UNIVERSITÄT KAISERSLAUTERN

MAXIMAL COHEN-MACAULAY MODULES OVER ISOLATED SINGULARITIES

Dorin Popescu

Preprint Nr. 249

ISSN 0943-8874

FACHBEREICH MATHEMATIK

MAXIMAL COHEN–MACAULAY MODULES OVER ISOLATED SINGULARITIES

Dorin Popescu

Preprint Nr. 249

ISSN 0943–8874

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Erwin–Schrödinger–Straße
67663 Kaiserslautern
Februar 1994

Maximal Cohen-Macaulay modules over isolated singularities

(Improved version)

Dorin Popescu

Faculty of Mathematics, University of Bucharest,
Str. Academiei 14, Bucharest 70109, Romania

Introduction

The reduction ideals of Dieterich [5] and Yoshino [12] (see also [10], [11] or more generally [4]) are very useful for proving the first Brauer-Thrall Conjecture for maximal Cohen-Macaulay modules over an isolated singularity. Roughly speaking, for a large enough class of Cohen-Macaulay isolated singularities (R, m, K) (see (1.2)) there exists an m-primary ideal $a \subset R$ called reduction ideal (usually chosen to be a power of m or generated by a system of parameters) such that the assignment $L \to L/a$ defines an embedding from the category of maximal Cohen-Macaulay R-modules (shortly MCM(R)) to the category Mod(R/a) of finitely generated R/a-modules. This embedding maps injectively isomorphism classes of indecomposable modules from MCM(R) to the isomorphism classes of indecomposable R/a-modules and allows the application to MCM(R) of some results from the representation theory of artinian rings (like e.g. the Lemma Harada-Sai).

However the above embedding could be more useful if we could really describe its image, i.e. if we could characterize the finitely generated R/a-modules M which have the form L/aL for a certain $L \in$ MCM(R), or equivalently if we could describe the R/a-modules which are liftable to R in the sense of Auslander-Ding-Solberg [1]. It is the purpose of our paper to describe the liftable R/a-modules if a has the form $a = (x_1^2, \ldots, x_s^2)$ for a certain system of parameters $x = (x_1, \ldots, x_s)$ of R (we may always substitute a reduction ideal with a smaller one of this form). We show (see (1.21)) that a module $M \in$ Mod(R_2), $R_2 := R/(x_1^2, \ldots, x_s^2)$ is liftable to R, i.e. it has the form $R_2 \otimes_R L$ for a certain $L \in$ MCM(R) if and only if in M holds

(1) $$((x_1, \ldots, x_{j-1})M : x_j)_M = (x_1, \ldots, x_j)M$$

for all $j, 1 \leq j \leq s$. The isomorphism classes of these artinian modules , or of those which are indecomposable can perhaps be determined in some special cases using a computer or can be viewed as some orbits of certain affine groups acting on Grassmannian varieties. The second question is the subject of Section 2 and it is completely inspirated by the "geometric" describtion of torsion free modules of rank one over a local domain of dimension 1 given by (Greuel and Pfister [7]). Such describtions of maximal Cohen-Macaulay R-modules could be useful for defining the number of parameters following the idea of Knörrer [9].

Suppose that K is algebraically closed and let r, d be some positive integers with $r \leq d \leq r \dim_K R_2$. We show (see (2.10),(2.11)) that there exists an affine group G_r acting on a Grassmannian variety $X_{r,d}$, a quasi projective subvariety $S_{r,d} \subset X_{r,d}$ closed to the G_r-action and a bijection between the G_r-orbits of $S_{r,d}$ and the isomorphism classes of finitely generated R_2-modules M satisfying (1) with $\dim_K M = d$ and $\mu(M) = r$ ($\mu(M)$ denoting the minimal number of generators of M). In particular there exists a bijection between the G_r-orbits of $S_{r,d}$ and the isomorphism classes of maximal Cohen-Macaulay R-modules L such that $\mu(L) = r$ and $\dim_K R_2 \otimes_R L = d$. If (R, m) is an irreducible singularity and $L \in \mathrm{MCM}(R)$ then $\dim_K R_2 \otimes_R L = \dim_K R_2 \ \mathrm{rank}_R L$ and there exists a bijection between the G_r-orbits of $S_{r,uq}$, $u := \dim_K R_2$, $1 \leq q \leq r$ and the isomorphism classes of maximal Cohen-Macaulay R-modules L such that $\mu(L) = r$ and $\mathrm{rank}_R L = q$ (see (2.12)).

The idea to find maximal Cohen-Macaulay module varieties started while the author was visiting the University of Kaiserslautern after discussions with Greuel and Pfister. Some preparations were made when the author was visiting the University of Osnabrück, Standort Vechta with a grant from the Alexander von Humboldt Foundation, but the essential part was done when the author was visiting the Sonderforschungsbereich 170 in Göttingen. The author is grateful to these institutions for support and hospitality.

1 Maximal Cohen-Macaulay modules and liftings

(1.1). Let $\varphi : \Lambda \to \Gamma$ be a morphism of noetherian rings and M a finitely generated Γ-module. A finitely generated Λ-module L is a *lifting* of M to Λ (see [1]) if $M \cong \Gamma \otimes_\Lambda L$ and

$$1) \qquad \mathrm{Tor}_i^\Lambda(L,\Gamma) = 0$$

for all $i \geq 1$. A finitely generated Λ-module L is a *lifting with respect to* φ if it is a lifting of $\Gamma \otimes_\Lambda L$ to Λ, i.e. if 1) holds. Let Lift(φ) (or Lift(Λ,Γ) when the morphism is well known) be the category of liftings of Λ-modules with respect to φ. A Γ-module M is *liftable with respect to* φ if it has a lifting to Λ, i.e. if it has the form $\Gamma \otimes_\Lambda L$ for a certain $L \in$ Lift(φ). Let lift(φ) (or lift(Λ,Γ)) be the category of Γ-modules liftable with respect to φ.

Let (R,m,K) be a Cohen-Macaulay local ring , $x = (x_1,\ldots,x_s)$ a system of parameters in R and $R_t := R/(x_1^t,\ldots,x_s^t)$, $t \in \mathbf{N}$. Then x is regular in R and Lift(R,R_t) is exactly the category MCM(R) of maximal Cohen-Macaulay R-modules. The following Theorem is mainly a result of Yoshino ([12] Ch. 6), extended in [10](4.8),[11](2.8) or more generally in [4](3.15)).

Theorem (1.2). *Suppose that (R,m,K) is an excellent, henselian local Cohen-Macaulay ring,an isolated singularity containing a field and such that K is either perfect or $[K : K^p] < \infty$ if $p := \mathrm{char} K > 0$. Then there exists a number $t \in \mathbf{N}$ such that the base change functor*

$$F : MCM(R) \to Mod(R/m^t)$$

has the following properties:

i) if $L \in MCM(R)$ then L is indecomposable in $MCM(R)$ if and only if $F(L)$ is indecomposable in $Mod(R/m^t)$,
ii) if L,L' are indecomposable in $MCM(R)$ then $L \cong L'$ if and only if $F(L) \cong F(L')$.

From now on we choose a system of parameters x in m^t such that i) and ii) in Theorem (1.2) are satisfied. We preserve in the whole section (even paper with the exception of (2.10),(2.11)) the notations and assumptions of Theorem (1.2). The following lemma is not far from the last statement of [6] (2.1) b) and could be obtained directly from there using induction. However our proof uses the same idea.

Lemma (1.3). *Let $i \in \mathbf{N}$ be a positive integer. Then the base change functor $F_i : MCM(R) \to Mod(R_i)$ defines an injection from the isomorphism classes of maximal Cohen-Macaulay R-modules to the isomorphism classes of R_i-modules.*

Proof. Let $L, L' \in \mathrm{MCM}(R)$ such that $F_i(L) \cong F_i(L')$. Express L, L' as direct sums of indecomposable modules of $\mathrm{MCM}(R)$, let us say $L = \oplus_{j=1}^{u} L_j$, $L' = \oplus_{k=1}^{u'} L'_k$. Then $\oplus_{j=1}^{u} F_i(L_j) \cong F_i(L) \cong F_i(L') \cong \oplus_{k=1}^{u'} F_i(L'_k)$. By Theorem (1.2) i) $F_i(L_j), F_i(L'_k)$ are indecomposable and it follows $u = u'$ and $F_i(L_j) \cong F_i(L'_j)$ after perhaps a new numerotation of (L'_k). Using now Theorem (1.2) ii) we obtain $L_j \cong L'_j$ and thus $L \cong L'$.

