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Theo de Jong and Duco van Straten

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UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Erwin–Schrödinger–Straße
D–6750 Kaiserslautern

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by Theo de Jong and Duco van Straten.

Introduction.

The notion of Q-Gorenstein smoothings has been introduced by Kollár. ([Ko], 6.2.3). This notion is essential for formulating Kollár's conjectures on smoothing components for rational surface singularities. He conjectures, loosely speaking, that every smoothing of a rational surface singularity can be obtained by blowing down a deformation of a partial resolution, this partial resolution having the property (among others) that the singularities occuring on it all have qG-smoothings. (For more details and precise statements see [Ko], ch. 6.) It is therefore of interest to construct singularities having qG-smoothings. Let us recall the definition:

Definition [Ko]: Let X be a reduced surface singularity with X-{x} Gorenstein. Let $X_T \longrightarrow T$ be a one parameter smoothing. The smoothing is called Q-Gorenstein (qG for short) if some multiple of the canonical class of X_T is Cartier. X is called a qG-singularity if there exists a qG-smoothing of X.

The smallest natural number k such that k times the canonical class is Cartier is called the index. It is proved in [Ko] 6.2.4 that the index of X_T for a qG-smoothing of X is the same as the index of X. It should be remarked here that a qG-singularity can have more than one "essentially different" qG-smoothings. This will follow from our construction, but there is also an unpublished example of Wahl.
In this paper we construct qG-singularities of index two. The construction is motivated by a paper of Jan Stevens [St] in which he proves Kollar's conjectures for rational singularities of multiplicity four.

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§ 1. *Singularities of index two.*

Definition 1.1 ([Wa]): A one paramater deformation $X_T \longrightarrow T$ of a (germ of a) Cohen- Macaulay space X is called ω^* -constant if the natural restriction map

$$\omega^*_{X_T} \otimes \mathcal{O}_X \longrightarrow \omega^*_X$$

is surjective (and hence an isomorphism).

Our construction of qG-singularities of index two is based on the following:

Lemma 1.2 **:** Let $X_T \longrightarrow T$ be a one parameter ω^*-constant smoothing of a normal surface singularity X. If X is of index two, then the smoothing is Q-Gorenstein.

proof **:** We have to extend an isomorphism

$$\mathcal{O}_X \longrightarrow \omega_X^{[2]}$$

to the relative situation. Tensoring this with ω^*_X and taking reflexive hull this can be translated to lifting an isomorphism

$$\varphi : \omega^*_X \longrightarrow \omega_X$$

to an isomorphism over T. The ω^*-constancy gives us an exact sequence

$$0 \longrightarrow \omega^*_{X_T} \xrightarrow{.t} \omega^*_{X_T} \longrightarrow \omega^*_X \longrightarrow 0$$

and it follows that the depth of $\omega^*_{X_T}$ is three, so $\mathrm{Ext}^1(\omega^*_{X_T}, \omega_{X_T}) = 0$. From this fact we deduce the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\omega^*_{X_T}, \omega_{X_T}) \longrightarrow \mathrm{Hom}(\omega^*_{X_T}, \omega_{X_T}) \longrightarrow \mathrm{Hom}(\omega^*_X, \omega_X) \longrightarrow 0$$

Hence we can lift φ to a map $\varphi_T : \omega^*_{X_T} \longrightarrow \omega_{X_T}$. Let K resp. C be the kernel resp. the cokernel of φ_T. Because φ is an isomorphism one deduces from the snake lemma that $K \xrightarrow{t.} K$ and $C \xrightarrow{t.} C$ are both isomorphisms. So by Nakayama K and C are zero, and therefore φ_T is an isomorphism. ⊠

§2 *Triple points of projections*

We recall some results from [J-S2], which are essential for the rest of this article. Consider the following situation:

X: a (multi-) germ of a Cohen Macaulay surface singularity.

Y: the image of X under a generically 1-1 map to $\mathbb{C}^3$.

$I_Y = Hom(\mathcal{O}_X, \mathcal{O}_Y) \subset \mathcal{O}_Y \subset \mathcal{O}_X$, the conductor.

