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BICRITERIAL AND RESTRICTED PLANAR 2-MEDIAN PROBLEMS

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Bicriterial and restricted planar 2-Median Problems

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Abstract

Efficient algorithms and structural results are presented for median problems with 2 new facilities including the classical 2-Median problem, the 2-Median problem with forbidden regions and bicriterial 2-Median problems.

This is the first paper dealing with multi-facility multiobjective location problems.

The time complexity of all presented algorithms is $O(M \log M)$, where M is the number of existing facilities.

Keywords: Location Theory, Multi Criteria Problems, Restricted Location Problems, Multi-facility Location Problems

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1 Introduction

Much work has been done to extend 1-Median Problems with respect to forbidden regions and barriers. Works in this area are [Larson and Sadiq, 1983], [Batta et al., 1989], [Hamacher and Nickel, 1991] and [Hamacher and Nickel, 1992].

But very few has been done for the multifacility case, where the new facilities also have some interaction.

The same is true for multicriterial location problems. Here the first paper is from [Wendell et al., 1977] followed by a lot of others. A more general definition of multicriterial location problems, which allows to state multiobjective multifacility problems is given in [Hamacher and Nickel, 1993]. But multifacility problems haven't been discussed yet.

In this paper we will develop efficient algorithms which fill these two gaps a little bit.

We will present algorithms to solve the 2-Median problem for the l_1-norm with respect to a forbidden region. Also an algorithm for solving bicriterial 2-Median problems (with l_1-norm) is developed.

It should be noted that the results also hold for the l_∞-case, because of the norm-converting mapping $T(X)$, defined by

$$T(X) := \frac{1}{2}(x_1 + x_2, x_2 - x_1)$$

and the fact that

$$l_\infty(X,Y) = l_1(T(X),T(Y)) \quad (1.1)$$
$$l_1(X,Y) = l_\infty(T^{-1}(X),T^{-1}(Y))\,. \quad (1.2)$$

(see [Francis et al., 1992])

More formally we have given a set $Ex = \{Ex_1, \dots, Ex_M\}$ $(Ex_i = (a_i, b_i))$ of existing facilities and we are looking for a set $New = \{X_1, X_2\}$ $(X_i = (x_i, y_i))$ of new facilities to be located. The interaction between the new and the existing facilities is expressed by non-negative weights w_{mn} and the interaction between the two new facilities by the non-negative weight v.

The restricted 2-Median problem (R2M) can then be written as

$$\min_{New \subseteq F} \left(f(New) := \sum_{n=1}^{2} \sum_{m \in \mathcal{M}} w_{mn} l_1(Ex_m, X_n) + v l_1(X_1, X_2) \right),$$

where $\mathcal{M} := \{1, \dots, M\}$ and the feasible region $F := \mathbb{R}^2 \setminus int(\mathcal{R})$, with $\mathcal{R} \subseteq \mathbb{R}^2$ a connected set. If $\mathcal{R} = \emptyset$ we get the classical 2-Median problem (2M). The set of optimal locations for the restricted problem is called $\mathcal{X}^*_{\mathcal{R}}(f)$ and the set of optimal locations for the unrestricted problem is called $\mathcal{X}^*(f)$.

The bicriterial 2-Median problem (B2M) is written as

$$\min_{New \subseteq \mathbb{R}^2} F(New) := \begin{pmatrix} f^1(New) := \sum_{n=1}^{2} \sum_{m \in \mathcal{M}} w_{mn}^1 l_1(Ex_m, X_n) + v^1 l_1(X_1, X_2) \\ f^2(New) := \sum_{n=1}^{2} \sum_{m \in \mathcal{M}} w_{mn}^2 l_1(Ex_m, X_n) + v^2 l_1(X_1, X_2) \end{pmatrix},$$

where we mean minimize in the sense of finding pareto-optimal solutions. The set of pareto locations (i.e. $X \in \mathbb{R}^2$ with the property that there is no $Y \in \mathbb{R}^2$, $X \neq Y$ with $f_1(Y) \leq f_1(X)$ and $f_2(Y) \leq f_2(X)$) is denoted $\mathcal{X}_{par}(f_1, f_2)$.

The outline of the paper is as follows:
In Section 2 we develop basic properties of the objective function and Section 3 gives a solution procedure for (2M). Section 4 then applies these results to solve (R2M). The next section is devoted to a solution procedure for (B2M) using again the properties developed in Section 2. Finally Section 6 consists of conclusions and an outlook on further research activities.

It should be mentioned that the time complexity of all presented algorithms is $O(M log M)$.

2 Basic Concepts

First we want to reformulate the objective function. Note that although we concentrate on the single objective case the results apply to each of the objective functions in the multicriterial case.

$$\begin{aligned} f(New) &= \sum_{m \in \mathcal{M}} w_{m1} l_1(Ex_m, X_1) + \sum_{m \in \mathcal{M}} w_{m2} l_1(Ex_m, X_2) + v l_1(X_1, X_2) \\ &= \underbrace{\sum_{m \in \mathcal{M}} w_{m1} |a_m - x_1| + \sum_{m \in \mathcal{M}} w_{m2} |a_m - x_2| + v|x_1 - x_2|}_{:= f_1(New) = f_1(x_1, x_2) =: f_a(X)} + \\ &\quad \underbrace{\sum_{m \in \mathcal{M}} w_{m1} |b_m - y_1| + \sum_{m \in \mathcal{M}} w_{m2} |b_m - y_2| + v|y_1 - y_2|}_{:= f_2(New) = f_2(y_1, y_2) =: f_b(Y)} \end{aligned}$$

So we get two independent subproblems in the plane which are very similar to 1-Median problems and can be minimized separately.

Now we do a little further work to get rid of the absolute values in f_a and f_b.
To do so we sort the a_m and b_m in ascending order and delete duplicates. Further more we have to delete entries with $w_{m1} = 0$ and $w_{m2} = 0$, respectively and get two sequences for each component:

$$\begin{array}{ccc} a'_1 < a'_2 < \ldots < a'_{P_{a_1}} & , & a''_1 < a''_2 < \ldots < a''_{P_{a_2}} \\ b'_1 < b'_2 < \ldots < b'_{P_{b_1}} & , & b''_1 < b''_2 < \ldots < b''_{P_{b_2}} . \end{array}$$

Additionally we define $\mathcal{P}_{a_1} := \{1, \ldots, P_{a_1}\}$, $\mathcal{P}_{a_2} := \{1, \ldots, P_{a_2}\}$, $\mathcal{P}_{b_1} := \{1, \ldots, P_{b_1}\}$, $\mathcal{P}_{b_2} := \{1, \ldots, P_{b_2}\}$, $a'_0 := a''_0 := b'_0 := b''_0 := -\infty$ and $a'_{P_{a_1}+1} := a''_{P_{a_2}+1} := b'_{P_{b_1}+1} := b''_{P_{b_2}+1} := \infty$.

So we get two different subdivisions of the $\mathbb{R}^2$, namely

$$\begin{aligned}
\langle s,t\rangle_a &:= \{(x,y):\ a'_s \le x \le a'_{s+1}\ ,\ a''_t \le y \le a''_{t+1}\} \\
&\quad \text{for } s \in \mathcal{P}_{a_1} \cup \{0\}\ , t \in \mathcal{P}_{a_2} \cup \{0\} \\
\langle s,t\rangle_b &:= \{(x,y):\ b'_s \le x \le b'_{s+1}\ ,\ b''_t \le y \le b''_{t+1}\} \\
&\quad \text{for } s \in \mathcal{P}_{b_1} \cup \{0\}\ , t \in \mathcal{P}_{b_2} \cup \{0\}
\end{aligned}$$

It is clear that

$$\bigcup_{\substack{s\in\mathcal{P}_{a_1}\cup\{0\}\\ t\in\mathcal{P}_{a_2}\cup\{0\}}} \langle s,t\rangle_a = \mathbb{R}^2 = \bigcup_{\substack{s\in\mathcal{P}_{b_1}\cup\{0\}\\ t\in\mathcal{P}_{b_2}\cup\{0\}}} \langle s,t\rangle_b\ .$$

Since we may have dropped some duplicate values of the coordinates during this transformation we also have to modify the original weights to get the right objective value.

For this define

$$\begin{aligned}
C_{a_p} &:= \sum_{\{j:a'_p=a_j\}} w_{j1} \quad \text{for } p \in \mathcal{P}_{a_1} \\
D_{a_q} &:= \sum_{\{j:a''_q=a_j\}} w_{j2} \quad \text{for } q \in \mathcal{P}_{a_2}
\end{aligned}$$

and analogously C_{b_p} and D_{b_q}.

Now wet get regions

$$\begin{aligned}
\langle s,t\rangle_a^+ &:= \{(x,y) \in \langle s,t\rangle_a\ :\ x \ge y\} \\
\langle s,t\rangle_a^- &:= \{(x,y) \in \langle s,t\rangle_a\ :\ x \le y\}
\end{aligned}$$

and analogously $\langle s,t\rangle_b^+$, $\langle s,t\rangle_b^-$.

If we mean $\langle s,t\rangle_\nu^+$ or $\langle s,t\rangle_\nu^-$ for $\nu \in \{a,b\}$, we write $\langle s,t\rangle_\nu^{+,-}$.

For a function g from $\mathbb{R}^2$ to $\mathbb{R}$ we call

$$L_=(z) := \{(x,y) : g(x,y) = z\}$$

level curve and

$$L_\le(z) := \{(x,y) : g(x,y) \le z\}$$

level set of g with level z.

Now we can state the main result of this section.

