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Abstract

Given Q different objective functions, three types of single-facility problems are considered: Lexicographic, pareto and max ordering problems. After discussing the interrelation between the problem types, a complete characterization of lexicographic locations and some instances of pareto and max ordering locations is given. The characterizations result in efficient solution algorithms for finding these locations.

The paper relies heavily on the theory of restricted locations developed by the same authors, and can be further extended, for instance, to multi-facility problems with several objectives.

The proposed approach is more general than previously published results on multicriteria planar location problems and is particulary suited for modelling real-world problems.

Keywords: Location Theory, Multi Criteria Problems

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Contents

1 Introduction

The development of location theory has started with the optimal location of a single facility in the plane $\mathbb{R}^2$ subject to a given number of existing facilities, a problem already considered in the 17th century by Fermat. Since then, numerous publications are witness to the importance of this problem in modelling real world problems.

In this paper we investigate the single facility problem when several, in general conflicting, objective functions have to be considered. More precisely, let $Ex = \{Ex_1, \ldots, Ex_M\}$ be a set of existing locations $Ex_m = (a_{m1}, a_{m2}), m \in \mathcal{M} := \{1, \ldots, M\}$, in the plane. Each of the existing facilities has associated with it nonnegative weights $w_m^1, \ldots, w_m^Q$ representing, for instance, the frequency of transport of Q different commodities from a central warehouse of (unknown !) location $X = (x_1, x_2)$ to Ex_m. If $d(Ex_m, X)$ is the distance between Ex_m and X we consider the Q objective functions

$$f^q(X) := \sum_{i \in \mathcal{M}} w_m^q d(Ex_m, X) \quad (1.1)$$

$$g^q(X) := \max_{i \in \mathcal{M}} w_m^q d(Ex_m, X) \quad (1.2)$$

Another scenario where functions $f^q(X)$ and $g^q(X)$, $q \in \mathcal{Q} = \{1, \ldots, Q\}$ may be used as model is where each of Q decision makers is asked to give his personal view of a single facility location problem in terms of a sum or max objective by choosing "his" or "her" weights $w_m^q, m \in \mathcal{M}$.

For each $q \in \mathcal{Q}$ $\min_{X \in \mathbf{R}^2} f^q(X)$ and $\min_{X \in \mathbf{R}^2} g^q(X)$ is the well-known single objective planar median (or Weber or minisum) and center (or minmax or Weber-Rawls) problem (see, e.g., [Love et al., 1988] or [Francis et al., 1992]). We use in the following the denotation $1/P/\bullet/d/\sum$ and $1/P/\bullet/d/\max$, respectively, for these problems. Here, the five position classification for location problems

$$pos1/pos2/pos3/pos4/pos5$$

can be used to indicate the number of new facilities (pos 1), the type of problem as planar, network, discrete, etc (pos 2), any special assumptions and restrictions such as $w_m = 1$ for all $m \in \mathcal{M}$, etc. (pos 3), the type of distance function such as l_p, general d, etc (pos 4), and the type of objective function (pos 5) (see [Hamacher, 1992]).

If we know for all $q \in \mathcal{Q}$ the set $\mathcal{X}_q^*$ of all optimal locations of the median or center location problem with objective f^q or g^q we assume that

$$\bigcap_{q \in \mathcal{Q}} \mathcal{X}_q^* = \emptyset \quad (1.3)$$

This is what we mean by 'conflicting' objective functions. Without this assumption, which is very likely to be valid in most real-world situations, the multi-criteria approach reduces to a sequence of single criteria problems. In the following we always assume that condition (1.3) has been checked.

Planar location problems with multiple objectives have been considered, among others, by [Wendell et al., 1977], [Chalmet et al., 1981] and [McGinnis and White, 1978]. For an overview see [Current et al., 1990]. Parametric location problems which are closely related to multiple objective locations problems are discussed in [Brandeau and Chiu, 1991].

The aim of this paper is to give a very general approach to planar location problems with multiple objectives. In the next section we will introduce three different types of these problems, namely lexicographic, pareto and max ordering (MO) location problems, and discuss some relations between these problems. In Section 3 we develop algorithms to find all lexicographic locations for location problems with respect to l_p-distances. The next section gives a complete characterization of lexicographic, pareto and MO locations in the case where the distance function is the squared Euclidean distance. In Section 5 we discuss algorithms for finding all pareto and MO locations of bicriteria median problems with respect to rectilinear or Chebychev distances. After introducing a solution procedure for Q-criteria center problems in Section 6, we conclude the paper with a summary of results and further research topics.

2 Definitions and Basic Results

In this section we introduce three different orderings of $\mathbb{R}^Q$ which can be combined with the objective functions $f^q(X)$ and $g^q(X)$ of Section 1 resulting in different multi-criteria location problems. Moreover we discuss some relations between the resulting problems and establish a characterization for pareto locations.

In the following, let $z = (z_1, \ldots, z_Q)$, $z' = (z'_1, \ldots, z'_Q) \in \mathbb{R}^Q$.

- **Lexicographic Location Problem**

  The lex(icographical) ordering in $\mathbb{R}^Q$ is defined by

$$z \underset{lex}{\leq} z'$$
$$:\Longleftrightarrow$$
$$z = z' \quad \text{or} \quad z_i < z'_i \quad \text{for} \quad i := \min_{q \in \mathcal{Q}} \{q : z_q \neq z'_q\}$$

  If $\Pi(\mathcal{Q})$ is the set of all permutations of $\mathcal{Q} = \{1, \ldots, Q\}$, $X_\pi \in \mathbb{R}^2$ is called **lex**(icographic minimum) **location** or **lex optimal** (with respect to permutation $\pi \in \Pi(\mathcal{Q})$) if

$$\left(h^{\pi(1)}(X_\pi), \ldots, h^{\pi(Q)}(X_\pi)\right) \underset{lex}{\leq} \left(f^{\pi(1)}(Y), \ldots, h^{\pi(Q)}(Y)\right)$$

  for all $Y \in \mathbb{R}^2$. Here, h=f or h=g can be the median or center objective function introduced in Section 1, Notice that X_π is well-defined, since the lex ordering is a total ordering in $\mathbb{R}^Q$. Lex locations are useful if a preference in the Q objective functions h^q can be assumed. If this preference is known it can be modeled by choosing a specific permutation $\pi \in \Pi(\mathcal{Q})$. If the preference is not known in advance one may be interested to know the set of all lex locations which we denote by $\mathcal{X}_{lex}$.

- **Pareto Location Problem**
  Here, we compare two vectors $z, z' \in \mathbb{R}^Q$ using the componentwise ordering

  $$z \preceq z' :\Longleftrightarrow z_i \leq z'_i \quad \forall i \in \mathcal{Q}$$

  Notice that $\preceq$ defines only a partial ordering such that two vectors z and z' may not be comparable. X is called **pareto location** or **pareto optimal** if there is no $Y \in \mathbb{R}^2$ **dominating** X, i.e. satisfying

  $$h(Y) := (h^1(Y), \ldots, h^Q(Y)) \prec (h^1(X), \ldots, h^Q(X)) =: h(X)\,.$$

  Here $h(Y) \prec h(X)$ means $h(Y) \preceq h(X)$ and $h^q(Y) < h^q(X)$ for at least one $q \in \mathcal{Q}$. A pareto location X is one where an improvement with respect to one of the objective functions h^q can only be obtained by worsening one of the other objective functions h^p. Therefore, the set of all pareto locations, denoted $\mathcal{X}_{par}$, is a set of locations which are "good" candidates for placing a new facility.

- **MO Location Problems**
  Finally, we consider the max ordering in $\mathbb{R}^Q$ defined by

  $$z \underset{max}{\leq} z' \quad :\Longleftrightarrow \quad \max_{q\in\mathcal{Q}}\{z_q\} \leq \max_{q\in\mathcal{Q}}\{z'_q\}$$

  The max ordering is a total one. We call any minimizer X of $h(X) := (h^1(X), \ldots, h^Q(X))$ with respect to the max ordering, i.e. of $\max_{q\in\mathcal{Q}}\{h^q(X)$ an **MO location** or **MO optimal** and denote the set of all MO locations with $\mathcal{X}_{MO}$. MO location problems are good models if all objective functions have equal preference or if the importance of the different objective functions is not known in advance (see [Chung et al., 1993]). They can also be applied to combine equity measures with efficiency considerations ([Erkut, 1993], [Marsh and Schilling, 1993]).

Depending on whether the single objective functions h^q are all of the median type f^q or of the center type g^q, or $h^q = f^q$ for some $q \in \mathcal{Q}$ while $h^p = g^p$ for some other $p \in \mathcal{Q}$, we call the corresponding problems **multicriteria median, center**, or **mixed location problem** (see Figure 2.1).

In our scheme presented in Figure 2.1 the problems in [Chalmet et al., 1981] and [Wendell et al., 1977] are special cases of the type $1/P/\cdot/d/Q\text{-}\sum_{par}$ or $1/P/\cdot/d/Q\text{-}\sum_{par}$ with $M = Q$ and

$$w^q_m = \begin{cases} 1 & \text{if} \quad m = q \\ 0 & \text{else} \end{cases}\,.$$

The paper of [McGinnis and White, 1978] discusses a problem of the type $1/P/\cdot/d/2\text{-}(\sum, \max)_{par}$, where $w^1_m = v^2_m$ for all $m \in \mathcal{M}$.

The first result of this section establishes two relations between the three types of multicriteria location problems.

	$h^q = f^q, \forall q$	$h^q = g^q, \forall q$	$h^q \in \{f^q, g^q\}$
lex ordering	$1/P/\cdot/d/Q\text{-}\sum_{lex}$	$1/P/\cdot/d/Q\text{-}\max_{lex}$	$1/P/\cdot/d/Q\text{-}(\sum, \max)_{lex}$
componentwise ordering	$1/P/\cdot/d/Q\text{-}\sum_{par}$	$1/P/\cdot/d/Q\text{-}\max_{par}$	$1/P/\cdot/d/Q\text{-}(\sum, \max)_{par}$
max ordering	$1/P/\cdot/d/Q\text{-}\sum_{MO}$	$1/P/\cdot/d/Q\text{-}\max_{MO}$	$1/P/\cdot/d/Q\text{-}(\sum, \max)_{MO}$

Figure 2.1: **Classification scheme for multicriterial Location Problems**

Lemma 2.1.

1. *Any lex location is pareto optimal, i.e.,* $\mathcal{X}_{lex} \subseteq \mathcal{X}_{par}$.

2. *There is at least one pareto location X such that X is MO optimal, i.e.,* $\mathcal{X}_{par} \cap \mathcal{X}_{MO} \neq \emptyset$.

Proof. : 1. Suppose $\pi \in \Pi(Q)$ such that $X_\pi \in \mathcal{X}_{lex}$ is dominated by $Y \in \mathbb{R}^2$. Then $h^q(Y) \leq h^q(X_\pi)$, $\forall q \in Q$ and $h^p(Y) < h^p(X_\pi)$ for at least one $p \in Q$. Consequently,

$$h^{\pi(1)}(Y), \ldots, h^{\pi(Q)}(Y)) \underset{lex}{<} f^{\pi(1)}(X_\pi), \ldots, h^{\pi(Q)}(X_\pi))$$

contradicting the lex optimality of X_π.

