UNIVERSITÄT KAISERSLAUTERN

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Preprint No 204

FACHBEREICH MATHEMATIK

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August 1991

Limits of instantons

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Introduction. This is a continuation of our work [8]. In the previous article we studied Donaldson's compactification of the moduli space $I(n)$ of $SU(2)$-instantons on the 4-sphere with instanton number n by using an embedding of $I(n)$ into a complex projective moduli scheme of semi-stable sheaves of rank $2n$ and by taking its closure there.

More precisely let $MI(n)$ denote the complex moduli space of mathematical instanton bundles on $\mathbf{P}^3_{\mathbf{C}}$ with second Chern class $c_2 = n$. By the Atiyah-Ward correspondence $I(n)$ is an open subset in the classical topology in the set $MI(n)(\mathbf{R})$ of real valued points. Considering universal extensions $U(E)$ of members E of $MI(n)$ we got an embedding of $MI(n)$ into the moduli scheme $\overline{M}(n, 2n)$ of semi-stable sheaves of rank $2n$ on $\mathbf{P}^3_{\mathbf{C}}$ with Chern classes $(c_1, c_2, c_3) = (0, n, 0)$. The closure $\overline{\overline{I}}(n)$ of $I(n)$ in $\overline{M}(n, 2n)(\mathbf{R})$ turned out to be Donaldson's compactification and came equipped with a semi-algebraic structure.

The proof is, however, rather artificial. Actually we only showed that a special representative of an S-equivalence class in $\overline{\overline{I}}(n) \setminus I(n)$ is a limit of members of $I(n)$ embedded in $\overline{M}(n, 2n)(\mathbf{R})$, thereby forgetting important information contained in approximating families. This does not help the study of another compactification: the closure $\overline{I}(n)$ of $I(n)$ in $\overline{M}(n, 2)(\mathbf{R})$, where $\overline{M}(n, 2)$ is the moduli space of semi-stable sheaves of rank 2 with the same Chern classes $(0, n, 0)$, in which $MI(n)$ is an open subscheme. This compactification is more natural than the former. Its boundary $\overline{I}(n) \setminus I(n)$ is finer than $\overline{\overline{I}}(n) \setminus I(n)$ and the codimension is 1 whereas $\overline{\overline{I}}(n) \setminus I(n)$ is of codimension 4 except for the case $n = 1$.

The aim of this article is to prove that the essential part of $\overline{I}(n) \setminus I(n)$ consists of *good* quaternionic sheaves obtained by simple elementary transforms (see §1) from instantons in $I(n - k)$, and to give an interpretation of these objects as Yang-Mills potentials with poles. These still contain the information of the degeneration, thereby describing an algebraic *bubbling*. In case $n = 2$ we give as an example a complete description of the natural modification $\overline{I}(2) \to \overline{\overline{I}}(2)$, which is the identity on $I(2)$.

In this paper we first construct a universal space of monads and show that the action of a reductive group which identifies monads giving equivalent sheaves is in fact free in the strong sense. Then we see that the universal space is smooth at monads giving rise to *good* elementary transforms. In §2, §3 we characterize the monads representing elementary transforms on a single line and we compute the tangent space of the universal space at such a monad. Our computation provides us with a suitable tangent vector whose integral curve connects the elementary transforms to instantons in $I(n)$ (see §4). Thus we can prove that every elementary transform of an instanton along different simple lines is in the boundary $\overline{I}(n)\setminus I(n)$ (see Theorem 4.3).

In §5 we extend this result to the case of multiple lines. We study multiple extensions of the line bundle $\mathcal{O}_\ell(1)$ on a line ℓ and we prove that under suitable conditions there is a good resolution of the multiple extension as an $\mathcal{O}_{\mathbf{P}^3_{\mathbf{C}}}$-module. This is enough to show that the multiple extension can be deformed to a direct sum of smaller multiple extensions on different lines. Thus, by induction, all *good* elementary transforms of instantons are in our boundary (see Theorem 5.12). Moreover, this supplies us with a more natural but longer proof of the final section in [8].

The universal extension map $MI(n) \to \overline{M}(n, 2n)$ is unfortunately not defined over the whole closure $\overline{MI(n)}$ in $\overline{M}(n, 2)$. Thus it is not yet clear how to define a natural map $\overline{I}(n) \to \overline{\overline{I}}(n)$ at the moment, but it seems to be possible after studying $\overline{MI(n)}$ in more detail. Indeed, we see in §6 and §7 that we can apply the results on $\overline{MI(2)}$ in [9] to defining a natural map $\overline{I}(2) \to \overline{\overline{I}}(2)$, and to giving a characterization of $\overline{I}(2)$ and a classification of sheaves in it. It is a beautiful by-product that these sheaves can be characterized by geometric figures (see Proposition 7.6). Part of the results in §7 had been obtained in the dissertation [6].

Finally in §8 we interpret a simple elementary transform and all sheaves in $\overline{I}(2) \setminus I(2)$ as isomorphism classes of $SU(2)$-connections under the gauge transformation group of global C^∞-isomorphisms over the 4-sphere. The result is that we can distinguish equivalence classes of connections with the same pole and with the same local class outside the pole and moreover these classes correspond exactly to algebraic equivalence classes of elementary transforms with the same underlying instanton bundle and the same line. This shows on one hand how to extend the Atiyah-Ward correspondence to the degenerate cases, and on the other hand how to define algebraic *bubblings* of instantons. For $n = 2$ we get again a complete correspondence.

Acknowledgment. During the preparation of this paper both authors have been supported by the Deutsche Forschungsgemeinschaft, DFG.

Notation. We shall maintain the notation in the previous paper [8]. For a coherent sheaf E on a scheme T, $S(E)$ denotes the symmetric algebra of E over $\mathcal{O}_T$ and $\mathbf{P}(E)$ does the projective scheme $\mathrm{Proj}(S(E))$ over T. For a vector space F, $G_k(F)$ denotes the Grassmannian of k-dimensional vector subspaces of F.

Let $\mathcal{V}$ be a 4-dimensional vector space over the complex number field $\mathbf{C}$. Fixing a basis $\{Z_0, Z_1, Z_2, Z_3\}$ of $\mathcal{V}$, we define an antilinear automorphism σ of $\mathcal{V}$ by $\sigma(Z_0) = Z_1$, $\sigma(Z_1) = -Z_0$, $\sigma(Z_2) = Z_3$ and $\sigma(Z_3) = -Z_2$. Throughout this paper, X denotes the three-dimensional projective space $\mathbf{P}(\mathcal{V}^\vee) = \mathrm{Proj}(S(\mathcal{V}^\vee))$ and generally a point of X means a $\mathbf{C}$-valued point of X. Thus a point of X represents a line of $\mathcal{V}$ passing through the origin. Then σ gives rise to an involution, which is denoted also by σ, on X. σ is the involution obtained by the twistor space structure of $X \cong \mathbf{P}^3_{\mathbf{C}}$. Let ℓ be a line in X. By using a basis of the space of global sections of $\mathcal{O}_\ell(1)$, we obtain a surjection of $\mathcal{O}_X^{\oplus 2}$ to $\mathcal{O}_\ell(1)$. Q_ℓ denotes the kernel of this surjection. Q_ℓ is independent of the choice of the basis.

To make the partition of a matrix into blocks clear we use vertical and horizontal lines and we usually omit 0's in a matrix if they are clearly understood in the context. For an integer n, I_n means the identity matrix of degree n.

Let E be a coherent sheaf on a scheme. $r(E)$ and $c_i(E)$ denote the rank and the i-th Chern class of E respectively if they can be defined.

1 The universal space of monads

A $\mathbf{C}$-linear map $K : \mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge^4 \mathcal{V}^\vee \to \mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}^\vee$ can be regarded as an $n \times n$-matrix whose entries are in $\bigwedge^2 \mathcal{V}$. If $M : \mathbf{C}^n \to \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{V}$ is $\mathbf{C}$-linear, then it is represented by an $n \times (2n-2)$-matrix with entries in $\mathcal{V}$. We can define the matrix

$$K \wedge M$$

with entries in $\bigwedge^3 \mathcal{V}$ as the composition

$$\mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge^4 \mathcal{V}^\vee \xrightarrow{K} \mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}^\vee \xrightarrow{M \otimes \mathrm{id}} \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{V} \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}^\vee \longrightarrow \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{V}^\vee,$$

where the last map is defined by the contraction. The matrices K and M also define sheaf homomorphisms k and m respectively in the sequence

$$\mathbf{C}^n \otimes_{\mathbf{C}} \Omega^3_X(3) \xrightarrow{k} \mathbf{C}^n \otimes_{\mathbf{C}} \Omega^1_X(1) \xrightarrow{m} \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{O}_X.$$

Clearly this is a complex if and only if $K \wedge M = 0$.

Now let Y_n be the subscheme of

$$H_n = \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}) \times \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{V})$$

defined by the conditions

(1) $K \wedge M = 0$,

(2) m is surjective,

(3) k is injective,

(4) the cohomology sheaf $F = F(K, M) = \ker(m)/\operatorname{im}(k)$ is stable.

Note that the first condition gives rise to a closed subscheme and the other three define an open subscheme in it. Setting $\widetilde{X} = X \times Y_n$, we have the universal sequence

$$\text{(MD)}\quad \mathbf{C}^n \otimes_{\mathbf{C}} \Omega^3_{\widetilde{X}/Y_n}(3) \xrightarrow{\widetilde{k}} \mathbf{C}^n \otimes_{\mathbf{C}} \Omega^1_{\widetilde{X}/Y_n}(1) \xrightarrow{\widetilde{m}} \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{O}_{\widetilde{X}}.$$

This sequence is a complex because of the condition (1). By Nakayama's lemma and (2), $\widetilde{m}$ is surjective and hence $L = \ker(\widetilde{m})$ is flat over Y_n. The condition (3) implies that for every $\mathbf{C}$-valued point y, $\widetilde{k}(y)$ induces an injection of $\mathbf{C}^n \otimes_{\mathbf{C}} \Omega^3_X(3)$ to $L(y)$. Thus $\widetilde{k}$ is injective and the cohomology sheaf $\widetilde{F}$ of the complex (MD) is flat over Y_n. This means that (MD) is the universal monad over Y_n whose cohomology sheaf is flat over Y_n.

By Beilinson's spectral sequence we see the following.

(1.1) The couple $(Y_n, \widetilde{F})$ parametrizes all stable sheaves F of rank 2 with the following properties,

(a) $c_1(F) = 0$, $c_2(F) = n$ and $c_3(F) = 0$,

(b) F is stable,

(c) $H^i(X, F(-2)) = 0$ if $i = 1$ or 2 and $H^2(X, F) = 0$.

Let $\overline{M}(n, 2)$ be the moduli space of semi-stable sheaves E of rank 2 on X with $c_1(E) = 0$, $c_2(E) = n$ and $c_3(E) = 0$. Then the above (a) and (b) show that $\widetilde{F}$ provides us with a morphism

$$\psi_n : Y_n \to \overline{M}(n, 2)$$

whose image is the open subscheme of $\overline{M}(n, 2)$ consisting of the points corresponding to the sheaves with the above properties. Note that all mathematical instanton bundles are contained in $\operatorname{im}(\psi_n)$. It is easy to see the following.

Lemma 1.2. *Assume that E is in $\operatorname{im}(\psi_n)$. Let E' be the kernel of a surjection $\alpha : E \to \mathcal{O}_\ell(1)$, where ℓ is a line in X. Then E' is a member of $\operatorname{im}(\psi_{n+1})$.*

The linear algebraic group

$$G = GL(n, \mathbf{C}) \times GL(n, \mathbf{C}) \times GL(2n-2, \mathbf{C})$$

acts on H_n in the following way:

$$\text{for } (R,S,T) \in G, \ (K,M) \longmapsto (RKS^{-1}, SMT^{-1}).$$

Note that Y_n is a G-invariant subscheme of H_n and that the monad (MD) carries a natural G-linearization and hence so does $\widetilde{F}$. Since for every $t \in \mathbf{C}^*$, (tI_n, tI_n, tI_{2n-2}) acts trivially on H_n, we have an action of $\overline{G} = G/T$ on H_n and hence on Y_n, where T is the subgroup $\{(tI_n, tI_n, tI_{2n-2}) \mid t \in \mathbf{C}^*\}$ of the center of G. Obviously the morphism ψ_n is $\overline{G}$-equivariant with the trivial action of $\overline{G}$ on $\overline{M}(n,2)$.

Let Z be an analytic space and E a Z-flat coherent sheaf on $X_Z = X \times Z$ such that for every $z \in Z$, $E(z)$ has the properties (a), (b) and (c). Then, by the relative version of Beilinson's spectral sequence, we have a complex of coherent $\mathcal{O}_{X_S}$-modules

$$A \boxtimes_{\mathbf{C}} \Omega_X^3(3) \xrightarrow{f} B \boxtimes_{\mathbf{C}} \Omega_X^1(1) \xrightarrow{g} C \boxtimes_{\mathbf{C}} \mathcal{O}_X$$

such that (1) A, B and C are locally free $\mathcal{O}_S$-modules, (2) $r(A) = n$, $r(B) = n$ and $r(C) = 2n-2$, (3) f is injective and g is surjective and (4) the cohomolgy sheaf of the complex is isomorphic to the given E (see [8, §2]). Pick a point z in Z and an open neighborhood U of z, where A, B and C are free sheaves. Fixing free bases of $A|_U$, $B|_U$ and $C|_U$, we obtain a morphism θ of U to Y_n such that the restriction of the above complex to $X \times U$ is the pull-back of (MD) by $\mathrm{id}_X \times \theta$. If we change the free bases, then we have another morphism θ' and U-valued point γ of G such that $\mu(\gamma, \theta') = \theta$, where μ is the action of G on Y_n. Therefore, the couple $(Y_n, \widetilde{F})$ has the following universal property:

> **(1.3)** Let Z be an analytic space and E a Z-flat coherent sheaf on $X_Z = X \times Z$ such that for every $z \in Z$, $E(z)$ has the properties (a), (b) and (c). For every point z of Z, there is an open neighborhood U and a morphism θ of U to Y_n such that E is isomorphic to $(\mathrm{id}_X \times \theta)^*(\widetilde{F})$. Moreover, θ is unique up to the action of G.

We shall prove that the action of $\overline{G}$ on Y_n is free. For this purpose let us study a functor under a rather general setting. Let $f : Z \to S$ be a smooth projective morphism of noetherian schemes and let E_1 and E_2 be S-flat coherent $\mathcal{O}_Z$-sheaves. For a noetherian S-scheme T let $\mathit{Isom}(E_1, E_2)(T)$ be the set of isomorphisms of $E_1 \otimes_{\mathcal{O}_S} \mathcal{O}_T$ to $E_2 \otimes_{\mathcal{O}_S} \mathcal{O}_T$. Obviously $\mathit{Isom}(E_1, E_2)$ is a contravariant functor of the category of noetherian S-schemes to that of sets.

Lemma 1.4. *Assume that for every geometric point s of S, $E_1(s)$ and $E_2(s)$ are stable sheaves with the same Hilbert polynomial. Then $\mathit{Isom}(E_1, E_2)$ is represented by a G_m-bundle* $\mathbf{Isom}(E_1, E_2)$ *over a closed subscheme of S.*

Proof. By [5, 7.7.8 and 7.7.9] there is a coherent $\mathcal{O}_S$-module N such that we have an isomorphism of $f_*\mathcal{H}om_{\mathcal{O}_Z}(E_1, E_2 \otimes_{\mathcal{O}_S} M)$ to $\mathcal{H}om_{\mathcal{O}_S}(N, M)$ as functors on $\mathcal{O}_S$-modules M and that this formation is compatible with base change. We are going to apply these results to the case where $M = \mathcal{O}_S$. Let $\overline{S}$ be the closed subscheme of S defined by the coherent ideal $\mathrm{Ann}(N)$. For every point s of S, we have that

$$\mathrm{Hom}_{\mathcal{O}_{Z(s)}}(E_1(s), E_2(s)) \cong \mathrm{Hom}_{k(s)}(N(s), k(s)).$$

On the other hand, the left hand side of the above is $k(s)$ or 0 according as $E_1(s)$ is isomorphic to $E_2(s)$ or not, because of the stability of $E_i(s)$ and our assumption on the Hilbert polynomial. Thus we see that

$$\mathrm{Hom}_{\mathcal{O}_{Z(s)}}(E_1(s), E_2(s)) \cong \begin{cases} k(s) & \text{if } s \in \overline{S} \\ 0 & \text{if } s \notin \overline{S}. \end{cases}$$

This and the definition of $\overline{S}$ show that N is an invertible $\mathcal{O}_{\overline{S}}$-module. Let $\mathbf{N} = \mathrm{Spec}(S(N))$ be the total space of the line bundle N, where $S(N)$ is the symmetric algebra of N over $\mathcal{O}_{\overline{S}}$, We shall prove that $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$-section represents our functor. Pick a noetherian S-scheme T and a homomorphism u of $E_1 \otimes_{\mathcal{O}_S} \mathcal{O}_T$ to $E_2 \otimes_{\mathcal{O}_S} \mathcal{O}_T$. By Nakayama's lemma and the assumption on the stability, u is surjective if $u(t) \neq 0$ for every $t \in T$. If u is surjective, then $\ker(u)$ is flat over T and hence the equality of the Hilbert polynomials shows that $\ker(u) = 0$. Therefore, u is an isomorphism if and only if for every $t \in T$, $u(t)$ is not 0. Combining this consideration with the isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_{Z_T}}(E_1 \otimes_{\mathcal{O}_S} \mathcal{O}_T, E_2 \otimes_{\mathcal{O}_S} \mathcal{O}_T) &\cong \\ \mathrm{Hom}_{\mathcal{O}_T}(N \otimes_{\mathcal{O}_S} \mathcal{O}_T, \mathcal{O}_T) &\cong \mathrm{Hom}_S(T, \mathbf{N}),\end{aligned}$$

we see that $\mathbf{N}^*$ represents our functor. Q.E.D.

Let us apply the above lemma to the case where $Z = X \times Y_n \times Y_n$, $S = Y_n \times Y_n$ and $E_i = F_i = p_i^*(\widetilde{F})$, where p_i is the i-th projection of $X \times Y_n \times Y_n$ to $X \times Y_n$. For a noetherian scheme T over $Y_n \times Y_n$, pick an element a of $G_m(T)$ and a member u of $Isom(F_1, F_2)(T)$. Regarding a as the multiplication by a, $au = ua$ is again in $Isom(F_1, F_2)(T)$ and hence we have a functorial map

$$Isom(F_1, F_2)(T) \times G_m(T) \ni (u, a) \mapsto ua \in Isom(F_1, F_2)(T).$$

Thus we have an action of G_m on $\mathbf{I} = \mathbf{Isom}(F_1, F_2)$. By the above lemma $\mathbf{I}$ is a G_m-bundle over a closed subscheme $\overline{Y}$ of $Y_n \times Y_n$. It is clear that $\overline{Y}$ is the quotient of $\mathbf{I}$ by G_m.

Fixing a basis of each term of the universal monad (MD) and pulling it back to $X \times \mathbf{I}$ by the two projections to $\widetilde{X} = X \times Y_n$, we obtain two monads $(\mathrm{MD})_1$

and $(\mathrm{MD})_2$ on $X \times \mathbf{I}$ with fixed bases. Since the cohomology sheaf of $(\mathrm{MD})_i$ is the pull-back F_i' of F_i to $X \times \mathbf{I}$, the universal isomorphism of F_1' to F_2' gives rise to an isomorphism η of $(\mathrm{MD})_1$ to $(\mathrm{MD})_2$. By using fixed bases, η is represented by an $\mathbf{I}$-valued point $\tilde{g}$ of G. $\tilde{g}$ and the first projection q of $\mathbf{I}$ to Y_n provide us with a morphism $(q, \tilde{g})$ of $\mathbf{I}$ to $Y_n \times G$. On the other hand, the natural G-linearization on $\tilde{F}$ provides us with a morphism h of $Y_n \times G$ to $\mathbf{I}$. It is easy to see that $(q, \tilde{g}) \cdot h = \mathrm{id}$ and $h \cdot (q, \tilde{g}) = \mathrm{id}$ and hence they are isomorphisms. Moreover, these isomorphism are compatible with the actions of G_m on G and $\mathbf{I}$. Thus we get an isomorphic morphism $\overline{h}$ of $Y_n \times \overline{G}$ to $\overline{Y}$. It is now clear that the composition of $\overline{h}$ with the closed immersion of $\overline{Y}$ into $Y_n \times Y_n$ is nothing but the morphism $\Phi = (p, \mu)$, where p is the first projection of $Y_n \times \overline{G}$ and μ is the action of $\overline{G}$ on Y_n. This means that Φ is a closed immersion and thus completes the proof of the following.

Proposition 1.5. *The action μ of $\overline{G}$ on Y_n is free.*

Since H_n and $\overline{G}$ are smooth, the action of $\overline{G}$ on H_n is flat. On the other hand, this action gives rise to the action μ as a base change. Thus μ is a flat morphism. This and the above proposition show that the relation on Y_n given by the action μ is flat. Then the quotient Q_n of Y_n by $\overline{G}$ in the flat topology is an algebraic space (see [1, Corollary 6.3]). By virtue of the property (1.3) Q_n is the moduli space of sheaves on X with the properties (a), (b)and (c). Since Q_n and a suitable open subscheme of $\overline{M}(n,2)$ represent the sheafification of the same functor in the étale topology, Q_n is isomorphic to the open subscheme of $\overline{M}(n,2)$ and hence Q_n is an algebraic scheme. By Proposition 1.5 the base change of the quotient morphism π of Y_n to Q_n by Y_n is isomorphic to the first projection of $Y_n \times \overline{G}$ which is smooth.

$$\begin{array}{ccccc} Y_n \times \overline{G} & \cong & Y_n \times_{Q_n} Y_n & \longrightarrow & Y_n \\ & & \pi \downarrow & & \downarrow \\ & & Y_n & \xrightarrow{\pi} & Q_n \end{array}$$

On the other hand, we know that π is faithfully flat [7, Proposition 5.6]. Thus we obtain the following lemma.

Lemma 1.6. *Q_n is a scheme and the quotient morphism $\pi : Y_n \to Q_n$ is smooth.*

Let E be a stable sheaf of rank 2 on X with the property (c). Assume that $c_1(E) = 0$, $c_2(E) = n - a$ and $c_3(E) = 0$. Pick mutually disjoint lines $\ell_1, \dots, \ell_a$ and assume that there are surjections α_i of E to $\mathcal{O}_{\ell_i}(1)$. Then we have a surjection $\alpha : E \to \bigoplus_{i=1}^{a} \mathcal{O}_{\ell_i}(1)$.

Definition 1.7. We denote $\ker(\alpha)$ by $E(\ell_1, \dots, \ell_a; \alpha_1, \dots, \alpha_a)$ and call it the elementary transform of E along the lines $\ell_1, \dots, \ell_a$. The procedure to obtain $E(\ell_1, \dots, \ell_a; \alpha_1, \dots, \alpha_a)$ from E is said to be the elementary transformation along $\ell_1, \dots, \ell_a$.

As we have seen in [8, Lemma 3.6], if $\mathrm{Ext}^2_{\mathcal{O}_X}(E,E)=0$ and $E|_\ell \cong \mathcal{O}_\ell^{\oplus 2}$ with ℓ a line in X, then $\mathrm{Ext}^2_{\mathcal{O}_X}(E(\ell;\alpha),E(\ell;\alpha))=0$ and hence $E(\ell;\alpha)$ gives rise to a non-singular point of a moduli space of stable sheaves. Combining this and Lemma 1.6, we obtain our main result in this section.

Theorem 1.8. *Let E, $\ell_1,\dots,\ell_a$ and $\alpha_1,\dots,\alpha_a$ be as in Definition 1.7. Assume that for every i, $E|_{\ell_i}$ is isomorphic to the trivial bundle $\mathcal{O}_{\ell_i}^{\oplus 2}$ and that $\mathrm{Ext}^2_{\mathcal{O}_X}(E,E)=0$. Then Y_n is non-singular at the point corresponding to a monad whose cohomology sheaf is isomorphic to $E(\ell_1,\dots,\ell_a;\alpha_1,\dots,\alpha_a)$.*

Let E and $F=E(\ell_1,\dots,\ell_a;\alpha_1,\dots,\alpha_a)$ be as in the above theorem. Using Riemann-Roch and the spectral sequence of local and global Ext, it is not hard to see

$$\dim \mathrm{Ext}^1_{\mathcal{O}_X}(F,F)=8n-3.$$

This means that the moduli space $\overline{M}(n,2)$ is non-singular and of dimension $8n-3$ at the point corresponding to F. As was seen in the above, Y_n is a principal fiber bundle with group $\overline{G}$ over an open subscheme of $\overline{M}(n,2)$. On the other hand, we know that

$$\dim \overline{G}=n^2+n^2+(2n-2)^2-1=6n^2-8n+3.$$

We have therefore

Proposition 1.9. *Let E and $E(\ell_1,\dots,\ell_a;\alpha_1,\dots,\alpha_a)$ be as in Theorem 1.8. Then Y_n is of dimension $6n^2$ at the point corresponding to a monad whose cohomology sheaf is isomorphic to $E(\ell_1,\dots,\ell_a;\alpha_1,\dots,\alpha_a)$.*

2 Monads of elementary transforms

We are going to study a monad whose cohomology sheaf is an elementary transform of a mathematical instanton bundle along a line. A line ℓ in X is determined by a two-dimensional quotient space of $\mathcal{V}^\vee$ or equivalently a two-dimensional subspace W of $\mathcal{V}$. When W is spanned by x and y, ℓ is represented by $x\wedge y$ and we sometimes identify ℓ with $x\wedge y$. Throughout this section we shall fix a mathematical instanton bundle E with $c_2(E)=n-1$.

