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FACHBEREICH MATHEMATIK

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## 1 Introduction

Consider a diagram:

$$(\Sigma,0) \hookrightarrow (X,0)$$

of germs of analytic spaces, with $\Sigma \hookrightarrow Sing(X)$. Such a diagram we call admissible. In [2] the functor of admissible deformations $Def(\Sigma\ X)$ is introduced. The values on a space (S,0) consists of isomorphism classes of diagrams:

$$\begin{array}{ccccc} & \Sigma_{S,0} & \hookrightarrow & X_{S,0} & \\ flat & \downarrow & & \downarrow & flat \\ & (S,0) & = & (S,0) & \end{array}$$

such that $\Sigma_{S,0} \hookrightarrow X_{S,0}$ is again admissible, and with the obvious specialization property. In case of a germ of a holomorphic function:

$$(f,0) : (\mathbb{C}^n,0) \to (\mathbb{C},0)$$

with $(\Sigma,0) \subset (Sing(f),0)$ one similarly has a deformation functor $Def(\Sigma,f)$ and a smooth forgetful morphism:

$$Def(\Sigma,f) \to Def(\Sigma,X)$$

where $X$ is defined by $f = 0$. For details we refer to [2].

In this article we prove a globalization property of admissible deformations in case $(\Sigma,0)$ is a reduced curve singularity and $(f,0)$ (or $(X,0)$) has generically $A_\infty$ singularities along $(\Sigma,0)$. By using the main result of [3] we deduce a globalization property of deformations of normal surface singularities as well. In case of a *smoothing* of a normal surface singularity the existence of a globalization has been proved by Looijenga [4]. The motivation for proving such a globalization result comes from the fact that one can prove theorems about deformations of singularities using global methods. For example, in a forthcoming paper of Steenbrink [8] the globalization is used to prove a semicontinuity property of the spectrum for germs of holomorphic functions with a one dimensional singular locus $\Sigma$ and generically $A_\infty$ singularities along $\Sigma$.

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# 2 Compactification

The important result we need is the following Theorem of Pellikaan:

**Theorem 1 ([6], 5.2, 7.1, 7.4)** *Suppose $(f,0) : (\mathbf{C}^n,0) \hookrightarrow (\mathbf{C},0)$ has a one dimensional singular locus $(\Sigma,0)$ and generically $A_\infty$ along $(\Sigma,0)$. Let $I$ be the defining ideal of $(\Sigma,0)$ in $(\mathbf{C},0)$. Then $f$ is finitely $I$-determined, i.e. for any holomorphic function $g$ with $f-g \in m^k \cap \int I$ for $k \gg 0$ we have that $g$ is right equivalent to $f$. Moreover $f$ is right equivalent to a polynomial.*

**Theorem 2** *Under the hypothesis of Theorem 1, there exists an*

$$F \in H^0(\mathbf{P}^n, O(l)) \text{ for } l \gg 0$$

*such that:*

1. *$(F,0) \sim (f,0)$ (right equivalent)*
2. *The reduced singular locus $\Sigma$ of $F$ is a completion of $(\Sigma,0)$*
3. *$F$ has only smooth points, $A_\infty$ or $D_\infty$ singularities off 0.*

*proof* As reduced one dimensional singularity, $(\Sigma,0)$ is algebraic. Hence we can complete $(\Sigma,0)$ to a curve $\Sigma$, smooth off 0. Let $\overline{I}$ be the homogeneous ideal of $\Sigma$. So $I$ is the stalk at 0 of $\overline{I}$. We have that $f \in \int I$ $(= I^{(2)})$. Pick a $k \gg 0$ such that $f$ is $k$-determined with respect to $I$. This is possible after Theorem 1. Consider the sheaf $\mathcal{J} := m_0^k \cap \overline{I}^{(2)}$ on $\mathbf{P}^n$. (here $m_0$ is the maximal ideal of the point 0.) Choose $l \gg 0$ such that $\mathcal{J}(l)$ is generated by global sections. Again by Theorem 1, we may assume that $f$ is a polynomial of degree $l \geq k$. Consider the homogenization $\overline{f}$ of $f$ and the linear system:

$$V := \overline{f} + H^0(\mathbf{P}^n, \mathcal{J}(l)) \subset H^0(\mathbf{P}^n, \overline{I}^{(2)}(l))$$

