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Let X and Y be Hilbert spaces and $T : X \longrightarrow Y$ be a bounded linear operator. The operator equation

$$(*) \qquad Tx = y \ , \ y \in R(T)$$

is *ill–posed* if R(T) is not closed in Y, and in that case regularization methods are employed for approximating the minimal norm solution $\hat{x}$ of (*). (c.f. [4],[6]). In *Tikhonov regularization* one solves

$$(1) \qquad (\alpha I+A)x_\alpha^\delta = T_n^* y^\delta$$

where $\alpha > 0$, $y^\delta \in Y$ with $\|y-y^\delta\| \leq \delta$ and $A = T^*T$. The following results are known (c.f. [6], [15]):

(i) $\|x_\alpha^o - x_\alpha^\delta\| \leq \delta/\sqrt{\alpha}$

(ii) $\|\hat{x}-x_\alpha^o\| \longrightarrow 0$ as $\alpha \longrightarrow 0$

(iii) $\hat{x} \in R(A^\nu)$, $0 < \nu \leq 1 \Longrightarrow \|\hat{x}-x_\alpha^o\| = 0(\alpha^\nu)$

(iv) $\alpha = c\delta^{2/(2\nu+1)}$, $\hat{x} \in R(A^\nu)$, $0 < \nu \leq 1 \Longrightarrow \|\hat{x}-x_\alpha^\delta\| = 0(\delta^{2\nu/(2\nu+1)})$

and for all y^δ with $\|y-y^\delta\| \leq \delta$.

$\|\hat{x}-x_\alpha^\delta\| = o(\delta^{2\nu/(2\nu+1)}) \Longrightarrow \hat{x} = 0.$

In a posteriori parameter choice strategy $\alpha = \alpha(\delta)$ is determined during the course of solving (1). For this "*discrepancy principles*" are often used:

Morozov [11]: $\|Tx_\alpha^\delta - y^\delta\| = \delta$

Arcangeli [1]: $\|Tx_\alpha^\delta - y^\delta\| = \dfrac{\delta}{\sqrt{\alpha}}$

Schock [16]: $\|Tx_\alpha^\delta - y^\delta\| = \dfrac{\delta^p}{\alpha^q}$, $p \geq 0$, $q > 0$

Engl [2]: $\|Ax_\alpha^\delta - T^* y^\delta\| = \dfrac{\delta^p}{\alpha^q}$, $p \geq 0$, $q > 0$.

The following results are known about the above discrepancy principles:

(i) Morozov's method can not give a better rate than $0(\delta^{1/2})$ and it is attained for $\hat{x} \in R(A^{1/2})$. (c.f. [5]).

(ii) For Arcangeli's method, $\hat{x} \in R(T^*) \Rightarrow \|\hat{x} - x_\alpha^\delta\| = 0(\delta^{1/3})$, and $\|\hat{x} - x_\alpha^\delta\| = o(\delta^{2/3}) \Rightarrow \hat{x} = 0$. (c.f. [9].)

(iii) For Schock's general class of discrepancy principles, $\hat{x} \in R(A^\nu)$, $0 < \nu \leq 1$, $p/(q+1) = 2/(2\nu+1+\frac{1}{q}) \Rightarrow$

$$\|\hat{x} - x_\alpha^\delta\| = 0(\delta^s)\ ,\ s = 2/(2\nu+1+\tfrac{1}{q})\ .$$

Here, $s \longrightarrow 2\nu/(2\nu+1)$ as $q \longrightarrow \infty$ (c.f. [16]) .

(iv) In Engl's modification of Schock's method, $\hat{x} \in R(A^{\nu})$, $0 < \nu \leq 1$, $p/(q+1) = 2/(2\nu+1) \Longrightarrow$

$$\|\hat{x}-x_{\alpha}^{\delta}\| = 0(\delta^{2\nu/(2\nu+1)}) .$$

(c.f. [2].)

(v) In [12], Nair considered Schock's method and proved that $\hat{x} \in R(A^{\nu})$, $0 < \nu \leq 1$, $p/(q+1) = 2/[2\nu+1+(1-\nu)/2q] \Longrightarrow \|\hat{x}-x_{\alpha}^{\delta}\| = 0(\delta^{s})$, $s = 2\nu/[2\nu+1+(1-\nu)/2q]$.

This result improves the result of Schock [16] and gives the rate $o(\delta^{2/3})$ for $p/(q+1) = 2/3$ and $\nu = 1$, and also settles the open question of obtaining optimal rate for Arcangeli's method.