Proposition 2.1. *Suppose that E is trivial on a line ℓ in X and let F be an elementary transform $E(\ell;\alpha)$ of E. Then F is represented by a monad in Y_n of the form*

$$K=\left(\begin{array}{c|c}\ell & 0\\ \hline A & K'\end{array}\right),\quad M=\left(\begin{array}{c|c}x\ y & 0\cdots 0\\ \hline B\ C & M'\end{array}\right)$$

where $(K',M')\in Y_{n-1}$ is a monad of E, ℓ is the span of x, y and $A\wedge\ell=0$ (we identify ℓ with $x\wedge y$).

Proof. By [8, Proposition 3.1 and Lemma 6.4], there is an exact sequence of the universal extensions

$$0 \longrightarrow Q_\ell \longrightarrow U(F) \longrightarrow U(E) \longrightarrow 0$$

We can construct an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^3_X(3) & \longrightarrow & \mathbf{C}^n \otimes_{\mathbf{C}} \Omega^3_X(3) & \longrightarrow & \mathbf{C}^{n-1} \otimes_{\mathbf{C}} \Omega^3_X(3) & \longrightarrow & 0 \\
 & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \\
0 & \longrightarrow & \Omega^1_X(1) & \longrightarrow & \mathbf{C}^n \otimes_{\mathbf{C}} \Omega^1_X(1) & \longrightarrow & \mathbf{C}^{n-1} \otimes_{\mathbf{C}} \Omega^1_X(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_\ell & \longrightarrow & U(F) & \longrightarrow & U(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where β (γ or δ) is the multiplication by ℓ ($\begin{pmatrix} \ell & 0 \\ A & K' \end{pmatrix}$ or K', resp.). This is possible because the columns of the diagram are the monads of sheaves of the bottom sequence. Since $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X(1), Q_\ell) = 0$, the rightmost vertical surjection lifts to a homomorphism of $\mathbf{C}^{n-1} \otimes_{\mathbf{C}} \Omega^1_X(1)$ to $U(F)$. To show that A can be so chosen that $A \wedge \ell = 0$ is satisfied, we use

Lemma 2.2. *$E|_\ell$ is trivial if and only if $K' \wedge \ell$ is of maximal rank as a linear map of $\mathbf{C}^{n-1}$ to $\mathbf{C}^{n-1} \otimes_{\mathbf{C}} \bigwedge^4 V$.*

Proof. This is [8, Lemma 6.1, (3)] because $E|_\ell$ is a trivial bundle if and only if so is $U(E)|_\ell$.

Let us go back to the proof of Proposition 2.1. If $A \wedge \ell \neq 0$, then there is a column vector A' such that $A \wedge \ell = (K' \wedge \ell) \cdot A'$ because $K' \wedge \ell$ is of maximal rank by the above lemma. We can then replace A by $A - K' \cdot A'$.

Combining the above diagram with the diagrams defining universal extensions and Q_ℓ

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \rightarrow & F & \rightarrow & E & \rightarrow & \mathcal{O}_\ell(1) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & Q_\ell & \rightarrow & U(F) & \rightarrow & U(E) & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & \mathbf{C}^2 \otimes \mathcal{O}_X & \rightarrow & \mathbf{C}^{2n-2} \otimes \mathcal{O}_X & \rightarrow & \mathbf{C}^{2n-4} \otimes \mathcal{O}_X & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_\ell(1) & & 0 & & 0 & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}
$$

we obtain a monad of F and an exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_X(1) & \longrightarrow & \mathbf{C}^n \otimes_{\mathbf{C}} \Omega^1_X(1) & \longrightarrow & \mathbf{C}^{n-1} \otimes_{\mathbf{C}} \Omega^1_X(1) & \longrightarrow & 0 \\
 & & {\scriptstyle (x,y)}\big\downarrow & & \big\downarrow{\scriptstyle M} & & \big\downarrow{\scriptstyle M'} & & \\
0 & \longrightarrow & \mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{O}_X & \longrightarrow & \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{O}_X & \longrightarrow & \mathbf{C}^{2n-4} \otimes_{\mathbf{C}} \mathcal{O}_X & \longrightarrow & 0
\end{array}
$$

It follows that M can be given in the form of our proposition. Q.E.D.

We shall now study the real structure of Y_n which inherits the quaternionic structure σ on $\mathcal{V}$ (see [8, §4]). σ defines an antilinear map σ_n on H_n by

$$(K, M) \mapsto (\sigma K, \sigma M J),$$

where σK (or σM) is the matrix obtained by applying $\bigwedge^2 \sigma$ (or σ, resp.) to the entries of K (or M, resp), and where J is the square matrix of degree $2n-2$

$$
\begin{pmatrix}
J_2 & & & \\
 & J_2 & & \\
 & & \ddots & \\
 & & & \ddots & \\
 & & & & J_2
\end{pmatrix}
\qquad \text{with } J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

Y_n is invariant under the action of σ_n and $\sigma_n^2 = \mathrm{id}$. Thus we have a scheme $Y_{n,\mathbf{R}}$ over $\mathbf{R}$ such that $Y_n \cong Y_{n,\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$. We denote, by abuse of notation, the set of real points $Y_{n,\mathbf{R}}(\mathbf{R})$ by $Y_n(\mathbf{R})$. Then a $\mathbf{C}$-valued point (K, M) of Y_n is in $Y_n(\mathbf{R})$ if and only if

$$\sigma K = K \quad \text{and} \quad \sigma M J = M.$$

Lemma 2.3. *Let E and F be as in Proposition 2.1. F admits a quaternionic structure if and only if it is defined by a monad $(K, M) \in Y_n(\mathbf{R})$ such that*

$$K = \left(\begin{array}{c|c} \ell & 0 \\ \hline A & K' \end{array} \right), \quad M = \left(\begin{array}{c|c} x \ \sigma x & 0 \cdots 0 \\ \hline B \ \sigma B & M' \end{array} \right)$$

with $\bigwedge^2 \sigma(\ell) = \ell$, $\bigwedge^2 \sigma(A) = A$ and $(K', M') \in Y_{n-1}(\mathbf{R})$.

Proof. It is clear that if F is defined by (K, M) satisfying the condition in our lemma, it admits a quaternionic structure. Conversely, the quaternionic structure on F induces that of E because E is the double dual of F. Then the homomorphisms of the exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathcal{O}_\ell(1) \longrightarrow 0$$

are all real with respect to the quaternionic structures of E, F and $\mathcal{O}_\ell(1)$. By the construction of the matrices in Proposition 2.1 we get K and M which satisfy the condition of the lemma. Q. E. D.

3 Tangent space at an elementary transform

If (K, M) is a point of Y_n, then the tangent space $T_{(K,M)}$ of Y_n at this point can be described as the kernel of the linear operator $\rho_{(K,M)}$ of H_n to $\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \bigwedge^3 \mathcal{V})$:

$$\rho_{(K,M)} : (K_1, M_1) \mapsto K_1 \wedge M + K \wedge M_1.$$

Since we have

$$\begin{aligned} \dim H_n &= 6n^2 + 8n^2 - 8n \\ \dim \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \textstyle\bigwedge^3 \mathcal{V}) &= 8n^2 - 8n, \end{aligned}$$

we see that $\rho_{(K,M)}$ is surjective if and only if $\dim T_{(K,M)} = 6n^2$. This, Theorem 1.8 and Proposition 1.9 imply the following.

Proposition 3.1. *Let E and $E(\ell_1, \dots, \ell_a; \alpha_1, \dots, \alpha_a)$ be as in Theorem 1.8. If $(K, M) \in Y_n$ defines a monad of $E(\ell_1, \dots, \ell_a; \alpha_1, \dots, \alpha_a)$, then $\rho_{(K,M)}$ is surjective. If $a = 0$, then we understand that $E(\ell_1, \dots, \ell_a; \alpha_1, \dots, \alpha_a) = E$.*

To interpret the $\rho_{(K,M)}$ in terms of matrices let (K, M) be presented by a row vector

$$(a_1, \dots, a_n; b_1, \dots, b_n),$$

where $a_i \in \mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}$ (or $b_j \in \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{V}$) is a row vector of elements in $\bigwedge^2 \mathcal{V}$ (or $\mathcal{V}$, resp.) of degree n (or, $2n-2$, resp.). Setting $K = (k_{ij})$ and I to be the identity matrix of degree $2n-2$, we define

$$Q = \left(\begin{array}{ccccc} M & & & & \\ & M & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & M \\ \hline k_{11}I & k_{21}I & \cdots & \cdots & k_{n1}I \\ k_{12}I & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ k_{1n}I & k_{2n}I & \cdots & \cdots & k_{nn}I \end{array}\right).$$

Then the image of $(a_1, \ldots, a_n; b_1, \ldots, b_n)$ by $\rho_{(K,M)}$ is

$$(a_1, \ldots, a_n; b_1, \ldots, b_n)Q.$$

If we identify $(\bigwedge^3 \mathcal{V})^\vee$ with $\mathcal{V}$, then it is easily seen that

(3.2) $\rho_{(K,M)}$ is surjective if and only if for every column vector $c = {}^t(c_1, \ldots, c_n)$ with $c_j = (c_{1j}, \ldots, c_{2n-2,j}) \in \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{V}$, a relation $Q \wedge c = 0$ implies $c = 0$.

Let Y'_n be the closed subscheme of Y_n of all pairs (K, M) of the form

$$K = \left(\begin{array}{c|c} \ell & 0 \\ \hline A & K' \end{array}\right), \quad M = \left(\begin{array}{c|c} x \ y & 0 \cdots 0 \\ \hline B \ C & M' \end{array}\right)$$

Then the tangent space $T'_{(K,M)}$ of Y'_n at (K, M) is the linear space of all matrices (K_1, M_1) of the form

$$K_1 = \left(\begin{array}{c|c} \xi & 0 \\ \hline A_1 & K'_1 \end{array}\right), \quad M_1 = \left(\begin{array}{c|c} x_1 \ y_1 & 0 \cdots 0 \\ \hline B_1 \ C_1 & M'_1 \end{array}\right)$$

satisfying $K_1 \wedge M + K \wedge M_1 = 0$. We let

$$\mathrm{Hom}'_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}), \quad \mathrm{Hom}'_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{V})$$

denote the spaces of this type of matrices and set

$$H'_n = \mathrm{Hom}'_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}) \times \mathrm{Hom}'_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{V}).$$

Proposition 3.3. *Let E be a stable sheaf on X of rank 2 with the property (c) in §1 and with $c_1(E) = 0$, $c_2(E) = n-1$ and $c_3(E) = 0$. Assume that $\mathrm{Ext}^2_{\mathcal{O}_X}(E, E) = 0$ and for a line ℓ of X, $E|_\ell$ is a trivial vector bundle. If $(K, M) \in Y'_n$ gives rise to a monad of an elementary transform $F = E(\ell; \alpha)$ of E, then we have*

$$\dim T'_{(K,M)} = 6n^2 - 6n + 9.$$

Proof. Let us look at the sequence

$$0 \longrightarrow H'_n \xrightarrow{\rho'} \mathrm{Hom}'_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \bigwedge^3 \mathcal{V}) \xrightarrow{\gamma} \mathbf{C}^3 \otimes_{\mathbf{C}} \bigwedge^4 \mathcal{V} \longrightarrow 0,$$

where ρ' is defined by

$$(K_1, M_1) \mapsto K_1 \wedge M + K \wedge M_1$$

and γ is given by the matrix

$$\begin{pmatrix} x & y & 0 & \\ 0 & x & y & \\ & 0 & & \end{pmatrix}.$$

Since $T'_{(K,M)} = \ker(\rho')$, it is enough to prove that the above sequence is exact. As in the above we may assume that K and M have the following shape:

$$K = \begin{pmatrix} \ell & 0 & \cdots & \cdots & 0 \\ a_2 & k_{22} & \cdots & \cdots & k_{2n} \\ \vdots & \vdots & & & \vdots \\ a_n & k_{n2} & \cdots & \cdots & k_{nn} \end{pmatrix}, \quad M = \begin{pmatrix} x & y & 0\cdots 0 \\ B & C & M' \end{pmatrix}.$$

Setting I to be the identity matrix of degree $2n-2$ and A_i to be the following $2 \times (2n-2)$ matrix

$$A_i = \begin{pmatrix} a_i & 0 & \\ 0 & a_i & 0 \end{pmatrix}$$

then, by our convention right after Proposition 3.1, the operator ρ' can represented by the matrix

$$Q' = \left(\begin{array}{cc|cccc}
x & y & 0 & \cdots & \cdots & 0 \\ \hline
\multicolumn{2}{c|}{0} & M & & & \\
\multicolumn{2}{c|}{\vdots} & & M & & \\
\multicolumn{2}{c|}{\vdots} & & & \ddots & \\
\multicolumn{2}{c|}{0} & & & & M \\ \hline
\ell & & & & & \\
 & \ell & A_2 & A_3 & \cdots & A_n \\ \hline
\multicolumn{2}{c|}{0} & k_{22}I & k_{32}I & \cdots & k_{n2}I \\
\multicolumn{2}{c|}{\vdots} & k_{23}I & & & \vdots \\
\multicolumn{2}{c|}{\vdots} & \vdots & & & \vdots \\
\multicolumn{2}{c|}{0} & k_{2n}I & \cdots & \cdots & k_{nn}I
\end{array}\right)$$

As in (3.2) we have to show that for a column vector $c = {}^t(v, w, c_2, \ldots, c_n)$ with $v, w \in \mathcal{V}$ and with $c_j = (c_{1j}, \ldots, c_{2n-2,j}) \in \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{V}$, if we have $Q' \wedge c = 0$, then we have $c_{ij} = 0$ whereas ${}^t(v, w)$ is spanned by the column vectors of

$$\begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix}$$

because of the relation $x \wedge v + y \wedge w = 0$. By the equation $Q' \wedge c = 0$ we find that for all $j \geq 2$, $x \wedge c_{1j} + y \wedge c_{2j} = 0$ and hence c_{1j} and c_{2j} are in the subspace spanned by x, y. We put

$$c_{ij} = \alpha_{ij} x + \beta_{ij} y \quad \text{for } i = 1, 2 \text{ and } j \geq 2.$$

On the other hand, we also obtain that

$$\sum_{j \geq 2} k_{jt} c_{ij} = 0 \quad \text{for } i = 1, 2 \text{ and } t \geq 2,$$

which is equivalent to

$$(c_{i2}, \ldots, c_{in}) \wedge K' = 0 \quad \text{for } i = 1, 2,$$

where K' is the square matrix (k_{ij}) of degree $n - 1$. Multiplying by x and y, we get

$$\begin{pmatrix} \alpha_{i2} & \cdots & \alpha_{in} \\ \beta_{i2} & \cdots & \beta_{in} \end{pmatrix} K' \wedge \ell = 0.$$

Since $K' \wedge \ell$ is of maximal rank by Lemma 2.2, all these coefficients must be zero. Thus $c_{ij} = 0$ for $j \geq 2$ and $i = 1, 2$. Let c' be the column vector

${}^t(c_{32},\dots,c_{2n-2,2},\dots,c_{3n},\dots,c_{2n-2,n})$ and I' be the identity matrix of degree $2n-4$. Then for the matrix

$$Q'' = \left(\begin{array}{cccc} M' & & & \\ & M' & & \\ & & \ddots & \\ & & & M' \\ \hline k_{22}I' & k_{32}I' & \cdots & k_{n2}I' \\ k_{23}I' & & & \vdots \\ \vdots & & & \vdots \\ k_{2n}I' & \cdots & \cdots & k_{nn}I' \end{array}\right)$$

we have that $Q'' \wedge c' = 0$. Since Q'' represents the linear map $\rho_{(K',M')}$ and (K', M') gives rise to a non-singular point of Y_{n-1}, Proposition 3.1 and (3.2) show that $c' = 0$. Q.E.D.

Let (K, M) be a point of Y_n and let $S_{(K,M)}$ be the $\mathbf{C}$-vector subspace of $T_{(K,M)}$ spanned by the tangent vectors

$$(RK, 0), \quad (KS, -SM), \quad (0, MT),$$

where $R, S \in \mathfrak{gl}(n, \mathbf{C})$ and $T \in \mathfrak{gl}(2n-2, \mathbf{C})$. There is only one relation

$$(\lambda K, 0) - (K\lambda, -\lambda M) - (0, M\lambda) = 0, \quad \lambda \in \mathbf{C}$$

if (K, M) satisfies the conditions of Proposition 3.3 and hence

$$\begin{aligned} \dim S_{(K,M)} &= 2n^2 + (2n-2)^2 - 1 \\ &= 6n^2 - 8n + 3. \end{aligned}$$

$S_{(K,M)}$ is the tangent space of the $\overline{G}$-orbit of (K, M).

Proposition 3.4. *Let (K, M) be as in Proposition 3.3. Then we have*

$$\begin{aligned} \dim(T'_{(K,M)} \cap S_{(K,M)}) &= \dim S_{(K,M)} - 6n + 10 \\ \dim(T'_{(K,M)} + S_{(K,M)}) &= \dim T_{(K,M)} - 1. \end{aligned}$$

Proof. For R, S, T in the definition of $S_{(K,M)}$, we put $R = (r_{ij})$, $S = (s_{ij})$ and $T = (t_{ij})$. A member $(RK + KS, -SM + MT)$ of $S_{(K,M)}$ is in $T'_{(K,M)}$ if and only if the following conditions are satisfied:

$$\begin{aligned} (r_{12},\dots,r_{1n})K' + \ell(s_{12},\dots,s_{1n}) &= 0 \\ -(s_{12},\dots,s_{1n})M' + (x,y)\begin{pmatrix} t_{13} & \cdots & t_{1,2n-2} \\ t_{23} & \cdots & t_{2,2n-2} \end{pmatrix} &= 0. \end{aligned}$$

Since $K' \wedge \ell$ is of maximal rank, we obtain $r_{12} = \cdots = r_{1n} = 0$ and from that $s_{12} = \cdots = s_{1n} = 0$. This and the second equation imply that $t_{ij} = 0$ for $i = 1, 2$ and $j \geq 3$. Since these entries do not interfere with the diagonal relation, we get

$$\begin{aligned}\dim(T'_{(K,M)} \cap S_{(K,M)}) &= \dim S_{(K,M)} - 2(n-1) - 2(2n-4) \\ &= \dim S_{(K,M)} - 6n + 10\end{aligned}$$

By this and Proposition 3.3 we get that

$$\begin{aligned}\dim(T'_{(K,M)} + S_{(K,M)}) &= \dim T'_{(K,M)} + \dim S_{(K,M)} - \dim(T'_{(K,M)} \cap S_{(K,M)}) \\ &= \dim T'_{(K,M)} + 6n - 10 \\ &= 6n^2 - 6n + 9 + 6n - 10 \\ &= 6n^2 - 1.\end{aligned}$$

Q. E. D.

Indeed, $T'_{(K,M)} + S_{(K,M)}$ is contained in the tangent space of $\phi_n^{-1}(Z)$ at (K, M), where $Z \subset \overline{M}(n, 2)$ is the closed subscheme of elementary transforms, and we know that Z is of codimension 1 in the space of mathematical instanton bundles.

The above computations are valid over **R** if we take a real point (K, M) and replace Y_n by $Y_{n,\mathbf{R}}$, $\mathfrak{gl}(n, \mathbf{C})$ by $\mathfrak{gl}(n, \mathbf{R})$ and another $\mathfrak{gl}(n, \mathbf{C})$ by the Lie algebra $\{D \in M(n, \mathbf{C}) \mid DJ = J\overline{D}\}$. This is because if we extend the coefficient field to **C**, then we come to the situation over **C**.

Corollary 3.4.1. (1) *Let (K, M) be as in Proposition 3.3. There exists a tangent vector $(K_1, M_1) \in T_{(K,M)}$ with*

$$(K_1, M_1) \notin T'_{(K,M)} + S_{(K,M)}$$

(2) *If (K, M) is in $Y_n(\mathbf{R})$, then the same as in* (1) *holds for the real tangent spaces.*

Now we come to the main result of this section which plays a key role in the next section.

Proposition 3.5. *Let E and F be as in Proposition 3.3 and let F be represented by $(K, M) \in Y_n$ in the following form (see Proposition 2.1):*

$$K = \left(\begin{array}{c|c} \ell & 0 \\ \hline A & K' \end{array}\right), \quad M = \left(\begin{array}{c|c} x \ y & 0 \cdots 0 \\ \hline B \ C & M' \end{array}\right)$$

where $(K', M') \in Y_{n-1}$ is a monad of E, ℓ is the span of x, y and $A \wedge \ell = 0$. Let (K_1, M_1) be a tangent vector of Y_n at (K, M) and set $K_1 = (\xi_{ij})$. Then (K_1, M_1) is contained in $T'_{(K,M)} + S_{(K,M)}$ if and only if $\xi_{11} \wedge \ell = 0$.

To prove the proposition we need the following.

Lemma 3.6. *Under the assumption of Proposition 3.5 the sequence*

$$0 \longrightarrow \mathbf{C}^{2n-2} \xrightarrow{\theta} \mathbf{C}^{n-1} \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V} \xrightarrow{\tau} \mathbf{C}^{4n-4} \otimes_{\mathbf{C}} \bigwedge^4 \mathcal{V} \longrightarrow 0$$

is exact, where θ and τ are the multiplication by the following matrices, respectively

$$\left(\begin{array}{c} K' \\ \hline \ell I_{n-1} \end{array}\right), \quad \left(x \wedge (B\ C\ M') \mid y \wedge (B\ C\ M') \right).$$

Proof. We have $K' \wedge M' = 0$ and the relation $A \wedge (x, y) + K' \wedge (B, C) = 0$. Since $A \wedge x \wedge y = 0$, it follows that $x \wedge K' \wedge (B, C) = 0$ and $y \wedge K' \wedge (B, C) = 0$. Hence the sequence is a complex. Since θ is injective by Lemma 2.2, it is sufficient to prove that the matrix defining τ has linearly independent columns. For this purpose it is enough to show that $x \wedge y \wedge (B\ C\ M')$ has independent columns. Let $\lambda \in \mathbf{C}^{2n-2}$ be a column vector with

$$\ell \wedge (B\ C\ M')\lambda = 0.$$

Set

$$\begin{pmatrix} x & y & 0 \cdots 0 \\ B & C & M' \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{2n-2} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Since $\ell \wedge a_i = 0$, we can write $a_i = \alpha_i x + \beta_i y$. On the other hand, by $K \wedge M = 0$ we get in particular

$$A \wedge a_1 + K' \wedge \begin{pmatrix} a_2 \\ \vdots \\ a_n \end{pmatrix} = 0.$$

By using $A \wedge \ell = 0$ again, we obtain

$$K' \wedge x \wedge y \begin{pmatrix} \alpha_2 & \beta_2 \\ \vdots & \vdots \\ \alpha_n & \beta_n \end{pmatrix} = 0.$$

Since $K' \wedge \ell$ is of maximal rank, it follows that $\alpha_i = \beta_i = 0$ for $i \geq 2$, or $a_i = 0$ for $i \geq 2$. Now we have

$$M \cdot \lambda = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

But, if $a_1 \neq 0$, then this contradicts the surjectivity of m (see [9, Lemma 0.2]). Thus $a_1 = 0$ too, and then $\lambda = 0$ because the columns of M are independent.

