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Abstract

In this paper we investigate two optimization problems for matroids with multiple objective functions, namely finding the pareto set and the max-ordering problem which consists in finding a basis such that the largest objective value is minimal. We prove that the decision versions of both problems are NP-complete. A solution procedure for the max-ordering problem is presented and a result on the relation of the solution sets of the two problems is given. The main results are a characterization of pareto bases by a basis exchange property and finally a connectivity result for proper pareto solutions.

1 Introduction

Let $\mathcal{M} = (\mathcal{E}, \mathcal{I})$ be a matroid and $\mathcal{B}$ the set of bases of $\mathcal{M}$. With each $e \in \mathcal{E}$ we associate Q weights $w_q(e) \in \mathbb{R}_+$. For $X \subseteq \mathcal{E}$ let $w_q(X) = \sum_{e \in X} w_q(e)$ and $f(X) = (w_1(X), \dots w_Q(X))$ and $g(X) = \max_{q=1,\dots Q} w_q(X)$. For $a, b \in \mathbb{R}^Q$ we adopt the following notation: $a \leq b \Leftrightarrow a_q \leq b_q$ for $q = 1 \dots Q$ and $a < b \Leftrightarrow a \leq b, a \neq b$.

We consider the following problems:

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1. **Max-Ordering Matroid Optimization (MOMO)**

$$\min_{B \in \mathcal{B}} \{g(B)\} \tag{1}$$

2. **Multi Criteria Matroid Optimization (MCMO)**

$$\text{“}\min_{B \in \mathcal{B}}\text{”} \{f(B)\} \tag{2}$$

The sets of solutions of (1) and (2) are denoted by $\mathcal{B}_{MO}$ and $\mathcal{B}_{par}$ respectively. $B \in \mathcal{B}_{par}$ if there is no $B' \in \mathcal{B}$ such that $f(B') < f(B)$, i.e. "min" is understood in the sense of pareto-optimality. In MCMO we will say that basis B *dominates* basis B' if $f(B) < f(B')$.

The basic notions of Matroid Theory used can be found in [10]. For the foundations of Multicriteria Optimization we refer to [12].

Theorem 1 *The decision problems for* MOMO *and* MCMO *are NP-complete.*

The result for MOMO was stated without proof in [13]. For MCMO [11] gives a reference to [2]. The theorem is shown for the special case of spanning tree problems in [3]. We will prove the general result of Theorem 1 by means of Proposition 1, which shows that MOMO and MCMO are NP-complete even for the special case of uniform matroids $U_{m,n}$, where $n = |\mathcal{E}|$ and m is the cardinality of the bases of $U_{m,n}$. We will now define the three classes of matroids which will appear in this paper in terms of their bases, see also [10].

Definition 1

1. *A uniform matroid $U_{m,n}$ is defined by $\mathcal{E} = \{e_1, \dots e_n\}$ and $\mathcal{B} = \{B \subseteq \mathcal{E} \,||B| = m\}$.*
2. *A partition matroid is defined by $\mathcal{E} = \{e_1, \dots e_n\} = \cup_{i=1}^{r} \mathcal{E}_i$, where $\mathcal{E}_i$ are pairwise disjoint, nonempty, and $\mathcal{B} = \{B = \{f_i | f_i \in \mathcal{E}_i, i = 1, \dots r\}\}$.*
3. *A graphic matroid $\mathcal{M}(G)$ of a graph $G = (V, E)$ is defined by $\mathcal{E} = E(G)$ and $\mathcal{B} = \{B \subset E(G) | B$ defines a spanning tree of $G\}$.*

Proposition 1 MOMO *and* MCMO *are NP-complete for $\mathcal{M} = U_{m,n}$.*

Proof:

The proof is by reduction of PARTITION to MCMO and MOMO respectively. We consider the following version of PARTITION: Given a set $A, |A| = n$ with weights $w(a) \quad \forall a \in A$ such that $\sum_{a\in A} w(a) = 2W$. Then the decision poblem: "Is there a subset B of A, $|B| = m, \quad 1 < m < n$ such that $\sum_{a\in B} w(a) = W$?" is NP-complete. This is easily seen since solving this problem for $m = 2, \ldots n-1$ would solve PARTITION [5].

We will now construct an instance of MOMO and MCMO with weights $(w_1'(e), w_2'(e))$ for $U_{m,n}$ such that B solves PARTITION if and only if $f(B) \le (W + m\bar{w}, W + m\bar{w})$ if and only if $g(B) \le W + m\bar{w}$ where $\bar{w} > \max_{a\in A} w(a)$.

Let $\mathcal{E} = A$ and define $w_1'(a) = \bar{w} + \frac{2W}{m} - w(a)$ and $w_2'(a) = \bar{w} + w(a)$. For any basis B of $U_{m,n}$ obviously $f(B) \le (m\bar{w} + W, m\bar{w} + W)$ if and only if $g(B) \le m\bar{w} + W$.

