Planar Location Problems with Line Barriers

Kathrin Klamroth*
Universität Kaiserslautern

Abstract

The Weber Problem for a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, Ex_2, \ldots, Ex_M\} \subset I\!R^2$ with positive weights w_m $(m = 1, \ldots, M)$ is to find a new facility X^* such that $\sum_{m=1}^{M} w_m d(X, Ex_m)$ is minimized for some distance function d.
A variation of this problem is obtained if the existing facilities are situated on two sides of a linear barrier. Such barriers like rivers, highways, borders or mountain ranges are frequently encountered in practice.
Structural results as well as algorithms for this non-convex optimization problem depending on the distance function and on the number and location of passages through the barrier are presented. A reduction to convex optimization problems is used to derive efficient algorithms.

1 Introduction

Modern life encounters an ever growing concentration in many respects. Growing population, higher integration of electronic circuits or the economical need to choose an optimized site for new facilities have led to planar location problems with an ever growing number of obstacles (see e.g. [14]).

The classical *Weber Problem* (Median Problem, Minisum Problem) which is the basis for many developments is stated as follows:
Let

$$\mathcal{E}x = \{Ex_1, Ex_2, \ldots, Ex_M\}$$

be a finite set of existing facilities represented by points in $I\!R^2$. A positive weight $w_m = w(Ex_m)$ is associated with each existing facility Ex_m $(m \in \mathcal{M} := \{1, \ldots, M\})$ which can be interpreted as the demand of facility Ex_m. The objective is to find a new facility $X^* \in I\!R^2$ such that the weighted sum of distances between X^* and the existing facilities

$$f(X) = \sum_{m=1}^{M} w_m \, d(X, Ex_m)$$

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is minimized for some distance function d. $\mathcal{X}^*$ is the set of all optimal solutions of the Weber Problem.

This problem, which has the classification $1/P/\bullet/d/\sum$ with respect to the classification scheme for location problems proposed in [6, 7] ($1/P/\bullet/d/\sum$ is the classification of a single-facility location problem (1) in the Plane (P) with no special assumptions and constraints ($\bullet$), d as distance function and sum-objective function ($\sum$)), has already been thoroughly treated by many authors. For an overview see e.g. [5, 6, 13, 16]. Good algorithms have been developed for different distance functions d. In practice the modelling of the investigated region as the complete $\mathbb{R}^2$ is not realistic. There may be for example many areas where the positioning of a new facility is not allowed (see e.g. [4, 8, 9]) and of course there may be regions where trespassing is prohibited. Such barriers may be for example buildings, lakes or mountain ranges. The idealized case that the barriers are linear and have only a finite set of passages is a special case which is frequently encountered. Line barriers with passages may be rivers, border lines, highways, mountain ranges or on a smaller scale conveyer belts in an industrial plant. Here trespassing is only allowed through a finite set of passages.

For a given finite set of closed barriers

$$\mathcal{B} = \{B_1, B_2, \ldots, B_b\} \ \subset \mathbb{R}^2$$

let $F := \mathbb{R}^2 \setminus int(\bigcup_{i=1}^{b} B_i)$ be the feasible region where new facilities can be located. Furthermore let $d_{\mathcal{B}}(X, Y)$ be the length of a shortest path (with respect to d) from X to Y not crossing a barrier.

Thus the Weber Problem can be restated:

The *Weber Problem with Barriers* $1/P/\mathcal{B}/d_{\mathcal{B}}/\sum$ is to find a new facility $X^*_{\mathcal{B}} \in F$ such that

$$f(X) := \sum_{m=1}^{M} w_m \, d_{\mathcal{B}}(X, Ex_m)$$

is minimized. Note that $1/P/\mathcal{B}/d_{\mathcal{B}}/\sum$ only has a solution if all existing facilities are located in the same *connected* component of F.

The introduction of barriers significantly entails different treatment because the objective function $f(X)$ is not convex as in the classical Weber Problem. Literature has so far only treated some particular types of metrics and barrier shapes, like one circle as a barrier and the Euclidean distance [11] or closed polygons as barriers and the l_p-metric [1, 3]. Especially line barriers with passages have so far only been treated in the case of the Manhattan metric l_1 [12, 2] for which arbitrarily shaped barriers can be handled.

This paper presents some general results as well as algorithms for the special case that the barriers $\mathcal{B} = \mathcal{B}_L$ are line barriers with passages, i.e. for $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$, and a large class of metrics including the class of l_p metrics. Different numbers of permitted passages are considered.

2 General Results

The planar location problem with line barriers $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ can be modelled as follows:

Let $L := \{(x,y) \in I\!R^2 \mid y = ax + b\}$ be a line and let $\{P_n \in L \mid n \in \mathcal{N} := \{1,\ldots,N\}\}$ be a set of points on L. Then