Proof of Proposition 3.5. Assume that $\xi_{11} \wedge \ell = 0$. Set

$$M_1 = \left(\begin{array}{cc|cc} x_1 & y_1 & x_2 & y_2 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \end{array} \right).$$

The relation $K_1 \wedge M + K \wedge M_1 = 0$ implies in particular that

$$\xi_{11} \wedge (x, y) + (\xi_{12}, \ldots, \xi_{1n}) \wedge (B, C) + x \wedge y \wedge (x_1, y_1) = 0.$$

Since $\xi_{11} \wedge x \wedge y = 0$, we get

$$\begin{aligned} x \wedge (\xi_{12}, \ldots, \xi_{1n}) \wedge (B, C) &= 0 \\ y \wedge (\xi_{12}, \ldots, \xi_{1n}) \wedge (B, C) &= 0. \end{aligned}$$

Moreover, we have

$$(\xi_{12}, \ldots, \xi_{1n}) \wedge M' + \ell \wedge (x_2, y_2, \cdots) = 0$$

and hence also

$$\begin{aligned} x \wedge (\xi_{12}, \ldots, \xi_{1n}) \wedge M' &= 0 \\ y \wedge (\xi_{12}, \ldots, \xi_{1n}) \wedge M' &= 0. \end{aligned}$$

Now by Lemma 3.6 the row $(\xi_{12}, \ldots, \xi_{1n})$ is spanned by K' and ℓI_{n-1}, that is,

$$(\xi_{12}, \ldots, \xi_{1n}) = (r_2, \ldots, r_n) K' + (s_2, \ldots, s_n) \ell.$$

Let $(\widetilde{K_1}, \widetilde{M_1}) \in S_{(K,M)}$ be given by

$$\begin{aligned} \widetilde{K_1} &= \begin{pmatrix} 0 & r_2 & \cdots & r_n \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} \left(\begin{array}{c|c} \ell & 0 \\ \hline A & K' \end{array} \right) + \left(\begin{array}{c|c} \ell & 0 \\ \hline A & K' \end{array} \right) \begin{pmatrix} 0 & s_2 & \cdots & s_n \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} \\ \widetilde{M_1} &= - \begin{pmatrix} 0 & s_2 & \cdots & s_n \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} \left(\begin{array}{c|c} x \; y & 0 \cdots 0 \\ \hline B \; C & M' \end{array} \right). \end{aligned}$$

Then $(K_1, M_1) - (\widetilde{K_1}, \widetilde{M_1})$ has the form

$$(K_2, M_2) = \left(\begin{pmatrix} \xi' & 0 \cdots 0 \\ A_2 & K_2' \end{pmatrix}, \left(\begin{array}{cc|cc} u_1 & u_2 & u_3 & u_4 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \end{array} \right) \right).$$

The relation $K_2 \wedge M + K \wedge M_2 = 0$ provides us with

$$\ell \wedge (u_3, u_4, \cdots) = 0$$

and hence we see that $u_j = \alpha_j x + \beta_j y$ for $j \geq 3$. If we add $(0, \widetilde{\widetilde{M}}_1) \in S_{(K,M)}$ to (K_2, M_2) with

$$\widetilde{\widetilde{M}}_1 = -\left(\begin{array}{c|c} x \; y & 0 \cdots 0 \\ \hline B\; C & M' \end{array}\right)\left(\begin{array}{c|c} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \alpha_3 & \alpha_4 & \cdots \\ \beta_3 & \beta_4 & \cdots \end{matrix} \\ \hline 0 & 0 \end{array}\right)$$

then we obtain an element of $T'_{(K,M)}$ and hence we see that (K_1, M_1) is an element of $T'_{(K,M)} + S_{(K,M)}$. The converse is obvious by the definition of $T'_{(K,M)}$ and $S_{(K,M)}$.

Q. E. D.

4 Elementary transforms along different lines

The moduli space $I(n)$ of instantons with instanton number n is naturally regarded as an open set (in the classical topology) of the "real part" $\overline{M}(n,2)_{\mathbf{R}}(\mathbf{R})$ of the moduli space of semi-stable sheaves F of rank 2 on X with $c_1(F) = 0$, $c_2(F) = n$ and $c_3(F) = 0$ (see [8, §4]). Since $\overline{M}(n,2)_{\mathbf{R}}(\mathbf{R})$ is a closed set in a real projective space, it is compact and hence so is the closure $\overline{I}(n)$ of $I(n)$. The aim of this section is to prove that elementary transforms of an instanton bundle along different lines are in $\overline{I}(n)$.

To fix the idea pick a member E of $I(n-a)$ and mutually distinct real lines $\ell_1, \ldots, \ell_a$. Since E is quaternionic (see [8, §4]) and since ℓ_i is real, both $E|_{\ell_i}$ and $\mathcal{O}_{\ell_i}(1)$ are quaternionic. Note that $E|_{\ell_i}$ is a trivial bundle [8, §4]. Let β_i be a non-zero real homomorphism of $E|_{\ell_i}$ to $\mathcal{O}_{\ell_i}(1)$, or a homomorphism compatible with the quaternionic structure of $E|_{\ell_i}$ and $\mathcal{O}_{\ell_i}(1)$. Then β_i is surjective and parametrized by the real projective space $\mathbf{P}^3_{\mathbf{R}}$. Set α_i to be the composition of the restriction map $E \to E|_{\ell_i}$ with β_i. Then $E(\ell_1, \ldots, \ell_a; \alpha_1, \ldots, \alpha_a)$ is a quaternionic stable sheaf and defines a point in $\overline{M}(n,2)_{\mathbf{R}}(\mathbf{R})$. As is well-known, $\mathrm{Ext}^2_{\mathcal{O}_X}(E,E) = 0$.

Proposition 4.1. *Under the above notation assume that $a = 1$ and put $F = E(\ell_1; \alpha_1)$. Then F is in $\overline{I}(n)$. More precisely, there is a real analytic curve $C(t)$ in $Y_n(\mathbf{R})$ such that for a suitable $(K, M) \in Y_n(\mathbf{R})$ which defines F, $C(0) = (K, M)$ and $\{\psi_n(C(t)) \mid t > 0\} \subset I(n)$.*

Proof. By Lemma 2.3 F can be presented by a monad $(K, M) \in Y_n(\mathbf{R})$ of the form

$$K = \left(\begin{array}{c|c} x \wedge \sigma x & 0 \\ \hline A & K' \end{array}\right), \qquad M = \left(\begin{array}{c|c} x \;\; \sigma x & 0 \cdots 0 \\ \hline B \;\; \sigma B & M' \end{array}\right),$$

where (K', M') defines E and $\ell = x \wedge \sigma x$ with $A \wedge \ell = 0$. The pair satisfies the conditions of Corollary 3.4.1 and Proposition 3.5. By Corollary 3.4.1 there is a real tangent vector (K_1, M_1) at (K, M) which is not contained in $T'_{(K,M)} + S_{(K,M)}$. By Proposition 3.5, if we write $K_1 = (\xi_{ij})$, then we have $\xi_{11} \wedge \ell \neq 0$. Since $Y_{n,\mathbf{R}}$ is smooth at (K, M), there is a real analytic curve $(K(t), M(t))$ through (K, M) with tangent vector (K_1, M_1) at this point. $K(t)$ then has a form

$$\begin{aligned} K(t) &= K + tK_1 + \text{higher order terms in } t \\ &= \left(\begin{array}{c|c} \ell + t\xi(t) & t\xi_{12}(t), \ldots, t\xi_{1n}(t) \\ \hline A + tA_1(t) & K' + tK'_1(t) \end{array} \right) \end{aligned}$$

with $\xi(0) = \xi_{11}$. Identifying $\bigwedge^2 \mathcal{V}$ with $\bigwedge^2 \mathcal{V}^\vee$, $\det K(t) \in S^n(\bigwedge^2 \mathcal{V})$ gives rise to an element of $S^n(\bigwedge^2 \mathcal{V}^\vee)$ and defines a hypersurface of $\mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)$. Under this convention, [10, Theorem 2.11] and [8, Lemma 6.1 and Proposition 6.2] imply that it is enough to show that if t is sufficiently small and positive, then $\{\det K(t) = 0\}$ does not meet the real part of the Grassmannian in $\mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)$.

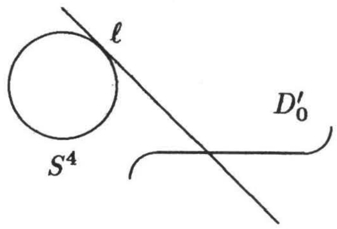
For real points p of $\bigwedge^2 \mathcal{V}$ or $\mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)$, we have (using $\bigwedge^4 \mathcal{V} \cong \mathbf{C}$ and $\xi(p) = \xi \wedge p$)

$$\begin{aligned} f(p,t) &:= \det(K(t) \wedge p) \\ &= (\ell + t\xi(t))(p) \cdot \det(K' + tK'_1(t))(p) \\ &\quad - \sum_{j \geq 2} (-1)^j t\xi_{1j}(t)(p) \cdot \det(A + tA_1(t) \mid \widehat{K_j}(t))(p), \end{aligned}$$

where $\widehat{K_j}(t)$ is the $(n-1) \times (n-2)$-matrix which is obtained from $K' + tK'_1(t)$ by deleting j-th column. $f(p,t)$ is well defined in an affine chart $\mathbf{R}^5$ in the real projective space $\mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)(\mathbf{R})$ containing the real part S^4 of the Grassmannian. Since $K'(\ell)$ is invertible, we can find an open neighborhood $U(\ell)$ of ℓ and an open interval $J \in 0$ in $\mathbf{R}$ such that $\det(K' + tK'_1(t))(p) \neq 0$ on $U(\ell) \times J$. Thus we can write

$$f(p,t) = (\ell(p) + tg(p,t)) \cdot \det(K' + tK'_1(t))(p).$$

We see that $g(\ell, 0) \neq 0$ because $\xi_{11} \wedge \ell \neq 0$ and $A \wedge \ell = 0$. Since $\{\ell = 0\}$ defines the tangent hyperplane of S^4 at ℓ, $\ell(p)$ has the same sign on S^4. Suppose, say, that $\ell(p) \geq 0$ on S^4. We may assume then that $\xi_{11}(\ell) > 0$ and hence that $g(\ell, 0) > 0$. By shrinking $U(\ell)$ and J if necessary, we may assume that $g(p,t) > 0$ on $U(\ell) \times J$. Let D_t be the divisor in $\mathbf{R}^5$ defined by the equation $\det K(t) = 0$. Then J can be



chosen so small that for all $t \in J$

$$D_t \cap (S^4 \setminus U(\ell)) = \emptyset.$$

This follows from the fact that

$$D_0 = \{\ell = 0\} \cup D'_0$$

meets S^4 only at ℓ because E has no real jumping line, where D'_0 is defined by $\det K' = 0$. On $(S^4 \cap U(\ell)) \times J$ however we have

$$\ell(p) \geq 0 \quad \text{and} \quad g(p,t) > 0.$$

It follows that for $t > 0$, $t \in J$ and $p \in S^4 \cap U(\ell)$,

$$\ell(p) + tg(p,t) > 0 \quad \text{and hence} \quad f(p,t) \neq 0.$$

This means that

$$D_t \cap S^4 \cap U(\ell) = \emptyset.$$

Since $D_t \cap (S^4 \setminus U(\ell)) = \emptyset$, this proves that

$$D_t \cap S^4 = \emptyset.$$

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Remark 4.2. In the above we assumed that E is stable. We can take, however, $\mathcal{O}_X^{\oplus 2}$ as E. In fact, F in this case is nothing but Q_ℓ and K is ℓ. Then we can find a family of real hyperplanes $\{D_t\}$ parametrized by a real analytic curve such that $D_0 = \{\ell = 0\}$ and $D_t \cap S^4 = \emptyset$ fo all $t > 0$. On the other hand, the moduli space $\overline{M}(1,2)_{\mathbf{R}}$ is exactly the space of real hyperplane in $\mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)$, which is isomorphic to $\mathbf{P}^5_{\mathbf{R}}$, and $Y_{1,\mathbf{R}}$ is a $\mathbf{R}^*$-bundle over it.

We can now prove one of the main results of this article.

Theorem 4.3. *Let a be a positive integer with $a \leq n$ and let E be a member of $I(n-a)$ or $\mathcal{O}_X^{\oplus 2}$ according to $a < n$ or $a = n$. Pick lines $\ell_1, \ldots, \ell_a$ and homomorphisms $\alpha_1, \ldots, \alpha_a$ as in the beginning of this section. Then the quaternionic sheaf $F = E(\ell_1, \ldots, \ell_a; \alpha_1, \ldots, \alpha_a)$ is in $\overline{I}(n)$.*

Proof. We proceed by induction on a. The case where $a = 1$ was proved in Proposition 4.1 and Remark 4.2. Setting $E' = E(\ell_1; \alpha_1)$, F is isomorphic to $F'(\ell_2, \ldots, \ell_a; \beta_2, \ldots, \beta_a)$ for suitable $\beta_2, \ldots, \beta_a$ because E and E' are isomorphic outside ℓ_1. By Proposition 4.1 and Remark 4.2, there exists a real analytic family $(K'(t), M'(t))$ of monads defining a monad (K'_0, M'_0) of E' and defining instantons $E'_t \in I(n-a+1)$ for $t > 0$. We can extend this family to a holomorphic deformation

$\widetilde{E}$ of E' over a disc $0 \in \Delta \subset \mathbf{C}$. Moreover, the action of σ on X extends to $X \times \Delta$ by using the conjugation on Δ and $\widetilde{E}$ is quaternionic with respect to this action. Since $\widetilde{E}|_{\ell_i \times \{0\}} = E'|_{\ell_i}$ is trivial for $i \geq 2$, we may even assume that $\widetilde{E}|_{\ell_i \times \{t\}}$ is trivial for all $t \in \Delta$.

$$\begin{array}{ccc} \ell_i \times \Delta & \xrightarrow{q_i} & \ell_i \\ \downarrow{\scriptstyle \pi_i} & & \\ \Delta & & \end{array}$$

Hence the direct images $\pi_{i*}(\widetilde{E}|_{\ell_i \times \Delta})$ are (locally) free of rank 2 and similarly $\pi_{i*}q_i^*(\mathcal{O}_{\ell_i}(1))$ are free. From this it follows that for $i \geq 2$, the given isomorphism

$$\Gamma(\beta_i) : \Gamma(E'|_{\ell_i}) \longrightarrow \Gamma(\mathcal{O}_{\ell_i}(1))$$

induced by β_i can be extended to a quaternionic isomorphism

$$\pi_{i*}(\widetilde{E}|_{\ell_i \times \Delta}) \longrightarrow \pi_{i*}q_i^*(\mathcal{O}_{\ell_i}(1)).$$

We get therefore a commutative diagram with surjective $\widetilde{\beta}_i$

$$\begin{array}{ccc} \pi^*\pi_{i*}(\widetilde{E}|_{\ell_i \times \Delta}) & \xrightarrow{\sim} & \pi_i^*\pi_{i*}q_i^*(\mathcal{O}_{\ell_i}(1)) \\ \| \wr & & \downarrow \\ \widetilde{E}|_{\ell_i \times \Delta} & \xrightarrow{\widetilde{\beta}_i} & q_i^*(\mathcal{O}_{\ell_i}(1)) \end{array}$$

and $\widetilde{\beta}_i$ extends the surjection $E'|_{\ell_i} \to \mathcal{O}_{\ell_i}(1)$. If we write

$$Z = \ell_2 \cup \cdots \cup \ell_a, \quad q : Z \times \Delta \longrightarrow Z,$$

we get a surjection

$$\widetilde{E} \longrightarrow \widetilde{E}|_{Z \times \Delta} \longrightarrow q * (\mathcal{O}_{\ell_2}(1) \oplus \cdots \oplus \mathcal{O}_{\ell_a}(1)).$$

Let $\widetilde{F}$ be its kernel. Then $\widetilde{F}$ is a flat deformation of F over Δ. For every real, positive $t \in \Delta$, we have an exact sequence

$$0 \longrightarrow \widetilde{F}_t \longrightarrow E'_t \longrightarrow \mathcal{O}_{\ell_2}(1) \oplus \cdots \oplus \mathcal{O}_{\ell_a}(1) \longrightarrow 0$$

with $E'_t \in I(n-a+1)$. Thus by our induction hypothesis we see that for every real, positive $t \in \Delta$, $\widetilde{F}_t$ is a member of $\overline{I}(n)$. It follows that F is contained in $\overline{I}(n)$

Q.E.D.

5 Splitting of m-fold extensions of $\mathcal{O}_\ell(1)$

Let ℓ be a line in X. An $\mathcal{O}_X$-module L is called an m-fold successive extension of $\mathcal{O}_\ell(1)$ if it is an extension

$$0 \longrightarrow \mathcal{O}_\ell(1) \longrightarrow L \longrightarrow L' \longrightarrow 0,$$

where L' is an $(m-1)$-fold extension of $\mathcal{O}_\ell(1)$ ($L' = \mathcal{O}_\ell(1)$ if $m = 2$). It appears as the cokernel of an m-fold successive elementary transformation along the line ℓ:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{O}_\ell(1) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & F' & \longrightarrow & E & \longrightarrow & L' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{O}_\ell(1) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where we consider F' as an $(m-1)$-fold successive elementary transformation with E a vector bundle.

Notation and Remark 5.1. In this section we fix a basis $\{z_0, z_1, w_0, w_1\}$ of $\mathcal{V}^\vee$ and take a line ℓ in X which is defined by the equation $z_0 = z_1 = 0$ with z_0 and z_1 elements of the symmetric algebra $S(\mathcal{V}^\vee)$. Then w_0 and w_1 can be regarded as linear forms on ℓ. Let $A = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ be a 2×1-matrix whose entries are in the algebra $S(\mathcal{V}^\vee)$. We define A' to be the matrix $(-a_1, a_0)$. In particular, we set $Z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$ and then $Z' = (-z_1, z_0)$. If both A and B are 2×1-matrices as in the above, then we have an identity

$$A'B + B'A = 0.$$

It is easy to see that if $A = \lambda B$, then we have $A'B = 0$. Conversely, if A (or B) is non-degenerate, that is, the linear map $\mathbf{C}^2 \to \mathcal{V}^\vee$ defined by A (or B, resp.) is injective, then the equality $A'B = 0$ implies that $B = \lambda A$ (or $A = \lambda B$, resp.) for some $\lambda \in \mathbf{C}$

We are going to study resolutions of m-fold extensions of $\mathcal{O}_\ell(1)$. Let us begin with a basic resolution.

Proposition 5.2. *Let L be an m-fold extension of $\mathcal{O}_\ell(1)$. Then L has a resolution*

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus m} \xrightarrow{q} \mathcal{O}_X^{\oplus 2m} \xrightarrow{p} \mathcal{O}_X(1)^{\oplus m} \longrightarrow L \longrightarrow 0$$

such that p and q are represented by the following matrices P and Q, respectively:

$$P = \begin{pmatrix} Z & & & & \\ A_{21} & Z & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{m1} & \cdots & \cdots & A_{m,m-1} & Z \end{pmatrix}$$

$$Q = \begin{pmatrix} Z' & & & & \\ A'_{21} & Z' & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A'_{m1} & \cdots & \cdots & A'_{m,m-1} & Z' \end{pmatrix}$$

where A_{ij} is a 2×1-matrix whose entries are linear forms in w_0, w_1 only.

Proof. Our proof is by induction on m. When $m = 1$, our assertion is nothing but a Koszul complex of $\mathcal{O}_\ell(1)$. Using the resolution in our proposition for an m-fold extension L', we have the standard construction of a resolution of an extension

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-1) & \longrightarrow & \mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1)^{\oplus m} & \longrightarrow & \mathcal{O}_X(-1)^{\oplus m} & \longrightarrow & 0 \\
 & & \downarrow Z' & & \downarrow \tilde{Q} & & \downarrow Q & & \\
0 & \longrightarrow & \mathcal{O}_X^{\oplus 2} & \longrightarrow & \mathcal{O}_X^{\oplus 2} \oplus \mathcal{O}_X^{\oplus 2m} & \longrightarrow & \mathcal{O}_X^{\oplus 2m} & \longrightarrow & 0 \\
 & & \downarrow Z & & \downarrow \tilde{P} & & \downarrow P & & \\
0 & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)^{\oplus m} & \longrightarrow & \mathcal{O}_X(1)^{\oplus m} & \longrightarrow & 0 \\
 & & \downarrow D & & \downarrow \tilde{C} & & \downarrow C & & \\
0 & \longrightarrow & \mathcal{O}_\ell(1) & \xrightarrow{\varepsilon} & L & \xrightarrow{\pi} & L' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

There is a homomorphism $\rho : \mathcal{O}_X(1)^{\oplus m} \to L$ lifting C. We have $\pi \cdot \rho \cdot P = 0$. Thus $\rho \cdot P$ is factored through ε or there is a homomorphism α of $\mathcal{O}_X^{\oplus 2m}$ to $\mathcal{O}_\ell(1)$

such that $\rho \cdot P = \varepsilon \cdot \alpha$. α has a lifting $-A$ which is represented by a 2×1-matrix whose entries are linear forms in $\mathcal{V}^\vee$. We may assume that A is independent of z_0, z_1, for otherwise we can modify it by using multiples of Z. Now $C = \varepsilon \cdot D + \rho$ is surjective and it is easy to see that its kernel is generated by the matrix

$$\widetilde{P} = \begin{pmatrix} Z & 0 \\ A & P \end{pmatrix}.$$

It follows then that the kernel of $\widetilde{P}$ is generated by

$$\widetilde{Q} = \begin{pmatrix} Z' & 0 \\ F & Q \end{pmatrix},$$

where $F : \mathcal{O}_X(-1)^{\oplus m} \to \mathcal{O}_X^{\oplus 2}$ satisfies the following matrix identity

$$FZ + QA = 0.$$

Using this we can reduce F as follows: Writing by induction hypothesis

$$P = \begin{pmatrix} Z & & & & \\ A_{21} & Z & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{m1} & \cdots & \cdots & A_{m,m-1} & Z \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$$

we have

$$F_\mu Z + \sum_{\nu<\mu} A'_{\mu\nu} A_\nu + Z' A_\mu = 0.$$

Since the entries of $A'_{\mu\nu}$ and A_ν are linear forms only in w_0, w_1, we must have

$$\sum_{\nu<\mu} A'_{\mu\nu} A_\nu = 0 \quad \text{and} \quad F_\mu Z + Z' A_\mu = 0.$$

Since the second equation is equivalent to $(F_\mu + A'_\mu)Z = 0$ and since Z is non-degenerate, we can find an α_μ in $\mathbf{C}$ such that $F_\mu = A'_\mu + \alpha_\mu Z'$. If we multiply $\widetilde{Q}$ by

$$\begin{pmatrix} 1 & & & & \\ -\alpha_1 & 1 & & & \\ -\alpha_2 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ -\alpha_m & 0 & \cdots & 0 & 1 \end{pmatrix}$$

from the left, the new $\widetilde{Q}$ has the form

$$\left(\begin{array}{c|c} Z' & 0\cdots 0 \\ \hline A'_1 & \\ \vdots & Q \\ A'_m & \end{array}\right).$$

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Let us make a remark which is quite critical in the sequel.

Remark 5.3. If a 2×1-matrix A whose entries are linear forms in w_0, w_1 is non-degenerate, then it induces a rank 1 matrix at each point of ℓ.

As a corollary to the above we obtain

Proposition 5.4. *Let L be an m-fold extension of $\mathcal{O}_\ell(1)$. Assume that we have a resolution as in Proposition 5.2 with all $A_{i,i-1}$ non-degenerate. Then $L\otimes\mathcal{O}_\ell\cong\mathcal{O}_\ell(1)$ and if $m\geq 2$, then $\mathcal{T}or_1^{\mathcal{O}_X}(L,\mathcal{O}_\ell)\cong\mathcal{O}_\ell(-1)\oplus\mathcal{O}_\ell(1)$.*

Proof. Restricting the resolution in Proposition 5.2 to the line ℓ, we obtain a complex

$$0\longrightarrow\mathcal{O}_\ell(-1)^{\oplus m}\xrightarrow{\bar{q}}\mathcal{O}_\ell^{\oplus 2m}\xrightarrow{\bar{p}}\mathcal{O}_\ell(1)^{\oplus m}\longrightarrow L\otimes\mathcal{O}_\ell\longrightarrow 0$$

where $\bar{p}$ and $\bar{q}$ are represented by the following matrices $\overline{P}$ and $\overline{Q}$, respectively:

$$\overline{P}=\begin{pmatrix} 0 & & & & \\ A_{21} & 0 & & & \\ \vdots & & \ddots & & \\ \vdots & & & 0 & \\ A_{m1} & \cdots & \cdots & A_{m,m-1} & 0 \end{pmatrix}$$

$$\overline{Q}=\begin{pmatrix} 0 & & & & \\ A'_{21} & 0 & & & \\ \vdots & & \ddots & & \\ \vdots & & & 0 & \\ A'_{m1} & \cdots & \cdots & A'_{m,m-1} & 0 \end{pmatrix}.$$

By our assumption these two matrices have a constant rank ($=m-1$) at any point on ℓ. It follows that $\mathcal{T}or_i^{\mathcal{O}_X}(L,\mathcal{O}_\ell)$ are locally free $\mathcal{O}_\ell$-modules. Furthermore, we see that $r(L\otimes\mathcal{O}_\ell)=1$ and $r(\mathcal{T}or_1^{\mathcal{O}_X}(L,\mathcal{O}_\ell))=2$, whereas $\mathcal{T}or_2^{\mathcal{O}_X}(L,\mathcal{O}_\ell)\cong\mathcal{O}_\ell(-1)$.