Theorem 2.1. *For $X = (x,y) \in \langle s,t\rangle_\nu^r$, with $r \in \{+,-\}$ and $\nu \in \{a,b\}$ the set $L_=(f_\nu(X))$ is the graph of a linear function.*

Proof. wlog $r = +$.

$$
\begin{aligned}
& f_\nu(X) =_r z \text{ with } z \in \mathbb{R}_+ \\
\Longleftrightarrow\quad & v(x_1 - x_2) + \sum_{p=1}^{s} C_{\nu_p}(x_1 - \nu'_p) - \sum_{p=s+1}^{P_{\nu_1}} C_{\nu_p}(x_1 - \nu'_p) + \\
& \sum_{q=1}^{t} D_{\nu_q}(x_2 - \nu''_q) - \sum_{q=t+1}^{P_{\nu_2}} D_{\nu_q}(x_2 - \nu''_q) = z \\
\Longleftrightarrow\quad & x_1 \underbrace{\left(v + \sum_{p=1}^{s} C_{\nu_p} - \sum_{p=s+1}^{P_{\nu_1}} C_{\nu_p} \right)}_{=:M_s^\nu} + \\
& x_2 \underbrace{\left(-v + \sum_{q=1}^{t} D_{\nu_q} - \sum_{q=s+1}^{P_{\nu_2}} D_{\nu_q} \right)}_{=:N_t^\nu} + const = 0 \\
\Longleftrightarrow\quad & x_2 = x_1 \left(-\frac{M_s^\nu}{N_t^\nu} \right) + const
\end{aligned}
$$

□

Corollary 2.2. *The slope of the level curves $L_=(f_\nu(X))$ for $\nu \in \{a, b\}$ is in*

$$\langle s,t \rangle_\nu^+ = -\frac{M_s^\nu}{N_t^\nu} ,$$

with

$$
\begin{aligned}
M_s^\nu &:= v + \sum_{p=1}^{s} C_{\nu_p} - \sum_{p=s+1}^{P_{\nu_1}} C_{\nu_p} , \\
N_t^\nu &:= -v + \sum_{q=1}^{t} D_{\nu_q} - \sum_{q=t+1}^{P_{\nu_2}} D_{\nu_q}
\end{aligned}
$$

and in

$$\langle s,t \rangle_\nu^- = -\frac{M_s^\nu}{N_t^\nu} ,$$

with

$$
\begin{aligned}
M_s^\nu &:= -v + \sum_{p=1}^{s} C_{\nu_p} - \sum_{p=s+1}^{P_{\nu_1}} C_{\nu_p} , \\
N_t^\nu &:= v + \sum_{q=1}^{t} D_{\nu_q} - \sum_{q=t+1}^{P_{\nu_2}} D_{\nu_q} .
\end{aligned}
$$

3 Solving (2M)

This section is devoted to the analysis of the structure of the set of optimal solutions of (2M), $\mathcal{X}^*(f)$. Based on these results an efficient solution procedure for (2M) is presented. So following our plan we characterize the set of optimal solutions for f_ν, $\nu \in \{a, b\}$.

Theorem 3.1. *$\mathcal{X}^*(f_\nu)$ is one of the following:*

a) *A corner point of $\langle s,t\rangle_\nu^{+,-}$.*

b) *A line segment joining two adjacent corners of $\langle s,t\rangle_\nu^{+,-}$.*

c) *A complete cell $\langle s,t\rangle_\nu^{+}$ or $\langle s,t\rangle_\nu^{-}$.*

Proof. Suppose a point $X \in \mathcal{X}^*(f_\nu)$ lies on the interior of a line segment joining to adjacent corner points of $\langle s,t\rangle_\nu^{+,-}$. We have shown in Section 2 that f_ν is a linear function on this line segment. So it follows that the whole line segment has to be optimal.

If a point $X \in \mathcal{X}^*(f_\nu)$ lies in the interior of $\langle s,t\rangle_\nu^{+,-}$, by the same arguments as above we get that the whole cell $\langle s,t\rangle_\nu^{+,-}$ has to be optimal.

By the convexity of the objective function we can not have the case of two optimal corner points and nothing else. (For the different cases see Figure 3.1) Finally, we know by construction of the cells that the level curve in the 4 adjacent cells to $\langle s,t\rangle_\nu^{+,-}$ has a slope different to the slope in $\langle s,t\rangle_\nu^{+,-}$ (see Corollary 2.2). So at most $\langle s,t\rangle_\nu^{+} \cup \langle s,t\rangle_\nu^{-}$ can be optimal.

□

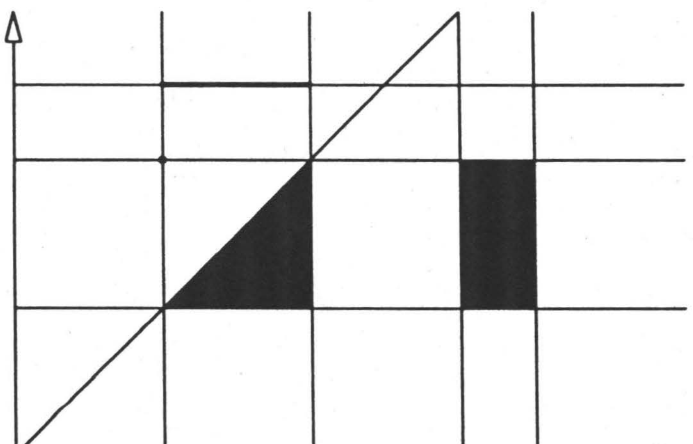

Figure 3.1: The possible shapes of $\mathcal{X}^*(f_\nu)$.

Since this result holds for both component functions f_ν we also have reproofed the following Corollary which has been stated for general l_1-multifacility problems by [Wendell et al., 1977].

Corollary 3.2. *An optimal solution of the two facility minisum problem can always be found on the intersection points of horizontal and vertical lines through the existing facilities.*

Now we use these results to give the following solution procedure for (2M).

First note that f_a and f_b can be optimized separately. Without loss of generality we restrict ourselves to the non-negative quadrant of the plane. We present the solution procedure for f_a. The same procedure is then applied to f_b.

The idea behind the procedure is that for $(x_1^*, x_2^*) \in \mathcal{X}^*(f_a)$, either $x_1^* \geq x_2^*$ or $x_2^* \geq x_1^*$.

Case I: $x_1 \geq x_2$**:**

$$
\begin{aligned}
f_a(x_1, x_2) &= \sum_{m \in \mathcal{M}} w_{m1}|a_m - x_1| + \sum_{m \in \mathcal{M}} w_{m2}|a_m - x_2| + v|x_1 - x_2| \\
&= \sum_{m \in \mathcal{M}} (w_{m1}|a_m - x_1|) + v|0 - x_1| + \sum_{m \in \mathcal{M}} (w_{m2}|a_m - x_2|) - vx_2 \\
&=: t_1(x_1) + t_2(x_2)
\end{aligned}
$$

So we have two independent subproblems, where $t_1(x_1)$ is the well-known median problem, which can be solved in $O((M+1)log(M+1))$ time (see, for example, [Francis et al., 1992]). The solution procedure just consists of finding the median of the weights w_{m1} and v, ordered according to the given values a_m and 0, for $m \in \mathcal{M}$. Because of the properties of median problems we know that $\mathcal{X}^*(t_1)$ is either a unique value in $\{0, a_1, \ldots, a_M\}$ or an interval $[a^*, b^*]$ with $a^*, b^* \in \{0, a_1, \ldots, a_M\}$. Then we substitute $x_1^* \in \mathcal{X}^*(t_1)$ in f_a and get

$$
\begin{aligned}
f_a(x_1^*, x_2) &= \sum_{m \in \mathcal{M}} w_{m1}|a_m - x_1^*| + \sum_{m \in \mathcal{M}} w_{m2}|a_m - x_2| + v|x_1^* - x_2| \\
&= Const + \sum_{m \in \mathcal{M}} (w_{m2}|a_m - x_2|) + v|x_1^* - x_2| \\
&=: Const + t_3(x_2)
\end{aligned}
$$

Now t_3 is a median problem with given values a_m, x_1^* and weights w_{m2} and v, for $m \in \mathcal{M}$. So we get for every x_1^* a set $\mathcal{X}^*(t_3)$ and these two solution sets together give us – if feasible – an optimal solution for f_a. Fortunately we don't have to repeat this procedure for all $x_1^* \in \mathcal{X}^*(t_1)$. Since we know from Theorem 3.1 that a non-unique optimum for f_a is either a line segment or a complete cell it suffices to solve min $f_a(a^*, x_2)$ and min $f_a(b^*, x_2)$ if $\mathcal{X}^*(t_1) = [a^*, b^*]$.

Case II: $x_1 \leq x_2$**:**
Is identical to Case I if we exchange the roles of x_1 and x_2.

The following lemma tells us that always one of the cases contains the complete solution.

Lemma 3.3. *For $\mathcal{X}^*(f_a)$ the following holds:*
$\mathcal{X}^*(f_a)$ never contains two elements (x_1^*, x_2^*) and (y_1^*, y_2^*) with $x_1^* > x_2^*$ and $y_1^* < y_2^*$.

Proof. Since $\mathcal{X}^*(f_a)$ is maximally a composed cell, the only case we must exclude is $\mathcal{X}^*(f_a) = \langle s,t\rangle_\nu^+ \cup \langle s,t\rangle_\nu^-$. Further we must have a constant slope in $\mathcal{X}^*(f_a)$. Looking at Corollary 2.2 we get

$$\frac{v + \sum_{m=1}^{s} C_{a_m} - \sum_{m=s+1}^{P_{a_1}} C_{a_m}}{-v + \sum_{m=1}^{t} D_{a_m} - \sum_{m=t+1}^{P_{a_2}} D_{a_m}} = \frac{-v + \sum_{m=1}^{s} C_{a_m} - \sum_{m=s+1}^{P_{a_1}} C_{a_m}}{v + \sum_{m=1}^{t} D_{a_m} - \sum_{m=t+1}^{P_{a_2}} D_{a_m}} \tag{3.1}$$

as a necessary condition for the whole cell to be optimal.