Now suppose B is a solution for these decision problems. Then

$$\begin{aligned} w_1'(B) &\le W + m\bar{w} \\ \Leftrightarrow m\bar{w} + 2W - \textstyle\sum_{a\in B} w(a) &\le W + m\bar{w} \\ \Leftrightarrow \textstyle\sum_{a\in B} w(a) &\ge W \end{aligned}$$

$$\begin{aligned} w_2'(B) &\le W + m\bar{w} \\ \Leftrightarrow \textstyle\sum_{a\in B} w(a) &\le W \end{aligned}$$

Therefore B solves PARTITION. Conversely any solution of PARTITION provides a yes-instance for the MOMO and MCMO instance.

□

The rest of the paper is organized as follows. Section 2 is devoted to the max-ordering problem. A general result and a theorem for the case of uniform matroids are proved. Section 3 and 4 are concerned with the multi-criteria problem. In Section 3 we will introduce the concept of basis exchanges and use it to characterize pareto bases. In Section 4 the basis- and paretograph are introduced and a result on the connectivity of a subgraph of the paretograph is given.

2 Max-ordering Matroid Optimization

For $\lambda \in [0,1]^Q$ such that $\sum_{q=1}^{Q} \lambda_q = 1$ let $w(\lambda, e) = \sum_{q=1}^{Q} \lambda_q w_q(e)$ and for $X \subseteq \mathcal{E}$ define

$$h(X) = \sum_{e \in X} w(\lambda, e) \tag{3}$$

Then $\min_{B \in \mathcal{B}} h(B)$ is a one criterion matroid basis problem which can be solved by the greedy algorithm. The following results have first been proved in [8] for the case of graphic matroids, where $\mathcal{B}$ is the set of spanning trees of a given connected graph G.

Lemma 1 $h(B) \leq g(B) \quad \forall B \in \mathcal{B}$

Proof:

$h(B) = \sum_{e \in B} w(\lambda, e) = \sum_{q=1}^{Q} \sum_{e \in B} \lambda_q w_q(e) \leq (\sum_{q=1}^{Q} \lambda_q) \max_{q=1,\ldots Q}\{w_q(B)\} = g(B)$

□

Therefore $\min_{B \in \mathcal{B}} h(B) \leq h(B^*) \leq g(B^*)$ for all $B^* \in \mathcal{B}_{MO}$. Now choose $B_1, \ldots B_K \in \mathcal{B}$ such that $h(B_1) \leq h(B_2) \leq \ldots \leq h(B_K) \leq h(B)$ for all $B \in \mathcal{B} \backslash \{B_1, \ldots B_K\}$.

Theorem 2 *Let K be the smallest index such that $\min_{k=1,\ldots K-1}\{g(B_k)\} \leq h(B_K)$ then $B^* \in argmin\{g(B_k) | k = 1, \ldots K-1\}$ is an optimal solution of* MOMO.

Proof:

For any $B \in \mathcal{B} \backslash \{B_1, \ldots B_K\} \quad g(B^*) \leq h(B_K) \leq h(B) \leq g(B)$.

□

There are examples showing that even for $K = |\mathcal{B}| \quad h(B_K) < \min_{B \in \mathcal{B}} g(B)$ is possible. The choice of λ is crucial for the performance of the ranking approach. Heuristics for its choice are proposed in [8]. An algorithm for MOMO for the special case of spanning trees was also given in that paper. We will now outline how to solve MOMO for uniform matroids. Based on the binary search-tree procedure of [7] for ranking problems it is sufficient to give an algorithm to find a second best solution B under the restriction that some set $I \subset \mathcal{E}$ is contained in B and some other set $O \subset \mathcal{E}$ disjoint from I has no element in common with B, where I and O are given subsets of $\mathcal{E}$.

Theorem 3 *Let B^* solve $\min\{h(B)|B \in \mathcal{B}, I \subset B, B \cap O = \emptyset\}$. Then if e^*, f^* are chosen such that $h(f^*) = \min\{h(f)|f \in \mathcal{E} \setminus (B^* \cup O)\}$ and $h(e^*) = \max\{h(e)|e \in B^* \setminus I\}$ it holds that $B^* \setminus \{e^*\} \cup \{f^*\}$ is a second best basis for objective function h.*

Proof:

Let B be a basis, $I \subset B, O \cap B = \emptyset$. Then $h(B^*) \leq h(B)$. Now $S_1 := B \setminus B^* \subseteq \mathcal{E} \setminus (B^* \cup O)$ and $S_2 := B^* \setminus B \subseteq B^* \setminus I$. Furthermore $h(B) = h(B^*) + \sum_{f \in S_1} h(f) - \sum_{e \in S_2} h(e)$. Since B^* contains I and the $m - |I|$ smallest elements of $\mathcal{E} \setminus O$ and due to the choice of e^* and f^* it follows that $h(f) - h(e) \geq h(f^*) - h(e^*) \geq 0$ for all $f \in \mathcal{E} \setminus (B^* \cup O)$ and $e \in B^* \setminus I$. Hence we conclude that choosing $S_1 = \{f^*\}, S_2 = \{e^*\}$ $B = B^* \cup S_1 \setminus S_2$ is a second best solution.