2. Let $X \in \mathcal{X}_{MO}$ be dominated by $Y \in \mathcal{X}_{par}$. Then $h^q(Y) \leq h^q(X), \forall q \in Q$ implies

$$\max_{q \in Q} h^q(Y) = h^p(Y) \leq h^p(X) \leq \max_{q \in Q} h^q(X)$$

such that Y is also an MO location. Hence $Y \in \mathcal{X}_{par} \cap \mathcal{X}_{MO}$.

□

Lemma 2.1 is illustrated in Figure 2.2.

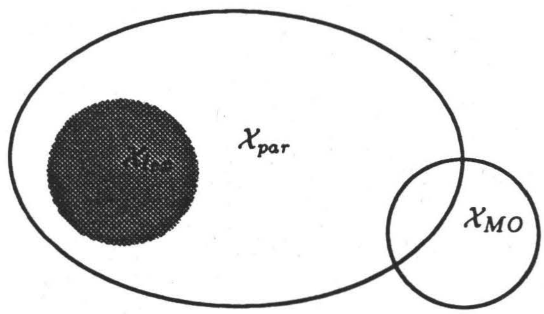

Figure 2.2: **Venn-Diagram illustrating the relation between the different multi-criteria location problems.**

Next, a characterization of pareto locations is given using level curves

$$L^q_=(z) := \{X \in \mathbb{R}^2 \quad : \quad h^q(X) = z\}$$

and level sets

$$L^q_\leq(z) := \{X \in \mathbb{R}^2 \quad : \quad h^q(X) \leq z\}$$

Theorem 2.2. *Let $X \in \mathbb{R}^2$ and $h^q(X) =: z_q$ for $q \in \mathcal{Q}$. Then the following holds:*

$$X \quad \text{is pareto location}$$

$$\Longleftrightarrow$$

$$\bigcap_{q \in \mathcal{Q}} L^q_\leq(z_q) \quad = \quad \bigcap_{q \in \mathcal{Q}} L^q_=(z_q)$$

Proof.

$$\begin{aligned}
X \quad \text{is pareto location} \quad &\Leftrightarrow \quad \nexists Y \in \mathbb{R}^2 \ : \ h(Y) \prec h(X) \\
&\Leftrightarrow \quad \nexists Y \in \mathbb{R}^2 \ : \ h^q(Y) \leq h^q(X) \text{ for all } q \in \mathcal{Q} \text{ and} \\
&\qquad h^p(Y) < h^p(X) \text{ for at least one } p \in \mathcal{Q} \\
&\Leftrightarrow \quad \nexists Y \in \mathbb{R}^2 \ : \ Y \in \bigcap_{q \in \mathcal{Q}} L^q_\leq(z_q) \text{ and} \\
&\qquad Y \in int\left(L^p_\leq(z_p)\right) \text{ for at least one } p \in \mathcal{Q} \\
&\Leftrightarrow \quad \bigcap_{q \in \mathcal{Q}} L^q_\leq(z_q) = \bigcap_{q \in \mathcal{Q}} L^q_=(z_q)
\end{aligned}$$

□

In the rest of the paper we consider single facility location problems where the distance function $d(Ex_m, X)$ for $Ex_m = (a_{m1}, a_{m2})$ and $X = (x_1, x_2)$ is given by the p-norm

$$l_p(Ex_m, X) = \begin{cases} (|a_{m1} - x_1|^p + ||a_{m2} - x_2|^p)^{1/p} & for \quad 1 \leq p < \infty \\ max\{|a_{m1} - x_1|, |a_{m2} - x_2|\} & for \quad p = \infty \end{cases}$$

An important tool in this and the following sections is the solution of restricted location problems defined as follows:

Let $\mathcal{R}$ be a subset of $\mathbb{R}^2$ with boundary $\partial\mathcal{R}$ and let $\mathcal{F} := (\mathbb{R}^2 \backslash \mathcal{R}) \cup \partial\mathcal{R}$ be the feasible set for placing a new facility. Then

$$\min_{X \in \mathcal{F}} f(X) \text{ and } \min_{X \in \mathcal{F}} g(X)$$

is a **restricted median** and **center** problem, denoted by $1/P/\mathcal{R}/l_p/\sum$ and $1/P/\mathcal{R}/l_p/\max$, respectively. Restricted location problems have various applications among others in the design of production processes, environmental planning and the placement of emergency facilities. Efficient solution procedures for solving these problems have been described in [Hamacher and Nickel, 1991] and [Hamacher and Nickel, 1992]. These algorithms are partially implemented in [Nickel and Hamacher, 1992].

3 Lex Location Problems

The definition of the lex ordering implies the validity of the following algorithm which finds all lex locations by solving sequences of restricted location problems:

Algorithm 3.1. *Solving $1/P/\bullet/Q$-$(\sum, \max)_{lex}$*

1. *For every* $\pi \in \Pi(\mathcal{Q})$ *do*
2. $\mathcal{X}_\pi := \mathbb{R}^2$
3. *For* $j := \pi(1)$ *to* $\pi(Q)$ *do*
4. *Solve* $\min_{X \in \mathcal{X}_\pi} h^j(X)$ *and set* $Y := \operatorname{argmin}_{X \in \mathcal{X}_\pi} h^j(X)$
5. $\mathcal{X}_\pi := \mathcal{X}_\pi \cap Y$
6. *If* $|X_\pi| = 1$ *then goto Step* 8.
7. *End*
8. *End*
9. **Output:** $\mathcal{X}_{lex} := \cup_{\pi \in \Pi(\mathcal{Q})} \mathcal{X}_\pi$.

In fact, the algorithm is valid for any distance function d. But since the restricted location problem in Step 4 can only be solved efficiently for $d = l_p$ we only consider this case.

Obviously, Algorithm 3.1 is in general non-polynomial since $Q!$ permutations have to be investigated. But we shall develop polynomial algorithms in the following.

3.1 Lex Median Problems with l_p-Distances for $1 < p < \infty$

We distinguish the case where the Ex_m, $m \in \mathcal{M}$, are colinear (i.e. there exists a line L in $\mathbb{R}^2$ such that $Ex_m \in L, \forall m \in \mathcal{M}$) and not.

Theorem 3.1. *If $1 < p < \infty$ and the Ex_m $m \in \mathcal{M}$ are not colinear, then the optimal location of the single facility, single criterion median problem $1/P/\bullet/l_p/\sum$ is unique.*

Proof. The proof is done by analyzing the proof for the convexity of the l_p-norms. In this convexity proof the Hölder-inquality is used and under the assumptions of Theorem 3.1 the Hölder-inequality is strict. So the objective function is strictly convex and has therefore a unique optimum. Since $f(X)$ is bounded below by 0 the proof is completed.
For the properties of the Hölder-inequality see [Hardy et al., 1967].

□

By Theorem 3.1 and Step 6 of Algorithm 3.1 $\mathcal{X}_{lex}$ consists in this case just of the single criterion optimal locations.

Algorithm 3.2.

Solving $1/P/\bullet/l_p/Q\text{-}\sum_{lex}$ for $1 < p < \infty$ and non-colinear Ex_m, $m \in \mathcal{M}$

1. *For* $q = 1$ *to* Q *do*
2. $\mathcal{X}_j^* = \underset{X \in \mathbf{R}^2}{\operatorname{argmin}} f^q(X)$.
3. *End*
4. **Output:** $\mathcal{X}_{lex} = \cup_{j \in \mathcal{Q}} \mathcal{X}_j^*$.

The complexity of the algorithm is Q times the complexity for solving a single criterion problem $1/P/\bullet/l_p/\sum$.

Next, we consider the (from a practical point of view) unlikely case that all Ex_m, $m \in \mathcal{M}$, are on one line. Without loss of generality we assume that this line is the x-axis, such that $Ex_m = (a_{m_1}, 0)$, $\forall m \in \mathcal{M}$.

The convexity of the objective function implies that the set of all optimal locations of $1/P/\bullet/l_p/\sum$ with respect to objective function f^q is for all $q \in \mathcal{Q}$ a (possibly degenerated) interval $\mathcal{X}_q^* =: [l_q, r_q]$.

Theorem 3.2. *In $1/P/\bullet/l_p/\sum_{lex}$ with $1 < p < \infty$ and $Ex_m = (a_{m_1}, 0)$, $m \in \mathcal{M}$, the set of lex locations is*

$$\mathcal{X}_{lex} = L_r \cup R_l$$

where

$$r := \min_{q \in \mathcal{Q}} r_q \quad , \quad l := \max_{q \in \mathcal{Q}} l_q$$

$$L_r := \{l_q : q \in \mathcal{Q}, l_q \geq r\}$$

and

$$R_l := \{r_q : q \in \mathcal{Q}, r_q \leq l\} \; .$$

Proof. Since Algorithm 3.1 computes $\mathcal{X}_{lex}$, the convexity of the objective function f^q implies that in each step of the algorithm

$$X_\pi = [l_p, r_q] \quad \text{for some } p, q \in \mathcal{Q} \text{ , or} \tag{3.1}$$

$$X_\pi = \{l_q\} \quad \text{for some } q \in \mathcal{Q} \text{ with } l_q \geq r \text{ , or} \tag{3.2}$$

$$X_\pi = \{r_q\} \quad \text{for some } q \in \mathcal{Q} \text{ with} r_q \leq l \; . \tag{3.3}$$

By our general assumption (1.3)

$$\bigcap_{q \in \mathcal{Q}} \mathcal{X}_q^* = \bigcap_{q \in \mathcal{Q}} [l_q, r_q] = \emptyset$$

such that ea^h iteration of the algorithm will end with one of the cases (3.2) or (3.3). Hence

$$\mathcal{X}_{lex} \subseteq L_r \cup R_l \; .$$

If, on the other hand, $q \in \mathcal{Q}$ with $l_q \geq r = r_p$, then we consider the permutation π with $\pi(1) = q$ and $\pi(2) = p$.

Using again the convexity of f^q we obtain in the second iteration of Algorithm 3.1 $\mathcal{X}_\pi = \{l_q\}$. (for instance $\pi(1) = q = 3$ and $\pi(2) = p = 1$ in Figure 3.1).

Correspondingly, $r_q < l = l_{\tilde{p}}$ yields with $\tilde{\pi}(1) = q$, $\tilde{\pi}(2) = \tilde{p}$ $\mathcal{X}_{\tilde{\pi}} = \{r_q\}$ in the second iteration of Algorithm 3.1. (see, $\tilde{\pi}(1) = q = 2$, $\tilde{\pi}(2) = \tilde{p} = 5$ in Figure 3.1). Hence

$$\mathcal{X}_{lex} \supseteq L_r \cup R_l \,.$$

and the theorem is proved.