Σ : the subvariety of Y defined by I_Y. We assume Σ to be ***reduced.***

I : the ideal in $\mathcal{O}_{\mathbb{C}^3}$ of Σ.

Δ : the subvariety of X defined by I_Y.

It is proved in [J-S2] that under these circumstances one has $\mathcal{O}_X = Hom_Y(I_Y, I_Y)$, so $X \to Y$ is determined by the pair $\Sigma \hookrightarrow Y$. Introduced in [J-S1] is the functor of admissible deformations $Def(\Sigma, Y)$ and in [J-S2] it is proved that there is a natural equivalence between $Def(\Sigma, Y)$ and $Def(X \to Y)$. In particular any deformation of $X \to Y$ induces a deformation of Σ.

A geometric interpretation of the ω^*- constant condition is given in the following theorem, which is [J-S2], theorem (2.1).

Proposition 2.1: Let $X_T \to Y_T$ be a one parameter deformation of $X \to Y$ over T, and $\Sigma_T \to T$ the induced one parameter deformation of Σ. Then:

$$\dim(Cok(\omega^*_{X_T} \otimes \mathcal{O}_X \to \omega^*_X)) = \dim(Cok(\int I_T / I_T^2 \otimes \mathcal{O}_\Sigma \to \int I / I^2))$$

Here $\int I = \{ f \in I : J(f) \subset I\}$ and $\int I_T$ is defined similarly.

In particular, a deformation of X is ω^*-constant if the induced deformation of the curve is "$\int I/I^2$-constant". We remark that if X is Gorenstein and we have a so-called disentanglement of Y, (see [J-S3]), then $\dim(\int I/I^2)$ equals the number of triple points. ([M-P], [J-S2], 2.2).

Corollary 2.2: A rational surface singularity of multiplicity four and index two is a qG-singularity.

proof: In view of lemma 1.1 it is enough to show that every rational quadruple point has an ω^*-constant smoothing. This is stated as corollary (2.5) of [J-S2], but no proof was given.

A generic projection Y of X has as reduced singular locus a curve Σ of multiplicity three and type two. $\int I/I^2$ is a cyclic module generated by the class of a certain $\Phi \in \int I$. Let $f = 0$ be a defining equation for Y, and let $f_t = f + t\Phi$ (t small). Then $Y_t := (f_t = 0)$ has smooth normalization. For all these facts we refer to [J-S4]. §1.

Now $\Sigma \subset Y_t$ is an admissible deformation, in which Σ is not deformed at all, so certainly $\int I/I^2$- constant. So the induced deformation of X is a smoothing and is ω^*-constant by (2.1). ⊠

This corollary was proved by J. Stevens [St], who used a different method.

§ 3. *The construction.*

In this paragraph we compare surfaces which have projections as in § 2 with the ***same*** Σ. We fix the notation in the following

Diagram 3.1

$$\begin{array}{ccccccccc}
 & & X_1 & \supset & \Delta_1 & & \Delta_2 & \subset & X_2 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{f_1=0\} & = & Y_1 & \supset & \Sigma & ==== & \Sigma & \subset & Y_2 = \{f_2=0\}
\end{array}$$

Furthermore, let I_k be the ideal of Σ in $\mathcal{O}_{Y_k}$, k=1,2.

Proposition 3.2 : Suppose $\int I/I^2$ is a cyclic module $\mathcal{O}_\Sigma$ -module. Then there is a 1-1 correspondence between ω^*-constant smoothing components of X_1 and X_2. Moreover, corresponding components are isomorphic up to a smooth factor.