$$\mathcal{B}_L := L \setminus \{P_1,\ldots,P_N\}$$

is called a *line barrier with passages* or shortly *line barrier*. The case that the barrier is a vertical line which is not included in this description can easily be transformed to this definition.
The feasible region F for new locations is defined as the union of the two closed half-planes F^1 and F^2 on both sides of $\mathcal{B}_L$. Here $F^1 \cup F^2 = I\!R^2$ because the line $y = ax + b$ belongs to both half-planes F^1 and F^2. As all results can easily be transfered to the case that the line barrier has a finite width, for simplification this model will be used in the following although a new location placed directly on the barrier is not allowed in reality. In the case that $\mathcal{B}_L$ has a finite width, only the boundary of $\mathcal{B}_L$ (except the passages) belongs to F.
Furthermore a finite number of existing facilities $Ex^i_m \in F^i$, $m \in \mathcal{M}^i := \{1,\ldots,M^i\}$ is given in each half-plane F^i, $i = 1,2$, represented by points in $I\!R^2$. A positive weight $w^i_m := w(Ex^i_m) \in I\!R_+$ is associated with each existing facility Ex^i_m representing the demand of Ex^i_m.
The major difference between this problem formulation and planar location problems without barriers is the modified distance function. If a distance function d is given for the unrestricted problem the distance function $d_{\mathcal{B}_L}$ of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ is defined as

$$d_{\mathcal{B}_L}(X,Y) := \min_{\substack{k \in I\!N \\ T_1,\ldots,T_k \in F}} \sum_{i=1}^{k-1} d(T_i, T_{i+1})$$

with $T_1 = X$, $T_k = Y$ and $T_i \in F$ $(i = 1,\ldots,k)$ such that there exists a shortest path (with respect to d) from T_i to T_{i+1} that does not cross $\mathcal{B}_L$. This definition leads to the following description of $d_{\mathcal{B}_L}$:

Lemma 1 *Let d be a metric derived from a norm and $i,j \in \{1,2\}$, $i \neq j$. Then*

$$d_{\mathcal{B}_L}(X,Y) = \begin{cases} d(X,Y) & \text{if} \quad X,Y \in F^i \\ d(X,P_{n_{X,Y}}) + d(P_{n_{X,Y}},Y) \quad \text{for some } n_{X,Y} \in \mathcal{N} & \text{if} \quad X \in F^i,\ Y \in F^j. \end{cases}$$

Proof: As d is a metric derived from a norm the shortest distance between any two points X and Y in the same half-plane F^i $(i \in \{1,2\})$ is not influenced by $\mathcal{B}_L$. Furthermore due to the triangle inequality every shortest path from a point $X \in F^1$ to another point $Y \in F^2$ has to pass through exactly one of the passages P_n, $n \in \mathcal{N}$. In this case $d_{\mathcal{B}_L}(X,Y) = d(X,P_{n_{X,Y}}) + d(P_{n_{X,Y}},Y)$ holds for $n_{X,Y} \in \mathcal{N}$.

□

Note that for $d_{\mathcal{B}_L}$ the triangle inequality is satisfied but that $d_{\mathcal{B}_L}$ is not positively homogenous. Thus $d_{\mathcal{B}_L}$ is not derived from a norm. As a consequence the objective function of this restricted location problem is usually not convex.
We use Lemma 1 to rewrite the objective function for a point $X \in F^i$:

Lemma 2 *Let d be a metric derived from a norm, $X \in F^i$ and $i, j \in \{1,2\}$, $i \neq j$. Then there exist passages $P_{n_1}, \ldots, P_{n_{M^j}}$ such that*

$$\begin{aligned} f(X) &= \sum_{m=1}^{M^i} w_m^i d_{\mathcal{B}_L}(X, Ex_m^i) + \sum_{m=1}^{M^j} w_m^j d_{\mathcal{B}_L}(X, Ex_m^j) \\ &= \underbrace{\sum_{m=1}^{M^i} w_m^i d(X, Ex_m^i) + \sum_{m=1}^{M^j} w_m^j d(X, P_{n_m})}_{=: f_X^i(X)} + \underbrace{\sum_{m=1}^{M^j} w_m^j d(Ex_m^j, P_{n_m})}_{=: g_X^j}. \end{aligned} \tag{1}$$

Here $f_X^i(Y) := \sum_{m=1}^{M^i} w_m^i d(Y, Ex_m^i) + \sum_{m=1}^{M^j} w_m^j d(Y, P_{n_m})$, $Y \in F^i$, is the objective function of an unrestricted Weber Problem ($1/P/\bullet/d/\sum$) in the half-plane F^i with existing facilities $Ex_1^i, \ldots, Ex_{M^i}^i, P_1, \ldots, P_N$.

A similar description of the objective function was given in [3], where the corner points of polygonal obstacles replace the passages. The next result limits for a large class of metrics the set of optimal solutions $\mathcal{X}_{\mathcal{B}}^*$ of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ to the union of the convex hulls of the existing facilities together with the set of passages on either side of $\mathcal{B}_L$.

Theorem 1 *Let d be a metric derived from a norm such that $\mathcal{X}^* \subseteq \text{conv}\{Ex_m \mid m \in \mathcal{M}\}$ holds for $1/P/\bullet/d/\sum$ (the analogous Weber Problem without the barrier $\mathcal{B}_L$). Then*

$$\begin{aligned} \mathcal{X}_{\mathcal{B}}^* \subseteq\ & \text{conv}\{Ex_m^1, P_n \mid m \in \mathcal{M}^1,\ n \in \mathcal{N}\} \\ & \cup\ \text{conv}\{Ex_m^2, P_n \mid m \in \mathcal{M}^2,\ n \in \mathcal{N}\} \end{aligned}$$

holds for $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$.