□

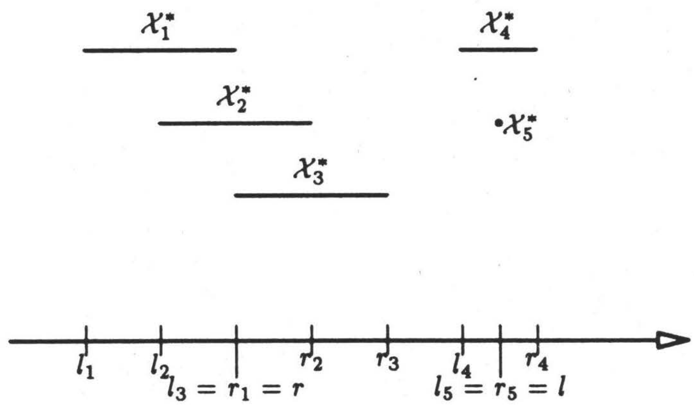

Figure 3.1: $\mathcal{X}_q^* = [l_q, r_q]$ as set of optimal locations with respect to objectives f^q, $q \in \{1, 2, 3, 4, 5\}$. The lex locations are $\{l_3, l_4, l_5\}$ and $\{r_1, r_2, r_3, r_5\}$.

3.2 Lex Median Problems with l_1 or l_∞-Distances

Using the well-known (see, e.g. [Francis et al., 1992]) transformation

$$T(y_1, y_2) = \frac{1}{2}(y_1 + y_2, -y_1 + y_2)$$

we get $l_\infty(Ex_m, X) = l_1(T(Ex_m), T(X))$ such that we can concentrate on the case of the rectilinear distance function l_1.

Lex locations can be computed using similar arguments as in the (proof of) Theorem 3.2. The single criterion optimal locations consist for each $q \in \mathcal{Q}$ of a (possibly degenerated) rectangle

$$\mathcal{X}_q^* = [l_q, r_q] \times [b_q, a_q] \,.$$

If these rectangles overlap vertically (Figure 3.2) or horizontally (Figure 3.3) one of the coordinates is characterized by the overlapping region while the other is defined analogous to Theorem 3.2. In the case where the sets $\mathcal{X}_q^*$, $q \in \mathcal{Q}$, do not overlap horizontally nor vertically, $\mathcal{X}_{lex}$ consists only of corner points and projections of corner points of $\mathcal{X}_q^*$, $q \in \mathcal{Q}$ (see Figure 3.4).

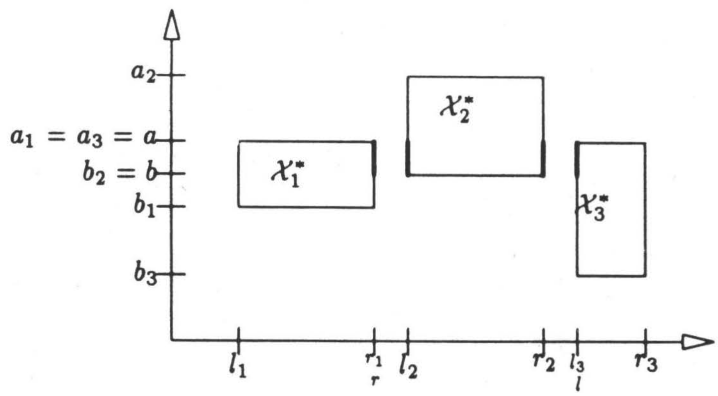

Figure 3.2: Solutions for $1/P/\bullet/l_1/\sum_{lex}$ if $b < a$

Algorithm 3.3. *Solving $1/P/\bullet/l_1/Q\text{-}\sum_{lex}$*

1. *For $q = 1, \ldots, Q$ compute the sets $\mathcal{X}_q^* = [l_q, r_q] \times [b_q, a_q]$ of optimal locations of the single criterion median problem $1/P/\bullet/l_1/\sum$ with objective function f^q.*

2. *If*

$$\bigcap_{q \in Q} \mathcal{X}_q^* \neq \emptyset$$

then **output:**

$$\mathcal{X}_{lex} = \mathcal{X}_{par} = \bigcap_{q \in Q} \mathcal{X}_q^*$$

3. *Compute $r := \min_{q \in Q} r_q$, $l := \max_{q \in Q} l_q$, $a := \min_{q \in Q} a_q$, $b := \max_{q \in Q} b_q$ and $L_r := \{l_q \; : \; q \in Q, l_q \geq r\}$, $R_l := \{r_q \; : \; q \in Q, r_q \leq l\}$, $B_a := \{b_q \; : \; q \in Q, b_q \geq a\}$, $A_b := \{a_q \; : \; q \in Q, a_q \leq b\}$.*

4. (a) *If $b < a$ then*
   output: $\mathcal{X}_{lex} = \{(x_1, x_2) \; : \; x_1 \in L_r \cup R_l, x_2 \in [b, a]\}$.

   (b) *If $l < r$ then*
   output: $\mathcal{X}_{lex} = \{(x_1, x_2) \; : \; x_1 \in [l, r], x_2 \in B_a \cup A_b\}$.

   (c) *If $b \geq a$ and $l \geq r$ then*
   output:

$$\mathcal{X}_{lex} = \bigcup_{q \in Q} (\{(x_1, x_2) \; : \; x_1 \in L_r \cup R_l, x_2 \in B_a \cup A_b | \; (x_1, x_2) \text{ is corner point of some } \mathcal{X}_q^* \text{ or } (x_1, x_2) \text{ is the projection of a corner point of some } \mathcal{X}_q^*.\})$$

   (see Figure 3.2,3.3,3.4, respectively).

The algorithm is the result of a detailed analysis of the restricted location problems (see [Hamacher and Nickel, 1991], [Hamacher and Nickel, 1992]) that have to be solved in Step 4 of Algorithm 3.1 for finding the lex locations for π with $\pi(1) = q$, $\forall q \in Q$. The same arguments as in the proof of Theorem 3.2 are used to get the description of the lex locations presented in Algorithm 3.3.

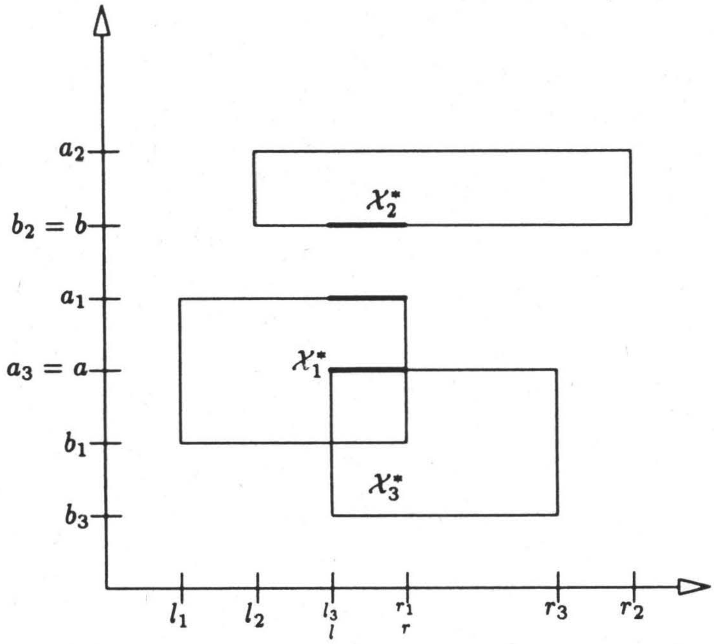

Figure 3.3: Solutions for $1/P/\bullet/l_1/\sum_{lex}$ if $l < r$

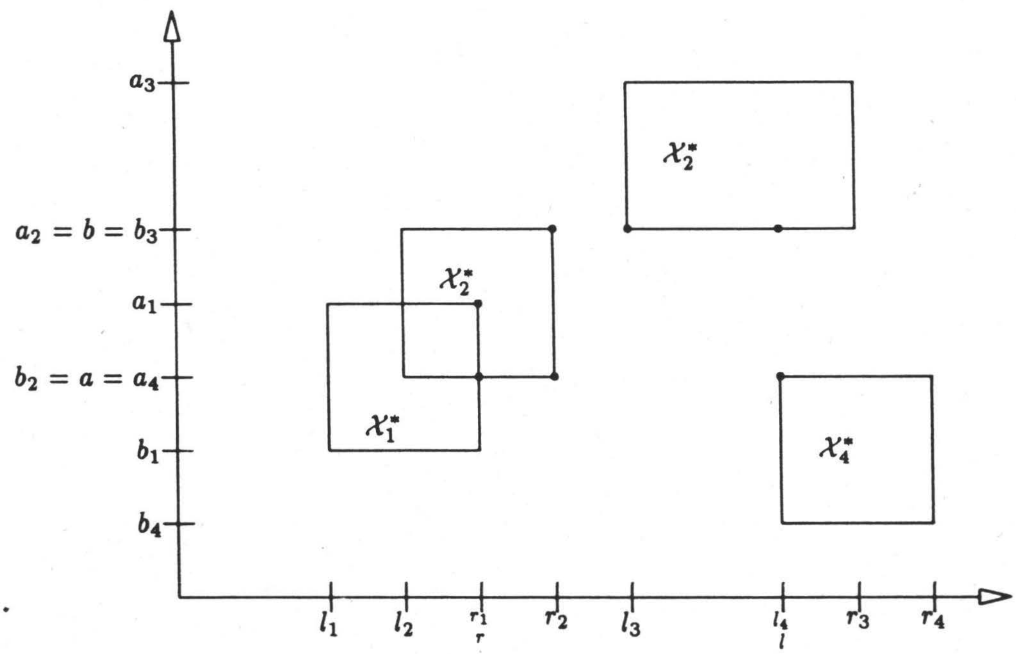

Figure 3.4: Solutions for $1/P/\bullet/l_1/\sum_{lex}$ if $l \geq r$ and $b \geq a$

3.3 Lex Center Problems with l_1 or l_∞-Distances

As in the last subsection we use the Transformation T and therefore only have to treat the l_∞ case explicitely.

First note that the single criterion optimal locations consists for each $q \in \mathcal{Q}$ of a horizontal or vertical line (possibly degenerated), i.e. $\mathcal{X}_q^* = [l_q, r_q] \times [b_q, a_q]$, with $l_q = r_q$ and/or $b_q = a_q$.

Figure 3.5: The possible arrangements of the solutions for $1/P/\bullet/l_1/\sum_{lex}$

Algorithm 3.4. *Solving $1/P/\bullet/l_\infty/\mathcal{Q}$-$\max_{lex}$*

1. *If*

$$\bigcap_{q \in \mathcal{Q}} \mathcal{X}_q^* \neq \emptyset$$

then **output:**

$$\mathcal{X}_{lex} = \mathcal{X}_{par} = \bigcap_{q \in \mathcal{Q}} \mathcal{X}_q^*$$

2. $\mathcal{X}_{lex} := \emptyset$
3. $q := 1$
4. *Compute* $\mathcal{X}_q^* = [l_q, r_q] \times [b_q, a_q]$ *, the set of optimal locations of the single criterion center problem* $1/P/\bullet/l_\infty/\max$ *with objective function* g^q.
5. *If the optimal location is unique then* $\mathcal{X}_{lex} := \mathcal{X}_{lex} \cup \{(l_q, b_q)\}$, $q := q + 1$, *goto Step 4*
6. *Else*

   (a) *Solve for all* $p \neq q$, $p \in \mathcal{Q}$ *the restricted location problem*

$$\min_{X \in \mathcal{X}_p^*} g^p(X)$$

   and get subsegments $[s_p, e_p]$ *of* $\mathcal{X}_q^*$.