But from the median conditions we must have for $x_1 \geq x_2$ that

$$v + \sum_{m=1}^{s} C_{a_m} - \sum_{m=s+1}^{P_{a_1}} C_{a_m} = 0$$

in order to have a non-unique optimum and analogously for $x_2 \geq x_1$ that

$$v + \sum_{m=1}^{t} D_{a_m} - \sum_{m=t+1}^{P_{a_2}} D_{a_m} = 0\,.$$

So we see that the slope condition in (3.1) can never be fulfilled and the lemma is proved.

□

Remark. Note that Lemma 3.3 strengthens Theorem 3.1 since the case of an optimal solution with $\mathcal{X}^*(f_a) = \langle s,t\rangle_\nu^+ \cup \langle s,t\rangle_\nu^-$ is excluded.

To formulate the solution algorithm we first define formally the median problem.

$$Med\,((a_1, w_1), \ldots, (a_Q, w_Q)) := \operatorname*{argmin}_{x \in \mathbf{R}} \sum_{q=1}^{Q} w_i |a_i - x|$$

Algorithm for finding $\mathcal{X}^*(f_\nu)$ with $\nu \in \{a, b\}$

1. $[a^*, b^*] = Med\,((0, v), (\nu_1, w_{11}), \ldots, (\nu_M, w_{M1}))$ (Possibly $a^* = b^*$.)
2. $[c^*, d^*] = Med\,((a^*, v), (\nu_1, w_{12}), \ldots, (\nu_M, w_{M2}))$ and
   $[c^*, e^*] = Med\,((a^*, v), (\nu_1, w_{12}), \ldots, (\nu_M, w_{M2}))$ with $d^* = c^* \vee e^* = c^* \vee d^* = e^*$.
3. $$\mathcal{X}_1^*(f_\nu) = \{(x, y) \;:\; (x, y) \in Conv\{(a^*, c^*), (a^*, d^*), (b^*, c^*), (b^*, e^*)\} \wedge (x \geq y)\}\,.$$
4. $[a^*, b^*] = Med\,((0, v), (\nu_1, w_{12}), \ldots, (\nu_M, w_{M2}))$ (Possibly $a^* = b^*$.)
5. $[c^*, d^*] = Med\,((a^*, v), (\nu_1, w_{11}), \ldots, (\nu_M, w_{M1}))$ and

$[c^*, e^*] = Med((a^*, v), (\nu_1, w_{11}), \ldots, (\nu_M, w_{M1}))$ with $d^* = c^* \vee e^* = c^* \vee d^* = e^*$.

6.
$$\mathcal{X}_2^*(f_\nu) = \{(x, y) \; : \; (x, y) \in Conv\{(a^*, c^*), (a^*, d^*), (b^*, c^*), (b^*, e^*)\} \wedge (x \geq y)\} \; .$$

7. **Output:** $\mathcal{X}^*(f_\nu) = \begin{cases} \mathcal{X}_1^*(f_\nu) & : \text{ if } \mathcal{X}_1^*(f_\nu) \supseteq \mathcal{X}_2^*(f_\nu) \\ \mathcal{X}_2^*(f_\nu) & : \text{ otherwise} \end{cases}$

Obviously the complexity of the algorithm is dominated by the complexity of the median problems. So the algorithm has a complexity of $O((M+1)log(M+1))$.

Algorithm for solving (2M)

1. Compute $\mathcal{X}^*(f_a)$.
2. Compute $\mathcal{X}^*(f_b)$.
3. **Output:** $\mathcal{X}^*(f) = \mathcal{X}^*(f_a) \times \mathcal{X}^*(f_b)$.

To illustrate the algorithm we present a little example:

Example 3.1. *We are given three existing facilities* $Ex_1 = (1, 2)$, $Ex_2 = (3, 4)$ and $Ex_3 = (5, 1)$. *The weights are given as* $w = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 5 & 1 \end{pmatrix}^T$ and $v = 2$.

1. *Computing* $\mathcal{X}^*(f_a)$.
   (a) $Med((0, 2), (1, 4), (3, 1), (5, 1)) = 1$.
   (b) $Med((1, 2), (1, 2), (3, 5), (5, 1)) = 3$
   $\Rightarrow \mathcal{X}_1^*(f_a) = \emptyset$.
   (c) $Med((0, 2), (1, 2), (3, 5), (5, 1)) = 3$.
   (d) $Med((3, 2), (1, 4), (3, 1), (5, 1)) = [1, 3]$
   $\Rightarrow \mathcal{X}_2^*(f_a) = \mathcal{X}^*(f_a) = \{(x_1, x_2) \; : \; x_1 \in [1, 3] \wedge x_2 = 3\}$.
2. *Computing* $\mathcal{X}^*(f_b)$.
   (a) $Med((0, 2), (2, 4), (4, 1), (1, 1)) = 2$.
   (b) $Med((2, 2), (2, 2), (4, 5), (1, 1)) = [2, 4]$
   $\Rightarrow \mathcal{X}_1^*(f_b) = \{(2, 2)\}$.
   (c) $Med((0, 2), (2, 2), (4, 5), (1, 1)) = [2, 4]$.
   (d) $Med((2, 2), (2, 4), (4, 1), (1, 1)) = 2$.
   (e) $Med((4, 2), (2, 4), (4, 1), (1, 1)) = 2$
   $\Rightarrow \mathcal{X}_2^*(f_b) = \mathcal{X}^*(f_b) = \{(y_1, y_2) \; : \; y_1 = 2 \wedge x_2 \in [2, 4]\}$.
3. *Therefore*
   $$\mathcal{X}^*(f) = \{(x_1, y_1), (x_2, y_2) \; : \; x_1 \in [1, 3] \wedge y_1 = 2 \wedge x_2 = 3 \wedge y_2 \in [2, 4]\} \; ,$$
   with objective value 30.

The example is illustrated in Figure 3.2.

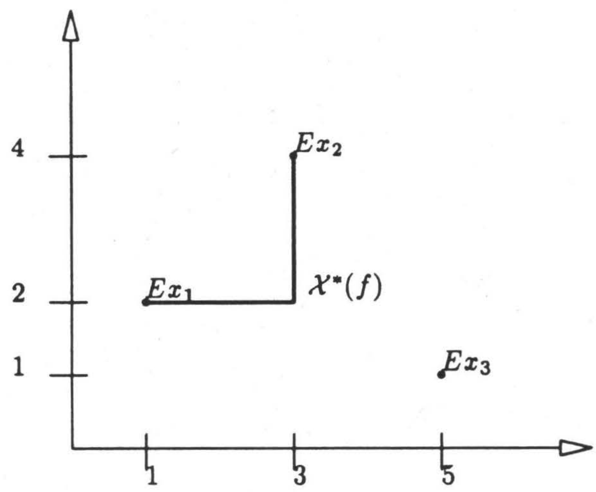

Figure 3.2: Illustration for Example 3.1

4 Solving (R2M)

As explained in Section 1 we are given a forbidden set $\mathcal{R}$. The objective is to find

$$\min_{New \subseteq F = \mathbb{R}^2 \setminus int(\mathcal{R})} f(New) .$$

For $\nu \in \{a, b\}$

- a vertical construction line is defined as $\left\{(x, y) \ : \ \exists p \in \mathcal{P}_{\nu_1} \text{ s.t. } x = \nu'_p\right\}$
- a horizontal construction line is defined as $\left\{(x, y) \ : \ \exists q \in \mathcal{P}_{\nu_2} \text{ s.t. } y = \nu''_q\right\}$
- a diagonal construction line is defined as $\{(x, y) \ : \ x = y\}$

and $\mathcal{H}_\nu$ is defined as the union of all possible construction lines.
The following result shows the combinatorial nature of our problem.

Theorem 4.1. *Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a convex set and $\mathcal{X}^*(f_\nu) \subseteq int(R)$. Then there is an optimal solution $X^*_\mathcal{R} \in \mathcal{X}^*_\mathcal{R}(f_\nu)$ satisfying*

1. $X^*_\mathcal{R} \in \partial\mathcal{R}$
2. $X^*_\mathcal{R} \in \mathcal{H}_\nu$
3. $\exists h \in \mathcal{H}_\nu \ : \ X^*_\mathcal{R} \in h \wedge dim(h \cap \partial\mathcal{R}) = 0.$

Moreover $\mathcal{X}^*_\mathcal{R}(f_\nu) \subseteq \partial\mathcal{R}$.

Proof. We first proof $\mathcal{X}^*_\mathcal{R}(f_\nu) \subseteq \partial\mathcal{R}$:
Let $Y \in (\mathbb{R}^2 \setminus \mathcal{R})$ and let X^* be an optimal solution of the unrestricted problem. Since $\mathcal{R}$ is convex and $X^* \in int(\mathcal{R})$ (by assumption) there exists an $X_{\partial\mathcal{R}} \in$

$\{tY + (1-t)X^* \ : \ 0 < t < 1\}$ with $X_{\partial\mathcal{R}} \in \partial\mathcal{R} \subseteq F$.
Since the f_ν are convex functions this means $f_\nu(Y) > f_\nu(X_{\partial\mathcal{R}})$ and so we have by contradiction $X^*_{\mathcal{R}} \in \partial\mathcal{R}$.
ad 2)
By the first part of the proof we can restrict ourselves to $X \in \partial\mathcal{R}$. If $X \in \partial\mathcal{R} \cap \langle t_0, s_0\rangle_\nu^{+,-}$ does not lie on the construction lines $\mathcal{H}_\nu$, the level curve through X will not change its slope $S_{t_0 s_0}$ in X. Let $L_{t_0 s_0}$ be the linear segment of the level curve through X with slope $S_{t_0 s_0}$.

