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ALGORITHM

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FACHBEREICH MATHEMATIK

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# On the Number of Pivot Steps Required by the Simplex Algorithm

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**Introduction:** Despite its very good empirical performance the known variants of the simplex algorithm require exponentially many pivot steps in terms of the problem dimensions of the given linear programming problem (LPP) in worst-case situation. The first to explain the large gap between practical experience and the disappointing worst-case was Borgwardt (1982a,b), who could prove polynomiality on the average for a certain variant of the algorithm—the "Schatteneckenalgorithmus (shadow vertex algorithm)"—using a stochastic problem simulation.

He studied LPP of type

$$(1) \quad \max_{x \in X} v^T x; \quad X := \{x \in \mathbb{R}^n \mid a_i^T x \leq 1; i = 1, \dots, m\}$$

with  $a_i, v, x \in \mathbb{R}^n, m \geq n \geq 2$ . The vectors  $a_i$  are supposed to be i.i.d.-variables on  $\mathbb{R}^n \setminus \{0\}$ , whose common distribution is invariant under rotations around the origin. The set  $X$  can be considered as a random nonempty polyhedron on  $\mathbb{R}^n$ . Introducing the notation

$$(2) \quad Y := \{y \in \mathbb{R}^n \mid x^T y \leq 1; x \in X\} = \text{convhull}(0, a_1, \dots, a_m)$$

for the polar polyhedron  $Y$  to  $X$  we define corresponding to Borgwardt the random variable  $s(X)$  by

$$(3) \quad s(X) := \int_{\omega_n} \int_{\omega_n} s_{u,v}(X) d_{\omega_n}^0(u) d_{\omega_n}^0(v)$$

where

$$(3.1) \quad s_{u,v}(X) := \begin{cases} \text{number of boundary simplices of } Y \text{ intersected by } \text{cone}(u, v) - 1 \\ \text{if } \mathbb{R}^+u \text{ intersects a boundary simplex of } Y \\ \text{number of boundary simplices of } Y \text{ intersected by } \text{cone}(u, v) \\ \text{if } \mathbb{R}^+u \text{ doesn't intersect a boundary simplex of } Y \end{cases}$$

$d_{\omega_n}^0(u)$  is the normed differential on the unit sphere  $\omega_n$  of  $\mathbb{R}^n$  in direction of  $u$ , i.e.  $\int_{\omega_n} d_{\omega_n}^0(u) = 1$ .

The above defined number  $s_{u,v}(X)$  equals the number of pivot steps, which phase II of the shadow vertex algorithm requires for maximizing the functional  $v^T x$  over  $X$  when the iteration starts with a vertex of  $X$ , whose polar vertex cone intersects  $\mathbb{R}^+u$ . So the random variable  $s(X)$  is the average number of pivot steps required by phase II of the algorithm to solve an LPP with domain  $X$  averaged on the choice of the starting vertex represented by  $u$  and the vector  $v$  defining the functional to maximize.

The most important tool in Borgwardts above mentioned polynomiality proof is the following estimation of the expectation value  $E(S)$  of the random variable  $s$ . Independent of the underlying rotationally invariant distribution holds, e.g. Borgwardt (1982a,b):

$$(4) \quad E(s) \leq \frac{e\pi}{4} \left( \frac{\pi}{2} + \frac{1}{e} \right) n^3 m^{1/(n-1)}.$$

But knowledge about the expectation value alone doesn't allow to quantify the probability of large deviations of the number of pivot steps from its expectation value. So, many researchers, e.g. Shamir (1987), raised the question for higher moments or even for the distribution of the random variable  $s$ .

The aim of this paper is to answer this question partially by estimating the quotient  $\frac{\text{Var}(s)}{E^2(s)}$  asymptotically for a subclass of the rotationally symmetric distributions—the distributions with compact domain.