   (b) *if*

$$\bigcap_{p \in \mathcal{Q} \setminus \{q\}} [s_p, e_p] \neq \emptyset$$

   then

$$\mathcal{X}_{lex} := \mathcal{X}_{lex} \cup \bigcap_{p \in \mathcal{Q} \setminus \{q\}} [s_p, e_p]$$

   (c) *else compute* $s := \max_{p \in \mathcal{Q} \setminus \{q\}} s_p$, $e := \min_{p \in \mathcal{Q} \setminus \{q\}} e_p$ *and* $S_e := \{s_p : p \in \mathcal{Q} \setminus \{q\} : s_p \geq e\}$. $E_s := \{e_p : p \in \mathcal{Q} \setminus \{q\} : e_p \geq s\}$.

      i. *if* $l_q = r_q$ *then*

$$\mathcal{X}_{lex} := \mathcal{X}_{lex} \cup \{(x_1, x_2) : x_1 = l_q, \ x_2 \in S_e \cup E_s\}$$

ii. *if* $a_q = b_q$ *then*

$$\mathcal{X}_{lex} := \mathcal{X}_{lex} \cup \{(x_1, x_2) \ : \ x_1 \in S_e \cup E_s, \ x_2 = a_q\}$$

7. *if* $q = Q$ *then*
   output: $\mathcal{X}_{lex}$

8. $q := q + 1$, *goto Step 4*

The validity proof of Algorithm 3.4 is analogous to the previous ones using the results of restricted location theory ([Hamacher and Nickel, 1991], [Hamacher and Nickel, 1992]) and is therefore omitted here.

3.4 Lex Center Problems with l_p-Distances for $1 < p < \infty$

Since the following theorem (see [Pelegrin et al., 1985]) holds, we can proceed as in the nonlinear case of Section 3.1 and use Algorithm 3.2.

Theorem 3.3. *If $1 < p < \infty$, then the optimal location of the single facility, single criterion center problem $1/P/\bullet/l_p/\max$ is unique.*

4 Multicriteria Median Problems with l_2^2-Distance

In this section we consider objective functions

$$f^q(X) = \sum_{m \in \mathcal{M}} w_m^q l_2^2(Ex_m, X) \tag{4.1}$$

where

$$l_2^2(Ex_m, X) = (a_{m_1} - x_1)^2 + (a_{m_2} - x_2)^2 \tag{4.2}$$

The following results are well-known (e.g. [Francis et al., 1992] or [Love et al., 1988]) for single-criterion problems.

Lemma 4.1. *1. The optimal location $X^* \in \mathbb{R}^2$ is uniquely defined, where*

(a) $X^* \in int(Conv\{Ex_1, \ldots, Ex_M\})$ if $w_m > 0, \forall m \in \mathcal{M}$ and

(b) $X^* \in Conv\{Ex_1, \ldots, Ex_M\}$ if $w_m \geq 0, \forall m \in \mathcal{M}$.

2. The level curve $L_=(z)$ is a circle centred $X^* = (x_1^*, x_2^*)$ with radius

$$r = \sqrt{\frac{z}{\sum_{m \in \mathcal{M}} w_m} - \frac{\sum_{m \in \mathcal{M}} w_m (a_{m_1}^2 + a_{m_2}^2)}{\sum_{m \in \mathcal{M}} w_m} + x_1^{*2} + x_2^{*2}}$$

4.1 Lex and Pareto Median Problems with l_2^2-Distance

An immediate consequence of Lemma 4.1 and Algorithm 3.4 is the following result.

Theorem 4.2.

$$\mathcal{X}_{lex} = \{X_1^*, \ldots, X_Q^*\} .$$

Next, we give a complete characterization of $\mathcal{X}_{par}$ as convex hull of $\{X_1^*, \ldots, X_Q^*\}$.

For that purpose we use the following scalarization of $1/P/\bullet/l_2^2/Q\text{-}\sum_{par}$ defined for all $\lambda \in \mathbb{R}^Q$ with $\lambda_q \geq 0 \ \forall q \in \mathcal{Q}$, $\sum \lambda_q = 1$.

$$S(\lambda) = \min_{X \in \mathbb{R}^2} f_\lambda(X) , \tag{4.3}$$

where $f_\lambda(X) := \sum_{q \in \mathcal{Q}} \lambda_q f^q(X)$.

The pareto locations are characterized using $S(\lambda)$ and results of [Geoffrion, 1968]. Before we state them we have to define a location $X \in \mathbb{R}^2$ as **proper pareto location** if it is a pareto location and if there exists some $k > 0$, such that for each $i, j \in \mathcal{Q}$ and $Y \in \mathbb{R}$

$$f^j(X) < f^j(Y) \text{ and} f^i(X) > f^i(Y)$$

implies

$$\frac{f_i(X) - f_i(Y)}{f_j(Y) - f_j(X)} \leq k .$$

$\mathcal{X}_{pro}$ is the set of all proper pareto locations.

Theorem 4.3.

1. $X \in \mathcal{X}_{pro} \iff X$ *solves* $S(\lambda)$ *for some* λ *with* $\lambda_q > 0, \forall q \in \mathcal{Q}$.
2. $X \in \mathcal{X}_{par} \implies X$ *solves* $S(\lambda)$ *for some* $\lambda \neq 0$.

We use Theorem 4.3 to show that $\mathcal{X}_{par} \subseteq Conv\{X_1^*, \ldots, X_q^*\}$. To that aim we prove that $S(\lambda)$ is equivalent to a 1-facility problem $1/P/\bullet/l_2^2/\sum$ where the set of existing facilities is $\{X_1^*, \ldots, X_Q^*\}$.

Lemma 4.4. *Let* $\lambda \in \mathbb{R}^Q$, *with* $\lambda_q \geq 0 \ \forall q \in \mathcal{Q}, \sum \lambda_q = 1$ *and*

$$S'(\lambda) := \min_{X \in \mathbb{R}^2} f'_\lambda(X)$$

where

$$f'_\lambda(X) := \sum_{q \in \mathcal{Q}} \left(\lambda_q \left(\sum_{m \in \mathcal{M}} w_m^q \right) l_2^2(X_q^*, X) \right) .$$

Then $S'(\lambda)$ *and* $S(\lambda)$ *have the same minimum i.e.,*

$$\operatorname{argmin} f'_\lambda(X) = \operatorname{argmin} f_\lambda(X) .$$

Proof.

By Lemma 4.1(2.) we get for all $q \in \mathcal{Q}$

$$f^q(X) = r_q^2(X) \sum_{m\in\mathcal{M}} w_m^q - x_{q_1}^* \sum_{m\in\mathcal{M}} w_m^q - x_{q_2}^* \sum_{i\in\mathcal{M}} w_m^q + \sum_{m\in\mathcal{M}} w_m^q(a_{m_1}^2 + a_{m_2}^2)\ ,$$

where $r_q(X) = l_2(X_q^*, X)$.
Therefore

$$f^q(X) - f^q(Y) = (\sum_{m\in\mathcal{M}} w_m^q) \left(l_2^2(X_q^*, X) - l_2^2(X_q^*, Y)\right)$$

and

$$\begin{aligned} f_\lambda(X) - f_\lambda(Y) & \\ &= \sum_{q\in\mathcal{Q}} \lambda_q(f^q(X) - f^q(Y)) \\ &= \sum_{q\in\mathcal{Q}} \lambda_q(\sum_{m\in\mathcal{M}} w_m^q) \left(l_2^2(X_q^*, X) - l_2^2(X_q^*, Y)\right) \\ &= f'_\lambda(X) - f'_\lambda(Y) \end{aligned}$$

□

Now we can state the main result of this section:

Theorem 4.5.

a) $\mathcal{X}_{pro} = int(Conv\{X_1^*, \ldots, X_Q^*\})$.

b) $\mathcal{X}_{par} = Conv\{X_1^*, \ldots, X_Q^*\}$.

Proof.

a)

$X \in \mathcal{X}_{pro} \iff X$ minimizes $S(\lambda)$ for some λ with $\lambda_q > 0\ \forall q \in \mathcal{Q}$ (Theorem 4.3)
$\iff X$ minimizes $S'(\lambda)$ (Lemma 4.4)
$\iff X \in Conv\{X_1^*, \ldots, X_Q^*\}$ (Lemma 4.1)

b)

$\mathcal{X}_{par} \subseteq Conv\{X_1^*, \ldots, X_Q^*\}$ follows as part a) where due to $\lambda \geq 0$ only one direction holds.

Since the level curves are circles (Lemma 4.1), any point $X \in \partial Conv\{X_1^*, \ldots, X_Q^*\}$ can be represented as unique intersection of two of these level curves. Henve $\partial Conv\{X_1^*, \ldots, X_Q^*\} \subseteq \mathcal{X}_{par}$ by Theorem 2.2. Since $\mathcal{X}_{pro} \subseteq \mathcal{X}_{par}$ b) follows from a)

□

Since the single criteria optima for the location problem can be computed analytically we only have to determine the convex hull of a given set of Q points in the plane. This can be done in $O(Q \log Q)$ time.

Theorem 4.6. $\mathcal{X}_{par} = \{X \in \mathbb{R}^2 \ : \ X$ *is the unique intersection of circles centered at* $X_1^*, \ldots, X_Q^*$, *respectively* $\}$

Proof. Follows immediately form Theorem 2.2 and Theorem 4.4.

□

Theorem 4.6 can be used for an alternative proof of Theorem 4.5 b) using geometric arguments.

4.2 The MO Median Problem with l_2^2-Distance

Using Lemma 2.1 (2) and Theorem 4.5 (b) we can solve the MO problem $1/P/\cdot/l_2^2/Q\text{-}\sum_{MO}$ by computing

$$\min_{X \in Conv\{X_1^*,\ldots,X_Q^*\}} \max_{q \in \mathcal{Q}} f^q(X) .$$

By Lemma 4.1 (2) we can rewrite

$$\begin{aligned} f^q(X) = z &= (\sum_{m \in \mathcal{M}} l_2^2(X, X_q^*) + \sum_{m \in \mathcal{M}} w_m^q (a_{m_1}^2 + a_{m_2}^2) + (\sum_{m \in \mathcal{M}} w_m^q)(x_{q_1} + x_{q_2}) \\ &= \tilde{w}_q l_2^2(X, X_q^*) + h_q , \end{aligned}$$

where $X_q^* := (x_{q_1}, x_{q_2})$, $\tilde{w}_q := \sum_{m \in \mathcal{M}} w_m^q$ and $h_q = \sum_{m \in \mathcal{M}} w_m^q (a_{m_1}^2 + a_{m_2}^2) + (\sum_{m \in \mathcal{M}} w_m^q)(x_{q_1} + x_{q_2})$.

The MO location problem is therefore equivalent to a center problem where the existing locations are $X_1^*, \ldots, X_Q^*$ and where the weighted distances to the $X_1^*, \ldots, X_Q^*$ also include an additional constant $h_1, \ldots, h_Q$, respectively. Problems of this type can be solved for l_1 and l_∞-distances (see [Francis et al., 1992]). We are not aware of a solution method for squared Euclidian distances.

5 Bicriterial Problems

Now we turn to problems of finding pareto and MO locations with respect to two criteria where we can make extensive use of Theorem 2.2. First we characterize a region containing the set $\mathcal{X}_{par}$.