Case 1 $L_{t_0 s_0}$ crosses $\partial\mathcal{R}$ in X.
Then $L_\leq(f(X)) \not\subseteq \mathcal{R}$. Consequently, X cannot be optimal, because we can find a level curve with a smaller level lying in the interior of $L_\leq(f(X))$ containing feasible points.

Case 2 $L_{t_0 s_0}$ is a supporting hyperplane of $\mathcal{R}$ in X.

1. If $L_{t_0 s_0} \subseteq \mathcal{R}$, then there is also a point $Y \in L_{t_0 s_0}$ where the level curve changes its slope. Since X has the same objective value as Y we can replace X by Y.
2. If $L_{t_0 s_0} \not\subseteq \mathcal{R}$, then, by the same arguments as in Case 1, X cannot be optimal.

To see that we only need to investigate cases where $\exists h \in \mathcal{H}_\nu \ : \ X^*_{\mathcal{R}} \in h \wedge dim(h \cap \partial\mathcal{R}) = 0$, suppose that for all $h \in \mathcal{H}_\nu$ with $X^*_{\mathcal{R}} \in h$ it holds that $dim(h \cap \partial\mathcal{R}) = 1$. This means that h coincides with one side of a cell $\langle s_0, t_0\rangle_\nu^{+,-}$. The level curve through $X^*_{\mathcal{R}}$ has to change it's slope in $X^*_{\mathcal{R}}$ and must not leave $\mathcal{R}$. But this implies that the level curve changes it's slope in $\langle s_0, t_0\rangle_\nu^{+,-}$, which is not possible. By the convexity of $\mathcal{R}$ a line can only have a 0- or 1-dimensional intersection with $\mathcal{R}$, but not both. So all is proved.

□

If $\mathcal{R}$ has a polyhedral structure with facets $s_1, \ldots, s_N$ Theorem 4.1 is also true if we additionally forbid a subset of the facets for locating the new facility.

Since we have decomposed our objective function in two functions f_a and f_b only dealing with the first and second coordinate, respectively, we also need to have forbidden sets $\mathcal{R}$ where we can also do such a decomposition.

The generic case is $\mathcal{R} = [a, b] \times [c, d]$ is a rectangle with sides parallel to the x- and y-axis. These case will be discussed in detail. But the approach can also be extended to general rectangles as well as to convex polyhedrons.

The first generalization can be handled by a rotation and the second by a combination of rotation and a sequence of rectangle problems.

But now we discuss the case where $\mathcal{R} = [a, b] \times [c, d]$:

Lemma 4.2. *The points (v_1, v_2) and (w_1, w_2) are on $\partial\mathcal{R}$, if and only if they fulfill at least one of the following conditions:*

- $a \le v_1 \le b$, $w_1 \in \{a, b\}$, $v_2 \in \{c, d\}$ and $c \le w_2 \le d$.
- $v_1 \in \{a, b\}$, $a \le w_1 \le b$, $c \le v_2 \le d$ and $w_2 \in \{c, d\}$.
- $a \le v_1 \le b$, $a \le w_1 \le b$, $v_2 \in \{c, d\}$ and $w_2 \in \{c, d\}$.
- $v_1 \in \{a, b\}$, $w_1 \in \{a, b\}$, $c \le v_2 \le d$ and $c \le w_2 \le d$.

The proof is just a simple testing of the possible cases and is therefore omitted here.

For our purposes we need a geometrical interpretation of the lemma above. The line segment connecting two points (x_1, y_1) and (x_2, y_2) is written as $\overline{(x_1, y_1)(x_2, y_2)}$.

Corollary 4.3. *Let*

$$\begin{aligned}
\mathcal{R}_a^1 &:= \overline{(a,a)(b,a)} \cup \overline{(a,b)(b,b)}\,, \\
\mathcal{R}_b^1 &:= \overline{(c,c)(d,c)} \cup \overline{(c,d)(d,d)}\,, \\
\mathcal{R}_a^2 &:= \overline{(a,a)(a,b)} \cup \overline{(b,a)(b,b)}\,, \\
\mathcal{R}_b^2 &:= \overline{(c,c)(c,d)} \cup \overline{(d,c)(d,d)}\,, \\
\mathcal{R}_a^3 &:= \mathcal{R}_a^1 \cup \mathcal{R}_a^2\,, \\
\mathcal{R}_b^3 &:= \mathcal{R}_b^1 \cup \mathcal{R}_b^2\,, \\
\mathcal{P}_a &:= \{(a,a),(a,b),(b,a),(b,b)\} \text{ and} \\
\mathcal{P}_b &:= \{(c,c),(c,d),(d,c),(d,d)\}\,.
\end{aligned}$$

The points (v_1, v_2) and (w_1, w_2) lie on $\partial\mathcal{R}$, if and only if they fulfil at least one of the following conditions:

- $(v_1, w_1) \in \mathcal{R}_a^1$ and $(v_2, w_2) \in \mathcal{R}_b^2$.
- $(v_1, w_1) \in \mathcal{R}_a^2$ and $(v_2, w_2) \in \mathcal{R}_b^1$.
- $(v_1, w_1) \in \mathcal{P}_a$ and $(v_2, w_2) \in \mathcal{R}_b^3$.
- $(v_1, w_1) \in \mathcal{R}_a^3$ and $(v_2, w_2) \in \mathcal{P}_b$.

Remark. For algorithmic purposes it is more convenient to interpret, $\mathcal{R}_\nu^1$ and $\mathcal{R}_\nu^2$ with $\nu \in \{a, b\}$ as the rectangles $\mathcal{R}_a^3$ and $\mathcal{R}_b^3$ with forbidden sides, respectively.

Now we can state an algorithm to solve (R2M):

<u>*Algorithm for solving (R2M) with* $\mathcal{R} = [a,b] \times [c,d]$</u>

1. Split $f(X)$ in $f_a(X)$ and $f_b(Y)$.
2. Generate $\mathcal{H}_a$ and $\mathcal{H}_b$.
3. Compute $\{Y^j_{1_i}, \ldots, Y^j_{K_i}\} = \mathcal{H}_\nu \cap \mathcal{R}^j_\nu$, for $\nu \in \{a,b\}$ and $j = 1,2,3$.
4. Let $X^*_{\nu_j} \in argmin\{f_\nu(Y^j_{1_\nu}), \ldots, f_\nu(Y^j_{K_\nu})\}$ and let L_{ν_j} be the level curve through $X^*_{\nu_j}$.
5. $\mathcal{X}^*_{\nu_j} = (L_{\nu_j} \cap \mathcal{R}^j_\nu)$.
6. $\mathcal{X}^*_{P_\nu} = argmin\{f_\nu(P_\nu)\}$.
7. <u>Output:</u> $\mathcal{X}^*_{\mathcal{R}} = argmin\{f((v_1,v_2),(w_1,w_2))\}$, where $(v_1,v_2),(w_1,w_2)$ have to fulfill the following conditions (see Corollary 4.3):
   - $(v_1,w_1) \in \mathcal{X}^*_{a_1}$ and $(v_2,w_2) \in \mathcal{X}^*_{b_2}$.
   - $(v_1,w_1) \in \mathcal{X}^*_{a_2}$ and $(v_2,w_2) \in \mathcal{X}^*_{b_1}$.
   - $(v_1,w_1) \in \mathcal{X}^*_{P_a}$ and $(v_2,w_2) \in \mathcal{X}^*_{b_3}$.
   - $(v_1,w_1) \in \mathcal{X}^*_{a_3}$ and $(v_2,w_2) \in \mathcal{X}^*_{P_b}$.

The validity of the algorithm follows directly from the results of this section.

Corollary 4.4. *If the input data is integral then the algorithm gives us always an integral solution (without any non-integral subresults).*

To illustrate this algorithm we present an example:

Example 4.1. *We use the same input data as in Example 3.1, :*
$Ex_1 = (1,2)$, $Ex_2 = (3,4)$, $Ex_3 = (5,1)$, $w = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 5 & 1 \end{pmatrix}^T$ and $v = 2$. *As forbidden rectangle we chose* $\mathcal{R} = [0,4] \times [1,5]$.

Now our objective function is

$$\begin{aligned} f(X) = \quad & 4|x_1-1| + 1|x_1-3| + 1|x_1-5| + 2|x_2-1| + 5|x_2-3| + |x_2-5| \\ & +2|x_1-x_2| \\ + \quad & 4|y_1-2| + |y_1-4| + |y_1-1| + 2|y_2-2| + 5|y_2-4| + |y_2-1| \\ & +|y_1-y_2| \,, \end{aligned}$$

which we split in

$$f_a(X) = 4|x_1-1| + 1|x_1-3| + 1|x_1-5| + 2|x_2-1| + 5|x_2-3| + |x_2-5| + 2|x_1-x_2|$$

and

$$f_b(Y) = 4|y_1-2| + |y_1-4| + |y_1-1| + 2|y_2-2| + 5|y_2-4| + |y_2-1| + |y_1-y_2| \,.$$

As restricted sets we get

$$\begin{aligned}
\mathcal{R}_a^1 &= \overline{(0,0)(4,0)} \cup \overline{(0,4)(4,4)}\,, \\
\mathcal{R}_b^1 &= \overline{(1,1)(5,1)} \cup \overline{(1,5)(5,5)}\,, \\
\mathcal{R}_a^2 &= \overline{(0,0)(0,4)} \cup \overline{(4,0)(4,4)} \text{ and} \\
\mathcal{R}_b^2 &= \overline{(1,1)(1,5)} \cup \overline{(5,1)(5,5)}\,.
\end{aligned}$$

The description of the other sets should be clear.