□

By Theorem 3 a second best solution can be found in $\mathcal{O}(m + n)$ time. Thus for a given K the complexity of finding the K-best solutions for $U_{m,n}$ is $\mathcal{O}(\min(n \log(n), nm) + (K - 1)(m + n))$, using the bound of [8] for general matroids. Indeed this bound is much better than the general bound.

The next section is concerned with the problem MCMO. MOMO and MCMO are related in the sense that there is at least one basis B such that B is optimal for both problems. To see this note that for any basis $B \in \mathcal{B}_{MO}$ not dominated by some other $B' \in \mathcal{B}_{MO}$ $B \in \mathcal{B}_{par}$ holds. In fact this property holds for general multi objective problems.

3 Multi Criteria Matroid Optimization

Let $B, B' \in \mathcal{B}$ such that $B \setminus B' = \{e_1, \ldots e_n\}$ and $B' \setminus B = \{f_1, \ldots f_n\}$.

Definition 2 *1. $\tau(B, B') := [\{e_1, \ldots e_n\}, \{f_1, \ldots f_n\}]$ is called $B - B'$-exchange and $B' = B[\tau] = B \setminus \{e_1, \ldots e_n\} \cup \{f_1, \ldots f_n\}$.*

2. The weight of τ is defined as $w(\tau(B, B')) := \sum_{i=1}^{n} (w(e_i) - w(f_i))$.

3. *Correspondingly* $\tau' := \tau(B', B) = [\{f_1, \ldots f_n\}, \{e_1, \ldots e_n\}]$ *is the* $B'-B$*-exchange. Its weight is* $w(\tau') = -w(\tau)$.

4. *Two bases* B *and* B' *are called neighbours iff* $B \setminus B' = \{e\}, B' \setminus B = \{f\}1$, *i.e.* $B' = B\backslash\{e\} \cup \{f\}$.

5. *Let* $B \in \mathcal{B}, \{e_1, \ldots e_n\} \subseteq B$ *and* $\{f_1, \ldots f_n\} \subset \mathcal{E}$ *such that* $\{e_1, \ldots e_n\} \cap \{f_1, \ldots f_n\} = \emptyset$. *If* $\tau = [\{e_1, \ldots e_n\}, \{f_1, \ldots f_n\}]$ *and* $B[\tau]$ *is a basis,* τ *is called a* basis-exchange w.r.t B.

Now if $B, B' \in \mathcal{B}$ and $\tau = [\{e_1, \ldots e_n\}, \{f_1, \ldots f_n\}]$ is the $B - B'$–exchange then

$$w(B') = \sum_{e \in B} w(e) - \sum_{i=1}^{n} w(e_i) + \sum_{i=1}^{n} w(f_i) = w(B) - w(\tau) \qquad (4)$$

Therefore

Lemma 2 1. $w(\tau(B, B')) > 0 \Leftrightarrow B'$ *dominates* B.

2. $w(\tau(B, B')) < 0 \Leftrightarrow B$ *dominates* B'.

3. $w(\tau(B, B')) = 0 \Leftrightarrow f(B) = f(B')$.

As a corollary we state

Corollary 1 *Let* $B, B' \in \mathcal{B}, f(B) \neq f(B')$. *Then* B *and* B' *do not dominate each other if and only if* $w(\tau(B, B')) \not\geq 0$ *and* $w(\tau(B, B') \not\leq 0$.

Definition 3 1. *A basis exchange* τ *such that* $w(\tau)$ *has positive and negative components is called* plus-minus exchange *or* pme.

2. *A pme* τ *w.r.t some basis* B *such that there is no pme* τ' *w.r.t.* B *with* $w(\tau) < w(\tau')$ *is an* efficient basis exchange.

Proposition 2 *If* $B, B' \in \mathcal{B}_{par}$ *then* $\tau(B, B')$ *as well as* $\tau(B', B)$ *are efficient w.r.t* B *and* B' *respectively or* $f(B) = f(B')$.

Proof:

Suppose $f(B) \neq f(B')$. By Corollary 1 $\tau(B, B')$ and $\tau(B', B)$ are pme's. Assume the existence of a pme $\tilde{\tau}$ w.r.t B such that $w(\tilde{\tau}) > w(\tau)$. Then by (4) $w(B[\tilde{\tau}]) = w(B) - w(\tilde{\tau}) < w(B) - w(\tau) = w(B')$, contradicting $B' \in \mathcal{B}_{par}$. Analogously it is seen that $\tau(B', B)$ is efficient w.r.t B'.

□

The converse of Proposition 2 is given in Proposition 3.

Proposition 3 *If $B \in \mathcal{B}_{par}$ and τ is an efficient exchange w.r.t B then $B[\tau] \in \mathcal{B}_{par}$.*

Proof:

As τ is a pme w.r.t B $B[\tau]$ is not dominated by B due to Corollary 1. Now suppose it exists $B^* \in \mathcal{B}_{par}$ such that $f(B^*) < f(B[\tau])$ and consider $\tau' = \tau(B, B^*)$ and $\tau^* = \tau(B[\tau], B^*)$ (see Figure 1).