Let $\mathcal{X}_{1,2}$ and $\mathcal{X}_{2,1}$ be the set of lex locations with respect to permutation $\pi(1) = 1$, $\pi(2) = 2$ and $\pi'(1) = 2$, $\pi'(2) = 1$, respectively. Then $\mathcal{X}_{lex} = \mathcal{X}_{1,2} \cup \mathcal{X}_{2,1}$.

Theorem 5.1. *Let* $X_1 \in \mathcal{X}_{1,2}$, $X_2 \in \mathcal{X}_{2,1}$. *Then*

$$\mathcal{X}_{par} \subseteq \mathcal{O}_{par} := L_{\leq}^1(h^1(X_2)) \cap L_{\leq}^2(h^2(X_1)) .$$

Proof. We show that there is no efficient $X \in \mathbb{R}^2 \backslash \mathcal{O}_{par}$.

1. $X \in (\mathcal{X}_1^* \backslash \mathcal{X}_{1,2})$ and $X \in (\mathcal{X}_2^* \backslash \mathcal{X}_{2,1})$ are dominated by X_1 and X_2, respectively, and are therefore not efficient.

2. Let $X \in \mathbb{R}^2 \setminus (\mathcal{O}_{par} \cup (\mathcal{X}_1^* \cup \mathcal{X}_2^*))$

   (a) For $X \notin L^1_{\leq}(h^1(X_2))$ it follows that $X_2 \in int(L^1_{\leq}(h^1(X))$. It also holds that $X_2 \in int(L^2_{\leq}(h^1(X)))$ because $X_2 \in \mathcal{X}_2^*$ and $X \notin \mathcal{X}_2^*$. Together we have

   $$X_2 \in int\left(L^1_{\leq}(h^1(X)) \cap L^2_{\leq}(h^2(X))\right)$$

   and Theorem 2.2 implies that X is not efficient.

   (b) Analogous for $X \in L^2_{\leq}(h^2(X_1))$.

□

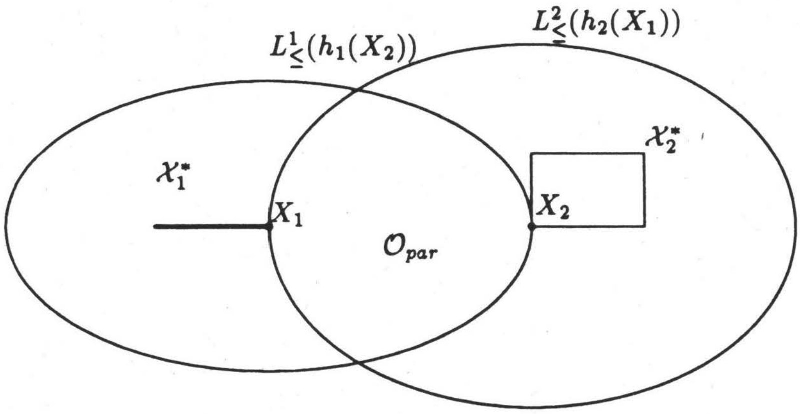

Figure 5.1: **Illustration of Theorem 5.1 (Notice that the form of the level sets is just a symbol for its correct form depending on the definition of h^q.)**

In the following we investigate bicriteria median problems for $d = l_1$ and $d = l_\infty$.

5.1 Solving $1/P/\bullet/l_1/2\text{-}\sum_{par}$

For getting special results on the l_1-metric we first have to characterize the level-curves more precisely (see [Francis et al., 1992]).

Let $a'_1, \ldots, a'_P$ be the different values of the first coordinates of the existing facilities in increasing order, such that

$$a'_1 < a'_2 < \ldots < a'_P .$$

$b'_1, \ldots, b'_Q$ are defined analogously with respect to the second coordinates of Ex_m for all $m \in \mathcal{M}$. The vertical and horizontal lines passing through $\{(a'_s, 0) : s = 1, \ldots, P\}$ and $\{(0, b'_t) : t = 1, \ldots, Q\}$ are called **construction lines**.

Additionally we define $a'_0 = b'_0 = -\infty$ and $a'_{P+1} = b'_{Q+1} = \infty$. We get a decomposition of $\mathbb{R}^2$ into rectangles

$$\langle s, t\rangle := \{(y_1, y_2) : a'_s \leq y_1 \leq a'_{s+1} \ , \ b'_t \leq y_2 \leq b'_{t+1}\} \ ,$$

for $s \in \mathcal{P}_0 := \{0, 1, 2, \ldots, P\}$ and $t \in \mathcal{Q}_0 := \{0, 1, 2, \ldots, Q\}$. From the definition we know that

$$\bigcup_{\substack{s \in \mathcal{P}_0 \\ t \in \mathcal{Q}_0}} \langle s, t \rangle = \mathbb{R}^2 .$$

If we define for all $s \in \mathcal{P}_0$, $t \in \mathcal{Q}_0$

$$M_s := \left(\sum_{p=1}^{s} \sum_{j: a'_p = a_j} w_j \right) - \left(\sum_{p=s+1}^{P} \sum_{j: a'_p = a_j} w_j \right)$$

$$N_t := \left(\sum_{q=1}^{t} \sum_{j: b'_q = b_j} w_j \right) - \left(\sum_{q=t+1}^{Q} \sum_{j: b'_q = b_j} w_j \right)$$

we are ready to state two key properties of the level curves (for a proof see [Francis et al., 1992]).

Theorem 5.2. *For all $s \in \mathcal{P}_0$, $t \in \mathcal{Q}_0$ the level curves of $f^i(X)$, $i = 1, 2$ are linear with slope $-\frac{M_s}{N_t}$ in $\langle s, t \rangle$.*

By analysing the change of the slope we can say more.

Corollary 5.3. *Let $\mathcal{X}^* = [x^*_{min_1}, x^*_{max_1}] \times [x^*_{min_2}, x^*_{max_2}]$ be the set of opimal solutions for $1/P/\bullet/l_1/\sum$.*
Define

- $D_{la} := \{(x_1, x_2) : x_1 < x^*_{min_1}, x_2 > x^*_{max_2}\}$
- $D_{ra} := \{(x_1, x_2) : x_1 > x^*_{max_1}, x_2 > x^*_{max_2}\}$
- $D_{rb} := \{(x_1, x_2) : x_1 > x^*_{max_1}, x_2 < x^*_{min_2}\}$
- $D_{lb} := \{(x_1, x_2) : x_1 < x^*_{min_1}, x_2 < x^*_{min_2}\}$
- $D_l := \{(x_1, x_2) : x_1 < x^*_{min_1}, x^*_{min_2} \le x_2 \le x^*_{max_2}\}$
- $D_a := \{(x_1, x_2) : x^*_{min_1} \le x_1 \le x^*_{max_1}, x_2 > x^*_{max_2}\}$
- $D_r := \{(x_1, x_2) : x_1 > x^*_{max_1}, x^*_{min_2} \le x_2 \le x^*_{max_2}\}$
- $D_b := \{(x_1, x_2) : x^*_{min_1} \le x_1 \le x^*_{max_1}, x_2 < x^*_{min_2}\}$,

(see Figure 5.2), then the following holds

1. *The slope of the level curve in D_{lu} and D_{rl} is strictly positiv.*
2. *The slope of the level curve in D_{ru} and D_{ll} is strictly negativ.*
3. *The slope of the level curve in D_a and D_b is 0.*
4. *The slope of the level curve in D_l and D_r is ∞.*

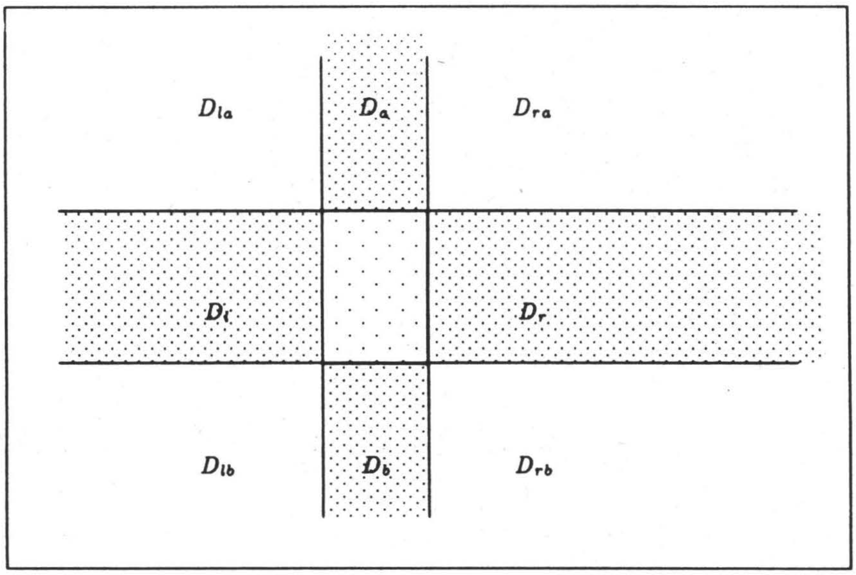

Figure 5.2: Illustration of Corollary 5.3

First we specialize the characterization of the set of efficient solutions given in Theorem 5.1. Wlog let $\mathcal{X}_1^* = [\alpha_1, \beta_1] \times [\gamma_1, \delta_1]$ and $\mathcal{X}_2^* = [\alpha_2, \beta_2] \times [\gamma_2, \delta_2]$, with

$$\alpha_1 \leq \alpha_2 \ , \ \beta_1 \leq \beta_2 \ , \ \gamma_1 \leq \gamma_2 \ , \ \delta_1 \leq \delta_2 \ . \tag{5.1}$$

Theorem 5.4.

1. $\mathcal{X}_{par} = \mathcal{X}_1^* \cap \mathcal{X}_2^*$
   if $\mathcal{X}_1^* \cap \mathcal{X}_2^* \neq \emptyset$.

2. $\mathcal{X}_{par} = [\alpha_1, \beta_1] \cap [\alpha_2, \beta_2] \times [\delta_1, \gamma_2]$
   if $[\alpha_1, \beta_1] \cap [\alpha_2, \beta_2] \neq \emptyset$ *and* $\mathcal{X}_1^* \cap \mathcal{X}_2^* = \emptyset$.

3. $\mathcal{X}_{par} = [\beta_1, \alpha_2] \times [\gamma_1, \delta_1] \cap [\gamma_2, \delta_2]$
   if $[\gamma_1, \delta_1] \cap [\gamma_2, \delta_2] \neq \emptyset$ *and* $\mathcal{X}_1^* \cap \mathcal{X}_2^* = \emptyset$.

4. $\mathcal{X}_{par} \subseteq [\beta_1, \alpha_2] \times [\delta_1, \gamma_2]$
   if $[\alpha_1, \beta_1] \cap [\alpha_2, \beta_2] = \emptyset$ *and* $[\gamma_1, \delta_1] \cap [\gamma_2, \delta_2] = \emptyset$.