By the shape of $\overline{P}$ we know that there is a surjection of $L \otimes \mathcal{O}_\ell$ to $\mathcal{O}_\ell(1)$, which must be isomorphism. From the defining complex and the above results we deduce that

$$\begin{aligned} \deg(\mathcal{T}or_1^{\mathcal{O}_X}(L,\mathcal{O}_\ell)) &= \deg(L \otimes \mathcal{O}_\ell) + \deg(\mathcal{T}or_2^{\mathcal{O}_X}(L,\mathcal{O}_\ell)) \\ &= 1 - 1 = 0. \end{aligned}$$

Put $K = \ker(\bar{p})$. We find that $K \cong \mathcal{O}_\ell^{\oplus 2} \oplus \mathcal{O}_\ell(-1)^{\oplus m-1}$. The homomorphism $\bar{q}$ induces r

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_\ell(-1)^{\oplus m-1} & \xrightarrow{r} & K & \longrightarrow & \mathcal{T}or_1^{\mathcal{O}_X}(L,\mathcal{O}_\ell) & \longrightarrow & 0 \\ & & & \searrow & \downarrow & & & & \\ & & & & \mathcal{O}_\ell^{\oplus 2m} & & & & \end{array}$$

where the map of $\mathcal{O}_\ell(-1)^{\oplus m-1}$ to $\mathcal{O}_\ell^{\oplus 2m}$ is represented by the matrix

$$\begin{pmatrix} A'_{21} & 0 & & \\ \vdots & \ddots & & \\ A'_{m1} & \cdots & A'_{m,m-1} & 0 \end{pmatrix}.$$

Now it is easy to see that the matrix for r has the shape

$$\left(\begin{array}{c|ccc|c} A'_{21} & 0 & \cdots & 0 & 0 \\ \hline A'_{31} & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ A'_{m1} & & & 1 & 0 \end{array}\right).$$

From this we get our assertion on $\mathcal{T}or_1^{\mathcal{O}_X}(L,\mathcal{O}_\ell)$. Q.E.D.

Here we make a remark which is simple but quite useful to understand the structure of m-fold extensions of $\mathcal{O}_\ell(1)$.

Lemma 5.5. *Let $\mathcal{I}$ be the ideal sheaf of ℓ in $\mathcal{O}_X$. Then $H^1(X,\mathcal{I}L) = 0$.*

Proof. Since we have a surjection $\mathcal{O}_X^{\oplus 2} \to \mathcal{I}(1)$, there is a surjective homomorphism of $L(-1)^{\oplus 2}$ to $\mathcal{I} \otimes_{\mathcal{O}_X} L$, a fortiori, to $\mathcal{I}L$. This supplies us with a surjection $H^1(X, L(-1)^{\oplus 2}) \to H^1(X,\mathcal{I}L)$ because $L(-1)^{\oplus 2}$ is supported by one-dimensional ℓ. On the other hand, by the definition of m-fold extension of $\mathcal{O}_\ell(1)$ we see that $H^1(X, L(-1)^{\oplus 2}) = 0$. These prove our assertion.

We are going to investigate m-fold extensions L of $\mathcal{O}_\ell(1)$ which admit a surjection $\mathcal{O}_X^{\oplus 2} \to L$. By the above lemma we know that the natural map $H^0(X,L) \to H^0(X, L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell)$ is surjective. Thus L is generated by two sections if and only if so is $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$.

Proposition 5.6. *Let L be an m-fold extension of $\mathcal{O}_\ell(1)$ which is generated by two sections. Consider a resolution in Proposition 5.2. Then $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ is a locally free $\mathcal{O}_\ell$-module if and only if every $A_{i,i-1}$ is non-degenerate.*

Proof. By virtue of Proposition 5.4 we have only to prove that every $A_{i,i-1}$ is non-degenerate if $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ is a locally free $\mathcal{O}_\ell$-module. By our assumption we have two surjections $\mathcal{O}_\ell(1)^{\oplus m} \to L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ and $\mathcal{O}_\ell^{\oplus 2} \to L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$. By the latter we see that $r(L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell) \leq 2$ and that $r(L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell) = 2$ only if $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell \cong \mathcal{O}_\ell^{\oplus 2}$. This is, however, impossible by the first surjection. Thus $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ is a line bundle. By looking at the shape of the resolution, we find a surjection of $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ to $\mathcal{O}_\ell(1)$. These show that $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell \cong \mathcal{O}_\ell(1)$.

Let $W = A_{m,m-1}$ and let L'' be the cokernel of

$$\begin{pmatrix} Z & \\ W & Z \end{pmatrix} : \mathcal{O}_X^{\oplus 4} \longrightarrow \mathcal{O}_X(1)^{\oplus 2}.$$

Restricting them to ℓ we have an exact sequence

$$\mathcal{O}_\ell^{\oplus 4} \xrightarrow{\begin{pmatrix} 0 & \\ W & 0 \end{pmatrix}} \mathcal{O}_\ell(1)^{\oplus 2} \longrightarrow L'' \otimes_{\mathcal{O}_X} \mathcal{O}_\ell \longrightarrow 0.$$

It follows from this that $L'' \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ is locally free if and only if W is non-degenerate. We have a surjection $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell \to L'' \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$. By this and our assumption there is a surjection $\mathcal{O}_\ell(1)^{\oplus 2} \to L'' \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ and hence we see that $W \neq 0$. Then $L'' \otimes_{\mathcal{O}_X} \mathcal{O}_\ell \cong \mathcal{T} \oplus \mathcal{O}_\ell(1)$, where $\mathcal{T}$ is a torsion $\mathcal{O}_\ell$-module. As we have seen in the above, we have a surjective map of $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ to $\mathcal{T} \oplus \mathcal{O}_\ell(1)$, which implies that $\mathcal{T} = 0$. This comletes the proof of the non-degeneracy of $A_{m,m-1}$.

Assume that there is a degenerate $A_{j,j-1}$ and let i be the biggest integer such that $A_{i,i-1}$ is degenerate. From the matrix identity $QP = 0$ we obtain

$$A'_{j+1,j-1} Z + A'_{j+1,j} A_{j,j-1} + Z' A_{j+1,j-1} = 0.$$

Since entries of A_{jk} are linear forms only in w_0, w_1, we have the relation

$$A'_{j+1,j} A_{j,j-1} = 0.$$

Thus the non-degeneracy of $A_{j,j-1}$ implies by Remark 5.1

$$A_{j,j-1} = \alpha_j W \quad \text{for } j > i \text{ and suitable } \alpha_j \in \mathbf{C}^*.$$

We can go one more step to obtain

$$A_{i,i-1} = \alpha_i W \quad \text{for a suitable } \alpha_i \in \mathbf{C},$$

and then by the choice of i, α_i must be 0. Consider the decomposition of the matrix P

$$
\left(\begin{array}{ccccc|ccccc}
Z & & & & & & & & & \\
A_{21} & Z & & & & & & & & \\
\vdots & & \ddots & & & & & & & \\
\vdots & & & Z & & & & & & \\
A_{i1} & \cdots & \cdots & 0 & Z & & & & & \\
\hline
A_{i+1,1} & \cdots & \cdots & \cdots & A_{i+1,1} & Z & & & & \\
\vdots & & & & \vdots & \vdots & \ddots & & & \\
\vdots & & & & \vdots & \vdots & & \ddots & & \\
\vdots & & & & \vdots & \vdots & & & Z & \\
A_{m,1} & \cdots & \cdots & \cdots & A_{m,i} & A_{m,i+1} & \cdots & \cdots & W & Z
\end{array}\right)
$$

and the corresponding extension

$$0 \longrightarrow L_1 \longrightarrow L \longrightarrow L_2 \longrightarrow 0.$$

In the induced sequence

$$
\begin{array}{ccccccccc}
\longrightarrow \mathcal{T}or_1^{\mathcal{O}_X}(L_2, \mathcal{O}_\ell) & \longrightarrow & L_1 \otimes_{\mathcal{O}_X} \mathcal{O}_\ell & \xrightarrow{\theta} & L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell & \xrightarrow{\pi} & L_2 \otimes_{\mathcal{O}_X} \mathcal{O}_\ell & \longrightarrow & 0 \\
& \searrow^{\beta} & \downarrow & & & & & & \\
& & \mathcal{O}_\ell(1)^{\oplus 2} & & & & & &
\end{array}
$$

π must be an isomorphism because $L_2 \otimes_{\mathcal{O}_X} \mathcal{O}_\ell \cong \mathcal{O}_\ell(1)$ by Proposition 5.4. Hence $\theta = 0$. There is a surjection $L_1 \to \mathcal{O}_\ell(1)^{\oplus 2}$ by the shape of the defining matrix of L_1. It follows that the induced homomorphism β is surjective. On the other hand, $\mathcal{T}or_1^{\mathcal{O}_X}(L_2, \mathcal{O}_\ell)$ is isomorphic to $\mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1)$ by Proposition 5.4 and there is no surjection of $\mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1)$ to $\mathcal{O}_\ell(1)^{\oplus 2}$. This is a contradiction. Q. E. D.

We can normalize the matrices P and Q in Proposition 5.2 under the assumption of Proposition 5.6. In fact

Proposition 5.7. *Let L be an m-fold extension of $\mathcal{O}_\ell(1)$ and assume that $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ is a locally free $\mathcal{O}_\ell$-module and generated by two sections. Then the matrices Q and P which appeared in the resolution of L in Proposition 5.2 can be* *normalized to the following forms, respectively*

$$\begin{pmatrix} Z' & & & & \\ A_1' & Z' & & & \\ A_2' & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ A_{m-1}' & \cdots & A_2' & A_1' & Z' \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} Z & & & & \\ A_1 & Z & & & \\ A_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ A_{m-1} & \cdots & A_2 & A_1 & Z \end{pmatrix}$$

with A_1 non-degenerate (the matrix relation $QP = 0$ is now trivially satisfied).

Proof. Our proof is by induction on m. Our assertion is trivial if $m = 1$. Assuming our proposition is true up to m, let us consider the case of $m+1$. By Proposition 5.2 and the induction hypothesis Q and P can be transformed to the following forms by choosing suitable bases of the terms in the resolution of L

$$Q = \left(\begin{array}{c|ccccc} Z' & & & & & \\ \hline B_1' & Z' & & & & \\ B_2' & A_1' & Z' & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ B_{m-1}' & \vdots & & \ddots & \ddots & \\ A_m' & A_{m-1}' & \cdots & \cdots & A_1' & Z' \end{array}\right)$$

$$P = \left(\begin{array}{c|ccccc} Z & & & & & \\ \hline B_1 & Z & & & & \\ B_2 & A_1 & Z & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ B_{m-1} & \vdots & & \ddots & \ddots & \\ A_m & A_{m-1} & \cdots & \cdots & A_1 & Z \end{array}\right)$$

with B_ν and A_ν matrices of linear forms only in w_0, w_1. Note that by Proposition 5.6 A_1 and B_1 are non-degenerate. Multiplying the third row of Q by the first column of P, we have

$$A_1' B_1 = 0 \quad \text{and hence} \quad B_1 = \lambda A_1$$

with $\lambda \neq 0$. Looking at the $(4,1)$-entry of QP, we see that

$$B_3' Z + A_2'(\lambda A_1) + A_1' B_2 + Z' B_3 = 0,$$

which implies that

$$A_1'(B_2 - \lambda A_2) = 0$$

or $B_2 = \lambda A_2 + \mu A_1$. We can modify the right hand matrix P by adding $-\mu$·second column to the first column without changing the extension. The term $-\mu Z$ at the $(2,1)$-entry can be compensated by adding μ-first row to the second row. This gives us a new resolution of the same type with $B_1 = \lambda A_1$ and $B_2 = \lambda A_2$. Now assume that P is transformed to the form with $B_i = \lambda A_i$ for $i \le \nu$. Computing the $(\nu+2,1)$-entry of QP, we obtain

$$B'_{\nu+2}Z + A'_{\nu+1}(\lambda A_1) + \cdots + A'_2(\lambda A_\nu) + A'_1 B_{\nu+1} + Z' B_{\nu+2} = 0,$$

where we understand $B'_m = A'_m$ and $B_m = A_m$. Since we have $A'_\nu A_2 + \cdots + A'_2 A_\nu = 0$, the above equality gives rise to

$$\begin{aligned} A'_{\nu+1}(\lambda A_1) + A'_1 B_{\nu+1} &= 0 \quad \text{or} \\ A'_1(B_{\nu+1} - \lambda A_{\nu+1}) &= 0. \end{aligned}$$

This provides us with a ρ in $\mathbf{C}$ such that $B_{\nu+1} = \lambda A_{\nu+1} + \rho A_1$. By using the $(\nu+1)$-th column and then the first row we can modify our matrix P to obtain $B_{\nu+1} = \lambda A_{\nu+1}$ without changing B_i for $i \le \nu$ as in the second step. By induction on ν, therefore, we come to the case where $B_i = \lambda A_i$ for some $\lambda \ne 0$ and for all $i \le m-1$. Finally taking

$$R = \begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad S = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad T = \begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

we can replace Q and P by

$$RQS \quad \text{and} \quad S^{-1}PT$$

which now have the desired form. Q.E.D.

Remark 5.8. Assume that L is generated by two sections. There are two surjections $\pi : \mathcal{O}_X(1)^{\oplus m} \to L$ and $\gamma : \mathcal{O}_X^{\oplus 2} \to L$. By the shape of the resolution in Proposition 5.2 we see that $H^1(X, \ker(\pi)) = 0$ and hence there is a homomorphism β of $\mathcal{O}_X^{\oplus 2}$ to $\mathcal{O}_X(1)^{\oplus m}$ which lifts γ. If a $(2 \times m)$-matrix $(B_1, \ldots, B_m)$ represents the map β, then we have a surjection

$$\mathcal{O}_X^{\oplus(2m+2)} \longrightarrow \mathcal{O}_X(1)^{\oplus m} \longrightarrow 0$$

which is represented by the matrix

$$\begin{pmatrix} Z & & & \\ A_{21} & \ddots & & \\ \vdots & \ddots & \ddots & \\ A_{m1} & \cdots & A_{m,m-1} & Z \\ B_1 & \cdots & B_{m-1} & B_m \end{pmatrix}.$$

This is the right part of a monad of the $\mathcal{O}_X$-module $\ker(\gamma)$. We can assume that every B_i is independent of Z_0, Z_1.

Remark 5.9. If L is generated by two sections, then $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ can have only *simple* torsion, that is, for the torsion part $\mathcal{T}$ of $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$, we have $\mathcal{T}_p \cong \mathbf{C}_p$ at every point p in the support of $\mathcal{T}$.

Proof. By the above remark we see that

$$\operatorname{rank} \begin{pmatrix} A_{21} & & & \\ \vdots & \ddots & & \\ A_{m1} & \cdots & A_{m,m-1} & \\ B_1 & \cdots & B_{m-1} & B_m \end{pmatrix}(p) = m$$

at every point p of ℓ. Thus

$$\operatorname{rank} \begin{pmatrix} A_{21} & & \\ \vdots & \ddots & \\ A_{m1} & \cdots & A_{m,m-1} \end{pmatrix}(p) \geq m-2$$

at every point p. Since $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell \cong \mathcal{T} \oplus \mathcal{O}_\ell(1)$, there is an exact sequence

$$\mathbf{C}_p^{2m} \xrightarrow{\delta} \mathbf{C}_p^m \longrightarrow \mathcal{T}_p \oplus \mathbf{C}_p \longrightarrow 0$$

at every point p in the support of $\mathcal{T}$, where δ is represented by the second matrix in the above. This completes the proof.

Now we come to the point to study the deformations of m-fold extensions of $\mathcal{O}_\ell(1)$ under the additional conditions in Proposition 5.7. Pick an m-fold extension L of $\mathcal{O}_\ell(1)$ such that L is generated by two sections and $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ is a locally free $\mathcal{O}_\ell$-module. Then L has a resolution as in Proposition 5.7 whose first matrix is given by

$$Z(A) = \begin{pmatrix} Z & & & & \\ A_1 & Z & & & \\ A_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ A_{m-1} & \cdots & A_2 & A_1 & Z \end{pmatrix}.$$

We define a new matrix $B = (B_{ij})$ of the same size as follows:

$$B_{ii} = \begin{cases} 0 & \text{if } i \text{ is odd} \\ -A_i & \text{if } i \text{ is even} \end{cases}$$

$B_{ij} = 0$ if $i < j$

$B_{ij} = 0$ if $i > j$ and i or j is even

$B_{ij} = A_{i-j+1}$ if $i > j$ and both i and j are odd.

Example: The case of $m = 7$.

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_1 & 0 & 0 & 0 & 0 & 0 \\ A_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & 0 & 0 \\ A_5 & 0 & A_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -A_1 & 0 \\ 0 & 0 & A_5 & 0 & A_3 & 0 & 0 \end{pmatrix}.$$

For a matrix $C = (C_{ij})$ consisting of (2×1)-cells C_{ij}, C' denotes the matrix (C'_{ij}). A denotes the matrix obtained from $Z(A)$ by substituting Z by 0. By checking the different cases of odd and even indices we get

$$A'B + B'A = 0 \quad \text{and} \quad B'B = 0.$$

Let T be an affine open neighborhood of 0 in the affine line $\mathbf{C}$. The above equalities provide us with

$$(Z(A)' + tB')(Z(A) + tB) = 0$$

for all t in T. Therefore, the matrices $Z(A)' + tB'$ and $Z(A) + tB$ define a resolution of a flat family $\tilde{L}$ over T. Since $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ is locally free,

$$\operatorname{rank} Z(A)(x) \geq m - 1$$

at every point x of X and therefore also

$$\operatorname{rank}(Z(A) + tB)(x) \geq m - 1$$

at every point $(x, t) \in X \times T$ if T is small enough.

Proposition 5.10. *Let ℓ_t be the line in X defined by the equation $\{Z - tA_1 = 0\}$. $\ell \cap \ell_t = \emptyset$ if $t \neq 0$. If $0 \neq t \in T$, then $\tilde{L}(t)$ decomposes into a direct sum $L_1 \oplus L_2$ of multiple extensions L_1 and L_2 of $\mathcal{O}_\ell(1)$ and $\mathcal{O}_{\ell_t}(1)$. Moreover, if T is* *sufficiently small, then both $L_1 \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ and $L_2 \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ are locally free and generated by two sections.*

Proof. Since entries of A_1 are linear forms only in w_0, w_1 and since A_1 is non-degenerate, it is obvious that $\ell \cap \ell_t = \emptyset$ if $t \neq 0$. The other assertions will be proved by induction on m. By the shape of $Z(A) + tB$ we obtain an extension

$$0 \longrightarrow L'_t \longrightarrow \tilde{L}(t) \longrightarrow L''_t \longrightarrow 0,$$

where L'_t is the cokernel of the homomorphism defined by the matrix

$$\begin{pmatrix} Z & 0 \\ A_1 & Z - tA_1 \end{pmatrix}.$$

Since $\ell \cap \ell_t = \emptyset$, L'_t is isomorphic to $\mathcal{O}_\ell(1) \oplus \mathcal{O}_{\ell_t}(1)$. Now we may assume that $L''_t \cong L''_1 \oplus L''_2$, where L''_1 and L''_2 are as in the proposition for $m-2$. By using the disjointness of lines again, we see that $\tilde{L}_t$ splits as desired. By the rank condition $L_1 \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ and $L_2 \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ are locally free. By the base change theorem $q_*(\tilde{L})$ is locally free on T, where q is the projection of $X \times T$ to T. Then two global sections which generate $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ extends to sections of $\tilde{L}$. These sections define a homomorphism δ of $\mathcal{O}_{X \times T}^{\oplus 2}$ to $\tilde{L}$. Since $\delta(0)$ is surjective, δ is surjective over $X \times T$ if T is small enough. Q. E. D.

Remark 5.11. Let ℓ be a real line. $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_\ell(1), \mathcal{O}_\ell(1))$ carries a natural real structure because $\mathcal{O}_\ell(1)$ is quaternionic. If we pick a real extension class, then the 2-fold extension defined by it has a quaternionic structure which extends the structure of $\mathcal{O}_\ell(1)$. Assume that an $(m-1)$-fold extension L' of $\mathcal{O}_\ell(1)$ carries a quaternionic structure. Then, we get a quaternionic structure on the m-fold extension of $\mathcal{O}_\ell(1)$ corresponding to a real point of $\mathrm{Ext}^1_{\mathcal{O}_X}(L', \mathcal{O}_\ell(1))$. Conversely, if an m-fold extension L has a quaternionic structure, $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ is a locally free $\mathcal{O}_\ell$-module and if L is generated by two sections, then L is obtained in the above way. In fact, $L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ is isomorphic to $\mathcal{O}_\ell(1)$ and the kernel of the natural map $L \to L \otimes_{\mathcal{O}_X} \mathcal{O}_\ell$ has the same properties as L.

We may assume that $\sigma^\vee(z_0) = z_1$ and $\sigma^\vee(w_0) = w_1$. Then the above results are valid under an additional assumption *real* or *quaternionic* in the following sense. All the (2×1)-matrices $C = {}^t(c_0, c_1)$ which appeared in Propositions 5.2, 5.4, 5.6 and 5.7 should satisfy an additional condition $\sigma^\vee(c_0) = c_1$. In Proposition 5.10 if we assume that L is quaternionic, then $\tilde{L}_t, L_1$ and L_2 can be quaternionic after we replace T by the real part of T. Moreover, we may assume that T is invariant under the action of the conjugation and then we see that $\tilde{L}$ is quaternionic with respect to the extension of σ to $X \times T$ as in the proof of Theorem 4.3.

Let E be an instanton bundle and ℓ a real line in X. Pick an m-fold extension L of $\mathcal{O}_\ell(1)$ such that $L\otimes_{\mathcal{O}_X}\mathcal{O}_\ell$ and L is generated by two sections. L is an extension

$$0 \longrightarrow \mathcal{O}_\ell(1) \longrightarrow L \longrightarrow L' \longrightarrow 0.$$

Since $E|_\ell \cong \mathcal{O}_\ell^{\oplus 2}$, there exists a surjection of E to $\mathcal{O}_\ell(1)$. Suppose that we have a surjection $\alpha' : E \to L'$. Since $H^1(X, E^\vee \otimes_{\mathcal{O}_X}\mathcal{O}_\ell(1)) = 0$, α lifts to a homomorphism of E to L, which is surjective by Nakayama's lemma. Thus we get a surjective homomorphism $\alpha : E \to L$. Obviously, α can be real.

Theorem 5.12. *Let $\ell_1, \dots, \ell_k$ be mutually distinct real lines in X and let L_i be m_i-fold extensions of $\mathcal{O}_{\ell_i}(1)$. Assume that every L_i carries a quaternionic structure, $L_i \otimes_{\mathcal{O}_X}\mathcal{O}_\ell$ is locally free and L_i is generated by two sections. Set $m = m_1 + \cdots + m_k$ and $L = L_1 \oplus \cdots \oplus L_k$. If E is an instanton bundle with instanton number $n - m$ and if $\alpha : E \to L$ is a real surjection, then $F = \ker(\alpha)$ is contained in $\overline{I}(n)$.*

Proof. We proceed by induction on $r = m - k$. If $r = 0$, then all the m_i's are 1 and hence our assertion is Theorem 4.3. We may now assume that $m_1 \geq 2$. By Proposition 5.10 there is a flat splitting deformation $\widetilde{L}_1$ of L_1 over an affine open neighborhood T of 0 in the affine line $\mathbf{C}$. Let q be the projection of $X \times T$ to T and let L_{iT} and E_T be the pull-backs of L_i and E by another projection $X \times T \to X$. Then E_T, $\widetilde{L}$ and L_{iT} are quaternionic with respect to the extended σ (see Remark 5.11). Setting $\widetilde{L} = \widetilde{L}_1 \oplus L_{2T} \oplus \cdots \oplus L_{kT}$, we obtain a flat family

$$\mathcal{H}om_{\mathcal{O}_{X\times T}}(E_T, \widetilde{L}) = E^\vee \otimes_{\mathcal{O}_X}\widetilde{L}$$

By the definition of an m_i-fold extension $\mathcal{O}_{\ell_i}(1)$ and induction on m_i, we see that $H^1(X, \widetilde{L}(0)) = 0$ and hence that if T is sufficiently small, then for all $t \in T$, $H^1(X, \widetilde{L}(t)) = 0$. Therefore, by the base change theorem we have that $q_*(E^\vee \otimes_{\mathcal{O}_X} \widetilde{L})$ is a free $\mathcal{O}_T$-module and the natural map

$$q_*(E^\vee \otimes_{\mathcal{O}_X}\widetilde{L})(t) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(E, \widetilde{L}(t))$$

is bijective for all $t \in T$. Then, noting that $q_*(E^\vee \otimes_{\mathcal{O}_X}\widetilde{L})$ is a real sheaf with respect to the natural real structure of the affine line, we find that the given real surjection α extends to such a homomorphism $\widetilde{\alpha}$ of E_T to $\widetilde{L}$ that $\widetilde{\alpha}(t)$ is real if t is a real point. Shrinking T if necessary, we may assume that $\widetilde{\alpha}$ is surjective. $\widetilde{F} = \ker(\widetilde{\alpha})$ is a flat family of elementary transforms of E. By the construction of $\widetilde{F}$, if t is real, then $\widetilde{F}(t)$ is quaternionic. On the other hand, for $t \neq 0$, then $\widetilde{L}(t)$ is the direct sum of m_i'-fold extension of $\mathcal{O}_{\ell_i}(1)$ $(i = 0, \dots, k)$, where $\ell_0 = \ell_t$ in Proposition 5.10, $m_0' + m_1' = m_1$ and where for $i \geq 2$, $m_i' = m_i$. Then r for $\widetilde{L}(t)$ is $m - k - 1$ if $t \neq 0$. By induction hypothesis if a real t is not 0, then $\widetilde{F}(t)$ is a member of $\overline{I}(n)$. Thus we see that $F = \widetilde{F}(0)$ is in $\overline{I}(n)$, too. Q.E.D.