By choosing $l$ to be a multiple of the degrees of generators of $\overline{I}$ we see that the base locus of the linear system $V$ is $\Sigma$. Hence by the Theorem of Bertini a generic element of $V$ is smooth outside $\Sigma$. To prove that a generic element of $V$ only has $A_\infty$ or $D_\infty$ singularities off 0, we cover $\mathbf{P}^n - \{0\}$ by open affines:

$$\mathbf{P}^n - \{0\} = \cup_{i=1}^{s} U_i$$

Because $\Sigma$ is smooth off zero we may assume that locally (i.e. in each $U_i$) there exist coordinates $x_1, \ldots, x_n$ such that $\Sigma$ is given by the ideal $(x_2, \ldots, x_n)$. Hence for every $F \in V$ we can write (locally):

$$F = \sum_{i,j=2}^{n} h_{ij}(x_1, \ldots, x_n) x_i x_j$$

Because $\mathcal{J}(l)$ is generated by global sections, we can conclude that for generic $F \in V$ $\det h_{ij}(x_1, 0, \ldots, 0)$ has only zeroes of multiplicity one. This means exactly that $F$ has only $A_\infty$ or $D_\infty$ singularities off 0, see [7]. □

**Corollary 1** *Let $(X, 0) \subset (\mathbf{C}^n, 0)$ be a hypersurface singularity with reduced one dimensional singular locus $(\Sigma, 0)$ and generically $A_\infty$ along $(\Sigma, 0)$. Then there exists a hypersurface $Y \subset \mathbf{P}^n$ such that*

1. *$(Y, 0) \simeq (X, 0)$*
2. *$Sing(Y)_{red}$ is a completion of $(\Sigma, 0)$*
3. *$Y$ has only smooth points, $A_\infty$ or $D_\infty$ singularities off 0.*

The proof is immediate from Theorem 2.

*Remark:* In general one has to allow $D_\infty$ singularities in the compactification. This happens already for the $D_\infty$ singularity itself. A formula for the total "virtual number of $D_\infty$ points" on a compactification is given in [1].

# 3 Globalization of deformations

We consider a function $(f, 0) : (\mathbf{C}^n) \to (\mathbf{C}, 0)$ with reduced one dimensional singular locus $(\Sigma, 0)$ and generically $A_\infty$ singularities along $(\Sigma, 0)$. Let $(S, 0)$ be a germ of a smooth one dimensional space and consider a deformation of $(\Sigma, 0)$:

$$(\Sigma_{S,0}) \to (S, 0) \qquad *)$$

Let $I_S$ be the defining ideal of $(\Sigma, 0)$ in $(\mathbf{C}^n, 0) \times (S, 0)$. Let $s$ be a parameter for (S,0) and define $C$ to be the cokernel of the injective map:

$$\cdot s : \int I_S \to \int I_S$$

where $\int I_S$ is the relative primitive ideal of $I_S$. Suppose $f \in C$ and take a lift $f_S \in \int I_S$. By [6], 7.1 $\int I / \int I \cap J(f)$ is finite dimensional. (Here $J(f)$ is the Jacobi ideal). Because $C \subset \int I$, also $C/C \cap J(f)$ is finite dimensional. We choose functions $f_1, \ldots, f_p \in C$ projecting onto a basis of this space. Choose moreover lifts $f_{1S}, \ldots, f_{pS} \in \int I_S$ of $f_1, \ldots, f_p$.

**Proposition 1** *Consider the $(p+1)$-parameter admissible deformation:*

$$(\mathcal{F}_S, 0) := (f_S + \sum s_i f_{iS}, 0)$$

*of $f$. Let $(f + sg, 0)$ be a one parameter admissible deformation of $f$. Then there exists a map:*

$$h : (\mathbf{C}, 0) \to (\mathbf{C}^{p+1}, 0)$$

*with $h^*(\mathcal{F}_S, 0) \simeq (f + sg, 0)$.*

*proof:* We can write $(f + sg, 0) = (f_S + s\bar{g}, 0)$ and we look at the $(p+2)$-parameter admissible deformation:

$$(\mathcal{G}, 0) = (f_S + \sum s_i f_{iS}, 0)$$

Because $\partial\mathcal{G}/\partial t|_{t=0} \in C$, and $\partial\mathcal{G}/\partial s_i|_{s_i=0}$ generate $C/C \cap J(f)$ we deduce that there are $\xi_i$ and $\eta_i$ such that:

$$\partial\mathcal{G}/\partial t = \sum \xi_i(s, s_i, t, x)\partial\mathcal{G}/\partial x_i + \sum \eta_i(s, s_i, t)\partial\mathcal{G}/\partial s_i$$