Figure 1 here

By Lemma 2 $w(\tau^*) > 0$. Therefore $w(\tau') = w(\tau) + w(\tau^*) > w(\tau)$. By Proposition 2 τ' is an efficient basis exchange w.r.t. B, in particular τ' is a pme. These facts contradict the choice of τ.

□

Proposition 4 is a basis-exchange property which is an extension of a theorem by Brualdi, [1]. The proof is omitted since it is a matter of matroid theory and not directly related to the topic of this paper, although Proposition 4 is often referred to in the sequel.

Proposition 4 *Let $B, B' \in \mathcal{B}$ and $B \setminus B' = \{e_1, \dots e_n\}, B' \setminus B = \{f_1, \dots f_n\}$. Then there is a permutation π of $\{1, \dots n\}$ such that*

- $B_0 := B, \quad B_i = B_{i-1} \setminus \{e_i\} \cup \{f_{\pi(i)}\} \quad i = 1, \dots n-1$ *and* $B_n = B'$ *are bases.*

- $B' \setminus \{f_{\pi(i)}\} \cup \{e_i\}$ *are bases for* $i = 1, \ldots n$.

- $B \setminus \{e_i\} \cup \{f_{\pi(i)}\}$ *are bases for* $i = 1, \ldots n$.

If B, B' are as in Proposition 4 we have that $\tau(B, B') = [\{e_1, \ldots e_n\}, \{f_1, \ldots f_n\}]$ and for $i = 1 \ldots n$ we define $\tau_i = [\{e_i\}, \{f_{\pi(i)}\}]$.

Proposition 5 *Let B, B' be as in Proposition 4 and let $J \subseteq \{1, \ldots n\}$. Define $\tau' := [\{e_i | i \in J\}, \{f_{\pi(i)} | i \in J\}]$ and $\tau'' := [\{f_{\pi(i)} | i \in J\}, \{e_i | i \in J\}]$. Then $B[\tau']$ and $B'[\tau'']$ are bases. Furthermore if $B, B' \in \mathcal{B}_{par}$ and $w(\tau') \neq 0$ then both τ' and τ'' are pme's.*

Proof:

The first part follows immediately from the properties given in Proposition 4. For the second assume the contrary. Then either $w(\tau') > 0$ or $w(\tau'') = -w(\tau') > 0$. Hence either $B[\tau']$ would dominate B or $B'[\tau'']$ would dominate B'.

□

4 Connectivity of $\mathcal{B}$ and $\mathcal{B}_{par}$

Definition 4 *1. For a matroid $\mathcal{M}$ the basisgraph $\mathcal{B}(\mathcal{M})$ is defined by its edge and node sets.*

$V(\mathcal{B}(\mathcal{M})) = \mathcal{B}$

$E(\mathcal{B}(\mathcal{M})) = \{[B, B'] | B, B' \in \mathcal{B}$ *are neighbours*$\}$

2. The paretograph $\mathcal{PB}(\mathcal{M})$ is defined by

$V(\mathcal{PB}(\mathcal{M})) = \mathcal{B}_{par}$

$E(\mathcal{PB}(\mathcal{M})) = \{[B, B'] | B, B' \in \mathcal{B}_{par}$ *are neighbours*$\}$

Therefore the paretograph is the subgraph of the basisgaph induced by $\mathcal{B}_{par}$. Immediate consequence of Proposition 4 is that $\mathcal{B}(\mathcal{M})$ is connected. Furthermore the sequence $B_0, \ldots B_n$ defines a shortest path (i.e. a path with as few edges as possible) from B to B' in $\mathcal{B}(\mathcal{M})$.

Example 1 *Consider the partition matroid on $\mathcal{E} = \{1, \ldots 6\}$ defined by the partition of $\mathcal{E}$ in $E_1 = \{1,5,6\}, E_2 = \{3\}, E_3 = \{2,4\}$ and weights as in Table 1.*

Table 1: Weights in Example 1

e	1	2	3	4	5	6
w(e)	(1,5)	(1,4)	(1,1)	(3,1)	(4,1)	(2,2)

Figure 2 shows $\mathcal{B}(\mathcal{M})$ and $\mathcal{PB}(\mathcal{M})$ where $\mathcal{PB}(\mathcal{M})$ consists of the bold face edges and the nodes incident to these edges.

Figure 2 here

The example shows that for $A, B \in \mathcal{B}_{par}$ no shortest paths from A to B in $\mathcal{B}(\mathcal{M})$ need be contained in $\mathcal{PB}(\mathcal{M})$.

Now we will once again focus on combined weights.

Lemma 3 *Let $\lambda_1, \ldots \lambda_Q \in (0,1), \sum_{q=1}^{Q} \lambda_q = 1$. Then for $B^* \in argmin\ \{h(B) | B \in \mathcal{B}\}$ it holds that $B^* \in \mathcal{B}_{par}$, where h is defined in (3).*

This result is well known, see [6]. Bases which are optimal solutions for $\min_{B \in \mathcal{B}} h(B)$ are called *proper pareto bases.* Furthermore it is known that all proper pareto bases lie on the boundary of $conv\{f(B) | B \in \mathcal{B}\}$. The part of the boundary of $conv\{f(B) | B \in \mathcal{B}\}$ which is the pareto set in the linear relaxation of MCMO is often referred to as *efficient frontier.* Proper pareto bases thus are the elements of $\mathcal{B}_{par}$ which lie on the efficient frontier.