The following is a direct corollary to Theorem 4.3 and Theorem 5.12.

Corollary 5.13. *Let U be the open set of $\overline{M}(n,2)$ formed by the semi-stable sheaves for which the universal extensions are defined. The morphism $\varphi(n,2)$ of $\overline{M}(n,2)$ to $\overline{M}(n,2n)$ defined by the universal extensions induces a surjection of $U \cap \overline{I}(n)$ to $\overline{\overline{I}}(n)$.*

6 Parametrization of $\overline{MI}(2)$ and its real structure

Let $\overline{MI}(2)$ be the closure of the moduli space of mathematical instantons in $\overline{M}(2,2)$ and let $\nu : \widetilde{MI}(2) \to \overline{MI}(2)$ be its normalization. In [9] there had been constructed a modification $\mu : Q \to \widetilde{MI}(2)$, where Q is a quadric bundle of Poncelet pairs of conics interpreting the sheaves geometrically, and it was shown that ν is a bijective morphism. We are going to show that the morphism $\varphi(2,2) : MI(2) \to \overline{M}(2,4)$ can be extended to $\widetilde{MI}(2)$ and that it also descends to the real forms.

It is necessary to recall the construction of Q and μ by Geometric Invariant Theory. As a preparation for this we introduce parametrizations of conics in a plane as follows. Let W be a three-dimensional $\mathbf{C}$-vector space and let Γ be a member of the Grassmannian $G_4(\mathbf{C}^2 \otimes_{\mathbf{C}} W^\vee)$ of four-dimensional subspaces in $\mathbf{C}^2 \otimes_{\mathbf{C}} W^\vee$. Setting $\Gamma' = \mathbf{C}^2 \otimes_{\mathbf{C}} W^\vee / \Gamma$, we have a homomorphism

$$\mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}(W)}(-1) \longrightarrow \mathbf{C}^2 \otimes_{\mathbf{C}} W^\vee \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}(W)} \longrightarrow \Gamma' \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}(W)}$$

and we let $S(\Gamma)$ denote its determinant. Similarly if $A \subset \mathbf{C}^2 \otimes_{\mathbf{C}} W^\vee$ is a two-dimensional subspace, then we denote by $C^\vee(A)$ the determinant of

$$A \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}(W^\vee)} \longrightarrow \mathbf{C}^2 \otimes_{\mathbf{C}} W^\vee \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}(W^\vee)} \longrightarrow \mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}(W^\vee)}(1)$$

and regard this as a conic in $\mathbf{P}(W^\vee)$.

Now let Z be the subvariety of

$$G_3(\bigwedge^2 \mathcal{V}) \times G_2(\mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}) \times G_4(\mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V})$$

formed by triples $(W^\vee, A, \Gamma)$ satisfying

(1) $A \subset \Gamma \subset \mathbf{C}^2 \otimes_{\mathbf{C}} W^\vee$

(2) the quadratic form $S(\Gamma)$ on $\mathbf{P}(W)$ is proportional to the quadratic form on $\mathbf{P}(W) \subset \mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)$ induced by the Plücker quadric $G_2(\mathcal{V})$ ($S(\Gamma)$ may be 0 and $\mathbf{P}(W) \subset G_2(\mathcal{V})$ is possible).

There is an $SL(2,\mathbf{C})$-action on the above product of the Grassmannians induced by the action on the factors $\mathbf{C}^2 \otimes_{\mathbf{C}}$ and the trivial action on $G_3(\bigwedge^2 \mathcal{V})$. It leaves Z invariant. The action is linearized by Plücker and Segre embeddings of the last two factors. Therefore we can consider the notion of semi-stable and stable points with respect to this action. In [9] it was proved

Proposition 6.1. *$Z^{ss} = Z \cap (G_3(\bigwedge^2 \mathcal{V}) \times G_2(\mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V})^{ss} \times G_4(\mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V})^{ss})$ and Z^{ss} is smooth. Moreover, for a semi-stable point $z = (W^\vee, A, \Gamma)$, the quadric forms $S(\Gamma)$ and $C^\vee(A)$ are non-zero and define a pair of conics in $\mathbf{P}(W)$ and $\mathbf{P}(W^\vee)$, respectively, which are Poncelet-related.*

The good quotient $Q = Z^{ss}//SL(2,\mathbf{C})$ is the set of Poncelet pairs $(S, C^\vee)$ with $S \subset G_2(\mathcal{V})$ and $C^\vee$ in the plane dual to that defined by S.

Over $X \times Z^{ss}$ we shall construct a flat family of semi-stable sheaves which gives rise to a morphism

$$Z^{ss} \longrightarrow \overline{MI}(2).$$

First there is a morphism $\beta : Z^{ss} \to G_2(\mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{V})^{ss}$ defined as follows. The subspace $\Gamma \subset \mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}$ of a point $z = (W^\vee, A, \Gamma)$ defines a linear map $\Gamma \otimes_{\mathbf{C}} \mathcal{V} \to \mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^3 \mathcal{V}$. We set $\beta(z) = B$ to be the kernel of the dual map

$$0 \longrightarrow B \longrightarrow \mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{V} \longrightarrow \Gamma^\vee \otimes_{\mathbf{C}} \mathcal{V}^\vee,$$

where $\bigwedge^3 \mathcal{V}$ is canonically identified with $\mathcal{V}^\vee$. It has been proved in [9, 5.8.1] that B is two-dimensional and semi-stable if z is a semi-stable point.

In addition let α be the projection of Z^{ss} to $G_2(\mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V})^{ss}$. We thus get a diagram

$$\begin{array}{ccccccc} & & & & & \overset{\alpha}{\nearrow} & G_2(\mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V})^{ss} \\ X & \overset{\eta}{\longleftarrow} & X \times Z^{ss} & \overset{\xi}{\longrightarrow} & Z^{ss} & & \\ & & & & & \underset{\beta}{\searrow} & G_2(\mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{V})^{ss} \end{array}$$

Now let $\mathbf{A} \subset \mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V} \otimes_{\mathbf{C}} \mathcal{O}_{G_2(\mathbf{C}^2 \otimes \bigwedge^2 \mathcal{V})}$ and $\mathbf{B} \subset \mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{V} \otimes_{\mathbf{C}} \mathcal{O}_{G_2(\mathbf{C}^2 \otimes \mathcal{V})}$ be the universal subbundles on the Grassmannians. We obtain a commutative diagram on $X \times Z^{ss}$

$$\begin{array}{ccccc} & & \mathbf{C}^2\otimes\bigwedge^2\mathcal{V}\otimes\eta^*\Omega_X^3(3) & & \\ & \nearrow & \downarrow & & \\ 0\to\xi^*\alpha^*\mathcal{A}\otimes\eta^*\Omega_X^3(3) & \xrightarrow{a} & \mathbf{C}^2\otimes\eta^*\Omega_X^1(1) & \xrightarrow{b} & \mathbf{C}^2\otimes\xi^*\beta^*\mathbf{B}^\vee\to D\to 0 \\ & & \downarrow & \nearrow & \\ & & \mathbf{C}^2\otimes\mathcal{V}^\vee\otimes\mathcal{O}_{X\times Z^{ss}} & & \end{array}$$

where $\bigwedge^2\mathcal{V}\otimes_{\mathbf{C}}\Omega_X^3(3)\to\Omega_X^1(1)$ is the canonical evaluation map with the identification $\bigwedge^2\mathcal{V}=\mathrm{Hom}_{\mathcal{O}_X}(\Omega_X^3(3),\Omega_X^1(1))$. We denote by D the cokernel of the induced homomorphism b. By [9] we have the following results.

Theorem 6.2. (1) *a is injective and $b\cdot a=0$.*

(2) *The cohomology sheaf $F=\ker(b)/\mathrm{im}(a)$ on $X\times Z^{ss}$ is flat over Z^{ss}.*

(3) *For every point z of Z^{ss}, the sheaf $F(z)$ on X is a semi-stable sheaf of rank 2 with $c_1(F(z))=0$, $c_2(F(z))=2$ and $c_3(F(z))=0$.*

(4) *The morphism $\theta: Z^{ss}\to\overline{M}(2,2)$ defined by F has the closure $\overline{MI}(2)$ as its image. It is factored through*

$$Z^{ss}\longrightarrow Q\xrightarrow{\mu}\widetilde{MI}(2)\xrightarrow{\nu}\overline{MI}(2)$$

where Q is the good quotient $Z^{ss}//SL(2,\mathbf{C})$ of Poncelet pairs and $\nu:\widetilde{MI}(2)\to\overline{MI}(2)$ is the normalization of $\overline{MI}(2)$. These data have the following properties:

(i) *ν is bijective,*

(ii) *Q is normal,*

(iii) *μ is a modification which blows down a 5-dimensional subvariety $Q_{exc}\subset Q$ along a $\mathbf{P}^1$-fibration $Q_{exc}\to\mu(Q_{exc})=M_{exc}$ such that*

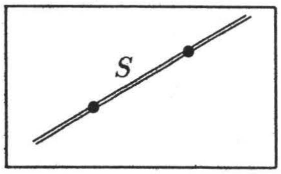
$$Q\setminus Q_{exc}\longrightarrow\overline{MI}(2)\setminus M_{exc}$$

is an isomorphism and

$$\begin{array}{ccc} Q_{exc} & \hookrightarrow & Q \\ \downarrow & & \downarrow \\ M_{exc} & \hookrightarrow & \widetilde{MI}(2) \end{array}$$

is a pushout diagram *with $\mu_*(\mathcal{O}_Q)=\mathcal{O}_{\widetilde{MI}(2)}$.*

(*The subvariety* Q_{exc} *consists of pairs* $(S, C^\vee) \in Q$ *such that* S *is a double line with its structure determined by the plane* $\mathbf{P}(W)$ *and that* $C^\vee$ *is given by a pair of points on* S. *The morphism* $\mu|_{Q_{exc}}$ *forgets the double structure or the plane of* S *and its fiber through* S *is the pencil of such double structures or planes.*)



$\mathbf{P}(W)$

Let us collect here a more detailed explanation of the structures of Poncelet related conics and their relationship with the sheaves $F(z)$.

Remark 6.3. If a point $z = (W^\vee, A, \Gamma)$ is given, then in addition to the conics $S_z = S(\Gamma) \subset G_2(\mathcal{V}) \cap \mathbf{P}(W)$ and $C_z^\vee = C^\vee(A) \subset \mathbf{P}(W^\vee)$, we can consider the variety

$$\tilde{J}_z \subset \mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)$$

given by $\det(A) = 0$, which is the equation of $C_z^\vee$ considered as an element of $S^2(\bigwedge^2 \mathcal{V}^\vee) \cong S^2(\bigwedge^2 \mathcal{V})$. Choosing any splitting

$$W^\vee \leftrightarrows \bigwedge^2 \mathcal{V} \cong \bigwedge^2 \mathcal{V}^\vee,$$

we have $C_z^\vee = \tilde{J}_z \cap \mathbf{P}(W^\vee)$. Moreover, if S_z is regular, then the polar dual of $C_z^\vee$ with respect to S_z is just $C_z = \tilde{J}_z \cap \mathbf{P}(W)$.

Remark 6.4. Let us give a geometric interpretation of the sheaf $F(z)$ with z a point of Z^{ss}. In addition to S_z we are given the conjugate conic S_z^0: If $S_z = G_2(\mathcal{V}) \cap \mathbf{P}(W)$, then S_z^0 is defined to be $G_2(\mathcal{V}) \cap \mathbf{P}(W^\perp)$, where the orthogonal space $W^\perp$ is given by the quadratic form of $G_2(\mathcal{V}) \subset \mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)$. If $S_z \subset \mathbf{P}(W) \subset G_2(\mathcal{V})$, we define S_z^0 to be S_z [9, 7.6]. Then we see that $S_z^0 \subset J_z = \tilde{J}_z \cap G_2(\mathcal{V})$ and these sets are the sets of jumping lines of $F(z)$ of order 2 and 1, respectively. If

$$\begin{array}{ccc} & I & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & G_2(\mathcal{V}) \end{array}$$

is the incidence correspondence of points and lines in X, then one finds that $S_z^0 = \operatorname{Supp}(R^1 q_* p^*(F(z)))$ and $J_z = \operatorname{Supp}(R^1 q_* p^*(F(z)(-1)))$.

Remark 6.5. The sheaf $F(z)$ is a mathematical instanton bundle if and only if both S_z and $C_z^\vee$ (or C_z) are regular conics. In this case the point $z = (W^\vee, A, \Gamma)$

together with $B = \beta(z)$ can be represented by matrices as follows [9, §2].

$$\begin{array}{ccccccccc}
\Gamma & \hookrightarrow & \mathbf{C}^2 \otimes \bigwedge^2 \mathcal{V} & \longrightarrow & B^\vee \otimes \bigwedge^2 \mathcal{V} & & \mathbf{C}^2 \otimes \mathcal{V}^\vee & \longrightarrow & B^\vee \\
\| \wr & & \| & & \| \wr & & \| & & \| \wr \\
\mathbf{C}^4 & \hookrightarrow & \mathbf{C}^2 \otimes \bigwedge^2 \mathcal{V} & \longrightarrow & \mathbf{C}^2 \otimes \bigwedge^3 \mathcal{V} & & \mathbf{C}^2 \otimes \mathcal{V}^\vee & \longrightarrow & \mathbf{C}^2
\end{array}$$

$$\begin{pmatrix} \xi & 0 \\ \omega & \xi \\ \eta & \omega \\ 0 & \eta \end{pmatrix} \qquad \begin{pmatrix} v & v' \\ w & w' \end{pmatrix} \qquad M := \begin{pmatrix} v & v' \\ w & w' \end{pmatrix}$$

where $\xi = v \wedge v'$, $\omega = v \wedge w' - v' \wedge w$, $\eta = w \wedge w'$ and $W^\vee$ is the span of $\{\xi, \omega, \eta\}$ and where v, v', w, w' form a basis of $\mathcal{V}$. Then A has the representation

$$\begin{array}{ccccc}
A & \hookrightarrow & \Gamma & \hookrightarrow \mathbf{C}^2 \otimes W^\vee \hookrightarrow & \mathbf{C}^2 \otimes \bigwedge^2 \mathcal{V} \\
\| \wr & & \| \wr & & \| \\
\mathbf{C}^2 & \longrightarrow & \mathbf{C}^4 & \longrightarrow & \mathbf{C}^2 \otimes \bigwedge^2 \mathcal{V}
\end{array}$$

$$K := \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \alpha_0 & \cdots & \alpha_3 \\ \alpha_1 & \cdots & \alpha_4 \end{pmatrix} \begin{pmatrix} \xi & 0 \\ \omega & \xi \\ \eta & \omega \\ 0 & \eta \end{pmatrix}$$

with a symmetric matrix K (since Serre duality can be applied to the cohomology of the self-dual bundle $F(z)$). The symmetry of K implies that the matrix of the coefficients α_i is persymmetric. The bundle $F(z)$ then has the monad

$$\mathbf{C}^2 \otimes_{\mathbf{C}} \Omega^3_X(3) \xrightarrow{K} \mathbf{C}^2 \otimes_{\mathbf{C}} \Omega^1_X(1) \xrightarrow{M} \mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{O}_X.$$

Finally the equation of $\widetilde{J}_z$ or $C_z^\vee$ becomes $\det K = ac - b^2$. Calculation of its discriminant with respect to $S^2(W^\vee)$ in terms of the coefficients α_i gives

$$\operatorname{discr} C^\vee(A) = 2 \det \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix}^2 \neq 0.$$

If $\{\xi^\vee, \omega^\vee, \eta^\vee\}$ denotes the basis of W dual to $\{\xi, \omega, \eta\}$, then the equation of S_z in $\mathbf{P}(W)$ is $\omega^{\vee 2} - \xi^\vee \eta^\vee = 0$.

Let us now consider the cokernel $\mathcal{M}$ of the homomorphism a, that is, the exact sequence

$$0 \longrightarrow \xi^* \alpha^* \mathcal{A} \otimes_{\mathbf{C}} \eta^* \Omega^3_X(3) \xrightarrow{a} \mathbf{C}^2 \otimes_{\mathbf{C}} \eta^* \Omega^1_X(1) \longrightarrow \mathcal{M} \longrightarrow 0.$$

This is a flat family of coherent sheaves $\mathcal{M}(z)$ of rank 4 with $c_1(\mathcal{M}(z)) = 0$, $c_2(\mathcal{M}(z)) = 2$ and $c_3(\mathcal{M}(z)) = 0$. Moreover, each $\mathcal{M}(z)$ is torsion free [9, Proposition 8.1] and it is easily seen that it satisfies the conditions of Proposition 1.7 of [8]. Hence every $\mathcal{M}(z)$ is semi-stable. We thus obtain a morphism Φ of Z^{ss} to $\overline{M}(2,4)$ which is factorized through the quotient $Q = Z^{ss}//SL(2,\mathbf{C})$.

$$\begin{array}{ccc} Z^{ss} & \xrightarrow{\ \Phi\ } & \overline{M}(2,4) \\ & \searrow \quad \nearrow_{\varphi} & \\ & Q & \end{array}$$

Furthermore, φ is constant on the fibers of μ in Theorem 6.2 because A or $\mathcal{M}(z)$ contains only the information of $\tilde{J}_z$ or $C_z^\vee$ but not of S_z. ($J_z = \mathrm{Supp}(R^1 q_* p*(F(z)(-1))) = \mathrm{Supp}(R^1 q_* p * (\mathcal{M}(z)(-1)))$, see [9] 4.4 Remark 4).

If we consider the pushout diagram

$$\begin{array}{ccccc} Q_{exc} & \overset{j}{\hookrightarrow} & Q & & \\ \mu' \downarrow & & \downarrow \mu & \searrow^{\varphi} & \\ M_{exc} & \hookrightarrow & \widetilde{MI}(2) & \xrightarrow{\ \tilde{\varphi}\ } & \overline{M}(2,4) \\ & & \psi : M_{exc} \longrightarrow \overline{M}(2,4) & & \end{array}$$

with μ' a $\mathbf{P}^1$-bundle, then we see that $\varphi \cdot j$ is factorized through M_{exc} and hence there is a unique morphism $\tilde{\varphi}$ which makes the above diagram commutative. (Alternatively, φ is factored through a continuous map $\tilde{\varphi}$ first to obtain a continuous φ and then it must be a morphism because we have a natural homomorphism

$$\mathcal{O}_{\overline{M}(2,4)} \longrightarrow \varphi_*(\mathcal{O}_Q) = \tilde{\varphi}_* \mu_*(\mathcal{O}_Q) = \tilde{\varphi}_*(\mathcal{O}_{\widetilde{MI}(2)})).$$

Remark 6.6. We will see in §8 that $\overline{I}(2) \subset \overline{MI}(2) \setminus M_{exc}$ and hence, without using the morphism $\tilde{\varphi}$, we are given a map $\overline{I}(2) \to \overline{M}(2,4)$.

Lemma 6.7. *The morphism $\tilde{\varphi}$ extends the morphism $\varphi(2,2) : MI(2) \to \overline{M}(2,4)$ given by the universal extension. The image of the morphism $\tilde{\varphi}$ is the closure $\overline{\overline{MI}}(2)$ of $MI(2)$ in $\overline{M}(2,4)$.*

$$
\begin{array}{ccc}
Q & \xrightarrow{\ \mu\ } & \widetilde{MI}(2) \\
{\scriptstyle \varphi}\big\downarrow & \swarrow{\scriptstyle \tilde{\varphi}} & \\
\overline{\overline{MI}}(2) & &
\end{array}
$$

Proof. If E is a member of $MI(2)$, then the universal monad gives us a monad

$$0 \longrightarrow A \otimes_{\mathbf{C}} \Omega_X^3(3) \xrightarrow{\ a\ } \mathbf{C}^2 \otimes_{\mathbf{C}} \Omega_X^1(1) \xrightarrow{\ b\ } \mathbf{C}^2 \otimes_{\mathbf{C}} B^\vee \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow 0$$

with $A = H^2(X, E(-3))$ and $B^\vee = H^1(X, E)$ and the cokernel of a can be identified with the universal extension (see [8]), which is exactly our assertion.

Let ℓ be a line in X and $p \in \ell$ a multiple point. Consider the ideal sheaf $J_{\ell \cup p}$. Then we have an exact sequence

$$0 \longrightarrow J_{\ell \cup p} \longrightarrow J_\ell \longrightarrow \mathbf{C}_p \longrightarrow 0.$$

It is easy to see that $J_{\ell \cup p}$ has the property (UV) of [8, §1] and by the same argument as in [8, Example 1.4] we know that the universal extension $U(J_{\ell \cup p})$ is not semi-stable. Pick a semi-stable sheaf E in $\overline{MI}(2)$ which fits in an exact sequence

$$0 \longrightarrow J_{\ell \cup p} \longrightarrow E \longrightarrow J_{\ell \cup q} \longrightarrow 0.$$

Then we have an exact sequence of the universal extensions

$$0 \longrightarrow U(J_{\ell \cup p}) \longrightarrow U(E) \longrightarrow U(J_{\ell \cup q}) \longrightarrow 0.$$

Since $U(J_{\ell \cup p})$ and $U(J_{\ell \cup q})$ have the same Hilbert polynomial and rank and since $U(J_{\ell \cup p})$ is not semi-stable, $U(E)$ is not semi-stable, either.

The above E corresponds to a point z of Z^{ss} and hence it is a subsheaf of $\mathcal{M}(z)$. By its construction there is a natural homomorphism of $A = \mathcal{M}(z)/E$ to $\mathcal{O}_X^{\oplus 2}$ which is isomorphic over $X \setminus \{p, q\}$. This provides us with an exact commutative diagram

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & E & = & E & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & T' & \to & \mathcal{M}(z) & \xrightarrow{\ \delta\ } & U(E) & \to & C' & \to & 0 \\
 & & {\scriptstyle \tau}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \gamma} & & \\
0 & \to & T & \to & A & \longrightarrow & \mathcal{O}_X^{\oplus 2} & \to & C & \to & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

and $\mathcal{M}(z)$ is an extension of Q_ℓ by Q_ℓ (K in Remark 6.5 is of a form $\begin{pmatrix} \xi & 0 \\ & \eta \end{pmatrix}$ with ξ and η decomposable). Let us summarize these results.

Proposition 6.8. *Let E be an extension*

$$0 \longrightarrow J_{\ell \cup p} \longrightarrow E \longrightarrow J_{\ell \cup q} \longrightarrow 0.$$

(1) *E has the property* (UV) *of* [8, §1] *but the universal extension $U(E)$ is not semi-stable.*

(2) *E gives rise to a point z of Z^{ss} and hence it is a subsheaf of the semi-stable sheaf $\mathcal{M}(z)$. There is a natural homomorphism of $\mathcal{M}(z)$ to $U(E)$ which is isomorphic on $X \setminus \{p, q\}$ but cannot be an isomorphism.*

We shall next study the reality condition on our parameter spaces and morphisms which is induced by the real structure of X. We have the antilinear automorphism σ on $\mathcal{V} \cong \mathbf{H}^2$ induced by the multiplication by j (see [8, §4]). σ induces an involution on

$$G_3(\bigwedge^2 \mathcal{V}) \times G_2(\mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}) \times G_4(\mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V})$$

by sending $z = (W^\vee, A, \Gamma)$ to

$$((\bigwedge^2 \sigma)(W^\vee), (t \otimes \bigwedge^2 \sigma)(A), (t \otimes \bigwedge^2 \sigma)(\Gamma)),$$

where t is the conjugation of $\mathbf{C}^2$. This action of σ leaves Z and Z^{ss} invariant and hence we get an involution σ_Z on Z. Since σ defines an involution σ_X on X, we get an involution

$$\tilde{\sigma} = \sigma_X \times \sigma_Z$$

on $X \times Z^{ss}$.