Proposition (1.4). *F_1 induces a bijection from the isomorphism classes of maximal Cohen-Macaulay R-modules to the isomorphism classes of the modules of lift(R_2, R_1).*

Proof. Let $M \in \mathrm{lift}(R_2, R_1)$. We need the following variant of [1] (3.6)(a consequence of (1.16) for $e = s$).

Lemma (1.5). *Let A be a noetherian local ring, $y = (y_1, \ldots, y_s)$ a regular system of elements in A, $A_u := A/(y_1^u, \ldots, y_s^u)$ and N a finitely generated A-module. If $N \in$ lift(A_2, A_1) then N is a direct summand of a module of lift(A, A_1).*

By the above lemma M is a direct summand of a module from $\mathrm{lift}(R, R_1)$, i.e. there exists $L \in \mathrm{Lift}((R, R_1)$ such that $R_1 \otimes_R L \cong M \oplus P$ for a certain R_1-module P. Express L, M, P as direct sums of indecomposable modules from $\mathrm{MCM}(R)$, respectively $\mathrm{Mod}(R_1)$, let us say $L = \oplus_{i=1}^{u} L_i$, $M = \oplus_{j=1}^{v} M_j$, $P = \oplus_{k=1}^{w} P_k$. By Theorem (1.2) i), $F_1(L_i)$ are indecomposable. Since $F_1(L) = \oplus_{i=1}^{u} F_1(L_i) = (\oplus_{j=1}^{v} M_j) \oplus (\oplus_{k=1}^{w} P_k)$ and R_1 is local artinian it follows $u = v + w$ and $M_j \cong F_1(L_j)$ after a renumerotation of (L_i). Then $M = F_1(\oplus_{j=1}^{v} L_j)$, i.e. $M \in \mathrm{lift}(R, R_1)$. Now apply Lemma (1.3).

Remark (1.6). i) A similar proof to (1.4) using [1](3.6) instead of our Lemma (1.5) gives a bijection from the isomorphism classes of $\mathrm{MCM}(R)$-modules to the isomorphism classes of the modules of $\mathrm{lift}(R/(x_1, \ldots, x_s)^2, R_1)$. This is somehow done in [6] (2.1) b). Since the modules of $\mathrm{Lift}(R_2, R_1)$ are easier to describe, as we will see, we prefer our statement from (1.4).

ii) Using i) and (1.4) we get in particular

1) $$\mathrm{lift}(R_2, R_1) = \mathrm{lift}(R/(x_1, \ldots, x_s)^2, R_1).$$

Thus i) and (1.4) says in fact the same thing. Unfortunately, we do not know a direct proof of 1) without using our (1.5).

iii) Note that in the above proof F_1 maps the inclusion $L_i \subset L$ (resp. its retraction) in the inclusion $M_i \subset M \oplus P$ (resp. its retraction).

Next we try to describe the modules of $\mathrm{Lift}(R_2, R_1)$ but first we need some small preparations.

Lemma (1.7). *Let $\Lambda \to \Gamma \to \Delta$ be two ring morphisms and L a Λ-module such that $Tor_i^\Lambda(L, \Gamma) = 0$ for all $i > 0$. Then $Tor_j^\Lambda(L, \Delta) \cong Tor_j^\Gamma(L \otimes_\Lambda \Gamma, \Delta)$ as Δ-modules for every $j > 0$. In particular, if $L \in Lift(\Lambda, \Gamma)$ then $L \in Lift(\Lambda, \Delta)$ iff $L \otimes_\Lambda \Gamma \in Lift(\Gamma, \Delta)$.*

Proof. Let

1) $$\ldots \to F_n \to \ldots \to F_1 \to F_0 \to L \to 0$$

be a free resolution of L over Λ. Tensorizing by Γ over Λ we obtain a free resolution

2) $$\ldots \to \Gamma \otimes_\Lambda F_n \to \Gamma \otimes_\Lambda F_1 \to \Gamma \otimes_\Lambda F_0 \to \Gamma \otimes_\Lambda L \to 0$$

of $\Gamma \otimes_\Lambda L$ over Γ since $\mathrm{Tor}_i^\Lambda(L, \Gamma) = 0$ for all $i > 0$. Note that tensorizing 1) and 2) by Δ over Λ, respect. Γ we obtain the same complex. This means that $\mathrm{Tor}_j^\Lambda(L, \Delta) \cong \mathrm{Tor}_j^\Gamma(L \otimes_\Lambda \Gamma, \Delta)$ as Δ-modules for every $j > 0$.

Remark (1.8). i) If $L \in \mathrm{Lift}(\Lambda, \Delta)$ then L (respect. $L \otimes_\Lambda \Gamma$) is not necessarily in $\mathrm{Lift}(\Lambda, \Gamma)$ (respect. $\mathrm{Lift}(\Gamma, \Delta)$). Take for example $\Lambda = \mathbf{Z}[X]$, $\Gamma = \Lambda/(2X)$, $\Delta = \Lambda/(2) = \mathbf{Z}_2[X]$ and $L = \mathbf{Z}$. We have $L \in \mathrm{Lift}(\Lambda, \Delta)$ but $\mathrm{Tor}_1^\Lambda(L, \Gamma) = \mathbf{Z}$ and $\mathrm{Tor}_2^\Gamma(L \otimes_\Lambda \Gamma, \Delta) \cong \mathbf{Z}_2$.

ii) If $\Gamma \to \Delta$ is faithfully flat then certainly $\mathrm{Lift}(\Lambda, \Delta) \subset \mathrm{Lift}(\Lambda, \Gamma)$. Other results of this type are given in (1.10), (1.12) and (1.15).

iii) Because of i) it is not clear in the notations and hypothesis of (1.5) when it holds $\mathrm{Lift}(A_2, A_1) \subset \mathrm{Lift}(A_2, A/(y_1, \ldots, y_s)^2)$. Such a result would be useful to show our (1.5) as a direct consequence of [1](3.6). However [1](3.6) and our (1.5) implies that it holds

$$\operatorname{lift}(A_2, A_1) \subset \operatorname{lift}(A/(y_1, \ldots, y_s)^2, A_1).$$

Indeed if $L \in \operatorname{Lift}(A_2, A_1)$ then $A_1 \otimes L$ is a direct summand in a module of $\operatorname{lift}(A, A_1)$ by (1.5) and so it is enough to apply [1](3.6).

The following easy Lemma is a particular case of [1](1.1); we sketch here a proof just for completeness of our paper.

Lemma (1.9). *Let Λ be a ring, L a Λ-module and $y \in \Lambda$ an element such that $(0:y)_\Lambda = y\Lambda$. Then $Tor_i^\Lambda(L, \Lambda/(y)) \cong (0:y)_L/yL$ for all $i > 0$.*

Proof. A free resolution of $\Lambda/(y)$ over Λ is given by multiplication with y

$$\ldots \to \Lambda \to \Lambda \to \Lambda.$$

Tensorizing with L the above exact sequence we get the complex

$$\ldots \to L \to L \to L$$

where the maps are given by scalar multiplication with y. Taking the homology groups of this complex the conclusion follows.

Lemma (1.10). *Let $\varphi : \Lambda \to \Gamma$ be a morphism of noetherian rings, $y \in \Gamma$ an element, $\Gamma_1 := \Gamma/(y)$, E a finitely generated Γ-module and L a finitely generated Λ-module. Suppose that $(0:y)_\Gamma = y\Gamma$. Then*
i) $E \in Lift(\Gamma, \Gamma_1)$ iff $(0:y)_E = yE$.
ii) $L \in Lift(\Lambda, \Gamma_1)$ iff $L \in Lift(\Lambda, \Gamma)$ and $\Gamma \otimes_\Lambda L \in Lift(\Gamma, \Gamma_1)$.