We interprete this as having a vectorfield $\Xi$ with $\Xi\mathcal{G} = 0$. By the usual argument, see e.g [5] p42 we get a map:

$$\tilde{h} : (\mathbf{C}^{p+1} \times \mathbf{C}, 0) \to (\mathbf{C}^{p+1}, 0)$$

satisfying $\tilde{h}^*(\mathcal{F}_S, 0) \simeq \mathcal{G}$. Restricting $\tilde{h}$ to $t = s$ and $s_i = 0$ we get a map $h : (\mathbf{C}, 0) \to (\mathbf{C}^{p+1}, 0)$ satisfying $h^*(\mathcal{F}_S, 0) \simeq (f + sg, 0)$. □

**Theorem 3** *Let $(f,0) : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ as in Theorem 1. Consider a one parameter admissible deformation $f_S$ of $f$. Then $f_S$ can be globalized, i.e. there exist $F_S$ homogeneous with $(f_S, 0) \sim (F_S, 0)$ as admissible deformations, such that $F_S$ is trivial off 0.*

*proof:* By Theorem 2 we can globalize $f$ to $F$ such that $F$ has only $A_\infty$ or $D_\infty$ singularities off 0. We have an induced deformation $(\Sigma_{S,0}) \to (S, 0)$ which can be globalized ([9], 4.3). Hence we have the ideal sheaf $\overline{I}_S$ of $\Sigma_S \subset \mathbf{P} \times S$. By Proposition 1 we only have to show that we can lift $f$ and $f_i$ to polynomials $f_S$ and $f_{iS}$. We define $\overline{C}$ by the exact sequence of sheaves:

$$0 \to \int \overline{I}_S \to \int \overline{I}_S \to \overline{C} \to 0$$

Because $H^1(\int \overline{I}_S(l)) = 0$ for $l \gg 0$ the existence of those lifts follows, if one takes $f$ and $f_i$ to be polynomials of sufficiently high degree. The triviality off 0 follows from the fact that the $A_\infty$ or $D_\infty$ singularities are rigid for the admissible deformation theory. □

**Corollary 2** *For a germ of a hypersurface singularity $(X, 0)$ with one dimensional reduced singular locus $(\Sigma, 0)$ and generically $A_\infty$ singularities along $(\Sigma, 0)$ we have a globalization property, i.e. for any one parameter admissible deformation $X_{S,0} \to (S, 0)$ there exist $Y_S \subset \mathbf{P}^n \times S$ such that $Y_{S,0} \simeq X_{S,0}$ as admissible deformation and $Y_S \to S$ trivial off 0.*

The proof is immediate from Theorem 3.

**Corollary 3** *Let $(Z, 0)$ be a normal surface singularity, $Z_{S,0} \to (S, 0)$ be a one parameter deformation. Then there exist a compactification $\overline{Z}_S \to S$ with $\overline{Z}_{S,0} \simeq Z_{S,0}$ as deformations of (Z,0). Moreover the only singular point of the $Z_0$ (the fibre over $0 \in S$) is 0.*

*proof:* Consider a generic projection $(Z,0) \to (X,0) \subset (\mathbf{C}^3,0)$. Then (X,0) satisfies the conditions of Corollary 2. By [3] 1.3 and 1.4 $Def(Z \to X)$ is naturally equivalent to $Def(\Sigma, X)$, and $Def(Z \to X) \to Def(X)$ is smooth. Hence we can take $\overline{Z}_S \to S$ to be the normalization of $Y_S \to S$ of Corrolary 2. The fact that $Z_0$ is smooth off 0 follows because $A_\infty$ and $D_\infty$ have smooth normalization. □

# References


[1] Jong, A.J. de, Jong, T. de; *The Virtual Number of $D_\infty$-Points II* Topology 29, 185-188, (1990)

[2] Jong, T. de, Straten, D. van; *A Defomation Theory for Non-Isolated Singularities* Abh. Math. Sem. Hamburg 60, 177-208, (1990)

[3] Jong, T. de, Straten, D. van; *Deformations of the Normalization of Hypersurfaces* Math. Ann. 288, 527-547 (1990)

[4] Looijenga, E.; *Riemann-Roch and Smoothings of Singularities* Topology 25, 293-302, (1986)

[5] Martinet, J.;*Singularities of Smooth Functions and Maps.* London Math. Soc. Lecture Note Series 58, Cambridge University Press (1983)

[6] Pellikaan, R.; *Finite Determinacy of Functions with Non-Isolated Singularities* Proc. London. Math. Soc. (3) 57, 357-382 (1988)

[7] Siersma, D.; *Isolated Line Singularities* In: Singularities at Arcata, Proc. Sym. Pure Math. 40 (2), 485-496 (1983)

[8] Steenbrink, J.H.M; *In Preparation*

[9] Wahl, J.; *Smoothings of Normal Surface Singularities* Topology 20, 219-246 (1981)



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