(vi) Recently S. George and M.T. Nair [3] improved the result in Nair [12]: For $\hat{x} \in R(A^{\nu})$, $0 < \nu \leq 1$,

$$0 < \nu \leq \tfrac{1}{2} \Longrightarrow \|\hat{x}-x_{\alpha}^{\delta}\| = \begin{cases} 0(\delta^{m}), & p/(q+1) \leq 2/(2\nu+1+(1-2\nu)/2q) \\ 0(\delta^{\mu}), & p/(q+1) \geq 2/(2\nu+1+(1-2\nu)/2q) \end{cases}$$

$$\tfrac{1}{2} \leq \nu \leq 1 \Longrightarrow \|\hat{x}-x_{\alpha}^{\delta}\| = \begin{cases} 0(\delta^{m}), & p/(q+1) \leq 2/(2\nu+1) \\ 0(\delta^{\lambda}), & p/(q+1) \geq 2/(2\nu+1) \end{cases} ,$$

where $m = p\nu/(q+1)$, $\mu = 1-[1+(1-2\nu)/2q]p/2(q+1)$,
$\lambda = 1-p/2(q+1)$.
This result gives the optimal rate $0(\delta^{2\nu/(2\nu+1)})$ for $\hat{x} \in R(A^{\nu})$, $\nu \geq 1/2$, $p/(q+1) = 2/(2\nu+1)$.
(Engl and his collaborators remarked in many papers that Arcangeli's method cannot give optimal rate. This was based on an incorrect oberservation of results in [9].)

In applications one generally replaces the operator A and T in (1) by their "approximations" A_n and T_n respectively. In projection methods ([6], [8]) $A_n = P_nAP_n$ and $T_n = TP_n$, and in a degnerate kernel method of Groetsch [7] for solving integral equation of the first kind. A_n is obtained by approximating the kernel of the integral operator A by convergent quadrature rules, and $T_n = T$. In both the above cases $\|A-A_n\| \longrightarrow 0$ and A and A_n are non–negative and self–adjoint.

Recently Nair [13] considered a general type of approximation methods (for compact T):

(a) $\|(A-A_n)x\| \longrightarrow 0$ for every $x \in X$

(b) $\|(A-A_n)A_n\| \longrightarrow 0$.

If T is the integral operator,

$$(Tx)(s) = \int_0^1 k(s,t)x(t)dt \ , \quad 0 \leq s \leq 1 \ ,$$

with continuous kernel k(s,t), then one may consider the operator norm induced by uniform norm on C[0,1] also; in that case T^* is defined by

$$(T^*x)(s) = \int_0^1 k(s,t)x(t)dt \ , \quad 0 \leq s \leq 1 \ .$$

The above consideration includes norm approximation as well as collectively compact approximation (A_n) of A. If (A_n) is the Nyström approximation of the integral operator A, then $\|A-A_n\| \leq \|A_n\|$, but (A_n) is a collectively compact approximation of A (c.f. Kress [10]).

In place of (1) one has

(2) $$(\alpha I+A_n)x^\delta_{\alpha,n} = T^*y^\delta \ .$$

The main results of [13] are the following theorems.

THEOREM 1 (Hilbert space setting). For n=1,2,..., let $\alpha = \alpha(n)$ be such that

(i) $\|(A-A_n)A_n\| \le c_1\alpha^2$, $0 < c_1 < 1$

(ii) $\|(A-A_n)A\| \le (1-c_1)\alpha^2$

(iii) $\|(A-A_n)A\| \le c\alpha^3$

(iv) $\|(A-A_n)T^*\| \le c\alpha^{3/2}$.

Then

$$\|\hat{x}-x^{\delta}_{\alpha,n}\| \le c(\|\hat{x}-x_\alpha\| + \delta/\sqrt{\alpha}) ,$$

where c is a generic constant independent of δ,α,n.

THEOREM 2 (Uniform norm case for c[0,1]). For n=1,2,... let $\alpha = \alpha(n)$ be such that

(i) $\|(A-A_n)A_n\| \le c_1\alpha^3/(\alpha+M)$, $0 < c_1 < 1$

(ii) $\|(A-A_n)A\| \le [(1-c_1)\alpha+M]\alpha^2/(\alpha+M)$

(iii) $\|(A-A_n)A\| \le c\alpha^4$

(iv) $\|(A-A_n)T^*\| \le c\alpha^2$.

Then

$$\|\hat{x}-x^{\delta}_{\alpha,n}\| \le c(\|\hat{x}-x_\alpha\| + \delta/\alpha) .$$

where $M = \sup_{0\le s\le 1} \left(\int_0^1 \left|\int_0^1 k(\tau,s)k(\tau,t)d\tau\right|^2 dt\right)^{1/2}$ and $c > 0$ is a generic constant independent from δ,α,n.