proof: Let $f \in \int I$ project onto a generator of $\int I/I^2$, and let $X_{1,t}$ be an ω^*-constant smoothing of X_1. By projection we get an admissible deformation $\Sigma_t \hookrightarrow Y_{1,t}$. We can assume that $Y_{1,t} = \{f_{1,t} = 0\}, t \neq 0$, has only pinch points and triple points, and so the deformed curve Σ_t, $t \neq 0$ only has triple points. By assumption, we can write $f_k = q_k.f + r_k$, $k=1,2$, and $r_k \in I^2$. As the deformation of X_1 is ω^*-constant, the deformation of Σ is $\int I/I^2$-constant by 2.1, so we can lift f to an $f_t \in \int I_t$ Now define $f_{2,t} = q_{2,t}.f_t + r_{2,t}$, where $q_{2,t}$ is a generic perturbation of q_2, $r_{2,t} \in I_t^2$ is a general perturbation of r_2 and put $Y_{2,t} = \{f_{2,t} = 0\}$. Now the singular locus of $Y_{2,t}$ is Σ_t and by openness of versality we may assume that the normalization $X_{2,t}$ of $Y_{2,t}$ is smooth. By proposition 2.1 $X_{2,t}$ is an ω^*-constant smooting of X_2. The fact that these components are isomorphic up the a smooth factor follows from the principle of I^2-equivalence ([J-S1], 1.16). ⊠

Proposition 3.3 : Suppose the ideal (f_1,f_2) defines a multiplicity four structure on Σ. Then X_1 and X_2 have index ≤ 2.

proof : Because Y_1 and Y_2 are both singular along Σ, it follows from the assumption that the pullback of f_m on X_k vanishes with multiplicity exactly two along Δ_k $(m \neq k)$ and nowhere else. Hence we get an isomorphism $\mathcal{O}_{X_k} \longrightarrow I_k^{[2]}$, and as I_k is a canonical ideal we are done. ⊠

Theorem 3.4 : Suppose that (f_1, f_2) defines a multiplicity four structure on Σ and that $\int I/I^2$ is a cyclic $\mathcal{O}_\Sigma$-module. Suppose that X_1 and X_2 are normal. Then there is a 1-1 correspondence between qG-components of X_1 and X_2. Moreover, corresponding components are isomorphic up to smooth factors.

proof : Combine 3.2, 3.3, and 1.1 . ⊠

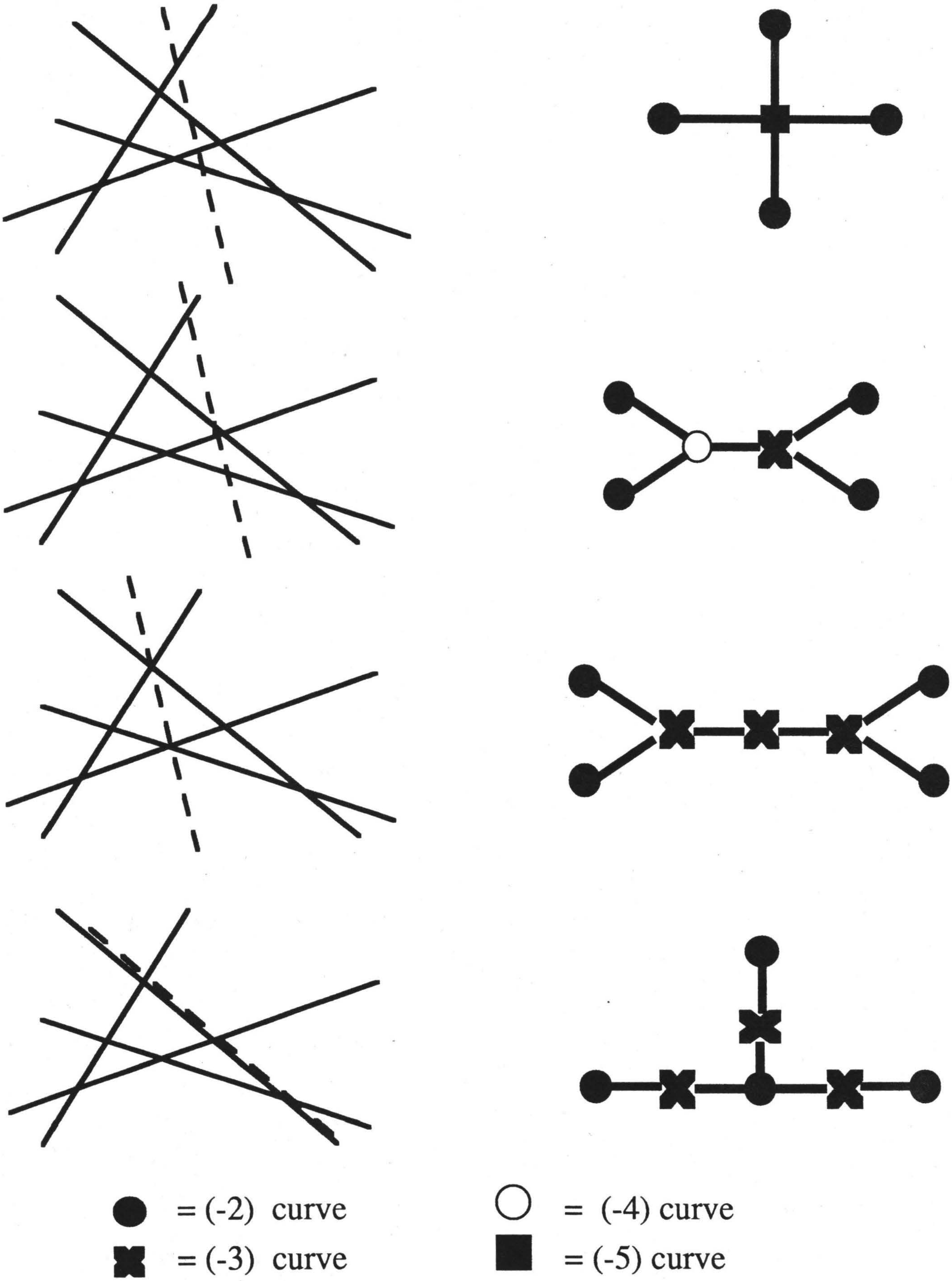
Remark 3.5 **:** In case that X_1 is Gorenstein, it is proved in [M-P] that $\int I/I^2$ is a cyclic module generated by the class of f_1. Moreover, the $\mathcal{O}_{X_1}$ ideal I_1 is *principal*. Any $g \in I$ whose class in I_1 is a generator we call an adjoint, and the surface $\{g=0\}$ an adjoint surface of Y_1. Now $f_2 = q.f_1 + u.g^2$, $q \in \mathcal{O}_{\mathbb{C}^3}$ and u a unit satisfies the condition of 3.3. So in this situation one can apply theorem 3.4. Remark that then qG-components of X_1 are simply smoothing components and also the condition of normality of X_1 can be dropped.