Now we construct $\mathcal{H}_a$ and $\mathcal{H}_b$ and compute the intersection points (see Figure 4.1 and 4.2). As resulting optima we get:

$$\begin{aligned}
\mathcal{X}_{a_1}^* &= \{\overline{(1,4)-(3,4)}\} \text{ with obj. value } 24 \\
\mathcal{X}_{b_1}^* &= \{(2,1)\} \text{ with obj. value } 22 \\
\mathcal{X}_{a_2}^* &= \{(4,3)\} \text{ with obj. value } 22 \\
\mathcal{X}_{b_2}^* &= \{\overline{(1,2)-(1,4)}\} \text{ with obj. value } 20 \\
\mathcal{X}_{a_3}^* &= \{(4,3)\} \text{ with obj. value } 22 \\
\mathcal{X}_{b_3}^* &= \{\overline{(1,2)-(1,4)}\} \text{ with obj. value } 20 \\
\mathcal{X}_{P_a}^* &= \{(4,4)\} \text{ with obj. value } 26 \\
\mathcal{X}_{P_b}^* &= \{(1,1)\} \text{ with obj. value } 24
\end{aligned}$$

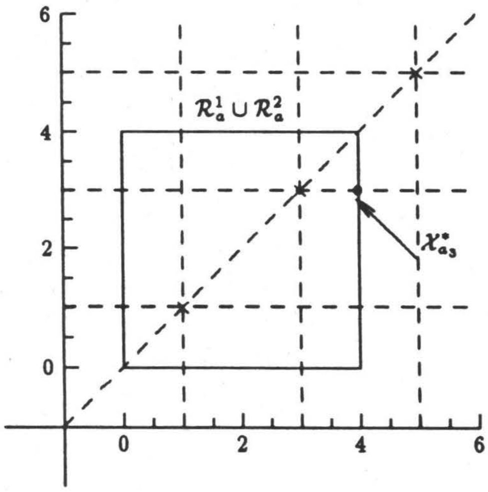

Figure 4.1: Illustration for Example 4.1

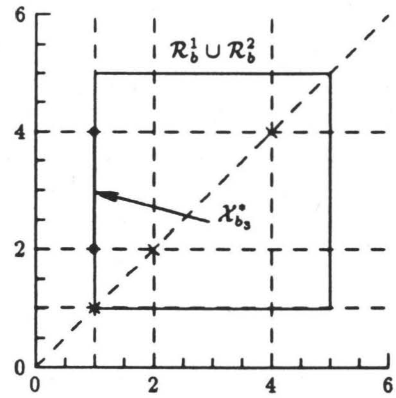

Figure 4.2: Illustration for Example 4.1

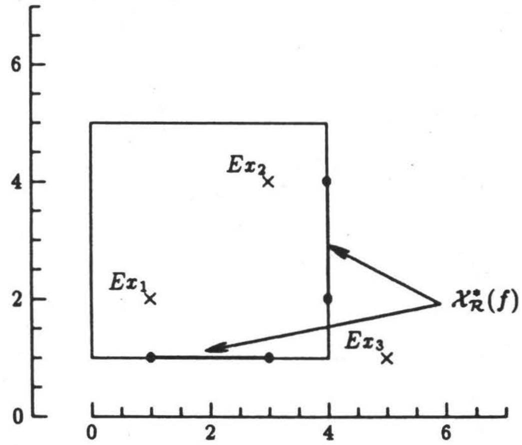

Figure 4.3: Illustration for Example 4.1

Using the criteria to select the optima in the last step of the algorithm yields:

$$\mathcal{X}_{\mathcal{R}}^*(f) = \{((([1,3],1),(4,[2,4])),((4,2),(1,3))\}, \text{ (see Figure 4.3).}$$

5 Solving (B2M)

We now want to give an efficient algorithm for finding all pareto locations for (B2M) without searching explicitly for points in $\mathbb{R}^4$.
The way we try to avoid the search in the $\mathbb{R}^4$ is as follows:

First we use the characterization of the set of optimal solutions for the four functions f_ν^i, $\nu \in \{a, b\}$ and $i = 1, 2$ given by Theorem 3.1 and develop a procedure for computing the pareto location sets $\mathcal{X}_{par}(f_a^1, f_a^2)$ and $\mathcal{X}_{par}(f_b^1, f_b^2)$. Then we show how these two sets can be combined such that we get $\mathcal{X}_{par}(f^1, f^2)$.

5.1 Computing $\mathcal{X}_{par}(f_a^1, f_a^2)$ and $\mathcal{X}_{par}(f_b^1, f_b^2)$

In the following we show the connection between this characterization of the optimal solution for the single criterion problems of Theorem 3.1 and the set of pareto locations for two criterion problems.
The following result which follows immediately from [Geoffrion, 1967] and [Isermann, 1974] establishes this relationship.

Theorem 5.1.

$$X_{par} \in \mathcal{X}_{par}(f_\nu^1, f_\nu^2)$$

$$\Longleftrightarrow$$

$$X_{par} \textit{ solves } \min_{X \in \mathbb{R}^2} (\lambda f_\nu^1(X) + (1-\lambda) f_\nu^2(X)) \textit{ for a } \lambda \textit{ with } 0 < \lambda < 1$$

.

This theorem means that we can find all pareto locations by solving all of these scalarizations for $0 < \lambda < 1$. But for a fixed λ we just get again a single criterion problem of the type f_ν^i: (For the sake of simplicity we show the calculation only for f_a^1 and f_a^2)

$$\begin{aligned}
& \lambda f_a^1(X) + (1-\lambda) f_a^2(X) \\
= \; & \lambda \left(\sum_{m \in \mathcal{M}} w_{m1}^1(|a_m - x_1|) + \sum_{m \in \mathcal{M}} w_{m2}^1(|a_m - x_2|) + v^1|x_1 - x_2| \right) + \\
& (1-\lambda) \left(\sum_{m \in \mathcal{M}} w_{m1}^2(|a_m - x_1|) + \sum_{m \in \mathcal{M}} w_{m2}^2(|a_m - x_2|) + v^2|x_1 - x_2| \right) \\
= \; & \sum_{m \in \mathcal{M}} (\lambda w_{m1}^1 + (1-\lambda) w_{m1}^2)(|a_m - x_1|) \\
& + \sum_{m \in \mathcal{M}} (\lambda w_{m2}^1 + (1-\lambda) w_{m2}^2)(|a_m - x_2|) + (\lambda v^1 + (1-\lambda) v^2)|x_1 - x_2|
\end{aligned}$$

In order to have the complete scenario of the f_ν^i-case we have to define construction lines. The equation above tells us that the construction lines ($\mathcal{H}_\nu^{1,2}$), defining the cells, for the scalarization are just the union of the construction lines $\mathcal{H}_a^1$ and $\mathcal{H}_a^2$ of the involved subproblems f_a^1 and f_a^2, respectively.

Define

$$\mathcal{X}_{1,2}(f_\nu^1, f_\nu^2) := \operatorname*{argmin}_{X \in \mathcal{X}^*(f_\nu^1)} f_\nu^2(X)$$
$$\mathcal{X}_{2,1}(f_\nu^1, f_\nu^2) := \operatorname*{argmin}_{X \in \mathcal{X}^*(f_\nu^2)} f_\nu^1(X)$$

as the two lexicographic optimal location sets. It is well known that the two lexicographic optimal location sets are pareto locations and contain the solutions of the scalarization where λ tends to 1 or 0, respectively.

So we have proved the following theorem (see also Figure 5.1):

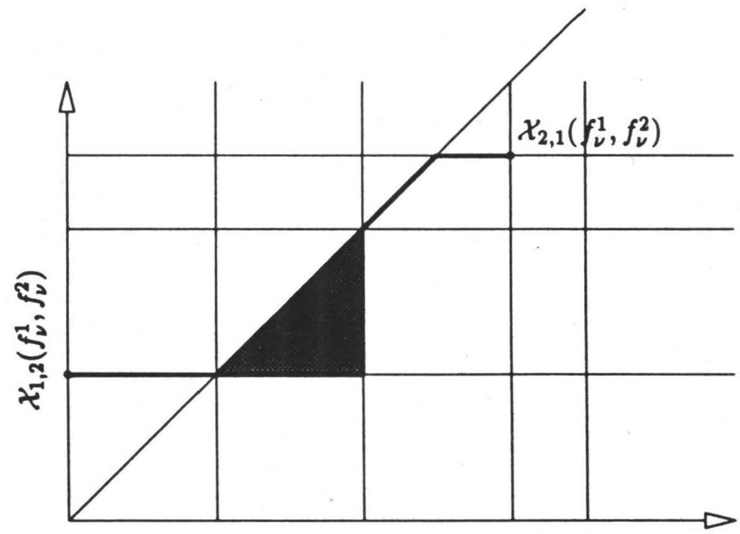

Figure 5.1: An example for the shape of $\mathcal{X}_{par}(f_\nu^1, f_\nu^2)$.

Theorem 5.2. *$\mathcal{X}_{par}(f_\nu^1, f_\nu^2)$ is a chain consisting of corner points, line segments and cells described in Theorem 3.1 connecting $\mathcal{X}_{1,2}(f_\nu^1, f_\nu^2)$ and $\mathcal{X}_{2,1}(f_\nu^1, f_\nu^2)$.*

One way to give an algorithm now would consist of computing all solutions for the scalarization and building the chain described in Theorem 5.2. But there is a much more easier way using level curves and level sets. The following two results are due to [Hamacher and Nickel, 1993].