Proof. By Lemma (1.9) it follows immediately i). The sufficiency of ii) follows from (1.7) and remains to show the necessity. We have the following exact sequence of Γ-modules

$$0 \to y\Gamma \to \Gamma \to y\Gamma \to 0$$

where the map $\Gamma \to y\Gamma$ is given by scalar multiplication with y. Thus $y\Gamma \cong \Gamma_1$. Tensorizing by L the above sequence we obtain the following exact sequence

$$\operatorname{Tor}_i^\Lambda(L, \Gamma_1) \to \operatorname{Tor}_i^\Lambda(L, \Gamma) \to \operatorname{Tor}_i^\Lambda(L, \Gamma_1),$$

$i > 0$. If $\mathrm{Tor}_i^\Lambda(L, \Gamma_1) = 0$, $i > 0$ then it follows $\mathrm{Tor}_i^\Lambda(L, \Gamma) = 0$, $i > 0$, i.e. $L \in \mathrm{Lift}(\Lambda, \Gamma)$. Now apply (1.7).

Proposition (1.11). *Let Λ be a noetherian ring, $y = (y_1, \ldots, y_s)$ a system of elements in Λ . $\Lambda_{[j]} := \Lambda/(y_1, \ldots, y_j)$ for $1 \leq j \leq s$ and L a finitely generated Λ-module. Suppose that for each j, $1 \leq j \leq s$ it holds*

$$((y_1, \ldots, y_{j-1}) : y_j) = (y_1, \ldots, y_j).$$

Then $L \in Lift(\Lambda, \Lambda_{[s]})$ iff for each j, $1 \leq j \leq s$ it holds

$$(\mathcal{L}_j) \qquad ((y_1, \ldots, y_{j-1})L : y_j)_L = (y_1, \ldots, y_j)L.$$

Proof. Apply induction on s, the case $s = 1$ being given by Lemma (1.10) i). Suppose that $s > 1$. If $L \in \mathrm{Lift}(\Lambda, \Lambda_{[s]})$ then $L \in \mathrm{Lift}(\Lambda, \Lambda_{[s-1]})$ and $\Lambda_{[s-1]} \otimes_\Lambda L \in \mathrm{Lift}(\Lambda_{[s-1]}, \Lambda_{[s]})$ by Lemma (1.10) ii). By induction hypothesis L satisfies $(\mathcal{L}_j)$ for all $1 \leq j < s$ and $(0 : y_s)_{\Lambda_{[s-1]} \otimes L} = y_s(\Lambda_{[s-1]} \otimes L)$, i.e. $(\mathcal{L}_j)$ holds in L for all $1 \leq j \leq s$. Conversely, if L satisfies $(\mathcal{L}_j)$ for all $1 \leq j \leq s$ we obtain $L \in \mathrm{Lift}(\Lambda, \Lambda_{[s-1]})$ and $\Lambda_{[s-1]} \otimes L \in \mathrm{Lift}(\Lambda_{[s-1]}, \Lambda_{[s]})$ again by induction hypothesis. Now applying Lemma (1.7) it follows $L \in \mathrm{Lift}(\Lambda, \Lambda_{[s]})$.

An extension of (1.10) ii) is given by the following:

Corollary (1.12). *Let $\varphi : \Lambda \to \Gamma$ be a morphism of noetherian rings, $y = (y_1, \ldots, y_s)$ a system of elements in Γ, $\Delta := \Gamma/(y)\Gamma$ and L a finitely generated Λ-module. Suppose that for each j, $1 \leq j \leq s$ it holds*

$$((y_1, \ldots, y_{j-1}) : y_j) = (y_1, \ldots, y_j).$$

Then $L \in Lift(\Lambda, \Delta)$ iff $L \in Lift(\Lambda, \Gamma)$ and $\Gamma \otimes_\Lambda L \in Lift(\Gamma, \Delta)$.

Proposition (1.13). *Conserving the notations and hypothesis from Proposition (1.11), let $\Omega^1_\Lambda(L) \subset \Lambda^{\mu(L)}$ be the first syzygy of L over Λ. Then the following statements are equivalent:*

i) $L \in Lift(\Lambda, \Lambda_{[s]})$,
ii) for each j, $1 \leq j \leq s$ $\Omega^1_\Lambda(L)$ satisfies

$$(\mathcal{L}\mathcal{L}_j) \qquad \Omega^1_\Lambda(L) \cap (y_1, \ldots, y_j)\Lambda^{\mu(L)} = (y_1, \ldots, y_j)\Omega^1_\Lambda(L),$$

iii) $\Omega^1_\Lambda(L) \in Lift(\Lambda, \Lambda_{[s]})$.

Proof. Apply induction on s. Let $f : \Lambda^{\mu(L)} \to L$ be a surjection such that $\mathrm{Ker} f = \Omega^1_\Lambda(L)$, i.e. the following sequence is exact

1) $$0 \to \Omega^1_\Lambda(L) \to \Lambda^{\mu(L)} \to L \to 0.$$

Tensorizing with $\Lambda_{[s]}$ we obtain the following exact sequences

2) $$0 \to \mathrm{Tor}_1^\Lambda(\Lambda_{[s]}, L) \to \Lambda_{[s]} \otimes \Omega^1_\Lambda(L) \to \Lambda_{[s]}^{\mu(L)} \to \Lambda_{[s]} \otimes L \to 0,$$

3) $$0 \to \mathrm{Tor}_2^\Lambda(\Lambda_{[s]}, L) \to \mathrm{Tor}_1^\Lambda(\Lambda_{[s]}, \Omega^1_\Lambda(L)) \to 0.$$

Let $s = 1$, $y = y_1$. Then i) holds exactly when

$$\mathrm{Tor}_j^\Lambda(\Lambda_{[1]}, L) \cong \mathrm{Tor}_1^\Lambda(\Lambda_{[1]}, L) = 0, \ j > 0$$

(see (1.9),(1.10)). By 3) this happens exactly when $\mathrm{Tor}_1^\Lambda(\Lambda_{[1]}, \Omega^1_\Lambda(L)) = 0$, i.e. i) $\Leftrightarrow$ iii) holds for $s = 1$. On the other hand $\mathrm{Tor}_1^\Lambda(\Lambda_{[1]}, L) = 0$ if and only if the canonical map $\Lambda_{[1]} \otimes \Omega^1_\Lambda(L) \to \Lambda_{[1]}^{\mu(L)}$ from 2) is injective, i.e. if and only if ii) holds for $s = 1$. Hence i) $\Leftrightarrow$ ii) holds for $s = 1$.

Suppose that $s > 1$. By (1.10) we have i) if and only if it holds

4) $$L \in \mathrm{Lift}(\Lambda, \Lambda_{[s-1]}) \text{ and}$$

5) $$\Lambda_{[s-1]} \otimes L \in \mathrm{Lift}(\Lambda_{[s-1]}, \Lambda_{[s]}).$$

By induction hypothesis we note that 4) holds if any one from i)—iii) holds. Hence we may suppose that 4) holds and it remains to show that then 5) holds if and only if $(\mathcal{LL}_s)$ holds or if and only if $\Lambda_{[s-1]} \otimes_\Lambda \Omega^1_\Lambda(L) \in \mathrm{Lift}(\Lambda_{[s-1]}, \Lambda_{[s]})$ (see (1.10)). Thus we may suppose $\mathrm{Tor}_j^\Lambda(\Lambda_{[s-1]}, L) = 0$, $j > 0$ and it follows $\mathrm{Tor}_1^\Lambda(\Lambda_{[s]}, L) \cong \mathrm{Tor}_1^{\Lambda_{[s-1]}}(\Lambda_{[s]}, \Lambda_{[s-1]} \otimes L)$ by (1.7). Then 5) holds iff $\mathrm{Tor}_1^\Lambda(\Lambda_{[s]}, L) = 0$ (see (1.9)) which happens exactly when $(\mathcal{LL}_s)$ holds. On the other hand we have $\Omega^1_{\Lambda_{[s-1]}}(\Lambda_{[s-1]} \otimes L) \cong \Lambda_{[s-1]} \otimes \Omega^1_\Lambda(L)$ because $\mathrm{Tor}_1^\Lambda(\Lambda_{[s-1]}, L) = 0$. But 5) holds if and only if $\Omega^1_{\Lambda_{[s-1]}}(\Lambda_{[s-1]} \otimes L) \in \mathrm{Lift}(\Lambda_{[s-1]}, \Lambda_{[s]})$ (case $s = 1$), which finishes our proof.