The vector space $\bigwedge^2 \mathcal{V}$ has a natural real structure coming from the quaternionic structure of $\mathcal{V}$ and then the Grassmannians used in the above are defined over $\mathbf{R}$. Thus Z^{ss} has a real form $Z^{ss}_{\mathbf{R}}$ and the universal subbundle A on $G_2(\mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V})$ is a real sheaf. On the other hand, $G_2(\mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{V})$ has also a natural real structure but the universal subbundle B on this space is quaternionic. It is clear that the morphisms α, β, ξ and η defined before Theorem 6.2 are all compatible with these real structures or the action of σ. This implies that $\xi^*\alpha^*\mathsf{A} \otimes_{\mathbf{C}} \eta^*\Omega^3_X(3)$, $\mathbf{C}^2 \otimes_{\mathbf{C}} \eta^*\Omega^1_X(1)$ and $\mathbf{C}^2 \otimes_{\mathbf{C}} \xi^*\beta^*\mathsf{B}^\vee$ are quaternionic sheaves with respect to $\tilde{\sigma}$. Moreover, the homomorphisms a and b in Theorem 6.2 are real. Thus we obtain the following.

Lemma 6.9. *The universal sheaf F over $X \times Z^{ss}$ is quaternionic with respect to $\tilde{\sigma}$. In particular, $\sigma_X^*(F(z)) \cong F(\sigma(z))$ for all points z of Z^{ss}.*

Since the sheaf $F \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)$ is a real sheaf, there is a coherent sheaf $F_{\mathbf{R}}$ on the real scheme $X_{\mathbf{R}} \times_{\mathbf{R}} Z^{ss}_{\mathbf{R}}$ such that $F_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} \cong F \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)$. Thus we have a morphism $\theta'_{\mathbf{R}}$ of $Z^{ss}_{\mathbf{R}}$ to $\overline{M}(2,3,0;2)_{\mathbf{R}}$. On the other hand, we have an $\mathbf{R}$-isomorphism λ of $\overline{M}(2,4)_{\mathbf{R}}$ to $\overline{M}(2,3,0;2)_{\mathbf{R}}$ such that $\lambda \otimes_{\mathbf{R}} \mathbf{C}$ is given by tensoring $\mathcal{O}_X(1)$. Thus $\theta_{\mathbf{R}} = \lambda^{-1} \cdot \theta'_{\mathbf{R}}$ is an $\mathbf{R}$-morphism of $Z^{ss}_{\mathbf{R}}$ to $\overline{M}(2,4)_{\mathbf{R}}$ such that $\theta_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = \theta$, where θ is the morphism given in Theorem 6.2.

$SL(2,\mathbf{R})$ acts naturally on the real form $Z_{\mathbf{R}}$ of Z and $Z^{ss}_{\mathbf{R}}$ is the open subscheme of $Z_{\mathbf{R}}$ whose geometric points are semi-stable points with respect to this action. Thus we have a good quotient $Q_{\mathbf{R}} = Z^{ss}_{\mathbf{R}}//SL(2,\mathbf{R})$ such that $Q_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ is isomorphic to Q and $\theta_{\mathbf{R}}$ is factored through the quotient morphism $Z^{ss}_{\mathbf{R}} \to Q_{\mathbf{R}}$. Since $Q_{\mathbf{R}}$ is normal, the morphism of $Q_{\mathbf{R}}$ to $\overline{M}(2,4)_{\mathbf{R}}$ passes through the normalization $\widetilde{MI}(2)_{\mathbf{R}}$ of $\overline{MI}(2)_{\mathbf{R}}$. We obtain therefore

$$\theta_{\mathbf{R}} : Z^{ss}_{\mathbf{R}} \longrightarrow Q_{\mathbf{R}} \xrightarrow{\mu_{\mathbf{R}}} \widetilde{MI}(2)_{\mathbf{R}} \xrightarrow{\nu_{\mathbf{R}}} \overline{MI}(2)_{\mathbf{R}}$$

such that by tensoring $\mathbf{C}$ over $\mathbf{R}$ the diagram provides us with that in Theorem 6.2, (4).

Since $\nu_{\mathbf{R}}$ is an isomorphism over $MI(2)_{\mathbf{R}}$, $I(2)$ is contained in $\widetilde{MI}(2)_{\mathbf{R}}(\mathbf{R})$ and then we denote the closure of $I(2)$ in $\widetilde{MI}(2)_{\mathbf{R}}(\mathbf{R})$ in the classical topology by $\widetilde{I}(2)$. Let $\tilde{\varphi}_{\mathbf{R}}$ be $\varphi(2,2)_{\mathbf{R}} \cdot \nu_{\mathbf{R}}$. Then it induces a continuous map of $\widetilde{I}(2)$ to $\overline{\overline{I}}(2)$, which is denoted by $\tilde{\varphi}_{\mathbf{R}}$, too. $\widetilde{I}(2)$ is compact because $\widetilde{MI}(2)_{\mathbf{R}}$ is projective. It follows that $\tilde{\varphi}_{\mathbf{R}}$ maps $\widetilde{I}(2)$ surjectively to $\overline{\overline{I}}(2)$.

$$\begin{array}{ccc} \widetilde{I}(2) & \xrightarrow{\tilde{\varphi}_{\mathbf{R}}} & \overline{\overline{I}}(2) \\ \cup & & \cup \\ I(2) & \xrightarrow{\sim} & I(2) \end{array}$$

Note that $\widetilde{I}(2)$ is in bijective correspondence with $\overline{I}(2)$ by $\nu_{\mathbf{R}}$. We are going to describe the fiber of $\tilde{\varphi}_{\mathbf{R}}$ in the next section.

7 Closure of $I(2)$

In the preceding section we have constructed a real morphism of $Z^{ss}_{\mathbf{R}}$ to $Q_{\mathbf{R}}$. This gives rise to a real analytic map

$$\varpi : Z^{ss}_{\mathbf{R}}(\mathbf{R}) \longrightarrow Q_{\mathbf{R}}(\mathbf{R}).$$

As we will see later (Remark 7.7) ϖ is not surjective though the complex analytic map $Z^{ss}(\mathbf{C}) \to Q(\mathbf{C})$ is, of course, surjective. Clearly a Poncelet pair $q = (S, C^\vee) \in Q(\mathbf{C})$ is in $Q_{\mathbf{R}}(\mathbf{R})$ if and only if both S and $C^\vee$ are invariant under the action of σ on $\mathbf{P}(\bigwedge^2 V^\vee)$. Then $\widetilde{J}_q$ is invariant under the action of σ, where $\widetilde{J}_q = \widetilde{J}_z$ with z a point of $Z^{ss}(\mathbf{C})$ over q. We have

$z = (W^\vee, A, \Gamma) \in Z_{\mathbf{R}}(\mathbf{R})$ if and only if $(\bigwedge^2 \sigma)(W^\vee) = W^\vee$, $(t \otimes \bigwedge^2 \sigma)(A) = A$ and $(t \otimes \bigwedge^2 \sigma)(\Gamma) = \Gamma$.

Lemma 7.1. *Let $E \in MI(2)$ be defined by a pair $q = (S, C^\vee) \in Q$ of regular conics. Then the following are equivalent:*

(1) *E is quaternionic.*

(2) *W is real, $S(\mathbf{R}) = S^4 \cap \mathbf{P}(W)$ is a circle cut out by the real 2-plane $\mathbf{P}(W)_{\mathbf{R}}(\mathbf{R})$ and $C^\vee(\mathbf{R})$ (or $C(\mathbf{R})$) is a regular real conic in $\mathbf{P}(W^\vee)_{\mathbf{R}}(\mathbf{R})$ (or $\mathbf{P}(W)_{\mathbf{R}}(\mathbf{R})$, resp.), that is, $C^\vee(\mathbf{R})$ is Poncelet related to $S(\mathbf{R})$.*

(3) *The entries of a monad (K, M) representing E (see Remark 6.5) can be so chosen that*

$$K = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \alpha_0 & \cdots & \alpha_3 \\ \alpha_1 & \cdots & \alpha_4 \end{pmatrix} \begin{pmatrix} \xi & 0 \\ \omega & \xi \\ \eta & \omega \\ 0 & \eta \end{pmatrix} \qquad M = \begin{pmatrix} v & \sigma(v) \\ w & \sigma(w) \end{pmatrix}$$

with $\alpha_i \in \mathbf{R}$ and $a, b, c, \xi, \omega, \eta \in \bigwedge^2 \mathcal{V}$ real.

(The equivalence (1) $\Longleftrightarrow$ (2) has been showed in [3])

Proof. Assuming (1) let us look at the Beilinson monad defining E :

$$H^2(E(-3)) \otimes_{\mathbf{C}} \Omega^3_X(3) \xrightarrow{K} H^1(E(-1)) \otimes_{\mathbf{C}} \Omega^1_X(1) \xrightarrow{M} H^1(E) \otimes_{\mathbf{C}} \mathcal{O}_X$$

The quaternionic structure of E induces real structures of $H^2(X, E(-3))$ and $H^1(X, E(-1))$ and a quaternionic one of $H^1(X, E)$. Moreover, $H^1(X, E(-1))$ is naturally dual to $H^2(X, E(-3))$ by Serre duality. Both K and M are compatible with these structures. Thus, if we fix a real basis of $H^2(X, E(-3))$ and a symplectic basis of $H^1(X, E)$ and take the basis of $H^1(X, E(-1))$ dual to that of $H^2(X, E(-3))$, then K and M can be written in the following forms:

$$K = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad M = \begin{pmatrix} v & \sigma(v) \\ w & \sigma(w) \end{pmatrix}$$

where a, b and c are real elements of $\bigwedge^2 \mathcal{V}$ and v, w are members of $\mathcal{V}$. If v, w are linearly dependent, then the map M can not be surjective. This implies that $\{v, w, \sigma(v), \sigma(w)\}$ forms a basis of $\mathcal{V}$. The equation $K \wedge M = 0$ is equivalent to

$$\begin{aligned} a \wedge v + b \wedge w &= 0 \\ b \wedge v + c \wedge w &= 0. \end{aligned}$$

From these we obtain that $a \wedge v \wedge w = b \wedge v \wedge w = c \wedge v \wedge w = d \wedge v \wedge w = 0$. This, the above equations and the fact that a, b, c and d are all real show that

a, b, c and d are linear combinations of $\xi = v \wedge \sigma(v)$, $\omega = v \wedge \sigma(w) - \sigma(v) \wedge w$ and $\eta = w \wedge \sigma(w)$. By using the above equations again we can find $\alpha_i \in \mathbf{R}$ which fulfill our requirement in (3).

If we assume (3), then $S(\mathbf{R}) = S^4 \cap \mathbf{P}(W)$ and $C^\vee(\mathbf{R}) = \tilde{J}_x \cap \mathbf{P}(W^\vee)_{\mathbf{R}}(\mathbf{R})$ have the real equations $\omega^{\vee 2} - \xi^\vee \eta^\vee = 0$ and $ac - b^2 = 0$, respectively. These quadratic forms are not definite and hence define regular conics. We have therefore (2).

Assume that the couple $(S, C^\vee)$ satisfies the condition of (2). The equations $f = \omega^{\vee 2} - \xi^\vee \eta^\vee = 0$ of S and $g = ac - b^2$ of C vanish on the circles $S(\mathbf{R})$ and $C(\mathbf{R})$, respectively. The same is true for the equations built up by $\bigwedge^2 \sigma(\xi), \ldots, \bigwedge^2 \sigma(c)$ which come from $\sigma^* E$ and σK, $\sigma M J$, where J is the same 2×2-matrix as J_2 in the proof of Lemma 2.2. It follows that $\bigwedge^2 \sigma(f)$ and f are the same up to a scalar multiple, and similarly for $\bigwedge^2 \sigma(g)$ and g. Thus S and C (or $C^\vee$) are invariant under the action of σ or $(S, C^\vee) \in Q_{\mathbf{R}}(\mathbf{R})$. It is easy to see that if E is real, then $S(\mathbf{R})$ can not be a circle. These prove that E is quaternionic. Q.E.D.

$I(2)$ can be regarded as a subset of $Q_{\mathbf{R}}(\mathbf{R})$ by the morphism $\mu_{\mathbf{R}}$ which is isomorphic over $MI(2)_{\mathbf{R}}$. We can now describe $I(2)$ as a subset of $Q_{\mathbf{R}}(\mathbf{R})$.

Proposition 7.2. *Let $q = (S, C^\vee) \in Q_{\mathbf{R}}(\mathbf{R})$ satisfy the condition (2) of Lemma 7.1. Then $(S, C^\vee)$ is a member of $I(2)$ if and only if for any real representation of K as in (3) of Lemma 7.1,*

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix}$$

is definite.

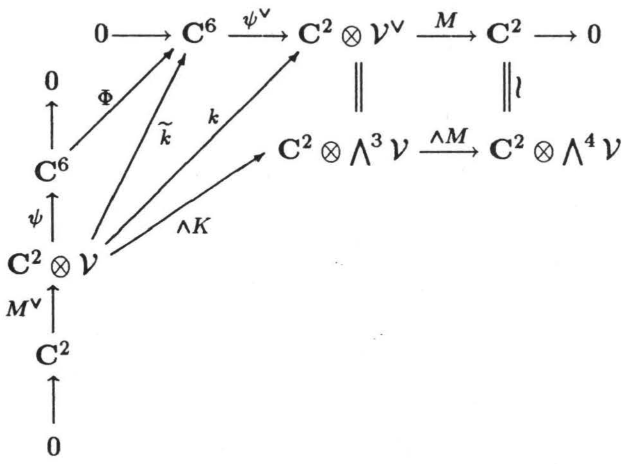
Proof. Let E be the sheaf in $MI(2)_{\mathbf{R}}(\mathbf{R})$ associated with $(S, C^\vee)$ by Lemma 7.1 and let (K, M) be a real monad as in (3) of Lemma 7.1 which corresponds to E. The equation

$$\det K = ac - b^2 = 0$$

is also the equation of $\tilde{J}_q$ (see Remark 6.3), where $\tilde{J}_q \cap G_2(\mathcal{V})$ is the set of jumping line of E. E is an instanton bundle if and only if it has no real jumping lines. We have to show therefore that our matrix A is definite if and only if $\tilde{J}_q \cap S^4 = \emptyset$. To prove this we associate (K, M) with another monad

$$\mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{O}_X(-1) \xrightarrow{\psi} \mathbf{C}^6 \otimes_{\mathbf{C}} \mathcal{O}_X \xrightarrow{\psi^\vee \Phi} \mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{O}_X(1)$$

as in [8, §4], where Φ is skew-hermitian. The matrices ψ and Φ are defined by a factorization of K as follows:



The map k is factorized through $\psi^\vee$ because the image of k is contained in $\ker(M) = \operatorname{im}(\psi^\vee)$. Since $\wedge K$ is symmetric, $\operatorname{im}(M^\vee)$ is a subspace of $\ker(\wedge K)$. Hence $\tilde{k} \cdot M^\vee = 0$ and then there is a homomorphism Φ of $\mathbf{C}^6$ to $\mathbf{C}^6$ such that we have $\tilde{k} = \Phi\psi$. Since K has entries in $\bigwedge^2 \mathcal{V}$, the map Φ is skew (see [8, after 4.8]). Moreover, since both K and M are real as in Lemma 7.1, (3), we find that Φ is skew-hermitian. This follows, however, also from direct calculation: Let $\{z_0, z_1, z_2, z_3\}$ form the basis of $\mathcal{V}^\vee$ dual to $\{v, w, \sigma(v), \sigma(w)\}$. Then by the shape of M we get

$$\psi = \begin{pmatrix} z_2 & z_3 & -z_0 & -z_1 & 0 & 0 \\ 0 & 0 & z_2 & z_3 & -z_0 & -z_1 \end{pmatrix}$$

and

$$\Phi = H\Sigma = \begin{pmatrix} \alpha_0 & 0 & \alpha_1 & 0 & \alpha_2 & 0 \\ 0 & \alpha_0 & 0 & \alpha_1 & 0 & \alpha_2 \\ \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & 0 \\ 0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 \\ \alpha_2 & 0 & \alpha_3 & 0 & \alpha_4 & 0 \\ 0 & \alpha_2 & 0 & \alpha_3 & 0 & \alpha_4 \end{pmatrix} \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}$$

where J is the 2×2-matrix which appeared in the proof of Lemma 7.1. It is easy to see from the above diagram that now the induced sequence

$$\mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{O}_X(-1) \xrightarrow{\psi} \mathbf{C}^6 \otimes_{\mathbf{C}} \mathcal{O}_X \xrightarrow{\psi^\vee H\Sigma} \mathbf{C}^2 \otimes_{\mathbf{C}} \mathcal{O}_X(1)$$

is a monad defining E. By the result of Verdier [10, Theorem 6.6], H is positive definite (up to sign) if E is an instanton bundle. Conversely, if H is positive definite, then for every $x \in \mathcal{V} \setminus \{0\}$,

$$K \wedge x \wedge \sigma(x) \quad = \quad \pm\psi(x) H \Sigma \psi(\sigma(x))$$

$$= \pm\psi(x)H\Sigma\Sigma^\vee\overline{\psi(x)}^\vee \quad \text{since } \psi(\sigma(x)) = -\overline{\psi(x)}\Sigma$$
$$= \pm\psi(x)H\overline{\psi(x)}^\vee$$

and this has maximal rank 2 because H is positive definite. This implies that $\widetilde{J}_q \cap S^4 = \emptyset$. Finally we see immediately that H is positive definite if and only if so is our matrix A. This completes our proof.

Example 7.3. Let $v, \sigma(v), w, \sigma(w) \in \mathcal{V}$ and ξ, ω, η be as before. If we take $(\alpha_0, \ldots, \alpha_4)$ to be $(0,0,1,0,0)$, then the resulting bundle E is a member of $MI(2)_{\mathbf{R}}(\mathbf{R})$ with quaternionic structure, but the matrix A is not definite. Indeed, the bundle E has the real jumping lines $\xi = v \wedge \sigma(v)$ and $\eta = w \wedge \sigma(w)$.

The following are obtained directly from Lemma 7.1 and Proposition 7.2.

Corollary 7.4. *The closures of $I(2)$ in $Q(\mathbf{R})$, $\widetilde{MI}(2)(\mathbf{R})$ and $\overline{MI}(2)(\mathbf{R})$ are semi-algebraic.*

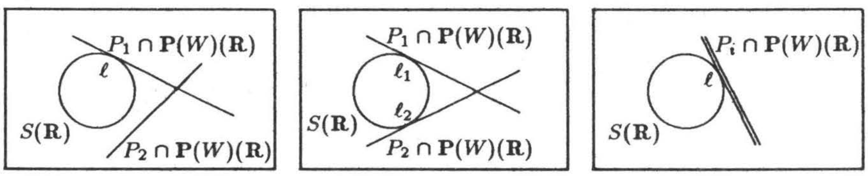
Corollary 7.5. *Let $(S, C^\vee)$ be a member of the closure $\overline{I}(2)$ of $I(2)$ in $Q(\mathbf{R})$. Then $S^4 \cap \mathbf{P}(W) \neq \emptyset$ and this is either a circle (Case I) or a point (Case II).*

If $q = (S, C^\vee)$ is on the boundary of $I(2)$ in $Q(\mathbf{R})$, then it is mapped to a point of $\overline{\overline{I}}(2) \setminus I(2)$ by $\varphi_{\mathbf{R}}\mu_{\mathbf{R}}$ (see the final part of §7). Then, by our results of [8, §4], we see that $\widetilde{J}_q$ is the union of two hyperplanes.

Proposition 7.6. *Let $\overline{I}(2)$ be as in Corollary 7.5 and $q = (S, C^\vee)$ a point of $\overline{I}(2) \setminus I(2)$. Then we have two cases, Case I and Case II, as stated in Corollary 7.5. In case*

(I) *$\widetilde{J}_q$ is a pair of hyperplanes $P_1, P_2 \subset \mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)$ one of which is touching $S(\mathbf{R})$ or S^4 at one point*

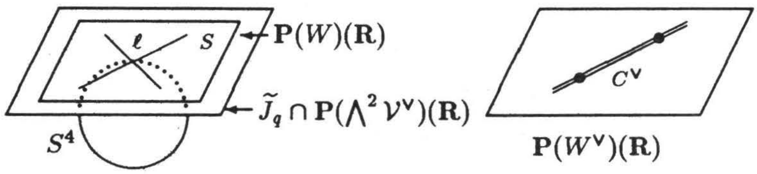
$\mathbf{P}(W)(\mathbf{R})$



(I_1) (I_2) (I_3)

and in case

(II) *$\widetilde{J}_q \cap \mathbf{P}(\bigwedge^2 \mathcal{V}^\vee)(\mathbf{R})$ is the double tangent hyperplane of S^4 at $\{\ell\} = S^4 \cap \mathbf{P}(W)$ and $S = G_2(\mathcal{V}) \cap \mathbf{P}(W)$ is a pair of distinct complex lines (invariant under the action of σ) with the intersection point ℓ being the only real point of S*



$C^\vee$ is a double line through two points determined by S and it is the intersection of the double hyperplane $\tilde{J}_q$ with $\mathbf{P}(W^\vee)$ in $\mathbf{P}(\bigwedge^2 \mathcal{V})$.

Proof. (I) In this case S is regular. Assume that $C^\vee$ is smooth. The $(S, C^\vee)$ corresponds to a mathematical instanton bundle E with quaternionic structure. E can be represented by a monad with

$$K = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} \xi & 0 \\ \omega & \xi \\ \eta & \omega \\ 0 & \eta \end{pmatrix}.$$

Since E is in $\overline{I}(2)$, the matrix

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix}$$

is positive semi-definite. It cannot be definite because the assumption $E \notin I(2)$ implies that $\tilde{J}_q \cap S^4 \neq \emptyset$. Thus $\det A = 0$. From this we obtain a decomposition of $\tilde{J}_q$ into $P_1 \cup P_2$ and then of $C^\vee$ or C as well (see Remark 6.3). Moreover, taking the unit ball B with boundary S^4, we see that $\tilde{J}_q \cap \overset{\circ}{B} \neq \emptyset$ because E is a limit of instantons. Hence one of the hyperplanes, say P_1, touches S^4 at a point ℓ and the real hyperplane $P_1(\mathbf{R})$ is the tangent hyperplane at ℓ.

Claim. ℓ is contained in $S(\mathbf{R})$.

In fact, for any $p \in \mathbf{P}(W^\perp)$, we see that the linear space spanned by ℓ and p is contained in $\tilde{J}_q$. Since $\tilde{J}_q$ does not meet the interior of B, it follows that $\mathbf{P}(W^\perp) \subset P_1$. On the other hand, $\ell \wedge p = 0$ for all $p \in \mathbf{P}(W^\perp)$. These mean that ℓ is a point of $\mathbf{P}(W) \cap S^4 = S(\mathbf{R})$.

(II) Let $\ell \in S^4 \cap \mathbf{P}(W)$ be the tangent point. Then $\mathbf{P}(W)$ is tangent to $G_2(\mathcal{V})$ at ℓ and hence $G_2(\mathcal{V}) \cap \mathbf{P}(W)$ decomposes into two lines e and f neither of which can have a real point other than ℓ (note that S^4 is the set of real points). Then neither e nor f can be invariant under the action of σ because the action of σ

on $\bigwedge^2 \mathcal{V}^\vee$ is just the complex conjugation. On the other hand, $G_2(\mathcal{V}) \cap \mathbf{P}(W)$ is σ-invariant. We see therefore that e and f are distinct and that $\sigma(e) = f$, $\sigma(f) = e$.

We can choose a basis $\{e_0, \dots, e_3\}$ of $\mathcal{V}^\vee$ with $\sigma(e_0) = e_1$, $\sigma(e_2) = e_3$ such that $\ell = e_{01}$, $W =< e_{01}, e_{02}, e_{13} >$, $e =< e_{01}, e_{02} >$ and $f =< e_{01}, e_{13} >$, where $e_{ij} = e_i \wedge e_j$. Then $W^\vee$ can be identified with $< e_{23}, e_{02}, e_{13} >$ by the pairing

$$W \times W^\vee \xrightarrow{\wedge} \bigwedge^4 \mathcal{V}^\vee \cong \mathbf{C}.$$

We then have $\mathbf{P}(W) \cap \mathbf{P}(W^\vee) =< e_{02}, e_{13} >$. We claim that

$$(7.6.1) \qquad \widetilde{J}_q(\mathbf{R}) \cap \mathbf{P}(W^\vee) \subset < e_{02}, e_{13} > (\mathbf{R}).$$

For if $p \in \widetilde{J}_q(\mathbf{R}) \cap \mathbf{P}(W^\vee)$ and $p \notin < e_{02}, e_{13} >$, then the line $< \ell, p > (\mathbf{R})$ is contained in $\widetilde{J}_q$ because $K \wedge \ell = 0$, where K is the matrix of the monad with $\{\det K = 0\} = \widetilde{J}_q$. On the other hand, since $p \wedge \ell \neq 0$, $< \ell, p > (\mathbf{R})$ meets the interior of S^4 ($p =< \xi >$, $\xi = e_{23} + \cdots$). This is impossible because $\widetilde{J}_q$ is allowed only to touch S^4.