From now on we will assume $Q = 2$ and that all pareto bases have different objective values. Let $B_1, \ldots B_K$ be all proper pareto bases ordered according to increasing $w_1(B)$. Then due to convexity the slopes of the line segments between $f(B_i)$ and $f(B_{i+1})$ are negative for all $i = 1, \ldots K-1$ and the sequence of slopes is non-decreasing.

Considering Example 1 in objective space, we see that all pareto bases are proper, see Figure 3.

Figure 3 here

Definition 5 *For $A \in \mathcal{B}$ let*

- $\mathcal{B}_A^+ = \{B \in \mathcal{B} | w_1(B) < w_1(A), w_2(B) > w_2(A)\}$
- $\mathcal{B}_A^- = \{B \in \mathcal{B} | w_1(B) > w_1(A), w_2(B) < w_2(A)\}$

Then if $A \in \mathcal{B}_{par}$ we have that $\mathcal{B}_{par} \backslash \{A\} \subseteq \mathcal{B}_A^+ \cup \mathcal{B}_A^-$.

E.g. in Example 1 $\mathcal{B}_{\{2,3,6\}}^- = \{\{2,3,5\},\{3,4,6\},\{3,4,5\}\}$.

The next result proves the connectivity of proper pareto bases in a combinatorial fashion. The result also follows from a connectivity result for basic solutions in multicriteria Linear Programming, see e.g. [9] or [12].

Proposition 6 *Let A be a proper pareto basis and $B' \in \mathcal{B}_A^-$ and $B'' \in \mathcal{B}_A^+$ such that*

$$\frac{w_2(\tau(A,B'))}{w_1(\tau(A,B'))} = \min_{B \in \mathcal{B}_A^-} \frac{w_2(\tau(A,B))}{w_1(\tau(A,B))} \quad (5)$$

$$\frac{-w_2(\tau(A,B''))}{w_1(\tau(A,B''))} = \min_{B \in \mathcal{B}_A^+} \frac{-w_2(\tau(A,B))}{w_1(\tau(A,B))} \quad (6)$$

and if the minimum is not unique such that $|A \backslash B|$ is minimal among all minimizers of (5) or (6). Then B' and B'' are proper pareto bases and neighbours of A.

Proof:

First note that if A is not one of the two lexicographical optimal bases B_1, optimal for the sequence (f_1, f_2) of objective functions, or B_2, optimal for (f_2, f_1) we have that $B_1 \in \mathcal{B}_A^+$ and $B_2 \in \mathcal{B}_A^-$. Hence both sets are nonempty. If $A = B_1$ $\mathcal{B}_A^-$ is nonempty, if $A = B_2$ $\mathcal{B}_A^+$ is nonempty and only B' respectively B'' are defined.

We prove the result for B'. For B'' similar arguments hold.

To show that B' is a proper pareto basis we have to find $\lambda \in (0,1)$ such that B' minimizes $\lambda w_1(B) + (1-\lambda)w_2(B)$. Therefore we set $\lambda = \frac{w_2(B')-w_2(A)}{w_1(A)-w_2(A)+w_2(B')-w_1(B')}$. By choice of B' it follows that $\lambda \in (0,1)$. Now suppose that for some $B \in \mathcal{B}_A^-$ we have that

$$(w_2(B') - w_2(A))\, w_1(B') + (w_1(A) - w_1(B'))\, w_2(B')$$

$$<$$

$$(w_2(B') - w_2(A))\, w_1(B) + (w_1(A) - w_1(B'))\, w_2(B)$$

(i.e. B' does not minimize $h(B)$ for the given λ) then easy calculations show that

$$\frac{w_2(A) - w_2(B')}{w_1(A) - w_1(B')} > \frac{w_2(A) - w_2(B)}{w_1(A) - w_1(B)}$$

contradicting the choice of B'.

Now if $|A \backslash B'| = n > 1$ we can apply the basis exchange property and find $\tau_1, \ldots \tau_n$ accordingly. Let τ' be the basis exchange w.r.t. A consisting of the elements of $\tau_1, \ldots \tau_{n-1}$. Considering all possible locations of $w(A[\tau'])$ in objective space we always derive contradictions (see Figure 4).