Proof. In the cases $1 - 3$ of the theorem we know from Corollary 5.3 that for $X \in [\alpha_1, \beta_1] \cap [\alpha_2, \beta_2] \times [\delta_1, \gamma_2]$ the slopes of $L^1_=(f^1(X))$ and $L^2_=(f^2(X))$ are ∞ and for $X \in [\alpha_1, \beta_1] \cap [\alpha_2, \beta_2] \times [\delta_1, \gamma_2]$ the slopes of $L^1_=(f^1(X))$ and $L^2_=(f^2(X))$ are 0. So the corresponding level-curves touch each other such that the conditions of Theorem 2.2 are fulfilled. For all other X the level-curves intersect so that

$$\left(L^1_=(f^1(X)) \cap L^2_=(f^2(X))\right) \subsetneq \left(L^1_\leq(f^1(X)) \cup L^2_\leq(f^2(X))\right)$$

and with Theorem 2.2 we know that X is not efficient.
In Case 4 where $\mathcal{X}^*{}_1$ and $\mathcal{X}^*{}_2$ have no x_1 or x_2 coordinate in common we get with the same arguments that no $X \notin [\beta_1, \alpha_2] \times [\delta_1, \gamma_2]$ can be efficient.

□

What is left now, is to investigate Case 4 of the theorem in more detail to characterize the complete set of efficient solutions.

First we determine $\mathcal{X}_{lex} \subseteq \mathcal{X}_{par}$. Since $L^1_=(f^1((\alpha_2, \gamma_2))$ has in (α_2, γ_2) a slope $\neq \infty$ we have $\mathcal{X}_{2,1} = \{(\alpha_2, \gamma_2)\}$ and analogously $\mathcal{X}_{1,2} = \{(\beta_1, \delta_1)\}$.

Now we construct a polypath on the construction lines going from (α_2, γ_2) to (β_1, δ_1). (We know that these endpoints are on the construction lines)

Let therefore $\langle s, t \rangle$ be a region determined through the construction lines and let $X \in \mathcal{X}_{par}$ be the upper-right corner point of $\langle s, t \rangle$, i.e. $X \in \langle s, t \rangle \cap \langle s+1, t+1 \rangle$ (see Figure 5.3).

Now we investigate the level curves $L^1_= := L^1_=(f^1(Y))$ and $L^2_= := L^2_=(f^2(Y))$ for a $Y \in int(\langle s, t \rangle)$. We have 3 cases

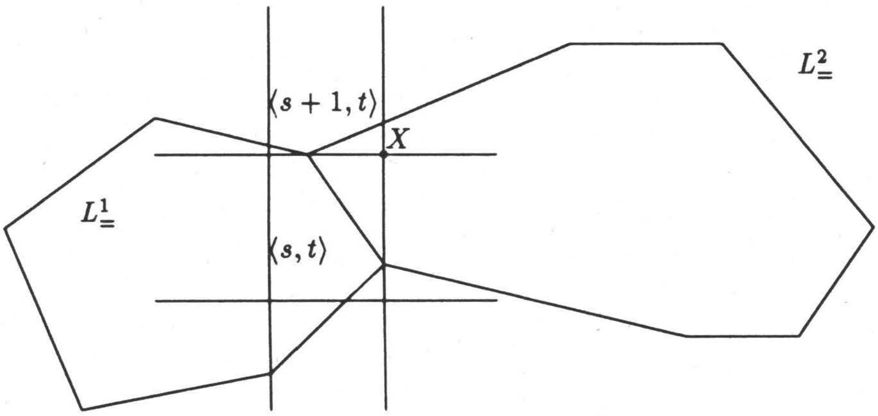

Figure 5.3: The slope of $L^1_=$ is equal to the slope of $L^2_=$. All $Y \in \langle s, t \rangle$ are pareto locations due to Theorem 2.2.

Case 1 (see Figure 5.3) $L^1_=$ and $L^2_=$ have the same slope in Y.
In this case $L^1_=(f^1(Y'))$ and $L^2_=(f^2(Y'))$ have the same slope for all $Y' \in \langle s, t \rangle$ and Theorem 2.2 is fulfilled. So $\langle s, t \rangle \subset \mathcal{X}_{par}$. Also, by Corollary 5.3, no $Y \in \langle s', t \rangle \cap \langle s, t' \rangle$ with $s \neq s'$, $t \neq t'$ can satisfy the conditions of Theorem 2.2 and are therefore not efficient.

Case 2 (see Figure 5.4) The slope of $L^1_=$ is smaller than the slope of $L^2_=$.
Again with Theorem 2.2 and Corollary 5.3 we have that all $Y \in (\langle s, t \rangle \cap \langle s, t+1 \rangle)$ are in $\mathcal{X}_{par}$ and all $Y \in \langle s', t \rangle \cap \langle s, t' \rangle$ with $s \neq s'$, $t \neq t'$ are not.

Case 3 (see Figure 5.5 The slope of $L^1_=$ is larger than the slope of $L^2_=$.
Analogous to Case 2 we observe that all $Y \in (\langle s, t \rangle \cap \langle s+1, t \rangle)$ are in $\mathcal{X}_{par}$ and all $Y \in \langle s', t \rangle \cap \langle s, t' \rangle$ with $s \neq s'$, $t \neq t'$ are not.

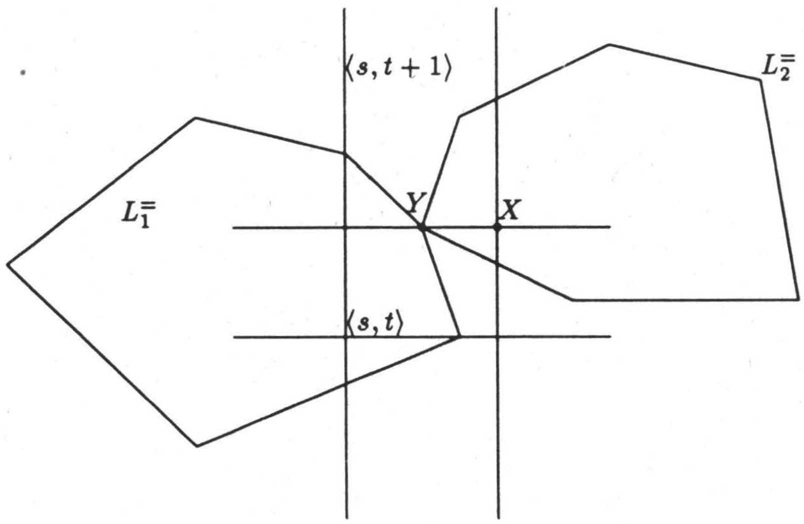

Figure 5.4: The slope of $L_=^1$ is smaller than the slope of $L_=^2$. Y is pareto location due to Theorem 2.2.

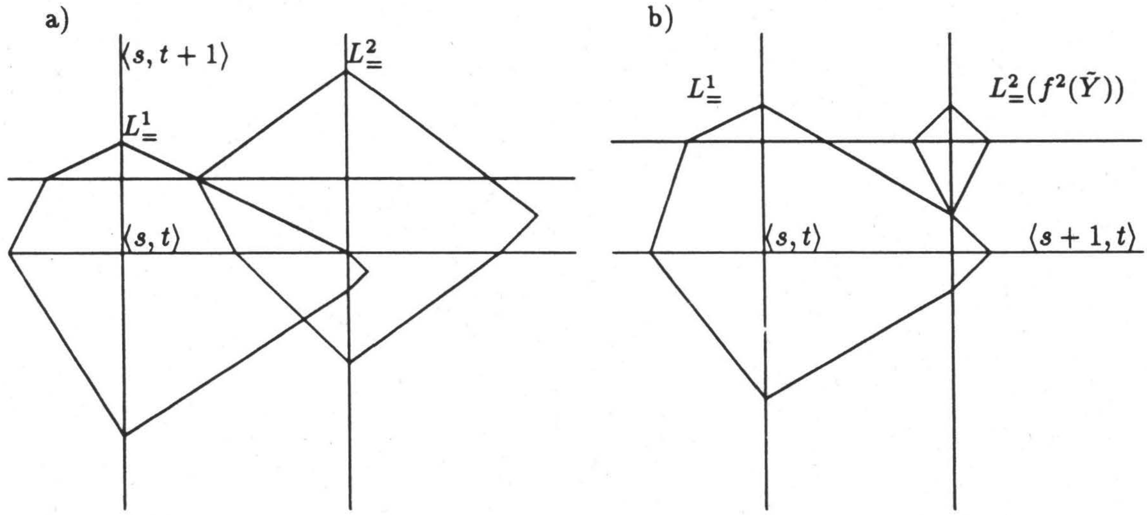

Figure 5.5: The slope of $L_=^1$ is larger than the slope of $L_=^2$. It is shown that $Y \in \langle s,t\rangle \cap \langle s,t+1\rangle$ are not pareto (a), while $Y \in \langle s,t\rangle \cap \langle s+1,t\rangle$ are pareto locations due to Theorem 2.2 (b).

After we have investigated the region $\langle s,t\rangle$ we continue with the next region corresponding to the cases above, that is $\begin{cases} \langle s-1,t-1\rangle & \text{in Case 1} \\ \langle s-1,t\rangle & \text{in Case 2} \\ \langle s,t-1\rangle & \text{in Case 3} \end{cases}$ and iterate the procedure until we reach (β_1,δ_1).

In summary we have proven the validity of the following algorithm.

Algorithm 5.1. *Algorithm for finding $\mathcal{X}_{par}$ for $1/P/\bullet/l_1/2\text{-}\sum_{par}$*

1. *Compute* $\mathcal{X}_1^* = [\alpha_1,\beta_1] \times [\gamma_1,\delta_1]$ *and* $\mathcal{X}_2^* = [\alpha_2,\beta_2] \times [\gamma_2,\delta_2]$
2. *If* $\mathcal{X}_1^* \cap \mathcal{X}_2^* \neq \emptyset$ *then*
   Output: $\mathcal{X}_{par} = \mathcal{X}_1^* \cap \mathcal{X}_2^* \longrightarrow$ **END**
3. *If* $[\alpha_1,\beta_1] \cap [\alpha_2,\beta_2] \neq \emptyset$ *then*
   Output: $\mathcal{X}_{par} = [\alpha_1,\beta_1] \cap [\alpha_2,\beta_2] \times [\delta_1,\gamma_2] \longrightarrow$ **END**
4. *If* $[\gamma_1,\delta_1] \cap [\gamma_2,\delta_2] \neq \emptyset$ *then*
   Output: $\mathcal{X}_{par} = [\beta_1,\alpha_2] \times [\gamma_1,\delta_1] \cap [\gamma_2,\delta_2] \longrightarrow$ **END**
5. *Let* $X = (\alpha_2,\gamma_2)$ *and determine* s *and* t *such that* $X = \langle s,t\rangle \cap \langle s+1,t+1\rangle$.
6. *Let* $\mathcal{X}_{par} = \emptyset$.
7. *While* $X \neq (\beta_1,\delta_1)$ *DO*
   (a) *Compute the slopes* S^1 *and* S^2 *for* $L^1_{=}(f^1(Y))$ *and* $L^2_{=}(f^2(Y))$ *respectively, for some* $Y \in (int\langle s,t\rangle)$.
   (b) *Select the appropriate of the following cases*
   - $\mathbf{S^1 = S^2}$ *Set* $\mathcal{X}_{par} = \mathcal{X}_{par} \cup \langle s,t\rangle$,
     $X = \langle s-1,t-1\rangle \cap \langle s,t\rangle$, $s = s-1$, $t = t-1$.
   - $\mathbf{S^1 < S^2}$ *Set* $\mathcal{X}_{par} = \mathcal{X}_{par} \cup (\langle s,t\rangle \cap \langle s,t+1\rangle)$,
     $X = \langle s-1,t\rangle \cap \langle s,t+1\rangle$, $s = s-1$.
   - $\mathbf{S^1 > S^2}$ *Set* $\mathcal{X}_{par} = \mathcal{X}_{par} \cup (\langle s,t\rangle \cap \langle s+1,t\rangle)$,
     $X = \langle s,t-1\rangle \cap \langle s+1,t\rangle$, $t = t-1$.