Proof: Without loss of generality let $X_{\mathcal{B}}^* \in F^1$ and $X_{\mathcal{B}}^* \notin \text{conv}\{Ex_m^1, P_n \mid m \in \mathcal{M}^1,\ n \in \mathcal{N}\}$ be an optimal solution of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$. It follows from Lemma 2 that the objective function for the point $X_{\mathcal{B}}^*$ can be written as

$$f(X_{\mathcal{B}}^*) = f_{X_{\mathcal{B}}^*}^1(X_{\mathcal{B}}^*) + g_{X_{\mathcal{B}}^*}^2$$

where $f_{X_{\mathcal{B}}^*}^1$ is the objective function of a problem of type $1/P/\bullet/d/\sum$. We know that $1/P/\bullet/d/\sum$ has all optimal solutions $X^* \in \mathcal{X}^* \subseteq \text{conv}\{Ex_m^1, P_n \mid m \in \mathcal{M}^1, n \in \mathcal{N}\}$. Since $X_{\mathcal{B}}^* \notin \text{conv}\{Ex_m^1, P_n \mid m \in \mathcal{M}^1, n \in \mathcal{N}\}$ any $X^* \in \mathcal{X}^*$ implies $f_{X_{\mathcal{B}}^*}^1(X^*) < f_{X_{\mathcal{B}}^*}^1(X_{\mathcal{B}}^*)$ and $f(X^*) \leq f_{X_{\mathcal{B}}^*}^1(X^*) + g_{X_{\mathcal{B}}^*}^2 < f_{X_{\mathcal{B}}^*}^1(X_{\mathcal{B}}^*) + g_{X_{\mathcal{B}}^*}^2 = f(X_{\mathcal{B}}^*)$, which is a contradiction to the optimality of $X_{\mathcal{B}}^*$.

□

$\mathcal{X}^* \subseteq \text{conv}\{Ex_m \mid m \in \mathcal{M}\}$ is satisfied for a large class of metrics, e.g. the class of l_p-metrics with $1 < p < \infty$ [10]. On the other hand there exist some metrics as e.g. the l_1 and l_∞ metric, for which only a weaker condition is true, namely that there exists at least one optimal solution $X^* \in \mathcal{X}^*$ of $1/P/\bullet/d/\sum$ with $X^* \in \text{conv}\{Ex_m \mid m \in \mathcal{M}\}$. Analogous to Theorem 1 a similar result can be shown for this case:

Corollary 1 *Let d be a metric derived from a norm such that exists $X^* \in \text{conv}\{Ex_m \mid m \in \mathcal{M}\}$ $(X^* \in \mathcal{X}^*)$ holds for $1/P/\bullet/d/\sum$. Then*

$$\begin{aligned} \exists\, X^*_{\mathcal{B}} \;\in\; & \text{conv}\{Ex^1_m, P_n \mid m \in \mathcal{M}^1\ ,\ n \in \mathcal{N}\} \\ & \cup\ \text{conv}\{Ex^2_m, P_n \mid m \in \mathcal{M}^2\ ,\ n \in \mathcal{N}\}. \end{aligned}$$

holds for $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$.

Unfortunately it is not possible to restrict $\mathcal{X}^*_{\mathcal{B}}$ e.g. to the half-plane F^1 or F^2 with the higher total weight as one may conjecture intuitively. This can be easily seen in Figure 1 where the weights of all sites are chosen equal to one and the distance is given by the Euclidean distance. The best solution $X^*_{\mathcal{B}}$ is located in F^1 whereas the higher total weight is found in F^2. Analogous it is easy to construct an example where $\mathcal{X}^*_{\mathcal{B}}$ is on the opposite side of $\mathcal{B}_L$ than $\mathcal{X}^*$.

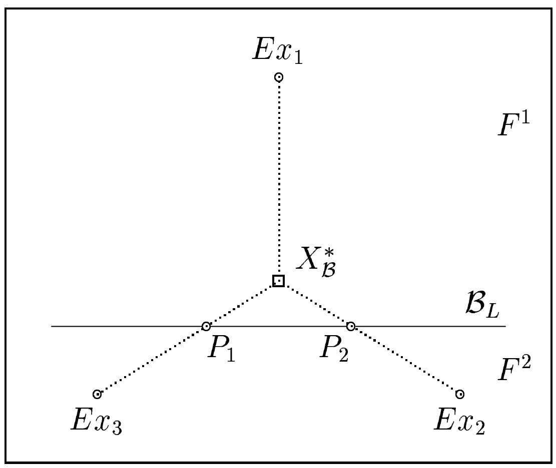

Figure 1: $d = l_2$, $w(Ex_m) = 1$, $m = 1, 2, 3$ and $\mathcal{X}^*_{\mathcal{B}} \subset F^1$.

In the following only such metrics d derived from a norm (and location problems $1/P/\bullet/d/\sum$) will be considered for which condition (2), i.e.

$$\mathcal{X}^* \;\subseteq\; \text{conv}\{Ex_m \mid m \in \mathcal{M}\}, \tag{2}$$

holds for the unrestricted location problem $1/P/\bullet/d/\sum$. All results can be easily transfered to the case that only the weaker condition (3) holds for $1/P/\bullet/d/\sum$, i.e.

$$\exists\, X^* \in \mathcal{X}^* \quad \text{with} \quad X^* \in \text{conv}\{Ex_m \mid m \in \mathcal{M}\}. \tag{3}$$

In the first case the complete set of optimal locations $\mathcal{X}^*_{\mathcal{B}}$ of the restricted location problem can be determined whereas in the latter case at least one optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ can be found. One of these two conditions is satisfied for all unrestricted Weber Problems with a metric derived from a norm [10, 15].