Theorem 5.3.

$$X \in \mathcal{X}_{par}(f_\nu^1, f_\nu^2)$$

$$\Longleftrightarrow$$

$$int\left(L_{\leq}(f_\nu^1(X))\right) \cap int\left(L_{\leq}(f_\nu^2(X))\right) = \emptyset \;\wedge\; L_{=}(f_\nu^1(X)) \cap L_{=}(f_\nu^2(X)) \neq \emptyset$$

As a direct consequence of this theorem we get the following lemma:

Lemma 5.4. *Let* $X^1 \in \mathcal{X}^*(f_\nu^1)$ *and* $X^2 \in \mathcal{X}^*(f_\nu^2)$. *Then*

$$\mathcal{X}_{par}(f_\nu^1, f_\nu^2) \subseteq L_{\leq}(f_\nu^1(X^1)) \cap L_{\leq}(f_\nu^2(X^2)) .$$

In the following we assume wlog, that

$$\max\{x \ : \ (x,y) \in \mathcal{X}^*(f_\nu^1)\} \leq \min\{x \ : \ (x,y) \in \mathcal{X}^*(f_\nu^2)\}$$

and

$$\max\{y \ : \ (x,y) \in \mathcal{X}^*(f_\nu^1)\} \leq \min\{y \ : \ (x,y) \in \mathcal{X}^*(f_\nu^2)\} .$$

This means in other words that the set of optimal solutions for the first objective function is in the lower left of the set of optimal solutions for the second objective function.

Now suppose we have an element $X \in \mathcal{X}_{par}(f_\nu^1, f_\nu^2)$ with $X = \langle s,t\rangle_\nu^{+,-} \cap \langle s+1,t+1\rangle_\nu^{+,-}$ (the existence of such points is guaranteed by Theorem 3.1 and Corollary 3.2) and $Y \in int(\langle s,t\rangle_\nu^{+,-})$. The cells $\langle s,t\rangle_\nu^{+,-}$ are given by the construction line sets $\mathcal{H}_\nu^1$ and $\mathcal{H}_\nu^2$.

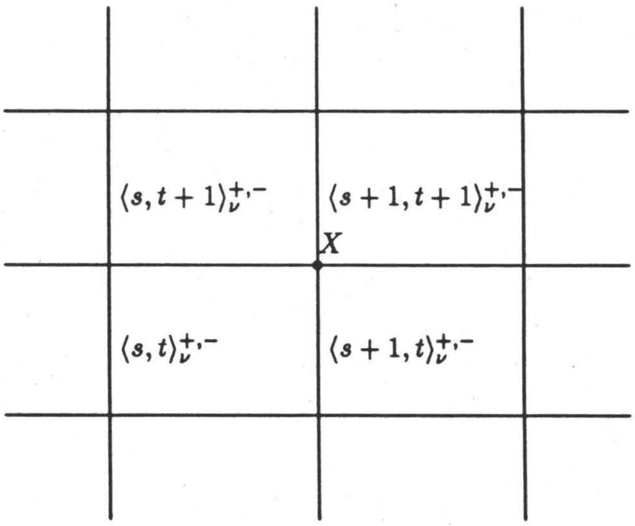

Figure 5.2: The cell $\langle s,t\rangle_\nu^{+,-}$ is uniquely defined.

We have two main cases:

Case I The cell $\langle s,t\rangle_\nu^{+,-}$ is uniquely defined (see Figure 5.2), which means that either $\langle s,t\rangle_\nu^{+} = \emptyset$ or $\langle s,t\rangle_\nu^{-} = \emptyset$.

Then we proceed as follows:

a) If the slopes of $L_=(f_\nu^1(Y))$ and $L_=(f_\nu^2(Y))$ are equal in Y then we know that the slope is equal for all points in $\langle s,t\rangle_\nu^{+,-}$. By Theorem 5.3 we have that all points in $\langle s,t\rangle_\nu^{+,-}$ are pareto locations. (see Figure 5.3).

b) If the slope of $L_=(f_\nu^1(Y))$ is smaller than the slope of $L_=(f_\nu^2(Y))$ and both are negative or the slope of $L_=(f_\nu^1(Y))$ is positive and the slope

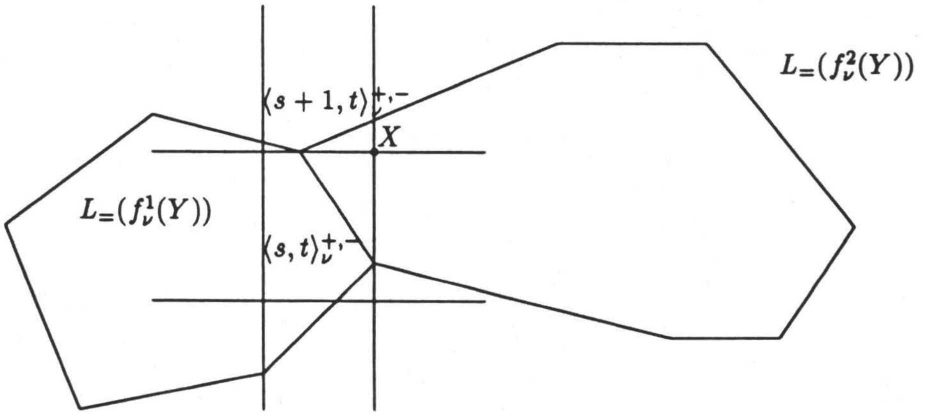

Figure 5.3: The slopes of $L_=(f_\nu^1(Y))$ and $L_=(f_\nu^2(Y))$ are equal. Therefore all $Y \in \langle s,t\rangle_\nu^{+,-}$ are pareto locations.

of $L_=(f_\nu^2(Y))$ is negative, then again with Theorem 5.3 we have that all $Y \in (\langle s,t\rangle_\nu^{+,-} \cap \langle s,t+1\rangle_\nu^{+,-})$ are pareto locations (see Figure 5.4).

c) If the slope of $L_=(f_\nu^1(Y))$ is bigger than the slope of $L_=(f_\nu^2(Y))$ and both are negative or the slope of $L_=(f_\nu^1(Y))$ is negative and the slope of $L_=(f_\nu^2(Y))$ is positive then analogously we have that all $Y \in (\langle s,t\rangle_\nu^{+,-} \cap \langle s+1,t\rangle_\nu^{+,-})$ are pareto locations (see Figure 5.5).

Depending on the appropriate case we continue with

$$X := \begin{cases} \langle s-1,t-1\rangle_\nu^{+,-} \cap \langle s,t\rangle_\nu^{+,-} & : \text{ in Case a)} \\ \langle s,t+1\rangle_\nu^{+,-} \cap \langle s-1,t\rangle_\nu^{+,-} & : \text{ in Case b)} \\ \langle s,t-1\rangle_\nu^{+,-} \cap \langle s+1,t\rangle_\nu^{+,-} & : \text{ in Case c)} \end{cases} .$$

Case II $X \in \langle s,t\rangle_\nu^{+} \cap \langle s,t\rangle_\nu^{-} \cap \langle s+1,t+1\rangle_\nu^{+,-}$ (see Figure 5.6).

Then let $Y_1 \in \langle s,t\rangle_\nu^{+}$ and $Y_2 \in int(\langle s,t\rangle_\nu^{-})$. Then we get the following cases which are proved analogously to the above ones.

a) The slopes of of $L_=(f_\nu^1(Y_1))$ and $L_=(f_\nu^2(Y_1))$ are equal.
Then all points in $\langle s,t\rangle_\nu^{+}$ are pareto locations.

b) The slopes of of $L_=(f_\nu^1(Y_2))$ and $L_=(f_\nu^2(Y_2))$ are equal.
Then all points in $\langle s,t\rangle_\nu^{-}$ are pareto locations.

c) The slope of $L_=(f_\nu^1(Y_1))$ is bigger than the slope of $L_=(f_\nu^2(Y_1))$ and both are negative or the slope of $L_=(f_\nu^1(Y_1))$ is positive and the slope of $L_=(f_\nu^2(Y_1))$ is negative .
Then all points in $\langle s,t\rangle_\nu^{+} \cap \langle s+1,t\rangle_\nu^{+}$ are pareto locations.

d) The slope of $L_=(f_\nu^1(Y_2))$ is smaller than the slope of $L_=(f_\nu^2(Y_2))$ and both are negative or the slope of $L_=(f_\nu^1(Y_2))$ is negative and the slope of $L_=(f_\nu^2(Y_2))$ is positive.
Then all points in $\langle s,t\rangle_\nu^{-} \cap \langle s,t+1\rangle_\nu^{-}$ are pareto locations.

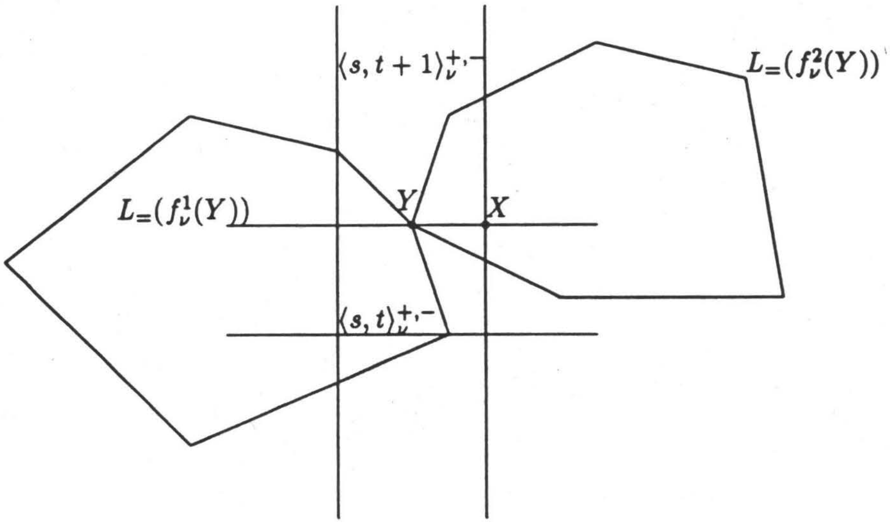

Figure 5.4: The slope of $L_=(f_\nu^1(Y))$ is smaller than the slope of $L_=(f_\nu^2(Y))$ and both are negative.

e) For $L_=(f_\nu^1(Y_1))$ and $L_=(f_\nu^2(Y_1))$ the opposite of Case c) is true and for $L_=(f_\nu^1(Y_2))$ and $L_=(f_\nu^2(Y_2))$ the opposite of Case d) is true.
Then all points in $\langle s,t\rangle_\nu^- \cap \langle s,t\rangle_\nu^+$ are pareto locations.

