;
4. there exists a strict set $\{M_i \mid i \in I, M_i \in CM(\Lambda, R_i)\}$ with rational algebras R_i such that for each rank $\underline{r}$ all indecomposable CM-Λ-modules of rank $\underline{r}$ except a finite number (up to isomorphism) are isomorphic to $M_i(L)$ for some $i \in I_{\underline{r}}$ and $L \in rep(R_i)$ where $I_{\underline{r}}$ is a finite subset of I (depending on $\underline{r}$).

If these conditions are satisfied, call Λ **CM-tame**.

Proof:

Again (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) is clear, so we have only to prove (1) $\Longrightarrow$ (4).

Fix an overring $\Lambda' \supset \Lambda$ and denote by $CM(\Lambda \mid \Lambda')$ the full subcategory in $CM(\Lambda)$ consisting of all modules M such that $\Lambda' M$ is Λ'-projective. Of course, if Λ' is hereditary (e.g. maximal), then $CM(\Lambda \mid \Lambda') = CM(\Lambda)$. Let $I \subset rad\Lambda$ be a two-sided Λ'-ideal such that $dim_K \Lambda'/I < \infty$ (it exists as Λ'/Λ is a finitely generated torsion Z-module). Then $IM \subset M \subset \Lambda' M$ for any CM-module M and any homomorphism $\varphi : M \longrightarrow N$ can be uniquely prolonged to $\varphi' : \Lambda' M \longrightarrow \Lambda' N$. Put

$$\Lambda_1 = \Lambda/I, \quad \Lambda_2 = \Lambda'/I$$

and consider a new category $C = C(\Lambda \mid \Lambda')$ whose objects are pairs (P, X) with P a (finitely generated) projective Λ_2-module, $X \subset P$ a Λ_1-submodule, and morphisms $(P, X) \longrightarrow (P_1, X_1)$ are Λ_2-homomorphisms $\varphi : P \longrightarrow P_1$ such that $\varphi(X) \subset X_1$. Define a functor $T : CM(\Lambda \mid \Lambda') \longrightarrow C$ putting $T(M) = (\Lambda' M/IM, M/IM)$ and let C_o be the full subcategory of C consisting of all such pairs (P, X) that $\Lambda_2 X = P$. Then the following lemma is evident:

Lemma 1

$T(M) \in C_o$ for any $M \in CM(\Lambda \mid \Lambda')$ and the functor $T : CM(\Lambda \mid \Lambda') \longrightarrow C_o$ is full, dense and reflects isomorphisms and indecomposability.

Now consider the Λ_1-Λ_2-bimodule $U = \Lambda_2$ and define a functor $\underline{Im} : U(K) \longrightarrow C$ putting, for $\varphi : P_1 \longrightarrow P_2$, $\underline{Im\varphi} = (P_2, Im\varphi)$. Denote $\mathbf{X}$ the full subcategory of $U(K)$ consisting of all such φ that $Ker\varphi \subset radP_1$ and $\Lambda_2 \cdot Im\varphi = P_2$. As Λ_1 is artinian, any Λ_1-module X possesses a projective cover whence we obtain the following lemma:

Lemma 2

If $\varphi \in \mathbf{X}$, then $\underline{Im\varphi} \in C_o$ and the functors $\underline{Im} : \mathbf{X} \longrightarrow C_o$ is full, dense and reflects isomorphisms and indecomposability.

Identify, according to Theorem 1, $U(K)$ with $Rep(\mathbf{a})$ for a free triangular bocs $\mathbf{a}$. Then $\mathbf{X}$ becomes an open subcategory in $Rep(\mathbf{a})$, thus Theorem 2 is applicable, i.e. $\mathbf{X}$ is either tame or wild.

Let $u \in \mathbf{X}(R)$ for some algebra R. Then $u : P_1 \longrightarrow P_2$ where P_i is a projective $\Lambda_i \odot R^{op}$-module. Call u **good** provided $P_i \simeq \check{P}_i/I\check{P}_i$ where $\check{P}_1 (resp. \check{P}_2)$ is a projective $\Lambda \odot R^{op}$-module (resp. $\Lambda' \odot R^{op}$-module) and $Cokeru$ is flat over R. In this case denote $\check{u} : \check{P}_1 \longrightarrow \check{P}_2$ some homomorphism for which $u = \check{u}(modI)$.