Lemma 2 and Theorem 1 can be used to derive the optimal solutions of the restricted Weber Problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ from the optimal solutions of a finite number of unrestricted Weber Problems $1/P/\bullet/d/\sum$. This result shows that it is possible to reduce the global optimization problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ to a finite number of convex optimization problems for which good solution concepts are known.

Theorem 2 *Let d be a metric derived from a norm and let $1/P/\bullet/d/\sum$ be a Weber Problem for which $\mathcal{X}^* \subseteq \text{conv}\{Ex_m \mid m \in \mathcal{M}\}$ is true.*
Every optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ is also an optimal solution of the analogous unrestricted Weber Problem $1/P/\bullet/d/\sum$ in F^i with existing facilities $Ex^i_1, \ldots, Ex^i_{M^i}, P_1, \ldots, P_N$ $(i \in \{1,2\})$ and objective function $f^i_{X^*_{\mathcal{B}}}$.

Proof: Let $X^*_{\mathcal{B}} \in F^i$ be an optimal solution of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$. From Lemma 2 we have that

$$f(X^*_{\mathcal{B}}) = f^i_{X^*_{\mathcal{B}}}(X^*_{\mathcal{B}}) + g^j_{X^*_{\mathcal{B}}}.$$

Here $g^j_{X^*_{\mathcal{B}}}$ is constant and $f^i_{X^*_{\mathcal{B}}}(Y)$ is the objective function of the analogous unrestricted Weber Problem. Assume that $\exists Y \in F^i$ with $f^i_{X^*_{\mathcal{B}}}(Y) < f^i_{X^*_{\mathcal{B}}}(X^*_{\mathcal{B}})$. Then $f(Y) \leq f^i_{X^*_{\mathcal{B}}}(Y) + g^j_{X^*_{\mathcal{B}}} < f^i_{X^*_{\mathcal{B}}}(X^*_{\mathcal{B}}) + g^j_{X^*_{\mathcal{B}}} = f(X^*_{\mathcal{B}})$, which is a contradiction to the optimality of $X^*_{\mathcal{B}}$.

□

Theorem 2 can be easily transformed to the case that only the weaker condition (3) is true for the unrestricted Weber Problem $1/P/\bullet/d/\sum$:

Corollary 2 *Let d be a metric derived from a norm and let $1/P/\bullet/d/\sum$ be a Weber Problem such that there exists an optimal solution $X^* \in \text{conv}\{Ex_m \mid m \in \mathcal{M}\}$.*
Then there exists at least one optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ which is also an optimal solution of the analogous unrestricted Weber Problem $1/P/\bullet/d/\sum$ in F^i with existing facilities $Ex^i_1, \ldots, Ex^i_{M^i}, P_1, \ldots, P_N$ $(i \in \{1,2\})$ and objective function $f^i_{X^*_{\mathcal{B}}}$.

Using Theorem 2 a very simple algorithm can be given to find an optimal solution of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ which checks for all existing facilities in one half-plane F^i all possible passages to the other half-plane F^j and determines the optimal solution of the corresponding unrestricted location problem in F^j. This procedure must be carried out for each half-plane. A necessary condition for the correctness of this procedure is that condition (2) or condition (3) is satisfied for the analogous unrestricted location problem $1/P/\bullet/d/\sum$ and therefore Theorem 2 or Corollary 2 can be applied.

In the following the idea of reducing the non-convex optimization problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ to a finite set of convex optimization problems will be used to derive efficient polynomial algorithms for solving $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$. They are based on the fact that only a small number of unrestricted location problems of the form $1/P/\bullet/d/\sum$ must be considered to obtain a candidate set for the optimal solutions of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$.

3 Algorithms

The case where only one passage P allows passing through $\mathcal{B}_L$ is trivial. Let $X \in F^i$ and $i, j \in \{1, 2\}$, $i \neq j$. Every path from an existing facility $Ex_m^j \in F^j$ to X has to pass through P. Thus the objective function for the point X can be written as

$$f(X) = \underbrace{\sum_{m=1}^{M^i} w_m^i d(X, Ex_m^i) \; + \; \left(\sum_{m=1}^{M^j} w_m^j \right) d(X, P)}_{=: f_X^i(X)} \; + \; \underbrace{\sum_{m=1}^{M^j} w_m^j d(P, Ex_m^j)}_{=: g_X^j}.$$

As the weight of P is the sum of weights of all existing facilities in one half-plane, $w(P) = \sum_{m=1}^{M^j} w_m^j$, in this case the optimal new location can be restricted to the half-plane with the higher total weight if it is not equal for both half-planes. (For the other half-plane the optimal location is always P.) In this case an unrestricted Weber Problem has to be solved at most twice (once for each half-plane). This can be done with a complexity of $O(T)$, where $O(T)$ is the complexity of the analogous unrestricted Weber Problem.

For the more realistic case that more than one passage allows passing through $\mathcal{B}_L$ the number of unrestricted Weber Problems to be solved for obtaining the optimal solution can be reduced considerably compared to the straight forward approach described in Section 2. In all cases the time complexity of the deduced algorithms is polynomial.