Corollary (1.14). *Let A be a noetherian local ring, $z = (z_1, \ldots, z_s)$ a regular system of elements in A, $A_u := A/(z_1^u, \ldots, z_s^u)$, $u \in \mathbf{N}$ and L a finitely generated A_2-module. Then the following statements are equivalents:*

i) $L \in \mathrm{Lift}(A_2, A_1)$,
ii) for each j, $1 \leq j \leq s$ $((z_1, \ldots, z_{j-1})L : z_j)_L = (z_1, \ldots, z_j)L$,

iii) for each j, $1 \le j \le s$ $\Omega^1_{A_2}(L) \cap (z_1,\ldots,z_j)A_2^{\mu(L)} = (z_1,\ldots,z_j)\Omega^1_{A_2}(L)$,
iv) $\Omega^1_{A_2}(L) \in Lift(A_2, A_1)$.

Proof. It is easy to see that ii) holds for $L = A_2$. Indeed if $(y_1,\ldots,y_s)$ is a regular system in A we have $((y_1,\ldots,y_{s-1},y_s^2) : y_s) = (y_1,\ldots,y_s)$. Applying this argument to the regular sequence $(z_1,\ldots,z_{j-1},z_{j+1}^2,\ldots,z_s^2,z_j)$ we obtain

$$((z_1,\ldots,z_{j-1},z_j^2,\ldots,z_s^2) : z_j) = (z_1,\ldots,z_j,z_{j+1}^2,\ldots,z_s^2)$$

which proves our claim. Now apply (1.11) and (1.13).

The following lemma is well known, it appears somehow in some proofs from [1].

Lemma (1.15). *Let (A,m) be a noetherian local ring, $\varphi : A \to B$ a ring morphism, $z \in m$ an element such that $\varphi(z)$ is regular in B, $B_1 := B/(\varphi(z))$ and L a finitely generated A-module. Then $L \in Lift(A,B_1)$ iff $L \in Lift(A,B)$ and $B \otimes_A L \in Lift(B,B_1)$.*

Proof. Tensorizing by L over A the exact sequence

$$0 \to B \to B \to B_1 \to 0$$

given by the scalar multiplication with $\varphi(z)$ we obtain the following exact sequences

$$\mathrm{Tor}_i^A(L,B) \to \mathrm{Tor}_i^A(L,B) \to \mathrm{Tor}_i^A(L,B_1),$$

$i \ge 1$, where the first maps are scalar multiplications with $\varphi(z)$. If $\mathrm{Tor}_i^A(L,B_1) = 0$ then $\varphi(z)\mathrm{Tor}_i^A(L,B) = \mathrm{Tor}_i^A(L,B)$ and by Nakayama's Lemma it follows $\mathrm{Tor}_i^A(L,B) = 0$. Thus $L \in \mathrm{Lift}(A,B_1)$ implies $L \in \mathrm{Lift}(A,B)$. Now apply (1.7).

The next proposition is a weak variant in our frame of [1](3.6).

Proposition (1.16). *Let (A,m) be a noetherian local ring, $(z_1,\ldots,z_s)$ a regular system of elements of A, $A_{u|j|} := A/(z_1^u,\ldots,z_j^u)$, $u \in \mathbf{N}$, $1 \le j \le s$, $A_1 := A_{1|s|} = A/(z)$, e, $1 \le e \le s$ an integer and L an $A_{2|e|}$-module from $Lift(A_{2|e|}, A_1)$. Then there exists a module $P \in Lift(A,A_1)$ such that $M := A_{1|e|} \otimes_{A_{2|e|}} L$ is a direct summand in $A_{1|e|} \otimes_A P$.*

Proof. Induct on e. If $e = 1$ then the result follows from [1] (3.2). Indeed let $\Omega^1_A(M)$ (resp.$\Omega^1_{A_{1|1|}}(M)$) be the first syzygy of M over A (resp. $A_{1|1|}$).

Tensorizing by $A_{1|1|}$ the exact sequence

$$0 \to \Omega^1_A(M) \to A^{\mu(M)} \to M \to 0$$

we obtain the following exact sequence

1) $\quad 0 \to M \cong \mathrm{Tor}_1^A(A_{1|1|}, M) \to A_{1|1|} \otimes_A \Omega^1_A(M) \to \Omega^1_{A_{1|1|}}(M) \to 0$

which splits exactly when $M \in \mathrm{lift}(A_{2|1|}, A_{1|1|})$ (see [1] (3.2)). But $M \in \mathrm{lift}(A_{2|1|}, A_{1|1|}) \cap \mathrm{Lift}(A_{1|1|}, A_1)$ because $L \in \mathrm{Lift}(A_{2|1|}, A_1)$ by assumptions (apply (1.15) several times). Thus 1) splits and $N := M \oplus \Omega^1_{A_{1|1|}}(M)$ is lifted by $\Omega^1_A(M)$ to A, i.e. $N \in \mathrm{lift}(A, A_{1|1|})$. As $M \in \mathrm{Lift}(A_{1|1|}, A_1)$ we have also $\Omega^1_{A_{1|1|}}(M) \in \mathrm{Lift}(A_{1|1|}, A_1)$ and thus $N \in \mathrm{Lift}(A_{1|1|}, A_1)$. Hence $P := \Omega^1_A(M) \in \mathrm{Lift}(A, A_1)$ by (1.7) and N is a direct summand of $A_{1|1|} \otimes_A P$.

Suppose that $e > 1$. Use the case $e = 1$ for

$$A_{2|e-1|} \to A_{2|e|} \to A_{2|e|}/z_e A_{2|e|} \to A_1$$

and L. This is possible because $(z_e, \ldots, z_s)$ is regular on $A_{2|e-1|}$ and $L \in \mathrm{Lift}(A_{2|e|}, A_1)$. Then there exists a module $Q \in \mathrm{Lift}(A_{2|e-1|}, A_1)$ such that $L/z_e L$ is a direct summand in $Q/z_e Q$ (case $e = 1$).

Now apply the induction hypothesis (case (s-1) !) to

$$A \to A_{2|e-1|} \to A_{1|e-1|} \to A_1$$

and Q. Then there exists a module P in $\mathrm{Lift}(A, A_1)$ such that $A_{1|e-1|} \otimes_{A_{2|e-1|}} Q$ is a direct summand in $A_{1|e-1|} \otimes_A P$. Thus $A_{1|e|} \otimes_{A_{2|e|}} L$ is a direct summand in $A_{1|e|} \otimes_{A_{2|e-1|}} Q$ which is also a direct summand in $A_{1|e|} \otimes_A P$.

(1.17). Let A be a noetherian ring, $y \in A$ a regular element in A, $A_j := A/(y^j)$ for $j \in \mathbf{N}$, M a finitely generated A_1-module, $\Omega^1_A(M)$ the first syzygy of M over A and $\bar{\alpha} : M \to A_1 \otimes \Omega^1_A(M)$ the canonical map from the following exact sequence which is similar to (1.16) 1)

$$0 \to M \cong \mathrm{Tor}_1^A(A_1, M) \to A_1 \otimes_A \Omega^1_A(M) \to \Omega^1_{A_1}(M) \to 0.$$

By [1] (3.2) $M \in \mathrm{lift}(A_2, A_1)$ if and only if $\bar{\alpha}$ has a retraction. If $\bar{\beta}$ is a retraction then a lifting of M to A_2 is given taking the image $(\xi_{\bar{\beta}})$ of the extension

$(\xi) \qquad 0 \to \Omega^1_A(M) \to A^{\mu(M)} \to M \to 0$

by $\mathrm{Ext}^1_A(M, \beta q)$, where $q : \Omega^1_A(M) \to A_1 \otimes_A \Omega^1_A(M)$ is the canonical surjection. Indeed, then (ξ_β) has the form

$$0 \to M \to L_2 \to M \to 0,$$

where L_2 is a lifting of M to A_2 (see [1] (1.5)).