   (c) *ENDDO*
8. **Output:** $\mathcal{X}_{par} \longrightarrow$ **END**.

For examples of the solution sets see Figure 5.6.

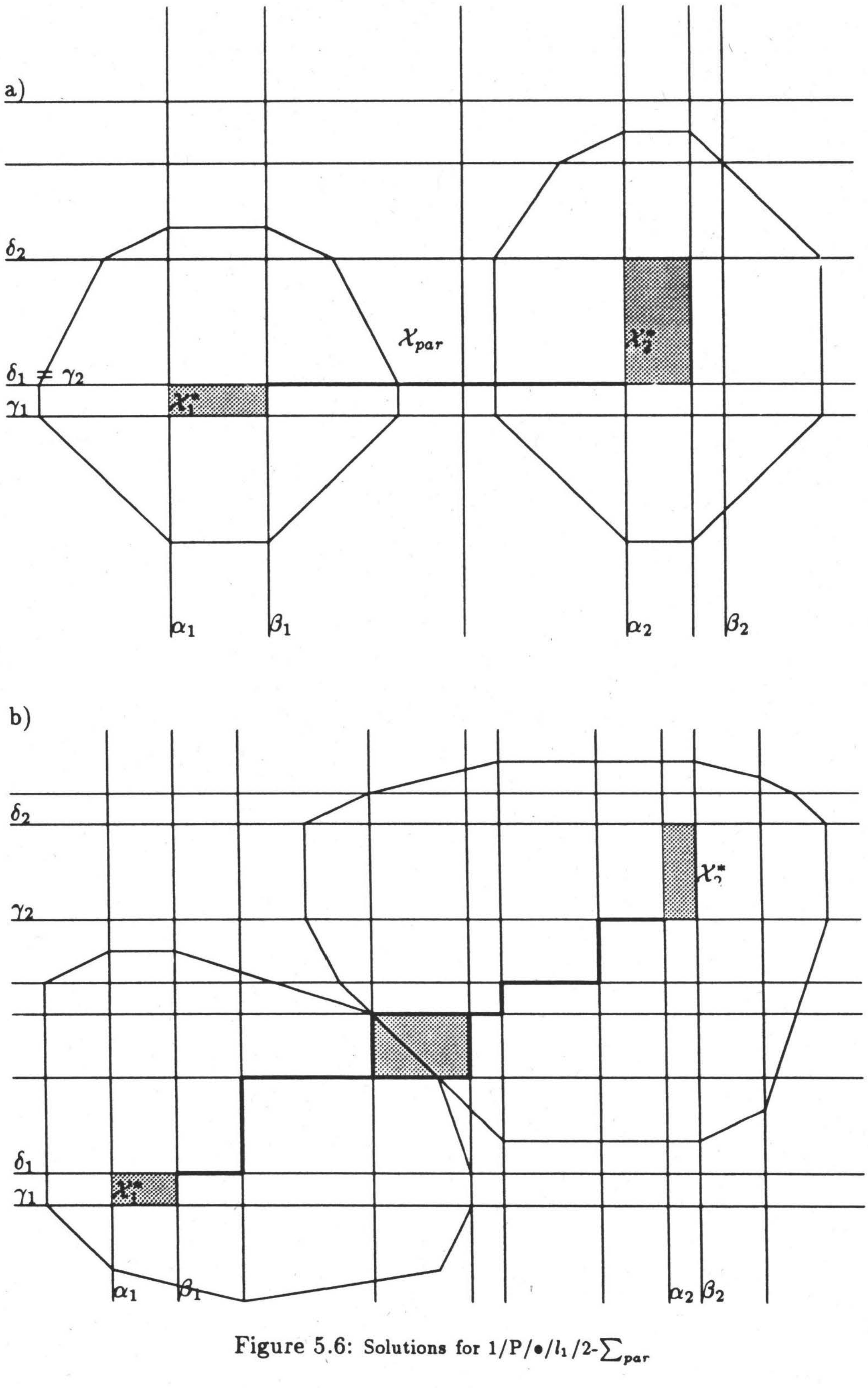

Figure 5.6: Solutions for $1/P/\bullet/l_1/2\text{-}\sum_{par}$

5.2 Solving $1/P/\cdot/l_1/2\text{-}\sum_{MO}$

Using Algorithm 5.1 and Lemma 2.1 (2) we can solve the MO location problem $1/P/\cdot/l_1/2\text{-}\sum_{MO}$.

For this purpose consider the two sets $\mathcal{X}_{1,2}$ and $\mathcal{X}_{2,1}$ of lex locations with respect to permutations $\pi(1) = 2$ and $\pi(1) = 1$, respectively.

Case 1 $f^1(X_{1,2}) \geq f^2(X_{1,2})\ \forall X_{1,2} \in \mathcal{X}_{1,2}$.

Then

$$\max_{q=1,2} f^q(X) \geq f^1(X) \geq f^1(X_{1,2}) = \max_{q=1,2} f^q(X_{1,2})\ \forall X \in \mathbb{R}^2$$

and

$$\mathcal{X}_{par} \cap \mathcal{X}_{MO} = \mathcal{X}_{1,2}\ .$$

Case 2 $f^2(X_{2,1}) \geq f^1(X_{2,1})\ \forall X_{2,1} \in \mathcal{X}_{2,1}$.

By symmetry we obtain as in Case 1

$$\mathcal{X}_{par} \cap \mathcal{X}_{MO} = \mathcal{X}_{2,1}\ .$$

Case 3 $\forall X = X_{1,2} \in \mathcal{X}_{1,2}$ and $Y = X_{2,1} \in \mathcal{X}_{2,1}$

$$f^1(X) < f^2(X) \text{ and } f^2(Y) < f^1(Y) \tag{5.2}$$

Consider any polygon POL through $\mathcal{X}_{par}$ with endpoints $\mathcal{X}_{1,2}$ and $\mathcal{X}_{2,1}$ and which consists of construction lines. With increasing distance from $Z \in POL$ to $\mathcal{X}_{1,2}$ $f^1(Z)$ increases while $f^2(Z)$ decreases. Therefore (5.2) implies the existence of a breakpoint X_{MO} such that $f^1(X_{MO}) = f^2(X_{MO})$ or of two adjacent breakpoints X and Y such that (5.2) holds. In the latter case X_{MO} is the unique point in the line segment $[X, Y]$ with $f^1(X_{MO}) = f^2(X_{MO})$. Notice that X_{MO} is easy to compute since $f^1(Z)$ and $f^2(Z)$ are linear on $[X, Y]$. In both cases $\mathcal{X}_{MO} = \{X_{MO}\}$.

6 Q-criteria Center Problems

6.1 Solving $1/P/\cdot/l_\infty/Q$-$\max_{par}$

We will again apply Theorem 2.2 and get a similar approach as in the previous section.

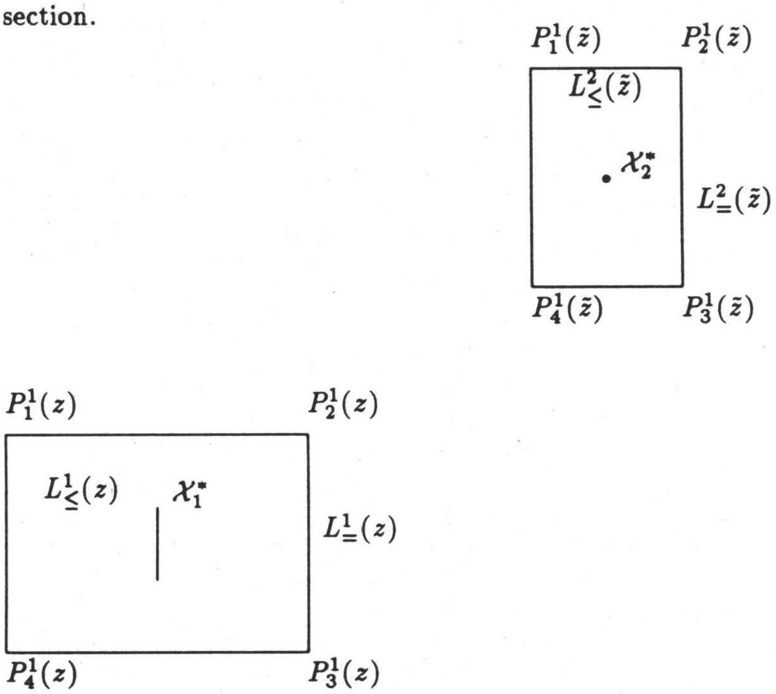

Figure 6.1: **An example for level curves and level sets in center problems.**

It is well-known that the level curves $L_=^q(z)$ of each of the single-criterion problems $1/P/\cdot/l_\infty/\max$ with objective g^q are rectangles with sides parallel to the x_1- and x_2-axis and corner points $P_1^q(z), \ldots, P_4^q(z)$, $q \in \mathcal{Q}$. Correspondingly, the level sets $L_\le^q(z)$ are boxes (see Figure 6.1).

Using the characterization in Theorem 2.2 we obtain pareto locations by intersecting rectangles in such a way that the intersections of the corresponding boxes have no interior point. In this way we obtain single points or line segments which are parallel to the x_1- or x_2-axis. Figure 6.2 illustrates for $Q = 2$ the four possible types of intersections if we assume that $\mathcal{X}_1^*$ and $\mathcal{X}_2^*$ are vertical line segements with $\mathcal{X}_1^*$ to the lower left of $\mathcal{X}_2^*$.

The algorithm for finding the complete set $\mathcal{X}_{par}$ of all pareto locations of $1/P/\cdot/l_\infty/2$-$\max_{par}$, consists in evaluating the trajectories of the corner points. They are known to be piecewise linear, such that we can evaluate in each breakpoint whether there exists pareto locations of type a), b), c) or d). The resulting set $\mathcal{X}_{par}$ has the butterfly form indicated in Figure 6.3. If all weights are 0 or 1 then the situation is even simpler because then the trajectory of each corner point is linear.

Details of the algorithm are omitted here.