§4. *Examples.*

Example 4.1 **:** Let $f_1 = xyz$, $Y_1 = \{f_1 = 0\}$ and let $X_1 = \mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \mathbb{C}^2$ be the normalization of Y_1. Then $I = (xy, yz, zx)$. X_1 is obviously Gorenstein and $g = xy + yz + zx$ can be taken as an adjoint of Y_1. We see that $f_2 := (xy)^2 + (yz)^2 + (zx)^2 \equiv g^2 \mod (f_1)$, so we can apply theorem 3.4 to conclude that the normalization X_2 of $Y_2 = \{f_2 = 0\}$ is a qG-singularity. Of course, this is well-known, as X_2 is just the cone over the rational normal curve of degree 4.
On the other hand, we can take $f_2 := (xy+yz+yx)^2 + xyz.(x^2 + y^2 + z^2)$ $\equiv g^2 \mod(f_1)$ and apply 3.4 to conclude that X_2, which has

as dual resolution graph, is a qG-singularity.

Example 4.2 **:** Let $f_1 = xyz(x+y+z)$, $Y_1 = \{f_1 = 0\}$ and let $X_1 = \coprod_{i=1}^{4} \mathbb{C}^2$ be the normalization of Y_1. We consider equations of the form $f_2 = L.f_1 + r$, where r is a general element of I^2, the corresponding $Y_2 = \{f_2 = 0\}$ and their normalizations X_2. These X_2 now all are qG-singularities by application of 3.4. Below we give pictures of $L.f_1 = 0$ in the projective plane (the dashed line is L=0, the solid ones $f_1 = 0$), and the corresponding dual resolution graphs.


= (-2) curve
= (-3) curve
= (-4) curve
= (-5) curve

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Adress: Fachbereich Mathematik
Univerität Kaiserslautern
Erwin Schrödingerstrasse
6750 Kaiserslautern
Germany.