Lemma (1.18). *Suppose that there exists a lifting U of M to A and some maps $\alpha : U \to \Omega^1_A(M)$, $\beta : \Omega^1_A(M) \to U$ such that $\beta\alpha = 1_U$, $A_1 \otimes \alpha = \bar{\alpha}$ and $A_1 \otimes \beta = \bar{\beta}$. Then $A_2 \otimes U \cong L_2$.*

Proof. Let $\gamma : A^{\mu(M)} \to U$ be a surjective morphism and $\Omega^1_A(U) = \mathrm{Ker}\,\gamma$ the first syzygy of U over A ($\mu(M) = \mu(U)$). We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \Omega^1_A(U) & & & & \\
 & & & & \downarrow & & & & \\
(\xi) & 0 & \longrightarrow & \Omega^1_A(M) & \longrightarrow & A^{\mu(M)} & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \| & & \\
(\eta) & 0 & \longrightarrow & U & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & & \downarrow & & & & \\
 & & & & & 0 & & & &
\end{array}$$

where the lines and the middle vertical are exact and the map $U \to U$ is given by scalar multiplication with y. By Snake Lemma we obtain surjective the left vertical map γ' induced by γ and $\mathrm{Ker}\,\gamma' = \Omega^1_A(U)$. Since β is also surjective and $\mathrm{Ker}\,\beta = \Omega^1_A(U)$ ($\Omega^1_A(U)$ is a lifting of $\Omega^1_{A_1}(M)$ and $\Omega^1_A(M) \cong U \oplus \Omega^1_A(U)$!) there exists an A-automorphism φ of U such that $\varphi\gamma' = \beta$. Changing γ by $\varphi\gamma$ we may suppose that $\gamma' = \beta$. Thus $\xi_{\bar{\beta}} =$

$\mathrm{Ext}^1_A(M,\beta q)(\xi) = \mathrm{Ext}^1_A(M,q'\beta)(\xi) = \mathrm{Ext}^1_A(M,q')(\eta) =: \eta'$, where $q' : U \to M$ is the canonical surjection. But η' is exactly the extension

$$0 \to M \cong y(A_2 \otimes_A U) \to A_2 \otimes_A U \to M \to 0$$

and it follows $L_2 \cong A_2 \otimes_A U$.

Proposition (1.19). *Conserving the notations and hypothesis of Theorem (1.2) F_2 defines a bijection from the isomorphism classes of maximal Cohen-Macaulay R-modules to the isomorphism classes of the modules from $Lift(R_2, R_1)$.*

Proof. Let $L \in \mathrm{Lift}(R_2, R_1)$. It is enough to show that $L \in \mathrm{lift}(R, R_2)$. Let $R_{2[j]} := R_2/(x_1, \ldots, x_j)$ for $0 \le j \le s$, $R_{2[0]} = R_2$, $R_{2[s]} = R_1$ and $L_j := R_{2[j]} \otimes_{R_2} L$. By decreasing induction on j, $0 \le j \le s$ we show that $L_j \in \mathrm{lift}(R, R_{2[j]})$ and in particular $L = L_0 \in \mathrm{lift}(R, R_2)$ which settles our question. If $j = s$ we have obviously $L_s = R_1 \otimes L \in \mathrm{lift}(R_2, R_1) = \mathrm{lift}(R, R_1)$ by Proposition (1.4).

Now suppose that $j < s$. By induction hypothesis we have $L_{j+1} \in \mathrm{lift}(R, R_{2[j+1]})$, let us say $L_{j+1} = R_{2[j+1]} \otimes V$ for a certain $V \in \mathrm{Lift}(R, R_{2[j+1]})$. Then x is regular on V (see (1.1)) and thus $x' = (x_1, \ldots, x_j, x_{j+2}^2, \ldots, x_s^2)$ is regular on $\bar{V} := V/x_{j+1}V$. Let $C_{j+1} := R/(x')$ and $W := \Omega^1_R(\bar{V})$. We have $C_{j+1} \otimes_R W \cong \Omega^1_{C_{j+1}}(C_{j+1} \otimes_R \bar{V}) = \Omega^1_{C_{j+1}}(L_{j+1})$ since x' is regular on $\bar{V}$. Moreover, W is a lifting of $\Omega^1_{C_{j+1}}(L_{j+1})$ to R. Indeed we have $\mathrm{grade}((x), W) = \mathrm{grade}((x), \bar{V}) + 1 = s$, i.e. $W \in \mathrm{MCM}(R)$ which is enough (see (1.1)).

Next we show that a direct summmand of W lifts L_j to R which finishes our proof. As $L_{j+1} = R_{2[j+1]} \otimes_{R_{2[j]}} L_j$ and $L_j \in \mathrm{Lift}(R_{2[j]}, R_{2[j+1]})$ (apply Corollary (1.12)), the following exact sequence splits (similar to the sequence 1) from the proof of (1.16))

$$0 \to L_{j+1} \to R_{2[j+1]} \otimes \Omega^1_{C_{j+1}}(L_{j+1}) \to \Omega^1_{R_{2[j+1]}}(L_{j+1}) \to 0.$$

Let $\bar{\beta}$ be a retraction of $\bar{\alpha} : L_{j+1} \to R_{2[j+1]} \otimes \Omega^1_{C_{j+1}}(L_{j+1})$ such that L_j is given by $\mathrm{Ext}^1_R(L_{j+1}, \bar{\beta} q)$ (see (1.17)),

$$q : \Omega^1_{C_{j+1}}(L_{j+1}) \to R_{2[j+1]} \otimes \Omega^1_{C_{j+1}}(L_{j+1})$$

being the canonical surjection. Express $W := \Omega^1_R(\bar{V})$ as a direct sum of indecomposable modules from $\mathrm{MCM}(R)$, let us say $W = \oplus_{i=1}^e W_i$. By

Theorem (1.2)' $F_1(W_i)$ is indecomposable and using Lemma ofNakayama we obtain also $R_{2[j+1]} \otimes W_i$ indecomposable. Then

$$R_{2[j+1]} \otimes \Omega^1_{C_{j+1}}(L_{j+1}) = R_{2[j+1]} \otimes_R W = \oplus_{i=1}^{e}(R_{2[j+1]} \otimes_R W_i)$$

has L_{j+1} as a direct summand and like in the proof of (1.4) it follows L_{j+1} isomorphic with a direct sum of some $R_{2[j+1]} \otimes W_i$, i.e. $L_{j+1} = R_{2[j+1]} \otimes U$ for a certain direct summand U of W ($U \cong V$). Moreover by Remark (1.6) iii) we can choose $U \subset W$ and $\beta : W \to U$ a retraction of the inclusion $\alpha : U \to W$ such that $R_{2[j+1]} \otimes \beta = \bar{\beta}$ and $R_{2[j+1]} \otimes \alpha = \bar{\alpha}$. Applying Lemma (1.18) to $C_{j+1} \otimes U$, $C_{j+1} \otimes \alpha$, $C_{j+1} \otimes \beta$, $C_{j+1} \otimes W \cong \Omega^1_{C_{j+1}}(L_{j+1})$ we achieve $L_j \cong R_{2[j]} \otimes U \in \text{lift}(R, R_{2[j]})$.

Corollary (1.20). *In the hypothesis of Theorem (1.2) it holds*

i) $Lift(R_2, R_1) = lift(R, R_2)$.

ii) Each module from $lift(R_2, R_1)$ *has an unique (up to an isomorphism) lifting to* R_2.

Proof. i) follows obviously from the above proposition. Let $L_1, L_2 \in \text{Lift}(R_2, R_1)$ be such that $R_1 \otimes L_1 \cong R_1 \otimes L_2$. By i) there exist $U_1, U_2 \in \text{Lift}(R, R_2)$ such that $R_2 \otimes U_1 \cong L_1$, $R_2 \otimes U_2 \cong L_2$. Then $F_1(U_1) \cong F_1(U_2)$ and so $U_1 \cong U_2$ by (1.3). Hence $L_1 \cong L_2$.

Theorem (1.21). *In the hypothesis of Theorem (1.2)* F_2 *defines a bijection from the isomorphism classes of modules of* $MCM(R)$ *to the isomorphism classes of finitely generated* R_2*-modules satisfying* $(\mathcal{L}_j)$ *from (1.11) for all* j, $1 \leq j \leq s$.

The proof follows from (1.19) and (1.14).

2 Searching module varieties

The aim of this section is to show that the isomorphism classes of the finitely generated R_2-modules satisfying $(\mathcal{L}_j)$, $1 \leq j \leq s$ from the previous section (and also the isomorphism classes of maximal Cohen-Macaulay R-modules by (1.21)) can be seen as orbits of an affine group acting on a Grassmannian variety when the residue field of R is algebraically closed.