3.1 Line barriers with two passages

For $i = 1, 2$ define the difference of distances between an existing facility and the two passages P_1 and P_2

$$D^i(m) := d(Ex_m^i, P_1) - d(Ex_m^i, P_2), \qquad m \in \mathcal{M}^i,$$

and without loss of generality assume that the existing facilities are ordered such that $D^i(1) \leq \cdots \leq D^i(M^i)$. Furthermore let $j \in \{1, 2\}$ with $j \neq i$. A shortest path SP from an existing facility $Ex_m^j \in F^j$ to a point $X \in F^i$ has to pass through one of the passages P_1 and P_2 depending on the following condition (see Figure 2):

$$\begin{aligned} P_1 \in SP \;&\Leftrightarrow\; d(Ex_m^j, P_1) + d(P_1, X) \; < \; d(Ex_m^j, P_2) + d(P_2, X) \\ P_2 \in SP \;&\Leftrightarrow\; d(Ex_m^j, P_1) + d(P_1, X) \; > \; d(Ex_m^j, P_2) + d(P_2, X). \end{aligned}$$

This is equivalent to condition (4):

$$
\begin{aligned}
P_1 \in SP \quad &\Leftrightarrow \quad D^j(m) \; < \; d(P_2, X) - d(P_1, X) \\
P_2 \in SP \quad &\Leftrightarrow \quad D^j(m) \; > \; d(P_2, X) - d(P_1, X).
\end{aligned} \tag{4}
$$

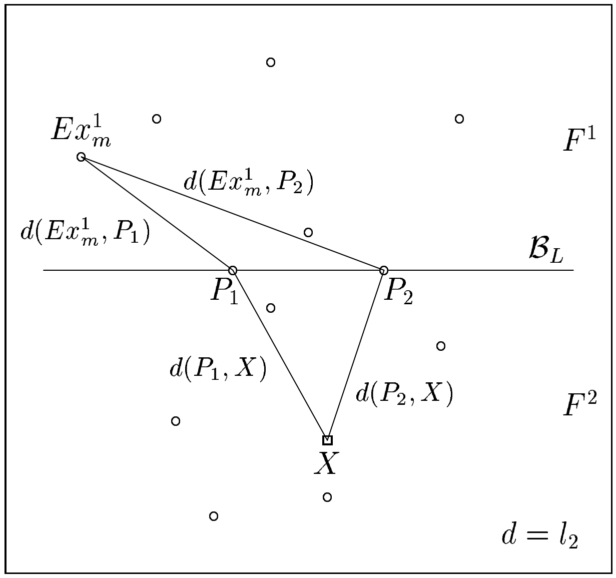

Figure 2: The shortest path from Ex^1_m to X depends on $D^1(m)$

If $D^j(m) = d(P_2, X) - d(P_1, X)$, then a shortest path may pass through either passage. With $k := \max\{m \in \{0, \ldots, M^j\} \mid D^j(m) < d(P_2, X) - d(P_1, X)\}$ the value of the objective function for the point X can be evaluated as

$$
\begin{aligned}
f(X) \;\; = \;\; & \underbrace{\sum_{m=1}^{M^i} w^i_m d(X, Ex^i_m) \; + \; \left(\sum_{m=1}^{k} w^j_m \right) d(X, P_1) + \left(\sum_{m=k+1}^{M^j} w^j_m \right) d(X, P_2)}_{=: f^i_k(X)} \\
& + \underbrace{\sum_{m=1}^{k} w^j_m d(P_1, Ex^j_m) \; + \; \sum_{m=k+1}^{M^j} w^j_m d(P_2, Ex^j_m)}_{=: g^j_k}.
\end{aligned} \tag{5}
$$

The only unknown parameters in (5) are the values of i and k because the coordinates of the optimal new locations are unknown. Therefore all possible values $i = 1, 2$ and $k = 0, \ldots, M^j$ are tested in Algorithm 1 to obtain the global optimum. This leads to a polynomial algorithm.

Algorithm 1 for solving $1/P/\mathcal{B}_L,\ N=2/d_{\mathcal{B}_L}/\sum$:

For $i=1,2$ do

1. Let $j\in\{1,2\}$ with $j\neq i$.
2. Let $D^j(m)\ :=\ d(Ex^j_m,P_1)\ -\ d(Ex^j_m,P_2);\ m\in\mathcal{M}^j$.
3. Sort the existing facilities such that $D^j(1)\leq\cdots\leq D^j(M^j)$.
4. For $k=0$ to M^j do
   (a) Let $w(P_1):=\sum_{m=1}^{k}w(Ex^j_m)$ and $w(P_2):=\sum_{m=k+1}^{M^j}w(Ex^j_m)$.
   (b) Determine the set of optimal solutions $\mathcal{X}^i_k$ of $1/P/\bullet/d/\sum$ with existing facilities $\mathcal{E}x:=\{P_1,P_2,Ex^i_1,\dots,Ex^i_{M^i}\}$ and the weights defined in (a).
   (c) For $X^i_k\in\mathcal{X}^i_k$ determine $\tilde{f}(X^i_k)\ =\ f^i_k(X^i_k)+g^j_k$.

Output: $\mathcal{X}^*_{\mathcal{B}}\ =\ \underset{X^i_k\in\mathcal{X}^i_k\ ;\ k\in\mathcal{M}^i\ ;\ i\in\{1,2\}}{\arg\min}\ \tilde{f}(X^i_k)$.