We first state that

$$s_u := \frac{w_2(\tau(B'', A))}{w_1(\tau(B'', A))} \leq \frac{w_2(\tau(A, B'))}{w_1(\tau(A, B'))} =: s_l$$

and that for all $B \in \mathcal{B}_A^-$ due to the choice of B'

$$s_l \leq \frac{w_2(\tau(A,B))}{w_1(\tau(A,B))} < 0$$

For all $B \in \mathcal{B}_A^+$ the choice of B'' implies

$$-\infty < \frac{w_2(\tau(A,B))}{w_1(\tau(A,B))} \leq s_u$$

From $s_l = \frac{w_2(\tau)}{w_1(\tau)} = \frac{w_2(\tau')+w_2(\tau_n)}{w_1(\tau')+w_1(\tau_n)}$ we have

$$s_l w_1(\tau') + s_l w_1(\tau_n) = w_2(\tau') + w_2(\tau_n) \tag{7}$$

Notice that it cannot hold that $\frac{w_2(\tau')}{w_1(\tau')} = s_l$ or that $\frac{w_2(\tau')}{w_1(\tau')} = s_u$ since $A[\tau']$ would then contradict the choice of B' or B''. This fact is used in the following arguments to derive the strict inequalities $\frac{w_2(\tau')}{w_1(\tau')} > s_l$ and $\frac{w_2(\tau')}{w_1(\tau')} < s_u$

By Proposition 5 we can distinguish two cases:

Case 1: $A[\tau'] \in \mathcal{B}_A^+ \Rightarrow A[\tau_n] \in \mathcal{B}_A^-$.

So $\frac{w_2(\tau(A[\tau'],A))}{w_1(\tau(A[\tau'],A))} = \frac{w_2(\tau')}{w_1(\tau')} < s_u \le s_l$ and $w_2(\tau') < s_l w_1(\tau')$. This and (7) imply $s_l w_1(\tau_n) < w_2(\tau_n)$ and hence $s_l > \frac{w_2(\tau_n)}{w_1(\tau_n)}$. Contradiction.

Case 2: $A[\tau'] \in \mathcal{B}_A^-$. Here we have two possibilities, because $A[\tau_n]$ may be either in $\mathcal{B}_A^-$ or in $\mathcal{B}_A^+$. First let $A[\tau_n] \in \mathcal{B}_A^-$. From $\frac{w_2(\tau')}{w_1(\tau')} > s_l$ we have $w_2(\tau') < s_l w_1(\tau')$. So (7) again implies $s_l w_1(\tau_n) < w_2(\tau_n)$ and hence $s_l > \frac{w_2(\tau_n)}{w_1(\tau_n)}$. Contradiction. Otherwise $A[\tau_n] \in \mathcal{B}_A^+$. As for $A[\tau_n] \in \mathcal{B}_A^-$: $w_2(\tau') < s_l w_1(\tau')$. In this case (7) implies $s_l w_1(\tau_n) < w_2(\tau_n)$ and $s_u \le s_l < \frac{w_2(\tau_n)}{w_1(\tau_n)}$, since $w_1(\tau_n) > 0$. Contradiction.

These contradictions imply that $n = 1$ and that B' is a neighbour of A. For the case that $A = B_1$ the cases $A[\tau'] \in \mathcal{B}_A^+$ and $A[\tau'] \in \mathcal{B}_A^-, A[\tau_n] \in \mathcal{B}_A^+$ can be dropped. The remaining case is as above and completes the proof.

□

Figure 4 here

Proposition 6 implies the connectivity of proper pareto solutions. We will show this also for the case of alternative solutions, i.e. pareto bases having the same objective value.

Theorem 4 *The set of proper pareto bases is connected.*

Proof:

In case all proper pareto solutions have different weights the theorem follows directly from Proposition 6.

In the general case we first prove that the sets of lexicographically optimal solutions are connected. Let $\mathcal{B}_1^* = \text{argmin}_{B \in \mathcal{B}}\{f_1(B)\}$ and $\mathcal{B}_2^* = \text{argmin}_{B \in \mathcal{B}}\{f_2(B)\}$. Then $\mathcal{B}_{1,2}^* := \text{argmin}_{B \in \mathcal{B}_1^*}\{f_2(B)\}$

and $\mathcal{B}^*_{2,1} := \text{argmin}_{B \in \mathcal{B}^*_2}\{f_1(B)\}$ are the sets of lexicographically optimal solutions for (f_1, f_2) and (f_2, f_1) respectively.

We show that $\mathcal{B}^*_{1,2}$ is connected. For $\mathcal{B}^*_{2,1}$ analoguous arguments hold. Let $B, B' \in \mathcal{B}^*_{1,2}$. For $\tau(B, B')$ we can define $\tau_1, \ldots \tau_n$ according to Proposition 4. It follows that $B[\tau_i] \in \mathcal{B}^+_B \cup \mathcal{B}^-_B \cup \mathcal{B}^*_{1,2}$ for $i = 1, \ldots n$ (otherwise $B[\tau_i]$ or $B'[\{f_{\pi(i)}\}, \{e_i\}]$ would dominate B and B'). But $\mathcal{B}^+_B = \emptyset$. Now suppose $B[\tau_i] \in \mathcal{B}^-_B$ for some i. Because $w(\tau(B, B')) = 0$ this would imply that $B[\tau'] \in \mathcal{B}^+_B$, where $\tau' = [\{e_j | j \neq i\}, \{f_{\pi(j)} | j \neq i\}]$. Hence we conclude $B[\tau_i] \in \mathcal{B}^*_{1,2}$. By Proposition 4 we conclude that $B_i \in \mathcal{B}^*_{1,2} \quad i = 1, \ldots n$, where B_i are defined as in Proposition 4. This implies connectivity of $\mathcal{B}^*_{1,2}$.