(2.1). Let (A, m) be a finite dimensional local algebra over an algebraically closed field K (for e.g. $A = R_2$), r,d two positive integers $r \leq d \leq r \dim_K A$ and $\mathrm{Gr}_m(A^r, d)$ the Grassmannian of all linear K-subspaces $V \subset mA^r$ such that $\dim_K(A^r/V) = d$, i.e. of *colength* d in A^r. In fact $\mathrm{Gr}_m(A^r, d)$ is exactly the Grassmannian $\mathrm{Gr}(mA^r, d-r)$ of all linear subspaces $V \subset mA^r$ of colength $d - r$ in mA^r. The inclusion $mA^r \subset A^r$ induces a closed embedding of $\mathrm{Gr}_m(A^r, d)$ in $\mathrm{Gr}(A^r, d)$. The group $G = \mathrm{GL}(r, A)$ acts obviously on $\mathrm{Gr}_m(A^r, d)$. The group $A^\star$ of invertible elements of A is embedded in G by $w \to wI_r$, where I_r is the identity $r \times r$-matrix. Thus the action of G induces by restriction an action of $A^\star$ on $\mathrm{Gr}_m(A^r, d)$. Let $\mathrm{M}(A; r, d)$ be the fixed point scheme of $A^\star$ on $\mathrm{Gr}_m(A^r, d)$ (see [8] (8.2)).

Lemma (2.2). *$\mathrm{M}(A; r, d)$ is a closed subscheme of $\mathrm{Gr}_m(A^r, d)$ (in particular of $\mathrm{Gr}(A^r, d)$), G acts regularly on $\mathrm{M}(A; r, d)$ and the assignment $L \to A^r/L$ defines a surjection ν from the set of closed points of $\mathrm{M}(A; r, d)$ onto the set of isomorphism classes of all finitely generated A-modules M with $\mu(M) = r$ and $\dim_K M = d$. Moreover two closed points $L, L' \in \mathrm{M}(A; r, d)$ are in the same G-orbit iff $A^r/L \cong A^r/L'$, i.e. iff $\nu(L) = \nu(L')$.*

Proof. By [8] (8.2), $\mathrm{M}(A; r, d)$ is a closed subscheme of $\mathrm{Gr}_m(A^r, d)$. G acts also on $\mathrm{M}(A; r, d)$ since the actions of $A^\star$ and G on $\mathrm{Gr}_m(A^r, d)$ commute. A linear space $L \in \mathrm{Gr}_m(A^r, d)$ is an A-module if and only if it is a fixed point under the action of $A^\star$. Indeed if L is fixed by $A^\star$ and $a \in m$ then $(1 + a) \in A^\star$ and so $(1 + a)L \subset L$, i.e. $aL \subset L$. Thus if L is a closed point in $\mathrm{M}(A; r, d)$ then L has a structure of A-submodule of mA^r, i.e. ν is well defined (we have $\mu(A^r/L) = r$, because $L \subset mA^r$!). If M is an A-module with the minimal number of generators $\mu(M) = r$ and $\dim_K M = d$ then $L = \Omega^1_R(M)$ the first syzygy of M is a closed point of $\mathrm{M}(A; R, d)$ such that $\nu(L)$ is exactly the isomorphism class of M.

If L,L' are in the same G-orbit then there exists $g \in G$ such that $g(L) = L'$. Thus g induces an A-isomorphism $A^r/L \cong A^r/L'$. Conversely, if $f : A^r/L \to A^r/L'$ is an A-isomorphism then there exists a linear endomorphism h of A^r lifting f. Since $L, L' \subset m A^r$ we have $K \otimes_A h = K \otimes_A f$ and so h must be surjective by Nakayama's Lemma. Then h is also injective and $h(L) = L'$, i.e. L, L' are in the same orbit.

Next we will be interested only in the reduced variety $\mathrm{M}(A; r, d)_{red}$ of $\mathrm{M}(A; r, d)$ and we would like to describe the closed points of $\mathrm{M}(A; r, d)_{red}$ corresponding to some liftings such as in (1.11), (1.13). We start with some preparations.

(2.3). Let V be a linear K-space of dimension s, d a positive integer $1 \leq d \leq s$ and $\mathrm{Gr}(V, d)$ be the Grassmannian of all linear subspaces $W \subset V$ of colength d in V. Let $g : V \to V'$ be a linear map of K-spaces, t an integer, $1 \leq t \leq c := s - d$ and $Z_{t,g}$ the subset of $\mathrm{Gr}(V, d)$ of all linear subspaces $W \subset V$ such that $\dim_K g(W) \leq t$. The following elementary Lemma we included here only for the completeness of the paper.

Lemma (2.4). *$Z_{t,g}$ is a closed subset of $Gr(V, d)$.*

Proof. First suppose that $t = c - 1$ ($Z_{c,g} = \mathrm{Gr}(V, d)$) and let $w_1, \ldots, w_c$ be a basis in W. Then $\dim_K g(W) \leq t$ if and only if $g(w_1) \wedge \ldots \wedge g(w_c) = 0$ in $\wedge V'$, i.e. if the Plücker coordinates of W satisfy a linear system of equations whose coefficients are $c \times c$- minors of the matrix associated to g in some basis chosen in V and V'. Clearly $Z_{t,g}$ is closed in this case.

Now suppose $t \leq c - 2$ and let $e = (e_1, \ldots, e_s)$ be a basis in V. Consider the linear map $h_{e,t,g}$: $V \to V' \oplus V$ given by $h_{e,t,g}(e_j) = g(e_j)$ if $1 \leq j \leq t + 1$ and $h_{e,t,g}(e_j) = e_j$ otherwise. We claim that $Z_{t,g} = \cap Z_{c-1, h_{e,t,g}}$, where the intersection is made over all basis e of V. Indeed, if $W \in Z_{t,g}$ then $\dim_K h_{e,t,g}(W) \leq \dim_K g(W) + c - t - 1 \leq c - 1$ and it follows $W \in Z_{c-1, h_{e,t,g}}$ for every basis e of V. Conversely, if W is not in $Z_{t,g}$ then there exist $e_1, \ldots, e_{t+1}$ in W such that the system $\{g(e_1), \ldots, g(e_{t+1})\}$ is linearly independent. Complete $e_1, \ldots, e_{t+1}$ to a basis of W and then to a basis of V. Then $\dim_K h_{e,t,g}(W) = c$, i.e. W is not in $Z_{c-1, h_{e,t,g}}$. As above $Z_{c-1, h_{e,t,g}}$ is closed and thus $Z_{t,g}$ is closed too.

Corollary (2.5). *Let V be a linear K-space of finite dimension and f an endomorphism of V. Suppose that $\mathrm{Ker}\, f = f(V)$. Then $\dim V = 2n$, $n := \dim f(V)$, $V/f(V) \cong f(V)$ and the closed points W of $Gr(V, 2d)$, $1 \leq d < n$ such that $\dim_K f(W) = n - d$ form a locally closed subset Z of $Gr(V, 2d)$.*

Proof. We have from hypothesis the following exact sequence

$$0 \to f(V) \to V \to f(V) \to 0,$$

where the map $V \to f(V)$ is induced by f. Thus $\dim_K V = 2\dim_K f(V) = 2n$ and $V/f(V) \cong f(V)$. Now by Lemma (2.4) the subsets $Z_{n-d,f}$ and $Z_{n-d-1,f}$ are closed in $\mathrm{Gr}(V, 2d)$. Then $Z = Z_{n-d,f} \setminus Z_{n-d-1,f}$ is locally closed.

Proposition (2.6). *Let x be an element of m. Suppose that $(0:x) = (x)$. Then $\dim_K A = 2n$, $n := \dim_K(x)$, $A/(x) \cong (x)$ and the closed points W of $M(A; r, 2d)_{red}$, $r \le d \le rn$ such that $(0:x)_W = xW$ form a locally closed subset Y of $M(A; r, 2d)_{red}$.*

Proof. Apply Corollary (2.5) to the case $V = A$ or A^r, f being the scalar multiplication of V by x. The set Z of closed points W of $\mathrm{Gr}(A^r, 2d)$ such that $\dim_K xW = rn - d$ is locally closed and thus $Z' := Z \cap M(A; r, 2d)_{red}$ is locally closed too in $\mathrm{Gr}(A^r, 2d)$ (see (2.2)). If W is a closed point of Z' then $xW \subset W$ because $W \in M(A; r, 2d)_{red}$ and the complex

1) $$0 \to xW \to W \xrightarrow{x} xW \to 0$$

is exact since $\dim_K W = 2rn - 2d = 2\dim_K xW$. Thus $(0:x)_W = xW$, i.e. $W \in Y$. Conversely, if $W \in Y$ then $(0:x)_W = xW$ and we have exact the sequence 1). It follows $\dim_K xW = \frac{1}{2}\dim_K W = rn - d$ and thus $W \in Z'$. Hence $Y = Z'$ is locally closed in $M(A; r, 2d)_{red}$.