We now choose a point $z = (W^\vee, A, \Gamma)$ in Z^{ss} defining $S = S(\Gamma)$ and $C^\vee = C^\vee(A)$ in such a way that $A \subset \Gamma \subset \mathbf{C}^2 \otimes W^\vee$ is represented by

$$A \hookrightarrow \Gamma \hookrightarrow \mathbf{C}^2 \otimes W^\vee$$

$$K := \begin{pmatrix} \alpha_0 & \cdots & \alpha_3 \\ \beta_0 & \cdots & \beta_3 \end{pmatrix} \begin{pmatrix} e_{02} & 0 \\ e_{01} & 0 \\ 0 & e_{01} \\ 0 & e_{13} \end{pmatrix}$$

The equation of $\widetilde{J}_q$ and $C^\vee$ is now

$$\det(K) = a_{02}e_{01}e_{02} + a_{03}e_{02}e_{13} + a_{12}e_{01}^2 + a_{13}e_{01}e_{13}$$

with $a_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$. We have to show that $\det(K) = \lambda e_{01}^2$, which is the equation of the double tangent hyperplane at e_{01}. Since C is invariant under the action of σ, we may assume that $\sigma(\det(K)) = \det(K)$, which means that $\bar{a}_{02} = a_{13}$, $\bar{a}_{03} = a_{03}$ and $\bar{a}_{12} = a_{12}$. Thus if we write $\det(K)$ by using the real basis $\{e_{01}, e_{02} + e_{13}, \sqrt{-1}(e_{02} - e_{13})\}$ of W, then the coefficients are real. Choosing three supplementary real vectors of $\bigwedge^2 \mathcal{V}^\vee$, the matrix of the quadratic form $\det(K)$ can be written in a form

$$\widetilde{\Phi}_K = \begin{pmatrix} \Phi_K & 0 \\ 0 & 0 \end{pmatrix}$$

such that Φ_K is real symmetric 3×3 matrix. By the original form, we have clearly

$$\det \Phi_K = \det \begin{pmatrix} 0 & a_{02} & a_{03} \\ a_{02} & 2a_{12} & a_{13} \\ a_{03} & a_{13} & 0 \end{pmatrix} = 2a_{03}(a_{02}a_{13} - a_{12}a_{03})$$

Since $(S, C^\vee)$ is a point of $\overline{I}(2)$, there is a sequence of real quadrics $\widetilde{\Phi}_i$ converging to $\widetilde{\Phi}_K$ and coming from bundles in $I(2)$. By Lemma 7.1, (2) we know that $\operatorname{rank} \widetilde{\Phi}_i = 3$ and $\operatorname{index} \widetilde{\Phi}_i \leq 2$ because $C_i^\vee(\mathbf{R})$ is a regular conic in $\mathbf{P}(W_i^\vee)(\mathbf{R})$. It follows that

$$\operatorname{rank} \Phi_K \leq 3 \text{ and } \operatorname{index} \Phi_K \leq 2.$$

We can now show that $\{\det K = 0\} = \widetilde{J}_q$ is the double tangent hyperplane of S^4 at ℓ. Assume first that $\det(\Phi_K) \neq 0$. Then $C^\vee(\mathbf{R}) \subset \mathbf{P}(W^\vee)(\mathbf{R})$ would be a regular real conic because $\operatorname{index} \Phi_K \leq 2$. This contradicts our claim (7.6.1). Hence $\det \Phi_K = 0$. Assume next that $a_{03} \neq 0$. Then $a_{02}a_{13} - a_{12}a_{03} = 0$ and it follows that for some $\lambda \in \mathbf{C}$,

$$(a_{12}, a_{13}) = \lambda(a_{02}, a_{03})$$

We may also assume that $a_{03} = 1$ and then we get

$$\det(K) = (e_{13} - a_{02}e_{01})(e_{02} - \lambda e_{01}).$$

By the invariance under the action of σ we have that $a_{02} = \bar{\lambda}$. Thus the equations $e_{13} \wedge \Xi - \bar{\lambda} e_{01} \wedge \Xi = 0$ and $e_{02} \wedge \Xi - \lambda e_{01} \wedge \Xi = 0$ in Ξ have a solution ξ with $\sigma(\xi) = \xi$ and $e_{01} \wedge \xi \neq 0$. This contradicts again (7.6.1). If $a_{03} = 0$, then $\det(K) = e_{01}(a_{02}e_{02} + a_{12}e_{01} + a_{13}e_{13})$. Since $\det(K)$ is σ-invariant, we may assume again without loosing any generality that $a_{12} = \bar{a}_{12}$ and $\bar{a}_{02} = a_{13}$. If $a_{02} \neq 0$, then the linear form $a_{02}e_{02} + \bar{a}_{02}e_{02} + a_{12}e_{01}$ has a real zero p with $e_{01}(p) \neq 0$, that is, p is in $\widetilde{J}_q(\mathbf{R}) \cap \mathbf{P}(W^\vee)$ but not in $< e_{02}, e_{13} >$, which is impossible. Therefore, we have that $a_{02} = 0$ and come to our requirement $\det(K) = a_{12}e_{01}^2$. Q.E.D.

Remark 7.7. The Γ in the above proof is not invariant under the action of $\bigwedge^2 \sigma$. Since z can be chosen in a minimal closed orbit, we see that no $z \in Z^{ss}$ over $q = (S, C^\vee)$ is σ-invariant. We have only the invariance of the orbit $o(z) = o(\sigma(z))$. This shows us that $Z^{ss}(\mathbf{R}) \to Q(\mathbf{R})$ is not surjective.

Corollary 7.8. *If we denote the closure of $I(2)$ in $Q(\mathbf{R})$ by $\overline{I}^Q(2)$, then $\overline{I}^Q(2) \subset Q(\mathbf{R}) \setminus Q_{exc}(\mathbf{R})$ and hence $\mu|_{Q \setminus Q_{exc}}$ induces an isomorphism of $\overline{I}^Q(2)$ to $\widetilde{I}(2)$*

$$
\begin{array}{ccccc}
Q(\mathbf{R}) \setminus Q_{exc}(\mathbf{R}) & \xrightarrow{\sim} & \widetilde{MI}(2)(\mathbf{R}) \setminus M_{exc}(\mathbf{R}) & & \\
\cup & & \cup & \searrow & \\
\overline{I}^{Q}(2) & \xrightarrow{\sim} & \tilde{I}(2) & & \overline{MI}(2)(\mathbf{R}) \\
 & & \downarrow l & \nearrow & \\
 & & \overline{I}(2) & &
\end{array}
$$

where $\tilde{I}(2)$ denotes the closure of $I(2)$ in $\widetilde{MI}(2)(\mathbf{R})$.

Proposition 7.6 enables us to study the boundary of $I(2)$ in $Q(\mathbf{R})$ in terms of the pair $(S, C^\vee)$. We define the subsets $I_1(2)$, $I_2(2)$, $I_3(2)$ and $I_4(2)$ of $Q(\mathbf{R})$ as the sets of pairs $(S, C^\vee) \in Q(\mathbf{R})$ described by the cases (I_1), (I_2), (I_3) and (II) of Proposition 7.6, respectively. By virtue of the geometric description we see immediately that each $I_\nu(2)$ is in the closure of the preceding one

$$\overline{I_\nu(2)} \supset I_{\nu+1}(2) \qquad \nu = 1, 2, 3.$$

Moreover, we see that the $I_\nu(2)$ are (locally closed) submanifolds. Indeed, the planes $\mathbf{P}(W)(\mathbf{R}) \subset \mathbf{P}(\bigwedge^2 V^\vee)(\mathbf{R})$ which cut S^4 in a circle form an open set $Z^\circ(\mathbf{R}) \subset G_3(\bigwedge^2 V)(\mathbf{R})$ and $I_1(2)$ is a fiber bundle over $Z^\circ(\mathbf{R})$ each fiber of which consists of pairs of real lines in $\mathbf{P}(W)(\mathbf{R})$, one touching $S^4 \cap \mathbf{P}(W)(\mathbf{R})$ and the other not. Analogous descriptions apply to $I_2(2)$ and $I_3(2)$.

The set $I_4(2)$ is a bundle over S^4 such that each fiber is the Grassmannian $G_2(T_\ell(S^4))$. By these descriptions of $I_\nu(2)$ we find at once that

$$
\begin{aligned}
\dim I_1(2) &= 12 \\
\dim I_2(2) &= 11 \\
\dim I_3(2) &= 10 \\
\dim I_4(2) &= 8
\end{aligned}
$$

Remark 7.9. The 2-dimensional jump from $I_3(2)$ to $I_4(2)$ corresponds to the fact $I_4(2)$ is contained in the singular locus of $Q(\mathbf{C})$, see [9, 3.12].

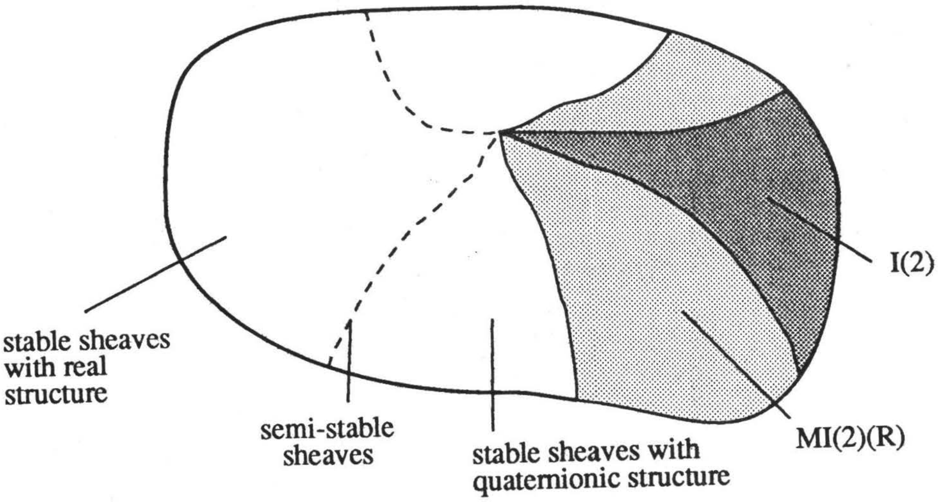
By [9, §9], the pairs $(S, C^\vee)$ in $I_1(2)$ correspond to quaternionic sheaves F which are obtained by elementary transformations

$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathcal{O}_\ell(1) \longrightarrow 0$$

from an instanton bundle $E \in I(1)$ by Theorem 4.3. Thus we have a stratification

$$\overline{I}(2) = I(2) \mathbin{\dot\cup} I_1(2) \mathbin{\dot\cup} I_2(2) \mathbin{\dot\cup} I_3(2) \mathbin{\dot\cup} I_4(2).$$

The positions of $I(2)$ or $\overline{I}(2)$ in $\overline{M}(2,2)(\mathbf{R})$ can be indicated by the following abstract picture.



The morphism $\varphi : \overline{I}(2) \to \overline{\overline{I}}(2)$ is now easily described in terms of the geometric description of the pairs $(S, C^\vee)$. If $z = (W^\vee, A, \Gamma) \in Z^{ss}$ represents $(S, C^\vee)$, then $\varphi(S, C^\vee)$ is the cokernel of the homomorphism

$$A \otimes_{\mathbf{C}} \Omega_X^3(3) \longrightarrow \mathbf{C}^2 \otimes_{\mathbf{C}} \Omega_X(1)$$

induced by the inclusion $A \subset \mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}$. Thus φ forgets the conic (or circle) S on the plane $\mathbf{P}(W)$.

If $(S, C^\vee)$ is in $I_1(2)$, then $A \subset \mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V} \cong \mathbf{C}^2 \otimes_{\mathbf{C}} \bigwedge^2 \mathcal{V}^\vee$ can be represented by a matrix

$$K = \begin{pmatrix} \ell & 0 \\ 0 & a \end{pmatrix}$$

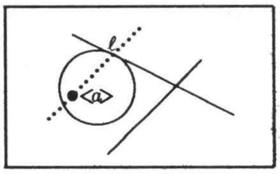
where ℓ is a point of S^4, a belongs to the ball with boundary S^4 and hence a can be regarded as the pole of a hyperplane which does not meet S^4. Thus we see that $< a > \in I(1)$ and

$$\varphi(S, C^\vee) = (< a >, \ell) \in I(1) \times S^4.$$

Similarly, for $(S, C^\vee) \in I_2(2)$, the image $\varphi(S, C^\vee)$ is the pair of two distinct points $(\ell_1, \ell_2) \in \Sigma^2(S^4)$. A pair $(S, C^\vee)$ in $I_3(2) \cup I_4(2)$ is mapped to the double point (ℓ, ℓ) in $\Sigma^2(S^4)$.

Now we can see the fibers of φ.

(1) If $(< a >, \ell)$ is a point of $I(1) \times S^4$, then the fiber of φ over the point consists of all pairs $(S, C^\vee)$ such that for a plane $\mathbf{P}(W)$ containing $< \ell >$ and $< a >$, $S = S^4 \cap \mathbf{P}(W)(\mathbf{R})$ and C consists of the tangent line of S at ℓ and the polar of $< a >$. Hence the fiber can be identified with the set of all planes $\mathbf{P}(W)(\mathbf{R}) \subset \mathbf{P}(\bigwedge^2 V^\vee)$ which contain $< a >$ and ℓ. This is a $\mathbf{P}^3(\mathbf{R})$.



$\mathbf{P}(W)(\mathbf{R})$

2) If $(\ell_1, \ell_2) \in \Sigma^2(S^4)$ is a pair of distinct points, the fiber is the $\mathbf{P}^3(\mathbf{R})$ formed by planes $\mathbf{P}(W)(\mathbf{R})$ through ℓ_1 and ℓ_2.

3) If ℓ represents a double line and if it is in the image of $I_3(2)$, then the fiber consists of all planes $\mathbf{P}(W)(\mathbf{R})$ through ℓ which is parametrized by the Grassmannian $G_{2,5}(\mathbf{R})$. When the double line is in the image of $I_4(2)$, then the fiber $G_{2,4}$ formed by planes $\mathbf{P}(W)(\mathbf{R})$ contained in $T_\ell(S^4)$.

Finally let us characterize the sheaves contained in $\overline{I}(2)$. According to the explicit study of the sheaves $F \in \overline{MI}(2)$ in [9, §9 and §10], the sheaves in $I_\nu(2)$ are, in particular, of the following type.

$I_1(2)$: Every sheaf $F \in I_1(2)$ is obtained by an elementary transformation

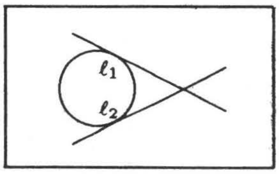
$$0 \longrightarrow F \longrightarrow E \xrightarrow{\pi} \mathcal{O}_\ell(1) \longrightarrow 0$$

from $E \in I(1)$ corresponding to the pole $< a >$, where π is defined by the corresponding $\mathbf{P}(W)$ (see [9, Proposition 9.1.1]). Moreover, the above sequence is compatible with the quaternionic structures on the sheaves. The morphism φ associates F with the pair (E, ℓ) and forgets the surjection π. Note also that every real, non-zero $\pi \in \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_\ell(1))(\mathbf{R})$ is compatible with the quaternionic structures and surjective. Thus we see again that the fiber is parametrized by $\mathbf{P}^3(\mathbf{R}) \cong \mathbf{P}(\mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_\ell(1))(\mathbf{R})^\vee)$.

$I_2(2)$: Here the sheaves are elementary transforms

$$0 \longrightarrow F \longrightarrow \mathcal{O}_X^{\oplus 2} \xrightarrow{\pi} \mathcal{O}_{\ell_1}(1) \oplus \mathcal{O}_{\ell_2}(1) \longrightarrow 0$$

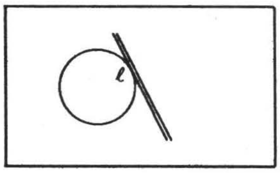
with two different lines ℓ_1, ℓ_2 and similarly to the previous case, π corresponds to the plane $\mathbf{P}(W)$.



$\mathbf{P}(W)(\mathbf{R})$

$I_3(2)$: The sheaves in this case are still elementary transforms but with a double structure on a line ℓ:

$$0 \longrightarrow F \longrightarrow \mathcal{O}_X^{\oplus 2} \xrightarrow{\pi} L \longrightarrow 0,$$



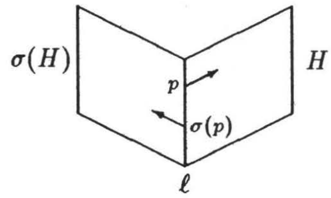
$\mathbf{P}(W)(\mathbf{R})$

where L is a non-trivial extension as an $\mathcal{O}_X$-module

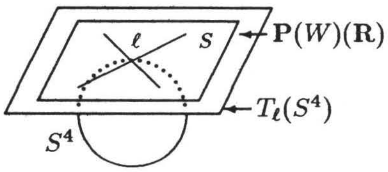
$$0 \longrightarrow \mathcal{O}_\ell(1) \longrightarrow L \longrightarrow \mathcal{O}_\ell(1) \longrightarrow 0$$

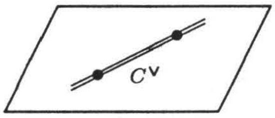
with $L \otimes \mathcal{O}_\ell \cong \mathcal{O}_\ell(1)$. Here the pair (π, L) corresponds to the plane $\mathbf{P}(W)$.

Whereas the cases $I_1(2)$, $I_2(2)$ and $I_3(2)$ are those occurring in Proposition 7.6, (I), $I_4(2)$ is different: In this case $\mathbf{P}(W)(\mathbf{R})$ is contained in the tangent space $T_\ell(S^4)$ and then F degenerates to an extension of ideal sheaves as follows. The two lines of $S \subset \mathbf{P}(W)(\mathbf{R})$ define two pencils each of which is formed by lines through a point in $\mathbf{P}(\mathcal{V}^\vee)$. Since S is invariant under the action of σ, the points are p and $\sigma(p)$ on ℓ and the planes spanned by pencils must be H and $\sigma(H)$. Let ℓ have the multiple structure at p defined by the direction H and we denote this by $\ell \cup p$. Similarly $\ell \cup \sigma(p)$ denotes the line ℓ with the multiple structure at $\sigma(p)$ defined by $\sigma(H)$.



By [9, 10.5 and 10.6] the sheaf F corresponding to the pair





$\mathbf{P}(W^\vee)(\mathbf{R})$

is equivalent to $I_{\ell \cup p} \oplus I_{\ell \cup \sigma(p)}$.

It is not hard to follow the degeneration from $I_3(2)$ to $I_4(2)$ in terms of sheaves. The two exact sequences

$$0 \longrightarrow \mathbf{C}(p) \longrightarrow \mathcal{O}_{\ell \cup p} \longrightarrow \mathcal{O}_\ell \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_\ell \longrightarrow \mathcal{O}_\ell(1) \longrightarrow \mathbf{C}(p) \longrightarrow 0$$

give rise to another exact sequence

$$0 \longrightarrow \mathbf{C}(p) \longrightarrow \mathcal{O}_{\ell \cup p} \longrightarrow \mathcal{O}_\ell(1) \longrightarrow \mathbf{C}(p) \longrightarrow 0.$$

Using this we can construct a *resolution* of the extension

$$0 \longrightarrow \mathcal{O}_\ell(1) \longrightarrow L \longrightarrow \mathcal{O}_\ell(1) \longrightarrow 0$$

to get an exact sequence

$$0 \to \mathbf{C}(p) \oplus \mathbf{C}(\sigma(p)) \to \mathcal{O}_{\ell\cup p} \oplus \mathcal{O}_{\ell\cup\sigma(p)} \to L \to \mathbf{C}(p) \oplus \mathbf{C}(\sigma(p)) \to 0.$$

Now we see how an elementary transform F of $\mathcal{O}_X^{\oplus 2}$ can degenerate to the direct sum $I_{\ell\cup p} \oplus I_{\ell\cup\sigma(p)}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \mathcal{O}_X^{\oplus 2} & \longrightarrow & L & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I_{\ell\cup p} \oplus I_{\ell\cup\sigma(p)} & \longrightarrow & \mathcal{O}_X^{\oplus 2} & \longrightarrow & \mathcal{O}_{\ell\cup p} \oplus \mathcal{O}_{\ell\cup\sigma(p)} & \longrightarrow & 0. \end{array}$$

Remark 7.10. The closure $\tilde{I}(2)$ (or, $\overline{I}^Q(2)$) of $I(2)$ meets the singular locus of $\widetilde{MI}(2)$ (or, Q, resp.) in the set described by $I_4(2)$. However, $\overline{I}^Q(2)$ does not meet neither the exceptional set Q_{exc} nor the divisor of Q corresponding to sheaves F whose double dual are not locally free.

8 Interpretation in terms of connections

The Atiyah-Ward-Penrose correspondence provides us with a bijective correspondence between $I(2)$ and the set of gauge equivalence classes of irreducible selfdual $SU(2)$-connections on S^4 with instanton number 2 [2]. If F is a member of $I_1(2) \subset \overline{I}(2)$, then there is an instanton bundle $E \in I(1)$, a real line ℓ and an exact sequence

$$0 \longrightarrow F \longrightarrow E \xrightarrow{\pi} \mathcal{O}_\ell(1) \longrightarrow 0.$$

Fixing E and ℓ, the set of isomorphism classes of F is in bijective correspondence with the set of the homothety classes of π, which is parametrized by $\mathbf{P}^3(\mathbf{R})$. This describes the fiber of the map φ of $\overline{I}(2)$ to Donaldson's compactification $\overline{\overline{I}}(2)$ over the point $(E, \ell) \in I(1) \times S^4$. We are now going to interpret this fiber as a set of equivalence classes of selfdual $SU(2)$-connections with *pole* at the fixed point $\ell \in S^4$. Physically these objects should interpret interactions between the field corresponding to E and the pole. Mathematically, in our description, globally non-equivalent connections arise as follows: Let F and E be $SU(2)$-vector bundles on S^4 with $c_2(F) = 2$ and $c_2(E) = 1$. There are inclusions $F \overset{\varepsilon}{\hookrightarrow} E$ which induce an isomorphism on $S^4 \setminus \ell$. If ∇_E is a regular connection on E , then we can obtain a connection ∇_ε with a pole at ℓ by the following diagram

$$\begin{array}{ccccc} E & \xrightarrow{\nabla_E} & E \otimes \mathcal{A}^1_{S^4} & \hookrightarrow & E \otimes \mathcal{A}^1_{S^4}(*\ell) \\ {\scriptstyle\varepsilon}\uparrow & & & & \| \\ F & & \xrightarrow[\nabla_\varepsilon]{} & & F \otimes \mathcal{A}^1_{S^4}(*\ell) \end{array}$$

where $\mathcal{A}^1_{S^4}$ denotes the C^∞-bundle of 1-forms on S^4 and $*\ell$ indicates forms or sections which may have a pole at ℓ (see the paragraph before Lemma 8.1). The inclusion ε provides us with an isomorphism $F(*\ell) \cong E(*\ell)$. We are proving essentially that if there is a global automorphism ψ of F with $\psi^*(\nabla_{\varepsilon_1}) = \nabla_{\varepsilon_2}$, then ε_1 and ε_2 are equivalent. Since the different inclusions may be considered as rotating the bundle F around the pole ℓ, the class of a pair (F, ∇_ε) interprets in a precise way the result of ***bubbling*** in the terminology of Freed-Uhlenbeck [4].

Let G be the Grassmannian $G_2(\mathcal{V})$, P be the subvariety of $X \times G$ which define the incidence correspondence between G and $X = \mathbf{P}(\mathcal{V}^\vee)$ and let p (or, q) be the projection of P to X (or, G). There is a canonical surjection of $q^*(\Omega^1_G)$ to $\Omega^1_{P/X}$ which induces an isomorphism

$$\Omega^1_G \cong q_*(\Omega^1_{P/X}).$$

If F is a coherent $\mathcal{O}_X$-module, then there are canonical homomorphisms

$$\begin{array}{ccccc} q_*(p^*(F)\otimes\Omega^1_{P/X}) & \longleftarrow & q_*(q^*q_*p^*(F)\otimes\Omega^1_{P/X}) & \longleftarrow & q_*p^*(F)\otimes q_*(\Omega^1_{P/X}) \\ & \nwarrow^{\kappa} & & & \Vert\wr \\ & & & & q_*p^*(F)\otimes\Omega^1_G \end{array}$$

Let us assume that there is a divisor $J(F) \subset G$ such that $F|_\ell \cong \mathcal{O}_\ell^{\oplus 2}$ for all $\ell \notin J(F)$. Then $q_*p^*(F)$ is locally free on $G \setminus J(F)$. It follows by the projection formula that κ is an isomorphism on $G \setminus J(F)$. If we denote by $\mathcal{O}_G(*H)$ the sheaf of the germs of meromorphic functions which may have a pole along the hypersurface H and if for an $\mathcal{O}_G$-module R, we set $R(*H) = R \otimes_{\mathcal{O}_G} \mathcal{O}_G(*H)$, then we have an isomorphism

$$q_*(p^*(F)\otimes\Omega^1_{P/X})(*J(F)) \overset{\sim}{\longleftarrow} q_*p^*(F)\otimes\Omega^1_G(*J(F))$$

because $R(*H) = 0$ if $\mathrm{Supp}(R) \subset H$.