Remark

(i) The approximation procedure does not reduce the accuracy guaranteed by the theoretical estimates.

(ii) In [7]., Groetsch requires $n\delta/\alpha^2 \to 0$ for the convergence of the method in uniform norm case, whereas the requirement in Theorem 2 is only $\delta/\alpha \to 0$

Next we suggest an a posteriori parameter choice strategy for determining α in (2) using a procedure adopted in [14].

Suppose (ε_n) is a positive real sequence such that $\varepsilon_n \longrightarrow 0$ and for each n=1,2,..., the conditions (i)–(iv) of Theorem 1 (resp. Theorem 2) are satisfied for all $\alpha \geq \varepsilon_n$.

For $\delta > 0$, n=1,2,..., cosider the discrepancy principle

$$D_n(\alpha,\delta) := \alpha^q \|A_n x^{\delta}_{\alpha,n} - T^* y^{\delta}\| = \delta^p \ , \ \alpha \geq \varepsilon_n \ ,$$

for some fixed $p \geq 0$, $q > 0$.

Let $\delta_o > 0$ be given, $\varepsilon = \sup \{\varepsilon_n : n=1,2,...,\}$.

$\|T^*y^{\delta}\| \geq c_2$, $\forall\ \delta \geq \delta_o$, $\|A_n\| \leq c_3$, $\forall$ n=1,2,..., $c_o = \dfrac{c_2}{1+c_3/\varepsilon}$ and

$\alpha_o = \max \{\varepsilon \ , (\delta_o^p/c_o)^{1/2}\}$.

PROPOSITION 3:

For $\delta \leq \delta_o$ there exists $\hat{n}(\delta) \in \mathbb{N}$, and for $n \geq \hat{n}(\delta)$ there exists $\alpha_n(\delta) > 0$ such that

(i) $\varepsilon_n \leq \alpha_n \leq x_o$, $D_n(\alpha_n(\delta),\delta) = \delta^p$, $\forall\ n \geq \hat{n}(\delta)$.

(ii) (Hilbert space setting)

$$\frac{p}{2(q+1)} \leq 1 \Longrightarrow \frac{\delta^p}{\alpha_n(\delta)^q} = 0(\delta^{p/(q+1)}) \text{ and } \frac{\delta}{\sqrt{\alpha_n(\delta)}} = 0(\delta^{1-(p/2(q+1))})$$

(iii) (Uniform norm casse for C[0,1])

$$\frac{p}{q+1} \leq 1 \Longrightarrow \frac{\delta^p}{\alpha_n(\delta)^q} = 0(\delta^{p/(q+1)}) \text{ and } \frac{\delta}{\alpha_n(\delta)} = 0(\delta^{1-(p/(q+1))}) \ .$$

THEOREM 4:

Let $\alpha = \alpha_n(\delta)$, $n \geq \hat{n}(\delta)$.

(i) (Hilbert space setting).

$\hat{x} \in R(A^{\nu})$, $0 < \nu \leq 1$, $p/2(q+1) \leq 1 \Longrightarrow$

$$\|\hat{x} - x^{\delta}_{\alpha,n}\| = 0(\delta^s), \ s = \begin{cases} \dfrac{p\nu}{q+1} \ , & \dfrac{p}{q+1} \leq \dfrac{2}{2\nu+1} \\ 1 - \dfrac{p}{2(q+1)}, & \dfrac{p}{q+1} \geq \dfrac{2}{2\nu+1} \end{cases} .$$

(ii) (Uniform norm case)

$\hat{x} \in R(A^{\nu}T^*)$, $0 < \nu \leq 1$, $p/(q+1) \leq 1 \Longrightarrow$

$$\|\hat{x}-x^{\delta}_{\alpha,n}\| = 0(\delta^{s}),\ s = \begin{cases} \frac{p\nu}{q+1}, & \frac{p}{q+1} \leq \frac{1}{\nu+1} \\ 1 - \frac{p}{q+1}, & \frac{p}{q+1} \geq \frac{1}{\nu+1} \end{cases}.$$

Proofs of Proposition 3 and Theorem 4 follows as in [14] by using the estimate $\|A_n x^{\delta}_{\alpha,n} - T^*y^{\delta}\| \leq c\sqrt{\alpha}(\delta+\sqrt{\alpha})$ (resp. $\|A_n x^{\delta}_{\alpha,n} - T^*y^{\delta}\|_{\infty} \leq c(\delta+\alpha)$).

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