Lemma 3

(a) If $u \in \mathbf{X}(R)$ is good and $M = Im\check{u}$, then $M \in CM(\Lambda, R)$.

(b) If $\{u_i \mid i \in I, u_i \in \mathbf{X}(R_i)\}$ is a strict set, all u_i are good and $M_i = Im\check{u}_i$, then $\{M_i \mid i \in I\}$ is also a strict set.

Proof

(a) Remark that $Coker u \simeq Coker\check{u}$, so we have an exact sequence

$$0 \longrightarrow M \longrightarrow \check{P}_2 \longrightarrow N \longrightarrow 0$$

with R-flat N and hence an exact sequence

$$0 \longrightarrow M \otimes_R L \longrightarrow \check{P}_2 \otimes_R L \longrightarrow N \otimes_R L \longrightarrow 0$$

for any $L \in Rep(R)$ where $\check{P}_2 \otimes_R L$ is Λ'-projective. This does imply all properties (M1) - (M4) for M.

(b) follows directly from Lemmas 1 and 2.

Lemma 4

Let $u \in \mathbf{X}(R)$ for a finitely generated commutative domain R. Then there exists a non-zero $f \in R$ such that $u_f \in \mathbf{X}(R_f)$ is good.

Proof

Denote by F the quotient field of R. Then $(\Lambda/rad\Lambda) \otimes F$ is semi-simple [B1], hence $rad(\Lambda \otimes F) = (rad\Lambda) \otimes F$ and $(\Lambda \otimes F)/rad(\Lambda \otimes F) \simeq (\Lambda/rad\Lambda) \otimes F$. Hence in $\Lambda \otimes F$ idempotents can be lifted modulo radical and any projective $(\Lambda \otimes F)$-module is of the form $P \otimes F$ for some projective Λ-module P. The same is true for the algebras Λ' and $\Lambda_i (i = 1, 2)$. As $\Lambda_1 = \Lambda/I$ and $I \subset rad\Lambda$, any projective $(\Lambda_1 \otimes F)$-module is of the form $(P \otimes F)/I(P \otimes F)$. Therefore, if P is a projective $\Lambda_1 \otimes R$-module, there exists a non-zero $f \in R$ such that $P_f \simeq \check{P}/I\check{P}$ for a projective $\Lambda_1 \otimes R_f$-module $\check{P}$. So if $u \in \mathbf{X}(R), u : P_1 \longrightarrow P_2$, we can find $f \in R$ for which $(P_i)_f \simeq \check{P}_i/I\check{P}_i$. But as Λ_i are finite-dimensional, $N = Coker u_f$ is finitely generated over R_f and there exists a non-zero $g \in R$ such that N_g is flat [B2], thus u_{fg} is good.

Corollary 1: If $\mathbf{X}$ is wild, then Λ is wild.

Proof

Let $u \in \mathbf{X}(R), R = K[x, y]$, be strict. Find $f \in R$ such that u_f is good and a maximal ideal $m \subset R$ such that $f \notin m$. As the m-adique completion of R is isomorphic to $\hat{R} = K[| x, y |]$ u_f provides a good and strict element $\hat{u} \in \mathbf{X}(\hat{R})$. Then lemma 3 implies that Λ is CM-wild.

Corollary 2 If Λ' is hereditary and $\mathbf{X}$ is tame, then Λ is CM-tame.

Proof

Let $\{u_i \mid i \in I, u_i \in \mathbf{X}(R_i)\}$ be a strict set satisfying conditions (4) of Theorem 2. Remark that if R is a rational algebra, then $Rep_d(R) - Rep_d(R_f)$ is finite for any non-zero $f \in R$ and any dimension d. Therefore, lemma 4 allows us to suppose all u_i good. But as Λ' is hereditary, $CM(\Lambda \mid \Lambda') = CM(\Lambda)$. Hence, lemmas 1-3 imply that the set $\{M_i \mid i \in I\}$ with $M_i = Im\check{u}_i$ satisfies condition (4) of Theorem 3.