For every $\mathcal{O}_X$-module F, there is the relative differential

$$d_{P/X} : p^*(F) \longrightarrow p^*(F)\otimes\Omega^1_{P/X}$$

which reduces to

$$\mathrm{id}\otimes d : F(x) \longrightarrow F(x)\otimes\Omega^1_{P_x}$$

on the fiber P_x over a point $x \in X$. Applying q_* to the differential and κ, we get a meromorphic connection

$$\begin{array}{ccccc} q_*p^*(F) & \longrightarrow & q_*(p^*(F)\otimes\Omega^1_{P/X}) & \hookrightarrow & q_*(p^*(F)\otimes\Omega^1_{P/X})(*J(F)) \\ \Big| & & \uparrow\kappa & & \|\wr \\ \Big| & & q(p^*(F)\otimes\Omega^1_G & \hookrightarrow & q_*p^*(F)\otimes\Omega^1_G(*J(F)) \\ \Big| & & & & \uparrow \\ \multicolumn{5}{c}{\underline{\qquad\qquad\qquad\qquad \nabla_F \qquad\qquad\qquad\qquad}} \end{array}$$

which is of course holomorphic on $G\setminus J(F)$. Clearly this construction is functorial.

Taking an instanton bundle $F \in I(n)$, we consider now the restriction of $q_*p^*(F)$ to $S^4 = G(\mathbf{R})$ as follows:

$$\mathcal{F} := q_*p^*(F)\otimes_{\mathcal{O}_{G(\mathbf{C})}} \mathcal{A}_{G(\mathbf{C})} \otimes_{\mathcal{A}_{G(\mathbf{C})}} \mathcal{A}_{G(\mathbf{R})},$$

where $\mathcal{A}$ denotes the sheaf of *complex-valued* C^∞ functions. If F is a member of $I(n)$, then $J(F)$ does not meet S^4 and hence $\mathcal{F}$ is a vector bundle on S^4. Moreover, we obtain a nowhere degenerate connection

$$\mathcal{F} \xrightarrow{\nabla_{\mathcal{F}}} \mathcal{F}\otimes\mathcal{A}^1_{S^4}$$

from ∇_F. The quaternionic structure $F \cong \sigma^*(F)$ gives rise to a compatibility diagram for $\nabla_{\mathcal{F}}$ by the functriality of our construction. It follows that $\nabla_{\mathcal{F}}$ in this case is an $SU(2)$-connection. The map $F \mapsto (\mathcal{F}, \nabla_{\mathcal{F}})$ is one direction of Atiyah-Ward correspondence for $I(n)$ ([2]).

Now let us assume that

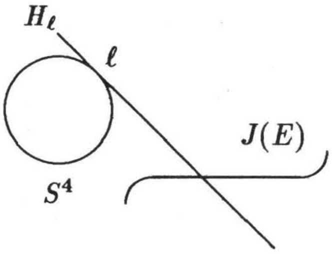
$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathcal{O}_\ell(1) \longrightarrow 0$$

is an elementary transformation of $E \in MI(n-1)$. In this case $J(F) = H_\ell \cup J(E)$, where H_ℓ is the cone of lines meeting the fixed line ℓ. We then obtain a commutative diagram

$$\begin{array}{ccccc} q_*p^*(E) & \xrightarrow{\nabla_E} & q_*p^*(E)\otimes\Omega^1_G(*J(E)) & \hookrightarrow & q_*p^*(E)\otimes\Omega^1_G(*J(F)) \\ \uparrow & & & & \| \\ q_*p^*(F) & & \xrightarrow[\nabla_F]{\qquad\qquad\qquad\qquad} & & q_*p^*(F)\otimes\Omega^1_G(*J(E)) \end{array}$$

From this ∇_F can be regarded as being induced from ∇_E with the additional pole set H_ℓ.

Let us further assume that $E \in I(n-1)$, $\ell \in G(\mathbf{R})$ and π is real. Then $J(E) \cap S^4 = \emptyset$ and only H_ℓ intersects S^4 at ℓ (see [8]).



By restricting ∇_F to S^4 we have a *connection with a pole* at ℓ

$$\mathcal{F} \xrightarrow{\nabla_{\mathcal{F}}} \mathcal{F} \otimes \mathcal{A}^1_{S^4}(*\ell)$$

and we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{\nabla_{\mathcal{E}}} & \mathcal{E} \otimes \mathcal{A}^1_{S^4} & \hookrightarrow & \mathcal{E} \otimes \mathcal{A}^1_{S^4}(*\ell) \\ \uparrow & & & & \| \\ \mathcal{F} & & \xrightarrow[\nabla_{\mathcal{F}}]{} & & \mathcal{F} \otimes \mathcal{A}^1_{S^4}(*\ell) \end{array}$$

such that $\nabla_{\mathcal{F}}$ becomes an induced $SU(2)$-connection which has a singularity at ℓ and which depends on the inclusion of F into E.

Lemma 8.1. *In the above situation $q_*p^*(F)$ is locally free in a neighborhood of $S^4 \subset G(\mathbf{C})$. In particular $\mathcal{F}$ is a vector bundle on S^4.*

Proof. F is the cohomology sheaf of a monad

$$\mathbf{C}^n \otimes_{\mathbf{C}} \Omega^3_X(3) \xrightarrow{\alpha} \mathbf{C}^n \otimes_{\mathbf{C}} \Omega^1_X(1) \xrightarrow{\beta} \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{O}_X.$$

Setting $N = \ker(\beta)$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus n} \longrightarrow N \longrightarrow F \longrightarrow 0$$

because $\Omega^3_X(3) \cong \mathcal{O}_X(-1)$. Since p is flat, this sequence yields another exact sequence

$$0 \longrightarrow q_*p^*(\mathcal{O}_X(-1)^{\oplus n}) \longrightarrow q_*p^*(N) \longrightarrow q_*p^*(F) \longrightarrow R^1q_*(p^*(\mathcal{O}_X(-1)^{\oplus n})).$$

Obviously the first and the last terms of the above sequence vanish and hence we see that $q_*p^*(F)$ is isomorphic to $q_*p^*(N)$. We are going to show that $q_*p^*(N)$ is locally free in a neighborhood of S^4. If we prove that for every $\ell' \in S^4$, $H^1(\ell', N|_{\ell'}) = 0$, then $\mathrm{Supp}(R^1q_*p^*(N))$ does not meet S^4 because N is locally free and q is flat. Then the exact sequence on the Grassmannian

$$0 \longrightarrow q_*p^*(N) \longrightarrow \mathbf{C}^n \otimes_{\mathbf{C}} Q^\vee \longrightarrow \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{O}_G \longrightarrow R^1q_*p^*(N) \longrightarrow 0$$

shows our assertion, where Q is the universal quotient bundle on G. Now let $\ell' \in S^4$ be different from ℓ. Restricting the above second sequence to ℓ', we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\ell'}(-1)^{\oplus n} \longrightarrow N|_{\ell'} \longrightarrow F|_{\ell'} \longrightarrow 0.$$

Since $F|_{\ell'} \cong \mathcal{O}_{\ell'}^{\oplus 2}$, we see that $H^1(\ell', N|_{\ell'}) = 0$. Next let us look at the case $\ell' = \ell$. Since F is an elementary transform of an instanton bundle E, there is an exact sequence

$$0 \longrightarrow \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_\ell(1), \mathcal{O}_\ell) \longrightarrow F \otimes \mathcal{O}_\ell \longrightarrow E \otimes \mathcal{O}_\ell \xrightarrow{\gamma} \mathcal{O}_\ell(1) \longrightarrow 0.$$

By using the Koszul complex of $\mathcal{O}_\ell$ we find that $\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_\ell(1), \mathcal{O}_\ell) \cong \mathcal{O}_\ell^{\oplus 2}$. On the other hand, $H^0(\gamma)$ must be surjective by the definition of the elementary transformation. Thus we see that $H^1(\ell, F \otimes \mathcal{O}_\ell) = 0$. By the above second sequence we have an exact sequence

$$0 \longrightarrow \mathcal{T}or_1^{\mathcal{O}_X}(F, \mathcal{O}_\ell) \longrightarrow \mathcal{O}_\ell(-1)^{\oplus n} \longrightarrow N \otimes \mathcal{O}_\ell \xrightarrow{\delta} F \otimes \mathcal{O}_\ell \longrightarrow 0.$$

For the kernel K of δ, we have a surjection $H^1(\ell, \mathcal{O}_\ell(-1)^{\oplus n}) \to H^1(\ell, K)$, which implies that $H^1(\ell, K) = 0$. We see therefore that $H^1(\ell, N|_\ell) = 0$ in this case, too.

Q.E.D.

Now we come to the main result of this section.

Proposition 8.2. *Let F_i be two elementary transforms with quaternionic structure of the same instanton bundle E along $\mathcal{O}_\ell(1)$:*

$$0 \longrightarrow F_i \longrightarrow E \xrightarrow{\pi_i} \mathcal{O}_\ell(1) \longrightarrow 0$$

and assume in addition that $\mathrm{Supp}(R^1 q_* p^*(F_i))$ *has dimension* ≤ 1. *Let $(\mathcal{F}_i, \nabla_i)$ be their induced connections on S^4 with a pole at ℓ. If $(\mathcal{F}_1, \nabla_1) \cong (\mathcal{F}_2, \nabla_2)$ by a global C^∞-isomorphism $\psi : \mathcal{F}_1 \to \mathcal{F}_2$ with $\nabla_1 = \psi^*(\nabla_2)$, then $F_1 \cong F_2$ as algebraic sheaves on X or equivalently π_1 and π_2 give rise to the same point in* $\mathbf{P}^3(\mathbf{R})$.

Remark 8.3. The condition on $R^1 q_* p^*(F_i)$ seems to be only of technical nature. We have no examples of sheaves in $\overline{MI}(n)$ which violate this condition. For every member F of $\overline{MI}(2)$, we know that $\mathrm{Supp}(R^1 q_* p^*(F))$ is a conic and hence F satisfies the condition.

Proof of Proposition 8.2. Our proof is divided into five parts.

1) Assume that ψ is given. Then $\psi|_{S^4 \setminus \{\ell\}}$ gives rise to an automorphism of $(\mathcal{E}, \nabla_{\mathcal{E}})|_{S^4 \setminus \{\ell\}}$. Since Atiyah-Ward correspondence is local ([2]), $\psi|_{S^4 \setminus \{\ell\}}$ corresponds to a holomorphic isomorphism

$$\vartheta : \mathcal{E}|_{S^4 \setminus \{\ell\}} \xrightarrow{\sim} \mathcal{E}|_{S^4 \setminus \{\ell\}}.$$

But, since ℓ is of codimension 2, the natural map

$$\mathrm{Hom}_{\mathcal{O}_X}(E,E) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X\setminus\{\ell\}}}(E|_{X\setminus\{\ell\}}, E|_{X\setminus\{\ell\}})$$

is an isomorphism. Thus ϑ extends to an endomorphism $\tilde{\vartheta}$ of E which must be a homothety λ. We may assume that $\lambda = 1$.

2) We now consider the diagram

$$\begin{array}{ccc} \mathcal{F}_1 & \overset{\epsilon_1}{\hookrightarrow} & \mathcal{E} \\ \psi\downarrow & & \| \\ \mathcal{F}_2 & \overset{\epsilon_2}{\hookrightarrow} & \mathcal{E} \end{array}$$

The equality $\epsilon_2 \cdot \psi = \epsilon_1$ on $S^4 \setminus \{\ell\}$ extends to the whole S^4 because the locally free C^∞ sheaf $\mathcal{H}om(\mathcal{F}_1, \mathcal{E})$ has no sections with 0-dimensional support. Now $\mathcal{F}_i$ and $\mathcal{E}$ carry real analytic structures, ϵ_i are real analytic and ϵ_2 induces a real analytic isomorphism between $\mathcal{F}_2$ and its image. Thus ψ is real analytic. Let $\tilde{\psi}$ be a *holomorphic* extension of ψ to a neighborhood U of S^4 in $G(\mathbf{C})$

$$\tilde{\psi} : q_*p^*(F_1)|_U \longrightarrow q_*p^*(F_2)|_U.$$

If U is small enough, then this extension is unique and an isomorphism. We have also a commutative diagram on U

$$\begin{array}{ccc} q_*p^*(F_1)|_U & \overset{\tilde{\epsilon}_1}{\hookrightarrow} & q_*p^*(E)|_U \\ \tilde{\psi}\downarrow & & \| \\ q_*p^*(F_2)|_U & \overset{\tilde{\epsilon}_2}{\hookrightarrow} & q_*p^*(E)|_U \end{array}$$

Since $\tilde{\epsilon}_2 \cdot \tilde{\psi} = \tilde{\epsilon}_1$ on S^4, we have the same equality on U. Hence $q_*p^*(F_1) = q_*p^*(F_2)$ as subsheaves of $q_*p^*(E)$. There is, at worst, a subvariety $A \subsetneqq H_\ell$ where the two subsheaves do not coincide. On one hand, $q_*p^*(\mathcal{O}_\ell(1))$ does not have torsion on its support H_ℓ (it is in fact the dual on H_ℓ of the ideal sheaf I_{P_α, H_ℓ} of an α-plane $P_\alpha \subset H_\ell$). On the other hand, the exact sequences

$$0 \longrightarrow q_*p^*(F_i) \longrightarrow q_*p^*(E) \longrightarrow q_*p^*(\mathcal{O}_\ell(1))$$

show that $\{q_*p^*(F_1) + q_*p^*(F_2)\}/q_*p^*(F_1)$ is a subsheaf of $q_*p^*(\mathcal{O}_\ell(1))$ supported by a subset of A. Thus $q_*p^*(F_2)$ must be contained in $q_*p^*(F_1)$. Similarly, we get $q_*p^*(F_1) \subset q_*p^*(F_2)$. These prove that $q_*p^*(F_1) = q_*p^*(F_2)$ globally on G as subsheaves of $q_*p^*(E)$.

3) By Proposition 2.1 each F_i can be presented by a monad of the form

$$K = \left(\begin{array}{c|c} \ell & 0 \\ \hline A_i & K' \end{array}\right), \qquad M = \left(\begin{array}{c|c} x\ y & 0\cdots 0 \\ \hline B_i\ C_i & M' \end{array}\right)$$

where (K', M') gives rise to a monad of E and $\ell = x \wedge y$. Let N_i be the kernel bundle of the monad of F_i, that is, we have an exact sequence

$$(8.2.1) \qquad 0 \longrightarrow N_i \longrightarrow \mathbf{C}^n \otimes_{\mathbf{C}} \Omega^1_X(1) \longrightarrow \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow 0.$$

As in the proof of Lemma 8.1 we have an isomorphism $q_*p^*(F_i) \cong q_*p^*(N_i)$ and for the kernel N' of M', $q_*p^*(E)$ is also isomorphic to $q_*p^*(N')$. By the part 2) we obtain isomorphic subsheaves

$$\begin{array}{ccc} q_*p^*(N_1) & \hookrightarrow & q_*p^*(N') \\ \downarrow\wr & & \| \\ q_*p^*(N_2) & \hookrightarrow & q_*p^*(N') \end{array}$$

4) In order to strip q_*p^* from the above diagram we shall show that under our assumption on the supports we get

$$N_i = (p_*q^*((q_*p^*(N_i))^\vee))^\vee \quad \text{and} \quad N' = (p_*q^*((q_*p^*(N'))^\vee))^\vee.$$

Applying the functor q_*p^* to (8.2.1) we obtain the exact sequence

$$0 \longrightarrow q_*p^*(N_i) \longrightarrow \mathbf{C}^n \otimes_{\mathbf{C}} Q^\vee \xrightarrow{m_i} \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{O}_G \longrightarrow R^1q_*p^*(N_i) \longrightarrow 0$$

on the Grassmannian G, where Q is the universal quotient bundle on G and where m_i is the homomorphism defined by M_i. By our assumption we see that for $i \leq 2$, $\mathcal{E}xt^i_{\mathcal{O}_G}(R^1q_*p^*(N_i), \mathcal{O}_G) = 0$ and hence we get the dual *exact* sequence

$$0 \longrightarrow \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{O}_G \xrightarrow{m_i^\vee} \mathbf{C}^n \otimes_{\mathbf{C}} Q \longrightarrow q_*p^*(N_i)^\vee \longrightarrow 0.$$

Applying the functor p_*q^*, we get

$$0 \longrightarrow \mathbf{C}^{2n-2} \otimes_{\mathbf{C}} \mathcal{O}_x \xrightarrow{m_i^\vee} \mathbf{C}^n \otimes_{\mathbf{C}} T_X(-1) \longrightarrow p_*q^*(q_*p^*(N_i)^\vee) \longrightarrow 0,$$

where $m_i^\vee$ is defined by ${}^tM_i$. This proves our first claim. The same applies to N' because we have a surjection $R^1q_*p^*(F_i) \to R^1q_*p^*(E)$. Combining the above claim and the result of 3) we get a commutative diagram

$$(8.2.2) \qquad \begin{array}{ccc} N_1 & \longrightarrow & N' \\ \downarrow\wr & & \| \\ N_2 & \longrightarrow & N' \end{array}$$

5) Finally we remark that the decomposition of the monad matrices K_i, M_i into blocks can be expressed in two exact commutative diagrams

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^3_X(3) & \rightarrow & N_i & \rightarrow & N' & \xrightarrow{\pi'_i} & \mathcal{O}_\ell(1) \rightarrow 0 \\
 & & \downarrow \ell & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^1_X(1) & \rightarrow & \mathbf{C}^n \otimes \Omega^1_X(1) & \rightarrow & \mathbf{C}^{n-1} \otimes \Omega^1_X(1) & \longrightarrow & 0 \\
 & & \downarrow (x,y) & & \downarrow M_i & & \downarrow M' & & \\
0 & \rightarrow & \mathbf{C}^2 \otimes \mathcal{O}_X & \rightarrow & \mathbf{C}^{2n-2} \otimes \mathcal{O}_X & \rightarrow & \mathbf{C}^{2n-4} \otimes \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_\ell(1) & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^3_X(3) & \rightarrow & \mathbf{C}^n \otimes \Omega^3_X(3) & \rightarrow & \mathbf{C}^{n-1} \otimes \Omega^3_X(3) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^3_X(3) & \rightarrow & N_i & \rightarrow & N' & \xrightarrow{\pi'_i} & \mathcal{O}_\ell(1) \rightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \| \\
 & & 0 & \rightarrow & F_i & \rightarrow & E & \xrightarrow{\pi'_i} & \mathcal{O}_\ell(1) \rightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

It follows from (8.2.2) and the first of the above two diagrams that $\pi'_1 \sim \pi'_2$ as cokernels, that is, we are given a commutative diagram

$$
\begin{array}{ccccccc}
N_1 & \longrightarrow & N' & \xrightarrow{\pi'_1} & \mathcal{O}_\ell(1) & \longrightarrow & 0 \\
\downarrow \wr & & \| & & \wr \downarrow \alpha & & \\
N_2 & \longrightarrow & N' & \xrightarrow{\pi'_1} & \mathcal{O}_\ell(1) & \longrightarrow & 0
\end{array}
$$

with $\alpha \neq 0$. Since by the second of the above two diagrams π'_i are factorized through E to give π_i, we get the desired result. Q.E.D.

Proposition 8.2 applies in particular to the component $I_1(2)$ of $\overline{I}(2) \setminus I(2)$. In case $n = 2$ we even have a complete Atiyah-Ward correspondence on the whole compactification $\overline{I}(2)$.

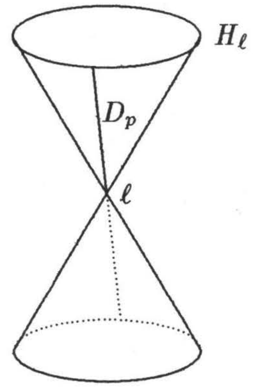
Proposition 8.4. *Let F_1 and F_2 be members of $I_\nu(2)$ for $1 \le \nu \le 4$ and let $(\mathcal{F}_i, \nabla_i)$ be their induced connections on S^4. If $(\mathcal{F}_1, \nabla_1) \cong (\mathcal{F}_2, \nabla_2)$ by a global C^∞-isomorphism $\psi : \mathcal{F}_1 \to \mathcal{F}_2$ with $\nabla_1 = \psi^*(\nabla_2)$, then $F_1 \cong F_2$ as algebraic sheaves on X.*

Proof. The case $I_1(2)$ has been treated in the above. In cases $I_2(2)$, $I_3(2)$ the kernel sheaves N_i correspond bijectively to the surjections π_i up to equivalence as can be seen analogously or obtained from [9]. In these cases $E = \mathcal{O}_X^{\oplus 2}$. As in the proof of Proposition 8.2 we find $N_1 \cong N_2$ and thus $\pi_1 \sim \pi_2$. To complete the proof we are left with the case of $I_4(2)$. Let $F_1 = I_{\ell \cup p_1} \oplus I_{\ell \cup \sigma(p_1)}$ and $F_2 = I_{\ell \cup p_2} \oplus I_{\ell \cup \sigma(p_2)}$, and assume that $(\mathcal{F}_1, \nabla_1) \cong (\mathcal{F}_2, \nabla_2)$ by a global C^∞-isomorphism. Note that $\mathcal{F}_1$, $\mathcal{F}_2$ are not locally free any more on S^4. As in the proof of Proposition 8.2 we prove that $q_*p^*(F_1) \cong q_*p^*(F_2)$ as subsheaves of $\mathcal{O}_G^{\oplus 2}$ in a neighborhood U of S^4. Let H_ℓ be the hyperplane cut of G formed by lines meeting ℓ and let D_p be the α-plane of lines passing through p. From the exact sequence

$$0 \longrightarrow \mathbf{C}_p \longrightarrow \mathcal{O}_{\ell \cup p} \longrightarrow \mathcal{O}_\ell \longrightarrow 0$$

we easily derive an isomorphism

$$q_*p^*(\mathcal{O}_{\ell \cup p}) \cong \mathcal{O}_{H_\ell \cup D_p},$$



where $\mathcal{O}_{H_\ell \cup D_p}$ is the structure sheaf of H_ℓ with a multiple structure along D_p. Since $q_*p^*(F_i)$ are equal in U and $G \setminus H_\ell$, there is a Zariski closed set $A \subsetneq H_\ell$ such that A does not contain ℓ and $q_*p^*(F_i)$ are equal on $G \setminus A$. On the other hand, we have $H^0_A(q_*p^*(\mathcal{O}_{\ell \cup p})) = 0$ because $D_p \not\subseteq A$. Then, as in the proof of Proposition 8.2, we conclude that $q_*p^*(F_1) = q_*p^*(F_2)$ globally as subsheaves of $\mathcal{O}_G^{\oplus 2}$. By Lemma 8.5 below we find that $\{p_1, \sigma(p_1)\} = \{p_2, \sigma(p_2)\}$, which proves that $F_1 = F_2$.

Lemma 8.5. *$q_*p^*(I_{\ell \cup a})$ is the subsheaf $I_{D_a}(-1) \subset \mathcal{O}_G(-1) \subset \mathcal{O}_G$ with I_{D_a} the ideal sheaf of the α-plain $D_a \subset G$.*

Proof. We have an exact sequence

$$0 \longrightarrow I_{\ell \cup a} \longrightarrow I_\ell \longrightarrow \mathbf{C}_a \longrightarrow 0$$

which provides us with an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & & & \\
 & & & & \downarrow & & & & & & \\
0 & \to & q_*p^*(I_{\ell\cup a}) & \to & q_*p^*(I_\ell) & \to & q_*p^*(\mathbf{C}_a) & \to & R^1q_*p^*(I_{\ell\cup a}) & \to & 0 \\
 & & & & \downarrow & & \| & & \| & & \\
 & & & & \mathcal{O}_G & & \mathcal{O}_{D_a} & & \mathcal{O}_g & & \\
 & & & & \downarrow & & & & & & \\
 & & & & \mathcal{O}_{H_\ell} & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & R^1q_*p^*(I_\ell) & = & 0 & & & &
\end{array}$$

where $g \subset G$ is the line formed by the pencil of lines through a in the plane in $\mathbf{P}^3$ spanned by ℓ and the tangent vector in the multiple point $a \in \ell$. Hence $q_*p^*(I_\ell) \cong \mathcal{O}_G(-1)$ and the homomorphism is factored through $\mathcal{O}_{D_a}(-1)$ to give $\mathcal{O}_g$ as a quotient. This proves our assertion.

$$\begin{array}{ccc}
\mathcal{O}_G(-1) & \longrightarrow & \mathcal{O}_{D_a} \\
 & \searrow \quad \nearrow & \\
 & \mathcal{O}_{D_a}(-1) &
\end{array}$$

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