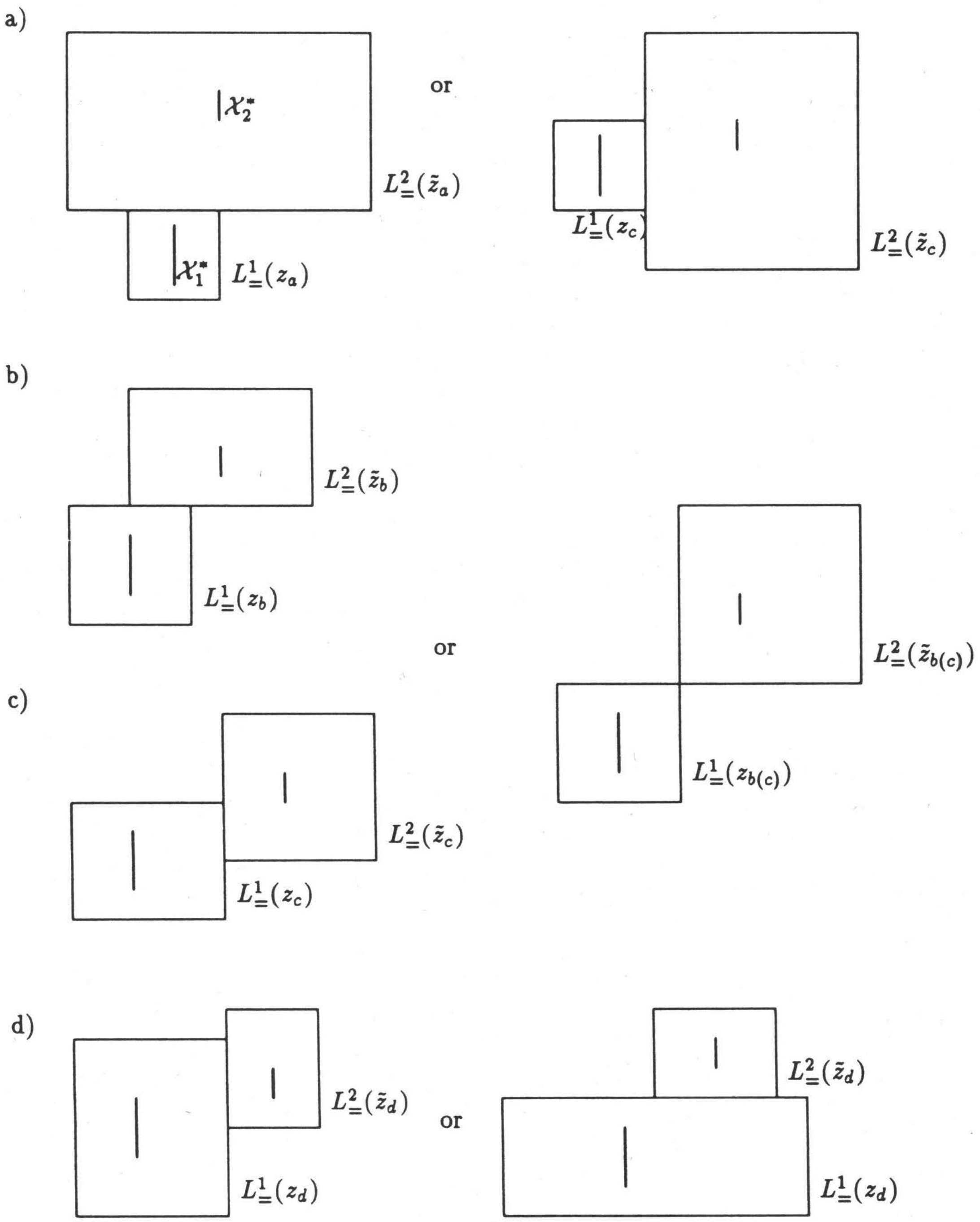

Figure 6.2: Possible instersections of level curves producing pareto locations in line segements a) $[P_1^1(z_a), P_2^1(z_a)]$ (or $[P_2^1(z_a), P_3^1(z_a)]$), b) $[P_4^2(\tilde{z}_b), P_2^1(z_b)]$, c) $[P_4^2(\tilde{z}_c), P_2^1(z_c)]$ and d) $[P_4^2(\tilde{z}_d), P_1^2(\tilde{z}_d)]$ (or $[P_3^2(\tilde{z}_d), P_4^2(\tilde{z}_d)]$). Notice that $z_a \leq z_b \leq z_c \leq z_d$ and $\tilde{z}_a \geq \tilde{z}_b \geq \tilde{z}_c \geq \tilde{z}_d$.

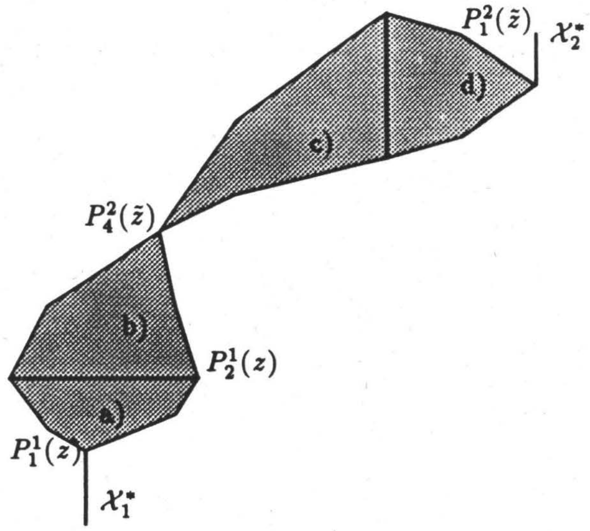

Figure 6.3: Butterfly form of $\mathcal{X}_{par}$ bounded by the trajectories of $P_1^1(z)$, $P_2^1(z)$, $P_4^2(\bar{z})$ and $P_1^2(\bar{z})$. The areas denoted with a) - d) correspond to the cases a) - d) in Figure 6.2.

The following theorem establishes an intersting relationship between 2- and Q-criteria problems.

Theorem 6.1. *Solving $1/P/\cdot/l_\infty/Q$-$\max_{par}$ is equivalent to solving $1/P/\cdot/l_\infty/2$-$\max_{par}$ for all Q^2 bicriteria problems with objectives g^{q_1} and g^{q_2} with $q_1 \neq q_2$, $q_1, q_2 \in \mathcal{Q}$.*

Proof. First note that a pareto location for $\mathcal{Q}_1 \subseteq \mathcal{Q}$ objectives will also be a pareto location for $\mathcal{Q}_2 \subseteq \mathcal{Q}$ objectives with $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$. (Since the nondominance of a solution is not influenced by adding other objectives.) Hence pareto locations of bicriteria problems are pareto locations of $1/P/\cdot/l_\infty/Q$-$\max_{par}$.

For the other direction of the proof we will use Theorem 2.2:
Suppose X_{par} is a pareto location with respect to $g^1, \ldots, g^Q$. By Theorem 2.2 we know

$$\bigcap_{q \in \mathcal{Q}} int(L_{\leq}^{q}(g^q(X_{par}))) = \emptyset$$

and since the level sets are boxes this means that the interior of the level sets do not overlap simultanously horizontally and vertically. But this means that there are at least two level sets $L_{\leq}^{q'}(g^{q'}(X_{par}))$ and $L_{\leq}^{\bar{q}}(g^{\bar{q}}(X_{par}))$ the interior of which does not overlap horizontally and vertically.
So we have

$$int(L_{\leq}^{q'}(g^{q'}(X_{par}))) \cap int(L_{\leq}^{\bar{q}}(g^{\bar{q}}(X_{par}))) = \emptyset\ .$$

Since X_{par} is a pareto location by Theorem 2.2

$$\bigcap_{q \in \mathcal{Q}} L_{=}^{q}(g^q(X_{par})) \neq \emptyset\ .$$

So

$$X_{par} \in L_{=}^{q'}(g^{q'}(X_{par})) \cap L_{=}^{\bar{q}}(g^{\bar{q}}(X_{par}))$$

and we have found a bicriterion problem for which X_{par} is a pareto location (see also Figure 6.4).

□

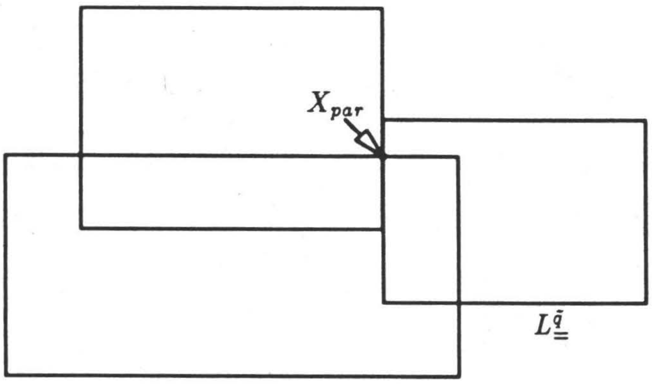

Figure 6.4: Illustration for the proof of Theorem 6.1.

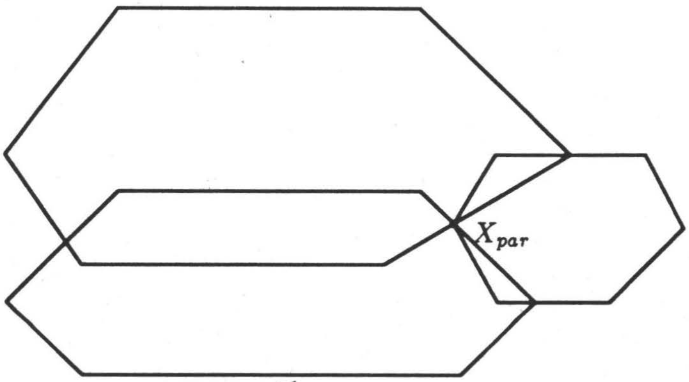

Figure 6.5: X_{par} is a pareto location for the 3 level-curves but not for any 2-subproblem.

Note that this result is not true for other problems as indicated for $1/P/\cdot/l_1/3\text{-}\sum_{par}$ in Figure 6.5.

With Theorem 6.1 and the normconverting mapping T introduced in Section 3.2 the multiobjective l_1-location problem presented by [Wendell et al., 1977] can be seen as a special case of $1/P/\cdot/l_\infty/Q\text{-max}_{par}$ and can therefore also be solved by our algorithm.

6.2 Solving $1/P/\cdot/d/Q\text{-max}_{MO}$

For any distance function d the max order center problem is as simple or as difficult as the corresponding single-criterion problem, since

$$\max_{q\in\mathcal{Q}}\left\{\max_{m\in\mathcal{M}} w_m^q d(Ex_m, X)\right\}$$

$$= \max_{m \in \mathcal{M}} \left(\max_{q \in \mathcal{Q}} w_m^q \right) d(Ex_m, X) \,.$$

The MO center problem is therefore equivalent to the single-criterion center problem with weights $\tilde{w}_m := \max_{q \in \mathcal{Q}} w_m^q$.

7 Conclusions

In this paper we developed some new concepts of multicriterial location problems. Several theoretical results including efficient algorithms for the stated problems (see Figure 2.1) are given.

We also demonstrated the tight relation between restrictive location problems (see [Hamacher and Nickel, 1991] and [Hamacher and Nickel, 1992]) and multiobjective location problems.

For the single-norm case a lot of the important cases are solved with polynomial algorithms. The lex locations can be determined efficently for all l_p-norms with center or median objectives. The pareto location problem for l_1 and l_∞ is completly solved using a combinatorial approach. We have also shown that the MO location problem for the center objective is nothing but a single objective center problem.

Additional topics which we have already solved, but which are beyond the scope of this paper, include multiple facility multicriteria location problems (see [Nickel, 1993]), multicriteria location problems with mixed norms and center/median objectives.

It is important to emphasize that our concept of multiobjective location problems generalizes the one known from the literature. In this generalized context multi-facility multiobjective problems are particulary suitable for modelling real-world problems.

Further extensions of our concepts to more general norms and to multifacility problems are under research. It also seems interesting to use the concepts presented in this paper for location problems on networks.

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