**Main results:**

**Theorem 1:**

For all  $a > 0$ , all rotationally symmetric distributions on the unit ball  $\Omega_n$  on  $\mathbb{R}^n$ ,  $n \geq 2$ :

$$(5) \quad P\left(\left|\frac{s}{E(s)} - 1\right| > a\right) \leq a^{-2} \frac{\text{Var}(s)}{E^2(s)} = o(1), \quad m \rightarrow \infty.$$

Remarks on theorem 1:

- i) The convergence rate on the right depends on the special choice of the distribution and it's no possibility to estimate the quotient with a common algebraic bound as theorem 2 ii) suggests.
- ii) As the most important consequence of theorem 1 we state that big relative deviations of  $s$  from the expectation value  $E(s)$  become rare as  $m$  increases.

For a more special class of distributions within the rotationally symmetric distributions on  $\Omega_n$  we can furthermore provide the convergence rate of the quotient  $\frac{\text{Var}(s)}{E^2(s)}$  in (5) more exactly.

First of all we need some more notation. Let

$$(6) \quad F(r) := \begin{cases} P(\|a\|_2 \leq r) & r \in [0, 1) \\ 1 & r \geq 1 \end{cases}$$

be the "radial distribution function (RDF)" associated with a rotationally symmetric distribution on  $\Omega_n$ , then  $F$  belongs to the class of "regular varying functions at 1" iff

$$(7) \quad 1 - F(1 - r) \sim r^\alpha L\left(\frac{1}{r}\right), \quad r \rightarrow 0+$$

where  $\alpha > 0$  and  $L$  is a "slowly varying function at infinity", i.e.

$$(7.1) \quad L \in \mathcal{L}^1[1, \infty); \quad \lim_{x \rightarrow \infty} \frac{L(tx)}{L(x)} = 1 \quad \forall t \in (0, \infty).$$

We call the class  $\mathcal{R}$  of rotationally symmetric distributions on  $\Omega_n$  with regular varying RDF "distributions with regular varying behaviour near the boundary of  $\Omega_n$ ". Now we are able to state

**Theorem 2:**

- i) For  $n \geq 2$  and a distribution of type  $\mathcal{R}$ :

$$(8) \quad \frac{\text{Var}(s)}{E^2(s)} = O\left(\left(1 - \tilde{G}\left(1 - \frac{1}{m}\right)\right)^{\frac{n-1}{2}}\right), \quad m \rightarrow \infty$$

where  $\tilde{G}$  is the inverse of the function

$$(8.1) \quad G(t) := P(a^{(n)} \leq t), \quad t \in [0, 1].$$

- ii) For  $n \geq 2$  and a distribution of type  $\mathcal{R}$ , where the function  $L$  of the RDF is a constant function, holds:

$$(9) \quad \frac{\text{Var}(s)}{E^2(s)} = O\left(m^{-\frac{n-1}{\alpha+n-1}}\right), \quad m \rightarrow \infty.$$

Remark on theorem 2:

For the distributions satisfying the propositions of part ii) of the theorem it is possible to evaluate an explicit estimation of the quotient studied with the same techniques the theorem is proved by. This estimation enables us to quantify e.g. the probability of LPP with worst-case complexity of the algorithm's phase II.

In addition to theorem 2 ii) we give two special cases the first having a RDF with  $\alpha = 1$  and  $L(t) = n$  the latter being a limiting case with  $\alpha \rightarrow 0$  and appropriately chosen functions  $L$ .

**Corollary:**

i) For the uniform distribution on  $\Omega_n$ ,  $n \geq 2$ :

$$(10) \quad \frac{\text{Var}(s)}{E^2(s)} = O(m^{-\frac{n-1}{n+1}}), m \rightarrow \infty.$$

ii) For the uniform distribution on  $\omega_n$ ,  $n \geq 2$ , holds:

$$(11) \quad \frac{\text{Var}(s)}{E^2(s)} = O\left(\frac{1}{m}\right), m \rightarrow \infty.$$

For further details and proofs of the results we refer to Küfer (1992a,b,c).

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