Now let A be a proper pareto basis, $A \notin \mathcal{B}^*_{1,2} \cup \mathcal{B}^*_{2,1}$. Proposition 6 can be applied iteratively to find proper pareto neighbours of A until bases in $\mathcal{B}^*_{1,2}$ and $\mathcal{B}^*_{2,1}$ are found. In such a way paths in $\mathcal{PB}(\mathcal{M})$ between A and these connected subgraphs of $\mathcal{PB}(\mathcal{M})$ are constructed.

□

Furthermore we note that in the case that all proper pareto solutions have different weights it follows that the cardinality of the set of proper pareto solutions is bounded by $\mathcal{O}(|E|^2)$. To see this consider again the sequence $B_1, \ldots B_K$ of all proper pareto bases as mentioned after Lemma 3. By Propositin 6 B_i and B_{i+1} are neighbours for $i = 1, \ldots K-1$. Thus they differ in one element. But there are not more than $\mathcal{O}(|E|^2)$ pairs of elements to define the non-decreasing sequence of slopes of line segments between $f(B_i)$ and $f(B_{i+1})$.

Unfortunately connectivity of proper pareto solutions is all that can be proved for pareto solutions. The paretograph $\mathcal{PB}(\mathcal{M})$ in general is not connected. A counterexample is given in [4]. We will conclude the paper with an example illustrating Theorem 4.

Example 2 *We consider the graphic matroid of the graph G of Figure 5 with edge weights as given in Table 2.*

Figure 5 here

Table 2: Weights in Example 2

e	[1,2]	[1,4]	[2,3]	[2,5]	[3,4]	[4,5]
w(e)	(0,0)	(0,0)	(2,4)	(3,7)	(3,3)	(4,6)

$\mathcal{M}(G)$ has 4 pareto bases (see Table 3), the paretograph is shown in Figure 6.

All pareto bases are proper pareto bases. The example shows that sets of bases having the same objective value are not connected in general, as can be seen by bases 2 and 3.

Figure 6 here

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Table 3: Pareto bases in Example 2

Node in $\mathcal{PB}(\mathcal{M})$	pareto basis B	f(B)
1	[1,2], [1,4], [2,3], [2,5]	(5, 11)
2	[1,2], [1,4], [2,5], [3,4]	(6, 10)
3	[1,2], [1,4], [2,3], [4,5]	(6, 10)
4	[1,2], [1,4], [3,4], [4,5]	(7, 9)

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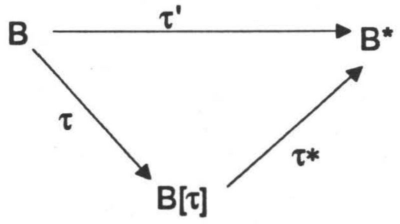