The time complexity of Algorithm 1 is $O(M\log M\ +\ MT)$, where $M:=M^1+M^2$ is the number of existing facilities and $O(T)$ is the time complexity of $1/P/\bullet/d/\sum$. In Step 3 the existing facilities must be sorted and in Step 4 an unrestricted Weber Problem with time complexity $O(T)$ has to be solved M times.
Note that $\tilde{f}(X)\geq f(X)$. Therefore during the algorithm the situation may occur that a solution X^i_k is evaluated for which the current value of k is not optimal and $\tilde{f}(X^i_k)>f(X^i_k)$. Anyhow this problem cannot arise for the global optimum as equality $\tilde{f}(X^*_{\mathcal{B}})=f(X^*_{\mathcal{B}})$ holds for $X^*_{\mathcal{B}}\in\mathcal{X}^*_{\mathcal{B}}$ and thus the global optima of f and $\tilde{f}$ coincide.

3.2 Line barriers with N passages, $N\geq 2$

Algorithm 1 can be generalized to the case that there is an arbitrary but finite number N of passages through the line barrier $\mathcal{B}_L$.

Without loss of generality we assume that the passages are in consecutive order, i.e. there is no other passage between P_i and P_{i+1} for $1\leq i\leq N-1$. Again the differences of distances between the existing facilities and every two adjacent passages P_n and P_{n+1} are needed: For $i,j\in\{1,2\}$, $i\neq j$, and $n=1,\dots,N-1$ define

$$D^i_n(m):=d(Ex^i_m,P_n)-d(Ex^i_m,P_{n+1}),\qquad m\in\mathcal{M}^i.$$

Since d is a metric derived from a norm a shortest path SP from an existing facility $Ex^j_m\in F^j$ to a point $X\in F^i$ has to pass through one of the passages $P_1,\dots,P_N$ depending on condition (6):

$$
\begin{array}{lll}
P_1 \in SP & \Leftrightarrow & D_1^j(m) < d(P_2, X) - d(P_1, X) \\
P_n \in SP & \Leftrightarrow & d(P_n, X) - d(P_{n-1}, X) < D_{n-1}^j(m) \;\wedge\; D_n^j(m) < d(P_{n+1}, X) - d(P_n, X) \\
P_N \in SP & \Leftrightarrow & d(P_N, X) - d(P_{N-1}, X) < D_{N-1}^j(m).
\end{array} \tag{6}
$$

To describe the objective function for a point $X \in F^i$ analogous to (5) some further notation will be useful:

For $n = 1, \dots, N-1$ let $\pi_n^j : \mathcal{M}^j \to \mathcal{M}^j$ be a permutation of $\mathcal{M}^j$ such that

$$D_n^j(\pi_n^j(1)) \leq \cdots \leq D_n^j(\pi_n^j(M^j)).$$

Furthermore define

$$k_n := \underset{m \in \mathcal{M}^j}{\operatorname{argmax}} \left\{0, \pi_n^j(m) \mid D_n^j(\pi_n^j(m)) < d(P_{n+1}, X) - d(P_n, X)\right\}, \qquad n = 1, \dots, N-1,$$

and $k_N := M^j$. Unfortunately two permutations π_n^j and $\pi_{\tilde{n}}^j$ need not be the same for $n \neq \tilde{n}$. Therefore let $\mathcal{M}_1^j := \mathcal{M}^j$ and

$$\mathcal{M}_n^j := \mathcal{M}_{n-1}^j \setminus \left\{\pi_{n-1}^j(m) \mid \pi_{n-1}^j(m) \leq k_{n-1}\right\}, \quad n = 2, \dots, N.$$

Then

$$
\begin{aligned}
f(X) \;=\; & \underbrace{\sum_{m=1}^{M^i} w_m^i d(X, Ex_m^i) + \sum_{n=1}^{N} \Bigg(\sum_{\substack{\pi_n^j(m) \in \mathcal{M}_n^j \\ \pi_n^j(m) \leq k_n}} w_{\pi_n^j(m)}^j \Bigg) d(X, P_n)}_{=: f_{k_1, \dots, k_N}^j(X)} \\
& + \underbrace{\sum_{n=1}^{N} \sum_{\substack{\pi_n^j(m) \in \mathcal{M}_n^j \\ \pi_n^j(m) \leq k_n}} w_{\pi_n^j(m)}^j d(P_n, Ex_{\pi_n^j(m)}^j)}_{=: g_{k_1, \dots, k_N}^j}.
\end{aligned}
$$

The unknown parameters are the values of i and $k_1, \dots, k_N$ as the coordinates of the optimal new locations are unknown. Therefore all possible values $i = 1, 2$ and $k_1, \dots, k_N \in \{0, \dots, M^j\}$ such that $k_1 + \cdots + k_N = M^j$ are tested in Algorithm 2 to obtain the global optimum. As in the special case of $N = 2$ this leads to a polynomial algorithm.

Algorithm 2 for solving $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$:

For $i = 1, 2$ do

1. Let $j \in \{1,2\}$ with $j \neq 2$ and
   $D_n^j(m) \;:=\; d(P_n, Ex_m^j) \;-\; d(P_{n+1}, Ex_m^j)$; $m \in \mathcal{M}^j$; $n = 1, \ldots, N-1$.
2. For $n = 1$ to $N-1$ find a permutation $\pi_n^i : \mathcal{M}^i \to \mathcal{M}^i$ such that
   $D_n^j(\pi_n^j(1)) \leq \cdots \leq D_n^j(\pi_n^j(M^j))$.
3. Let $\mathcal{M}_1^j := \mathcal{M}^j$.

   For $k_1 = 0$ to M^j do

   Determine $\mathcal{M}_2^j$.