Now (1) $\Longrightarrow$ (4) follows from corollaries 1 and 2.

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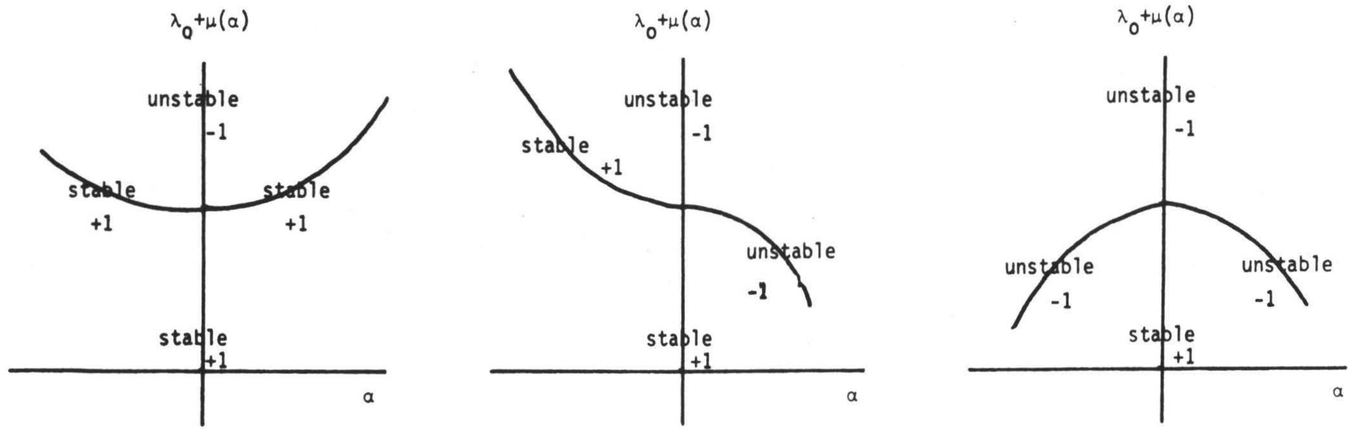

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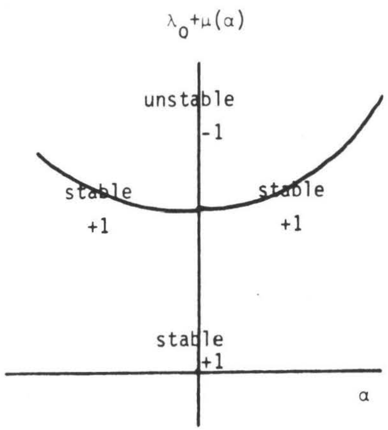
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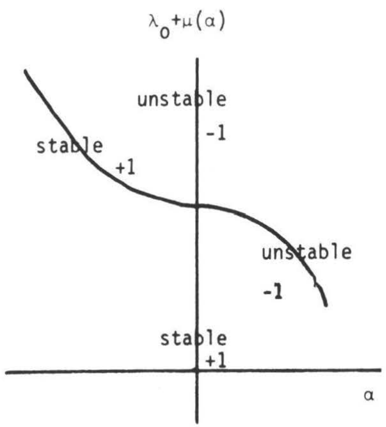
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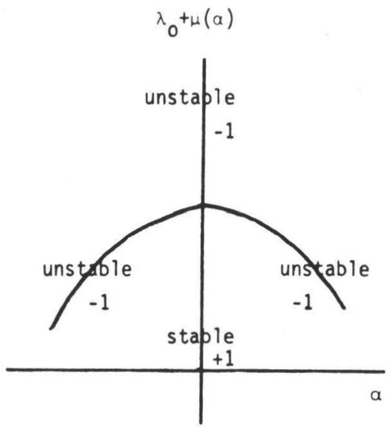
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