Depending on the appropriate case we continue with

$$X := \begin{cases} \langle s-1,t-1\rangle_\nu^{+,-} \cap \langle s,t\rangle_\nu^{+,-} & : \text{ in Case a),b),e)} \\ \langle s,t+1\rangle_\nu^{+,-} \cap \langle s-1,t\rangle_\nu^{+,-} & : \text{ in Case c)} \\ \langle s,t-1\rangle_\nu^{+,-} \cap \langle s+1,t\rangle_\nu^{+,-} & : \text{ in Case d)} \end{cases} .$$

Remark. If $\mathcal{X}^*(f_\nu^1)$ is not in the lower left of $\mathcal{X}^*(f_\nu^2)$ we have either to switch the roles of f_ν^1 and f_ν^2 in the cases, or we have in every step contrary slope conditions because we go from the upper left to the lower right.

Now it is clear how the chain of efficient locations is constructed: We start at one of the lexicographical optimal solutions, look for the right case and proceed with the next cell.

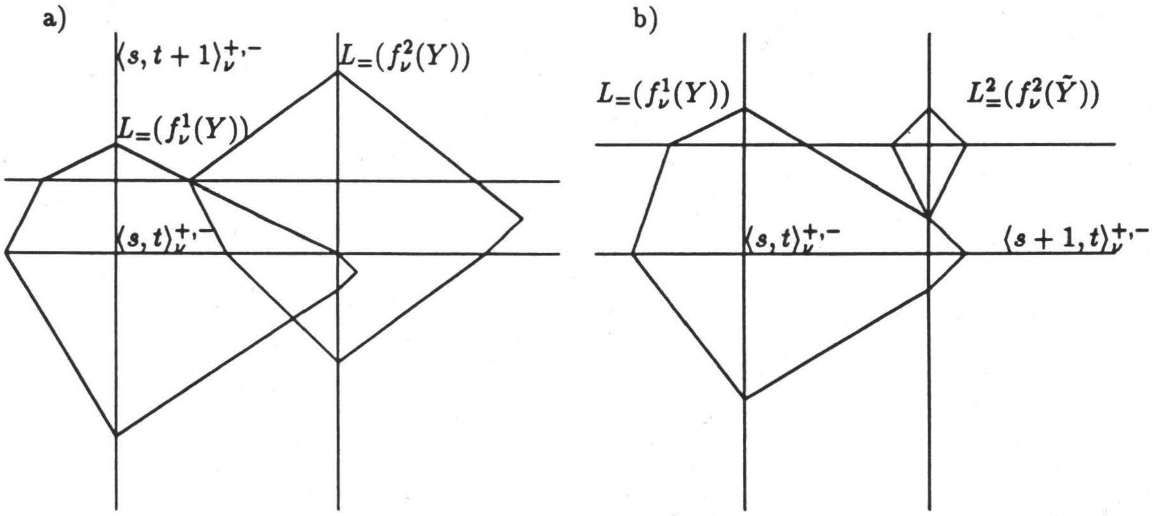

Figure 5.5: The slope of $L_=(f_\nu^1(Y))$ is bigger than the slope of $L_=(f_\nu^2(Y))$ and both are negative.

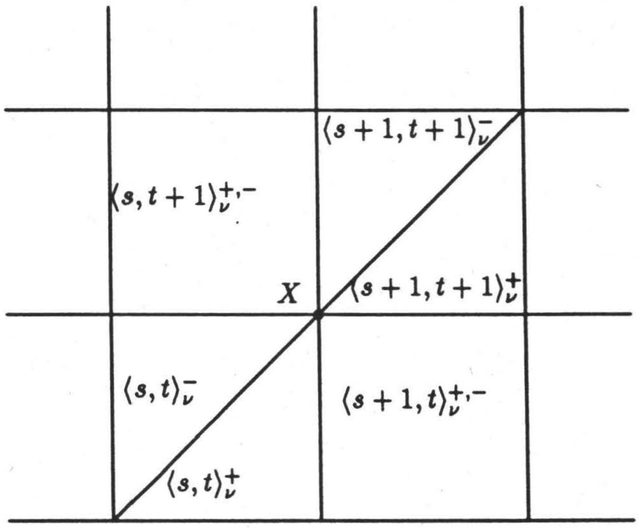

Figure 5.6: $\langle s, t\rangle_\nu^+ \neq \emptyset$ and $\langle s, t\rangle_\nu^- \neq \emptyset$.

5.2 Combining $\mathcal{X}_{par}(f_a^1, f_a^2)$ and $\mathcal{X}_{par}(f_b^1, f_b^2)$

Now given $\mathcal{X}_{par}(f_\nu^1, f_\nu^2)$ we want to investigate the change of the lambdas in the scalarization $\lambda f_\nu^1(X) + (1-\lambda) f_\nu^2(X)$ in more detail.

By Theorem 3.1 we know that the only possibility for a unique solution of the scalarization is when the solution is a corner point of a cell $\langle s,t\rangle_\nu^{+,-}$. If two corner points C_1 and C_2 are optimal we know from Theorem 3.1 that the whole line segment connecting these two corner points is optimal. But then the following equation holds

$$\lambda f_\nu^1(C_1) + (1-\lambda) f_\nu^2(C_1) = \lambda f_\nu^1(C_2) + (1-\lambda) f_\nu^2(C_2)$$

which is equivalent to

$$\lambda = \frac{f_\nu^2(C_2) - f_\nu^2(C_1)}{f_\nu^1(C_1) - f_\nu^1(C_2) + f_\nu^2(C_2) - f_\nu^2(C_1)},$$

where the denominator is not zero since $0 < \lambda < 1$.

So we have a unique λ corresponding to a scalarization in which optimal solution C_1 and C_2 are contained. We conclude that we can compute efficiently a sequence of λ_j $j = 1, \ldots, R$ with the following property.

For $\lambda \in (0, \lambda_1)$ the pareto location is the corner point C_1, for $\lambda = \lambda_1$ the pareto location is the line segment or cell defined by C_1 and C_2, etc. .

What is left now is the relationship between $\mathcal{X}_{par}(f_a^1, f_a^2)$, $\mathcal{X}_{par}(f_b^1, f_b^2)$ and $\mathcal{X}_{par}(f^1, f^2)$.

Lemma 5.5.

1. *$\mathcal{X}_{par}(f_a^1, f_a^2) \times \mathcal{X}_{par}(f_b^1, f_b^2) \supseteq \mathcal{X}_{par}(f^1, f^2)$.*
2. *$\forall X^a \in \mathcal{X}_{par}(f_a^1, f_a^2)$ exists a $X^b \in \mathcal{X}_{par}(f_b^1, f_b^2)$ with $(X^a, X^b) \in \mathcal{X}_{par}(f^1, f^2)$.*
3. *$\forall X^b \in \mathcal{X}_{par}(f_b^1, f_b^2)$ exists a $X^a \in \mathcal{X}_{par}(f_a^1, f_a^2)$ with $(X^a, X^b) \in \mathcal{X}_{par}(f^1, f^2)$.*

Proof.
ad 1)
Let $X = (X^a, X^b) \in \mathcal{X}_{par}(f^1, f^2)$ and wlog $X^a \notin \mathcal{X}_{par}(f_a^1, f_a^2) \Rightarrow \exists X^{a'} \neq X^a$ which dominates X^a (i.e. $f_a^1(X^{a'}) \leq f_a^1(X^a)$ and $f_a^2(X^{a'}) \leq f_a^2(X^a)$).
Since

$$f^1(X) = f_a^1(X^a) + f_b^1(X^b)$$

and

$$f^2(X) = f_a^2(X^a) + f_b^2(X^b)$$

by definition, it follows that $f^1(X^{a'}) \leq f^1(X^a)$ and $f^2(X^{a'}) \leq f^2(X^a)$, which means that $X' = (X^{a'}, X^b)$ dominates X. This is a contradiction to the assumption that X is a pareto location.

ad 2),3)
We use again Theorem 5.1 for f^1 and f^2. So $X = (X^a, X^b) \in \mathcal{X}_{par}(f^1, f^2)$ is equivalent to X is a solution for

$$\min_{X \in \mathbb{R}^4} \lambda f^1(X) + (1-\lambda) f^2(X)$$

for $0 < \lambda < 1$. This is equivalent to

$$\min_{X^a \in \mathbb{R}^2} \lambda f_a^1(X^a) + (1-\lambda) f_a^2(X^a) + \min_{X^b \in \mathbb{R}^2} \lambda f_b^1(X^b) + (1-\lambda) f_b^2(X^b)$$

for $0 < \lambda < 1$. But these are for a fixed λ just elements of $\mathcal{X}_{par}(f_a^1, f_a^2)$ and $\mathcal{X}_{par}(f_b^1, f_b^2)$, respectively. Since all lambdas occur the lemma is proved.

□

With these results we can give an algorithm which determines in $O(M \log M)$ time $\mathcal{X}_{par}(f_1, f_2)$.