Figure 1: Proof of Proposition 3

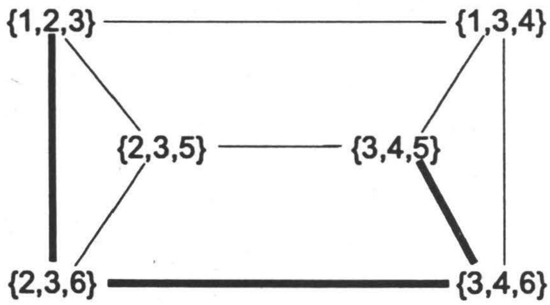

Figure 2: Illustration of Example 1

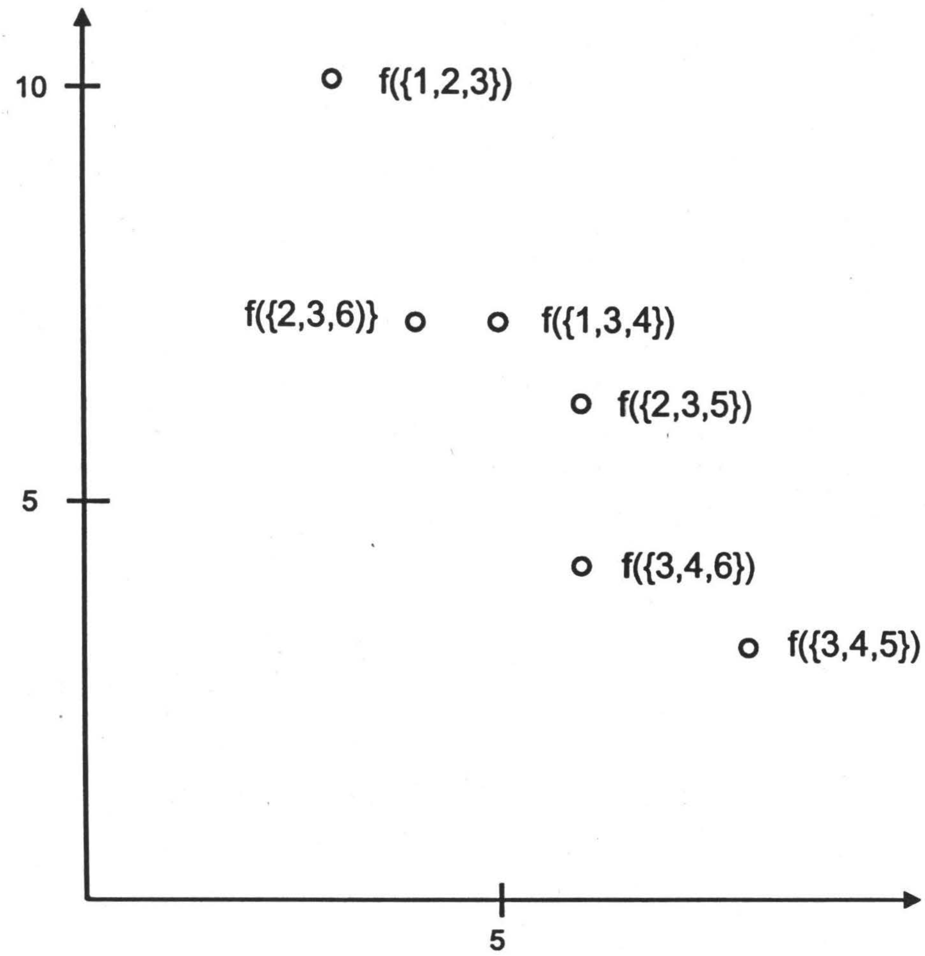

Figure 3: Example 1 in objective space

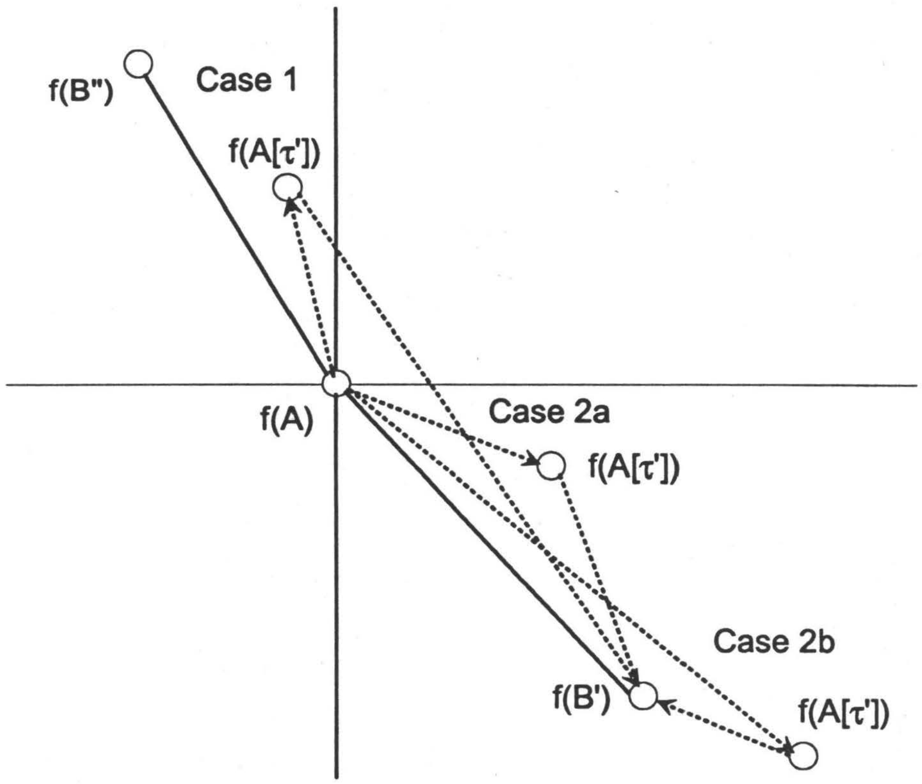

Figure 4: Proof of Proposition 6

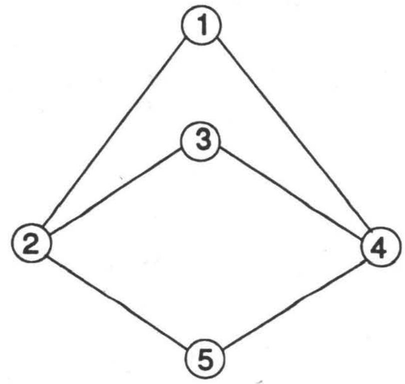

Figure 5: Graph for Example 2

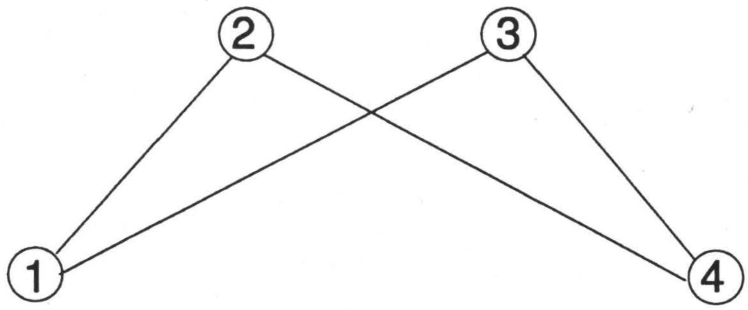

Figure 6: Pareto graph for Example 2