   For $k_2 = k_1$ to M^j do

   Determine $\mathcal{M}_3^j$.

   ...

   For $k_{N-1} = k_{N-2}$ to M^j do

   Determine $\mathcal{M}_N^j$ and let $k_N := M^j$.

   (a) For $n = 1$ to N let

   $$w(P_n) := \sum_{\substack{\pi_n^j(m) \in \mathcal{M}_n^j \\ \pi_n^j(m) \leq k_n}} w(Ex^j_{\pi_n{}^j(m)}).$$

   (b) Determine the set of optimal solutions $\mathcal{X}^i_{k_1,\ldots,k_N}$ of $1/P/\bullet/d/\sum$ with existing facilities $\mathcal{E}x = \{P_1, \ldots, P_N, Ex_1^i, \ldots, Ex^i_{M^i}\}$ and the weights defined in (a).

   (c) For $X^i_{k_1,\ldots,k_N} \in \mathcal{X}^i_{k_1,\ldots,k_N}$ determine

   $$\tilde{f}(X^i_{k_1,\ldots,k_N}) \;=\; f^i_{k_1,\ldots,k_N}(X^i_{k_1,\ldots,k_N}) + g^j_{k_1,\ldots,k_N}.$$

Output: $\mathcal{X}^*_{\mathcal{B}_L} \;= \operatorname{argmin} \tilde{f}(X^i_{k_1,\ldots,k_N})$.

With $M := M^1 + M^2$, the time complexity of Algorithm 2 is $O(N(M \log M) + \binom{M+N-1}{N-1}T)$, where $O(T)$ is the time complexity of $1/P/\bullet/d/\sum$. With increasing number of passages N the complexity grows exponentially but for a fixed number of passages it remains polynomial.

4 Example

To clearify the concepts described above the following example with the classification $1/P/\mathcal{B}/(l_2)_\mathcal{B}/\sum$ shall be considered: Let a line barrier

$$\mathcal{B}_L := \{(x,y) \in I\!R^2 \mid y = 5\} \setminus \{P_1 = (4,5), P_2 = (9,5)\}$$

divide the plane into two half-planes. Furthermore 6 existing facilities are given on both sides of $\mathcal{B}_L$ with coordinates and weights as listed in Table 1. Thus $\mathcal{M}^1 = \mathcal{M}^2 = \{1,2,3\}$ and $M^1 = M^2 = 3$.

Existing facility Ex_m^i		$w_m^i = w(Ex_m^i)$	D_m^i
Ex_1^1	$(5,7)$	1	-2.24
Ex_2^1	$(4.5,9)$	2	-1.99
Ex_3^1	$(10,7.5)$	2	3.81
Ex_1^2	$(3,3)$	2	-4.09
Ex_2^2	$(6,1)$	3	-0.53
Ex_3^2	$(8.5,4)$	2	3.49

Table 1: Existing facilities with their weights and the values of D_m^i

The 8 unrestricted Weber Problems which must be solved while applying Algorithm 1 are listed in Table 2. (The approximate solution values given in Table 2 where obtained using Weiszfeld's algorithm.)

subproblem	weights		optimal solutions of the subproblems			
(i,k)	$w(P_1)$	$w(P_2)$	$\mathcal{X}_k^i$	$f_k^i(X_k^i)$	g_k^j	$\tilde{f}(X_k^i)$
(1,0)	0	7	$(9,5)$	21.90	29.89	51.79
(1,1)	2	5	$(8.36,5.60)$	32.14	21.71	53.85
(1,2)	5	2	$(5.05,5.94)$	32.83	20.13	52.96
(1,3)	7	0	$(4,5)$	23.30	27.11	50.40
(2,0)	0	5	$(8.36,4.10)$	28.50	21.90	50.41
(2,1)	1	4	$(7.70,3.87)$	31.95	19.67	51.62
(2,2)	3	2	$(5.72,3.43)$	32.78	15.69	48.47
(2,3)	5	0	$(4.38,4.17)$	27.11	23.30	50.41

Table 2: Optimal solutions of the 8 subproblems

It is easy to see that $\mathcal{X}^*_\mathcal{B} = \{(5.72, 3.43)\}$ is the global optimal solution with $f(\mathcal{X}^*_\mathcal{B}) = \tilde{f}(\mathcal{X}^*_\mathcal{B}) = 48.47$. In comparison to this result the optimal solution $\mathcal{X}^*$ of the analogous unrestricted Weber Problem without considerion of the barrier $\mathcal{B}_L$ is given by $\mathcal{X}^* = (6.41, 4.40)$ with objective function value $f(\mathcal{X}^*) = 44.31$.