Next we need the following elementary

Lemma (2.7). *Let $x = (x_1, \ldots, x_s)$ be a system of elements from m and E a finitely generated A-module. Suppose that for each j, $1 \le j \le s$ it holds $x_j^2 = 0$ and*

$$((x_1, \ldots, x_{j-1}) : x_j) = (x_1, \ldots, x_j).$$

Then the following statements are equivalent:

i) $((x_1, \ldots, x_{j-1})E : x_j)_E = (x_1, \ldots, x_j)E$ *for each* j, $1 \le j \le s$,
ii) $(0 : (x_1 \ldots x_j))_E = (x_1, \ldots, x_j)E$ *for each* j, $1 \le j \le s$,
iii) $x_1 \ldots x_{j-1}E \cap (0 : x_j)_E = x_1 \ldots x_j E$ *for each* j, $1 \le j \le s$,
iv) $(0 : (x_1, \ldots, x_j))_E = x_1 \ldots x_j E$ *for each* j, $1 \le j \le s$.

Proof. Apply induction on s. For $s = 1$ there exist nothing to prove (in this case $x_1 \ldots x_{s-1}E$ denotes E and $(x_1, \ldots, x_{s-1})E = 0$). Suppose

$s > 1$. By induction hypothesis the equivalence for $j < s$ follows. Thus in all implications we may suppose that i)—iv) hold for $j < s$. Let $z \in E$. Then $x_s z \in (x_1, \ldots, x_{s-1})E$ if and only if $x_1 \ldots x_s z = 0$. The necessity follows because $x_j^2 = 0$ and if $x_1 \ldots x_s z = 0$ then by i) case $j < s$ we obtain by recurrence $x_2 \ldots x_s z \in x_1 E, \ldots, x_s z \in (x_1, \ldots, x_{s-1})E$. Thus $((x_1, \ldots, x_{s-1})E : x_s)_E = (0 : (x_1 \ldots x_s))_E$, i.e. i) $\Leftrightarrow$ ii) holds.

Now note that

1) $$x_1 \ldots x_{s-1} E \cap (0 : x_s)_E = x_1 \ldots x_{s-1} (0 : (x_1 \ldots x_s))_E.$$

If ii) holds we have

$$x_1 \ldots x_{s-1} (0 : (x_1, \ldots, x_s))_E = x_1 \ldots x_{s-1} (x_1, \ldots, x_s) E = x_1 \ldots x_s E$$

and thus iii) holds too. Conversely if iii) holds then by 1)

$$x_1 \ldots x_{s-1} (0 : (x_1 \ldots x_s))_E = x_1 \ldots x_s E.$$

It follows

$$(0 : (x_1 \ldots x_s))_E \subset x_s E + (0 : (x_1 \ldots x_{s-1}))_E = x_s E + (x_1, \ldots, x_{s-1})E =$$

$$(x_1, \ldots, x_s)E$$

because ii) holds for $j < s$. Hence ii) $\Leftrightarrow$ iii). Since iv) holds for $j < s$ we have

$$x_1 \ldots x_{s-1} E \cap (0 : x_s)_E = (0 : (x_1, \ldots, x_{s-1}))_E \cap (0 : x_s)_E =$$

$$(0 : (x_1, \ldots, x_s))_E$$

and thus iii) $\Leftrightarrow$ iv).

Remark (2.8). The conditions i)—iv) of the above lemma are also equivalent with the following one

$$\mathrm{Ext}^i_A(A/(x), E) = 0$$

for each $i > 0$. Thus if $m = (x)$ (this is not usually our case) then E satisfies this condition only if it is injective.

Proposition (2.9). *Let $x = (x_1, \ldots, x_s)$ be a system of elements from m. Suppose that for each j, $1 \leq j \leq s$, $x_j^2 = 0$ and*

$$((x_1, \ldots, x_{j-1}) : x_j) = (x_1, \ldots, x_j).$$

Then $\dim_K A = 2^s n$, $n := \dim_K A/(x)$ *and the closed points* W *of* $M(A;r,2^s d)_{red}$, $r \le d \le rn$ *such that for each* j, $1 \le j \le s$

$$((x_1,\dots,x_{j-1})W : x_j)_W = (x_1,\dots,x_j)W$$

form a locally closed subset of $M(A;r,2^s d)_{red}$ *(we denote it by* $\mathbf{Lift}(A;x,r,2^s d)$*).*

Proof. Let f_i be the scalar multiplication of $V = A^r$ by $x_1\dots x_i$, $1 \le i \le s$. By induction on s we obtain $\dim_K A = 2^s \dim_K A/(x)$, the case $s = 1$ being done in (2.5). Using Lemma (2.4) we note that the closed points W of $\mathrm{Gr}(V,2^s d)$ such that $\dim_K f_i(W) = 2^{s-i}(rn-d)$ form a locally closed set Z_i of $\mathrm{Gr}(V,2^s d)$. Then $Z' = \cap_{i=1}^s Z_i \cap \mathrm{M}(A;r,2^s d)_{red}$ is a locally closed subset. If W is a closed point of Z' then $x_i W \subset W$ because $W \in \mathrm{M}(A;r,2^s d)_{red}$ and the complexes

1) $$0 \to x_1\dots x_j W \to x_1\dots x_{j-1}W \xrightarrow{x_j} x_1\dots x_j W \to 0,$$

$1 \le j \le s$ are exact because

$$\dim_K x_1\dots x_{j-1}W = 2^{s-j+1}(rn-d) = 2\dim_K x_1\dots x_j W.$$

Thus

2) $$x_1\dots x_{j-1}W \cap (0:x_j)_W = x_1\dots x_j W$$

for each $1 \le j \le s$ and it follows $W \in \mathbf{Lift}(A;x,r,2^s d)$ by Lemma (2.7). Conversely if $W \in \mathbf{Lift}(A;x,r,2^s d)$ is a closed point then W satisfies 2) by Lemma (2.7) and thus the sequences 1) are exact. It follows

$$\dim_K x_1\dots x_{j-1}W = 2\dim_K x_1\dots x_j W.$$

Hence $\dim_K f_i(W) = 2^{s-i}(rn-d)$ for all i, i.e. $W \in Z'$. Consequently $\mathbf{Lift}(A;x,r,2^s d) = Z'$ is locally closed.

Theorem (2.10). *Conserving notations and hypothesis of Lemma (2.2) and Proposition (2.9) the following statements hold:*

i) $\mathbf{Lift}(A;x,r,2^s d)$ *is a quasi projective variety on which* G *acts regularly by restriction,*

ii) A closed point $U \in M(A;r,2^s d)_{red}$ *is in* $\mathbf{Lift}(A;x,r,2^s d)$ *if and only if* $U \in Lift(A,A/(x))$*,*

iii) the assignment $U \to A^r/U$ *defines a surjective map* $\rho_{x,r,2^s d}$ *from the set of closed points of* $\mathbf{Lift}(A;x,r,2^s d)$ *onto the set of isomorphism classes of all modules* M *of* $Lift(A,A/(x))$ *with* $\mu(M) = r$, $\dim_K M = 2^s d$,

iv) *two closed points $U, U' \in$ **Lift**$(A; x, r, 2^s d)$ are in the same G-orbit if and only if $\rho_{x,r,2^s d}(U) = \rho_{x,r,2^s d}(U')$.*

Proof. ii) follows from the equivalence i), ii) of Corollary (1.14). Let $g \in G$ and $U \in$ **Lift**$(A; x, r, 2^s d)$. Then $U \in \text{Lift}(A, A/(x)$ by ii). As g is an A-isomorphism we have also $g(U) \in \text{Lift}(A, A/(x))$ and thus $g(U) \in$ **Lift**$(A; x, r, 2^s d)$ by ii). Hence G acts by restriction on **Lift**$(A; x, r, 2^s d)$. Consequently i) holds with the help of Proposition (2.9).

iii) If $U \in$ **Lift**$(A; x, r, 2^s d)$ then $U \in \text{Lift}(A, A/(x))$ by ii) and thus by (1.13) $A^r/U \in \text{Lift}(A, A/(x))$ because $U = \Omega^1_A(A^r/U)$ ($U \subset mA^r$!), i.e. $\rho_{x,r,2^s d}$is well defined. If $M \in \text{Lift}(A, A/(x))$ satisfies $\mu(M) = r$ and $\dim_K M = 2^s d$ then $V := \Omega^1_A(M) \in \text{Lift}(A, A/(x))$ by (1.13) and $V \subset mA^r$ is of colength $2^s d$ in A^r. Thus $V \in$ **Lift**$(A; x, r, 2^s d)$ by ii) and $\rho_{x,r,2^s d}(V)$ is exactly the isomorphism class of M. Finally iv) follows from Lemma (2.2).