Define

$$\begin{aligned} \Lambda_\nu((\lambda_i, \lambda_{i+1})) &:= \left\{ \operatorname*{argmin}_{X \in \mathbb{R}^2} \lambda f_\nu^1(X) + (1-\lambda) f_\nu^2(X) \; : \; \lambda \in (\lambda_i, \lambda_{i+1}) \right\} \\ \Lambda_\nu((\lambda_i)) &:= \left\{ \operatorname*{argmin}_{X \in \mathbb{R}^2} \lambda_i f_\nu^1(X) + (1-\lambda_i) f_\nu^2(X) \right\} \end{aligned}$$

Algorithm for solving (B2M)

1. Compute $\mathcal{X}_{par}(f_a^1, f_a^2)$ and $\mathcal{X}_{par}(f_b^1, f_b^2)$.
2. Determine the sequences $\lambda_1^a, \ldots, \lambda_{R_a}^a$ and $\lambda_1^b, \ldots, \lambda_{R_b}^b$ representing $\mathcal{X}_{par}(f_a^1, f_a^2)$ and $\mathcal{X}_{par}(f_b^1, f_b^2)$, respectively.
3. Sort the λ_i^a and λ_i^b in increasing order, delete duplicates and get $\lambda_1, \ldots, \lambda_R$.
4. Set $\lambda_0 := 0$ and $\lambda_{R+1} := 1$.
5. **Output:**

$$\mathcal{X}_{par}(f_1, f_2) = \bigcup_{i=0}^{R} (\Lambda_a((\lambda_i, \lambda_{i+1})) \times \Lambda_b((\lambda_i, \lambda_{i+1}))) \cup \bigcup_{i=1}^{R} (\Lambda_a(\lambda_i) \times \Lambda_b(\lambda_i))$$

Example 5.1. *We are given four existing facilities:* $Ex_1 = (1,2), Ex_2 = (3,2),$ $Ex_3 = (5,5)$ *and* $Ex_4 = (7,4)$. *The weight tables are as follows*

$$w^1 = \begin{pmatrix} 4 & 1 & 1 & 0 \\ 2 & 5 & 1 & 0 \end{pmatrix}^T, \quad w^2 = \begin{pmatrix} 1 & 2 & 0 & 6 \\ 1 & 1 & 2 & 1 \end{pmatrix}^T,$$

$v^1 = 1$ and $v^2 = 2$.

With the algorithm of Section 3 for solving (2M) we get

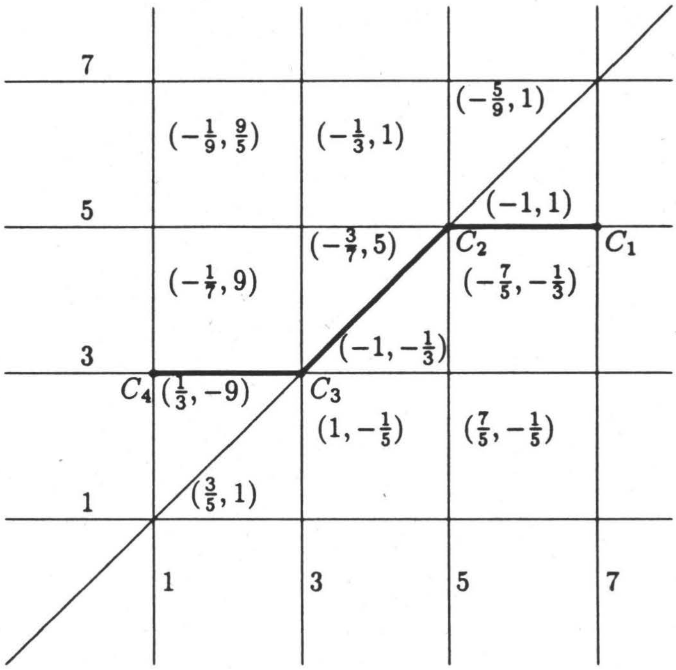

Figure 5.7: The set $\mathcal{X}_{par}(f_a^1, f_a^2)$ is the boldface line. The tuples in each cell represent the slopes of the level curve for f_a^1 and f_a^2 in this cell, respectively.

- $\mathcal{X}^*(f_a^1) = \{(1,3)\}$
- $\mathcal{X}^*(f_b^1) = \{(2,2)\}$
- $\mathcal{X}^*(f_a^2) = \{(7,5)\}$ and
- $\mathcal{X}^*(f_b^2) = \{(4,4)\}$.

First we determine $\mathcal{X}_{par}(f_a^1, f_a^2)$.

To do so we start with $X = (7,5) = \mathcal{X}^*(f_a^2)$, which is uniquely defined by $\langle 4,3\rangle_a^+ \cap \langle 3,2\rangle_a^+$. Then we compute with Corollary 2.2 the slopes of the level curves $S(L_=(f_a^1(Y))) = -\frac{7}{5}$ and $S(L_=(f_a^2(Y))) = -\frac{1}{3}$ for $Y \in int(\langle 3,2\rangle_a^+)$. Since $S(L_=(f_a^1(Y))) < S(L_=(f_a^2(Y)))$ and both are negative we know that all $Y \in \langle 3,2\rangle_a^+ \cap \langle 3,2\rangle_a^+$ are pareto locations and we continue with $X = (5,5)$.

Now $\langle 2,2\rangle_a^+ \neq \emptyset$ and $\langle 2,2\rangle_a^- \neq \emptyset$ so we are in Case II. Since $S(L_=(f_a^1(Y_1))) = -\frac{3}{7}$ is negative, $S(L_=(f_a^2(Y_1))) = 5$ is positive and $S(L_=(f_a^1(Y_2))) = -1 < S(L_=(f_a^2(Y_2))) = -\frac{1}{3}$, with both negative, we are in Case II e) and therefore all $Y \in \langle 2,2\rangle_a^+ \cap \langle 2,2\rangle_a^-$ are pareto locations and we continue with $X = (3,3)$.

Again we are in Case II. Since $S(L_=(f_a^1(Y_1))) = \frac{1}{3}$ is positive, $S(L_=(f_a^2(Y_1))) = -9$ is negative and $S(L_=(f_a^1(Y_2))) = \frac{3}{5} < S(L_=(f_a^2(Y_2))) = 1$, with both positive, all $Y \in \langle 1,1\rangle_a^- \cap \langle 1,2\rangle_a^-$ are pareto locations and we continue with $X = (1,3)$. But $(1,3) \in \mathcal{X}^*(f_a^1)$ and so we are done.

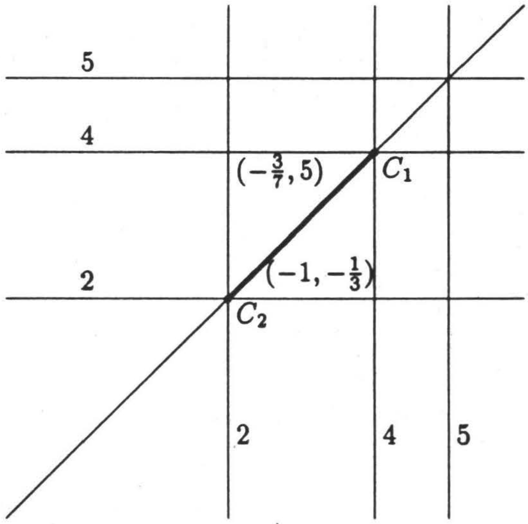

Figure 5.8: The set $\mathcal{X}_{par}(f_b^1, f_b^2)$ is the boldface line. The tuples in each cell represent the slopes of the level curve for f_b^1 and f_b^2 in this cell, respectively.

In summary we get

$$\mathcal{X}_{par}(f_a^1, f_a^2) = \overline{(7,5)-(5,5)} \cup \overline{(5,5)-(3,3)} \cup \overline{(3,3)-(1,3)}\,,$$

see also Figure 5.7.

By the same procedure we compute

$$\mathcal{X}_{par}(f_b^1, f_b^2) = \overline{(4,4)-(2,2)}\,,$$

which is shown in Figure 5.8.

Our next task is to compute the λ-sequences. We start with $\mathcal{X}_{par}(f_a^1, f_a^2)$ and get for the first two corner points

$$\lambda_1^a = \frac{f_a^2((5,5)) - f_a^2((7,5))}{f_a^1((7,5)) - f_a^1((5,5)) + f_a^2((5,5)) - f_a^2((7,5))} = \frac{1}{8}\,.$$

Doing this for all corner points we have the sequence $\frac{1}{8}$, $\frac{2}{7}$, $\frac{9}{10}$ for $\mathcal{X}_{par}(f_a^1, f_a^2)$ and a single element sequence $\frac{2}{7}$ for $\mathcal{X}_{par}(f_b^1, f_b^2)$. Adding 0 as first element, 1 as last element and mixing the two sequences in increasing order leads to the combined sequence 0, $\frac{1}{8}$, $\frac{2}{7}$, $\frac{9}{10}$, 1. Therefore

$$\begin{aligned}\mathcal{X}_{par}(f_1, f_2) = \quad & (7,5)\times(4,4) \cup (5,5)\times(4,4) \cup (3,3)\times(2,2) \cup (1,3)\times(2,2) \\ \bigcup \quad & \overline{(7,5)-(5,5)}\times(4,4) \cup \overline{(5,5)-(3,3)}\times\overline{(4,4)-(2,2)} \\ & \cup\overline{(3,3)-(1,3)}\times(2,2)\,.\end{aligned}$$

6 Conclusions

In this paper we developed $O(MlogM)$ algorithm to solve (2M), (R2M) and (B2M) for the l_1- and l_∞-metric. Also a lot of structural results giving deeper insight in this kind of problems have been presented.

It is important to emphasize that the algorithms in this paper give efficient methods for finding all solutions for the problems.

Since a pareto location with respect to q_1 objective functions is also a pareto location for $q_2 > q_1$ objective functions, the presented approach also leads to solution procedures for the general multiobjective case.

An additional topic which is already solved but is beyond the scope of this paper are solution methods for the general (RNM), where N is the number of new facilities. Also restricted multicriterial problems are under research.

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