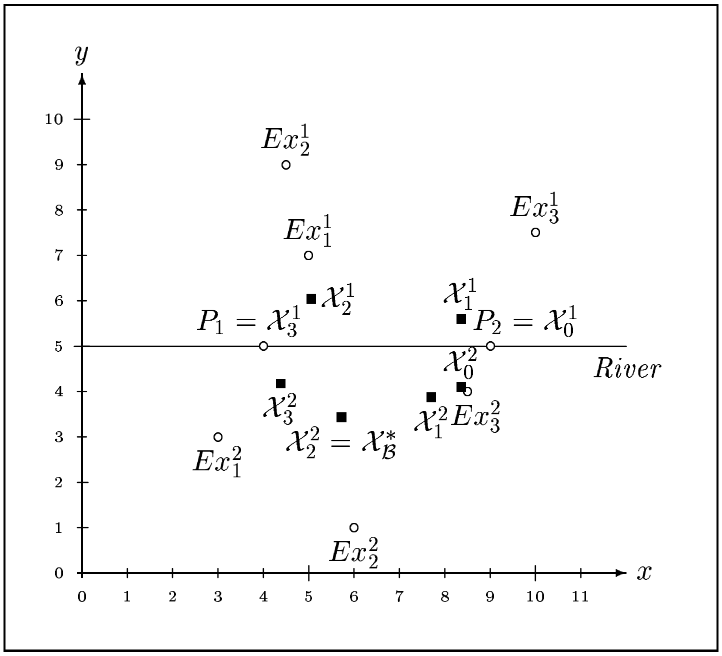

Figure 3: The example problem with the classification $1/P/\mathcal{B}/(l_2)_\mathcal{B}/\sum$.

5 Conclusions

The concepts developed in this paper allow the introduction of line barriers with a finite number of passages into the theory of planar location problems. For a broad class of location problems including the Weber Problem with the l_p-metric, the solution of the non-convex optimization problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/\sum$ can be reduced to the solution of a polynomial number of unrestricted location problems, i.e. to convex optimization problems.

These results hold for Weber Problems for which $\mathcal{X}^* \subseteq \text{conv}\{Ex_m \mid m \in \mathcal{M}\}$ or $\exists\ X^* \in \text{conv}\{Ex_m \mid m \in \mathcal{M}\}$ $(X^* \in \mathcal{X}^*)$, respectively, is fulfilled in the unrestricted case. This allows the simultaneous introduction of forbidden regions as well as the consideration of multicriteria location problems with Q different sum-objective functions. In the multicriteria case the determination of the global optimal solutions from the set of local optimal solutions of the individual unrestricted problems is not difficult if the given ordering of $\mathbb{R}^Q$ is either max-ordering or lexicographic ordering. If the required solutions are Pareto solutions, the global Pareto solutions must be determined from the union of all local Pareto solutions of the individual subproblems.

Moreover, no special treatment is necessary for the consideration of line barriers with a finite positive width W. In this case a point P_n^i is identified with the endpoint of each passage through the barrier in half-plane F^i ($i \in \{1, 2\}$). In each half-plane F^i these points are treated as the passages P_n. Furthermore the constant $C := M^j W$ is added to the objective function value where M^j is the number of existing facilities in the opposite half-plane.

A generalization to higher dimensional problems seems to be more of theoretical then of practical interest. Anyhow a generalization to the case that the barriers are hyperplanes in $\mathbb{R}^n$ which allow treshpassing only through a finite set of points can be done without using any supplementary considerations.

References

[1] Y.P. Aneja and M. Parlar. Algorithms for Weber facility location in the presence of forbidden regions and/or barriers to travel. *Transportation Science*, 28:70–76, 1994.

[2] R. Batta, A. Ghose, and U.S. Palekar. Locating facilities on the Manhatten metric with arbitrarily shaped barriers and convex forbidden regions. *Transportation Science*, 23:26–36, 1989.

[3] S.E. Butt and T.M. Cavalier. An efficient algorithm for facility location in the presence of forbidden regions. *European Journal of Operational Research*, 90:56–70, 1996.

[4] L.R. Foulds and H.W. Hamacher. Optimal bin location in printed circuit board assembly. *European Journal of Operational Research*, 66:279–290, 1993.

[5] R.L. Francis, F. Leon, J. McGinnis, and J.A. White. *Facility Layout and Location: An Analytical Approach, 2nd ed.* Prentice-Hall, New York, 1992.

[6] H.W. Hamacher. *Mathematische Verfahren der Planaren Standortplanung.* Vieweg Verlag, Braunschweig, 1995. 171 pages.

[7] H.W. Hamacher and S. Nickel. Multicriteria planar location problems. Technical Report 243, Universität Kaiserslautern, Department of Mathematics, 1993. accepted in European Journal of Operational Research.

[8] H.W. Hamacher and S. Nickel. Combinatorial algorithms for some 1-facility median problems in the plane. *European Journal of Operational Research*, 79:340–351, 1994.

[9] H.W. Hamacher and S. Nickel. Restricted planar location problems and applications. *Naval Research Logistics*, 42:967–992, 1995.

[10] H. Juel and R.F. Love. Hull properties in location problems. *European Journal of Operational Research*, 12:262–265, 1983.

[11] I.N. Katz and L. Cooper. Facility location in the presence of forbidden regions, I: Formulation and the case of Euclidean distance with one forbidden circle. *European Journal of Operational Research*, 6:166–173, 1981.

[12] R.C. Larson and G. Sadiq. Facility locations with the Manhattan metric in the presence of barriers to travel. *Operations Research*, 31:652–669, 1983.

[13] R.F. Love, J.G. Morris, and G.O. Wesolowsky. *Facilities Location: Models & Methods*. North Holland, New York, 1988.

[14] F. Plastria. Continuous location problems. In Z. Drezner, editor, *Facility Location*, pages 225–262. Springer Series in Operations Research, 1995.

[15] R.E. Wendell and A.P. Hurter. Location theory, dominance and convexity. *Operations Research*, 21:314–320, 1973.

[16] G.O. Wesolowsky. The Weber problem: History and perspectives. *Location Science*, 1:5–23, 1993.