Corollary (2.11). *In the notations and hypothesis of Theorem (1.21) the above $\rho_{x,r,2^s d}$ induces a bijection between the isomorphism classes of modules $L \in MCM(R)$ with $\mu(L) = r$ and $\dim_K R_2 \otimes_R L = 2^s d$ and the $G_r := GL(r, R_2)$-orbits of the quasi projective variety* **Lift**$(A; x, r, 2^s d)$.

Corollary (2.12). *In the notations and hypothesis of Theorem (1.21) suppose that R is a domain. Then the following statements hold:*

i) *for each maximal Cohen-Macaulay R-module M*

$$\dim_K R_2 \otimes_R M = u \ \text{rank}_R M$$

holds, where $R_2 := R/(x_1^2, \ldots, x_s^2)$ and $u := \dim_K R_2 \in 2^s \mathbf{N}$,

ii) *$\rho_{x,r,uq}$, $1 \le q \le r$ induces a bijection between the isomorphism classes of modules $M \in MCM(R)$ with $\mu(M) = r$, $\text{rank}_R M = q$ and the G_r-orbits of the quasi projective variety* $\mathbf{MCM}(R; x, r, q) :=$ **Lift**$(A; x, r, 2^s d)$.

Proof. ii) follows from (2.11) using i). By [3] (4.6.11) we have

$$l_R(R_2 \otimes_R M) = \ l_R(R_2) \ \text{rank}_R M$$

($l_R(N)$ denotes the length of the R-module N). Thus it is enough to note that $l_R(N) = \dim_K N$ for each R-module N of finite length. This equality holds by additivity because each such module N has a chain $0 = N_0 \subset N_1 \subset \ldots \subset N_e = N$ with $N_{i+1}/N_i \cong K$, $i \ge 0$.

We end our paper with two examples. The first one is of finite Cohen-Macaulay type and thus our quasi projective varieties have just finite orbits. This does not happen in the second one which is a strictly unimodular singularity (see [13]).

Example (2.13). Let k be an algebraically closed field of characteristic zero and A the complete local ring of the simple plane curve singularity $\mathbf{A_2}$, i.e. $A = k[[X,Y]]/(X^2 + Y^3)$. Then $X\ \mathrm{Ext}^1_A(L,-) = 0$ for every $L \in \mathrm{MCM}(A)$ (see [12] Ch. 6, or [4](4.2)) and so the element x given by X^2 in A generates a reduction ideal (see [12] (6.16),(6.18)). By Theorem (1.21) F_2 defines a bijection between the isomorphism classes of modules of $\mathrm{MCM}(A)$ and the isomorphism classes of finitely generated $A_2 = A/(X^4)$-modules M satisfying $((0) : X^2)_M = X^2M$. The indecomposable modules of $\mathrm{MCM}(A)$ are isomorphic with A and $L := (X,Y)A$ by [12](5.11). Clearly, $\dim_k A_2 = 12$, $\mu(L) = 2$ and $\dim_k A_2 \otimes_A L = 12$ since $\mathrm{rank}_A L = 1$. Thus $\mathbf{MCM}(A; x, 2, 1)$ contains just the $\mathrm{GL}(2, A_2)$-orbit of $\Omega^1_{A_2}(A_2 \otimes_A L) \cong A_2 \otimes_A L$ in $\mathrm{M}(A_2; 2, 12)$ and $\mathbf{MCM}(A; x, 2, 2)$ contains just (0) corresponding to the isomorphism class of A_2^2.

Example (2.14). Let B be the complete local ring of the plane curve singularity $\mathbf{W_{12}}$, i.e. $B = k[[X,Y]]/(X^4+Y^5)$ (see [13]). Then $X^3\mathrm{Ext}^1_B(L,-) = 0$ for every $L \in \mathrm{MCM}(B)$ because X^3 belongs to the Noetherian different $\mathcal{N}^B_{k[[Y]]}$(see [12](6.6)) as it is easy to check ($\mathrm{char}\,k = 0$!). Thus the element x given by X^6 in B generates a reduction ideal. By Theorem (1.21) F_2 defines a bijection between the isomorphism classes of $\mathrm{MCM}(B)$ and the isomorphism classes of finitely generated $B_2 = B/X^{12}B$-modules M satisfying $(0 : X^6)_M = X^6M$. Let $I_\lambda = (X + \lambda Y, Y^2)$, $\lambda \in k$. We have $f = x_{\lambda 1}y_{\lambda 1} + x_2 y_{\lambda 2} \in I_\lambda^2$ for $x_{\lambda 1} = X + \lambda Y$, $x_2 = Y^2$, $y_{\lambda 1} = X(X^2 - \lambda XY - 2\lambda^2 Y^2)$, $y_{\lambda 2} = 3\lambda^2 X^2 + 2\lambda^3 XY + Y^3$. Using [2] or [12](8.11)—(8.14) we see that the modules L_λ generated in B^4 by the elements $(0, x_{\lambda 1}, x_2, 0)$, $(y_{\lambda 1}, 0, 0, -x_2)$, $(y_{\lambda 2}, 0, 0, x_{\lambda 1})$, $(0, -y_{\lambda 2}, y_{\lambda 1}, 0)$ are nonisomorphic maximal Cohen-Macaulay B-modules with $\mu(L_\lambda) = 4$, $\mathrm{rank}_B L_\lambda = 2$ and such that $\Omega^1_B(L_\lambda) \cong L_\lambda$. Thus $\mathbf{MCM}(B; x, 4, 2)$ contains an infinite set of orbits.

References

[1] M.Auslander, S.Ding, O.Solberg, Liftings and Weak Liftings of Modules, J.of Algebra, 156(1993),273-317.

[2] R.-O.Buchweitz, G.-M.Greuel, F.-O.Schreyer, Cohen- Macaulay modules on hypersurface singularities II, Invent. Math., 88(1987)165-182.

[3] W.Bruns, J.Herzog,"Cohen-Macaulay rings", Cambridge University Press,1993.

[4] M.Cipu, J.Herzog, D.Popescu, Indecomposable generalized Cohen-Macaulay modules, Trans Amer. Math. Soc., to appear.

[5] E.Dieterich, Reduction of isolated singularities, Comment. Math. Helvetici 62(1987), 654-676.

[6] S.Ding, O.Solberg, The Maranda Theorem and Liftings of Modules, Communications in Alg., 21(1993),1161-1187.

[7] G.-M.Greuel, G.Pfister, Moduli spaces for torsion free modules on curve singularities I, J. of Algebraic Geometry 2(1993),81-135.

[8] J.E. Humphreys, "Linear Algebraic Groups" , Springer Graduate Texts in Math., Berlin-Heidelberg-New York, 1981.

[9] H. Knörrer, Torsionsfreie Moduln bei Deformation von Kurvensingularitäten, in: "Singularities, Representation of algebras and Vector bundles", Springer Lect. Notes in Math. 1273(1987),150-155.

[10] D.Popescu, Indecomposable Cohen-Macaulay modules and their multiplicities, Trans. Amer.Math.Soc. 323(1991),369-387.

[11] D.Popescu, M. Roczen, Indecomposable Cohen-Macaulay modules and irreducible maps, Compositio Math. 76(1990),277-294.

[12] Y. Yoshino, "Cohen–Macaulay Modules over Cohen–Macaulay Rings", London Math. Soc. Lect. Note Ser.,146,Cambridge, 1990.

[13] C.T.C Wall, Classification of Unimodal Isolated Singularities of Complete Intersections, in Proceedings of Symp. in Pure Mathematics, 40 (1983),625-640.