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In general, k is assumed to be an algebraically closed field of characteristic 0.

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1 Introduction

This article aims at investigating the relationship between a finitely generated commutative algebra A with affine variety X_A and the Lie algebra of all k-derivations of A denoted by $\mathrm{Der}(A)$. Translating this problem into the language of geometry means relating the behaviour of the variety X_A to that of the Lie algebra of all vector fields having polynomial coefficients with respect to a fixed embedding in an affine space.

Besides a general interest in studying this class of infinite-dimensional Lie algebras also in the algebraic case (see [11] Introduction 0.2), there is still another motivation to consider these types of Lie algebras derived from our investigation of finite-dimensional Lie algebras of the type

$$\mathrm{Der}(\mathcal{O}_X)/\mathrm{Der}_h(\mathcal{O}_X)$$

for a quasihomogeneous isolated complete intersection singularity (icis) $(X,0)$ with local algebra $\mathcal{O}_X$ (see [12], [3]). Here $\mathrm{Der}_h(\mathcal{O}_X)$ denotes the Lie subalgebra of the Hamiltonian or trivial derivations. In their paper [3] Laudal and Bjahr claim that this finite-dimensional Lie algebra is just the right object for determining $(X,0)$ up to isomorphism. Unfortunately, this is definitely wrong for low multiplicities because of a counterexample of a one-parametric family of hypersurface singularities of degree 4, which yields a trivial deformation of the Lie algebras (cf. [3]).

Thus, the question arises whether the Lie algebra of germs of vector fields $\mathrm{Der}(\mathcal{O}_X)$ or of vector fields with polynomial coefficients $\mathrm{Der}(A_X)$ on the variety X associated to A_X determines the germ $(X,0)$ or the algebraic variety X. A very powerful basis for handling this problem is an article of J. Grabowski ([4]), which continues works of I. Amemiya [2] and H. Omori [16] and develops a purely algebraic technique to describe these kinds of Lie algebras.

Here we can present the following main result:

Two normal affine varieties X and X' with rings of coordinates A_X and $A_{X'}$ are isomorphic if and only if the Lie algebras of k-derivations $\mathrm{Der}(A_X)$ and $\mathrm{Der}(A_{X'})$ are so.

To prove this, we try to analyse the algebraic structures occuring in an object of type $\mathrm{Der}(A_X)$ and to understand their interactions. Indeed, we obtain a more general result of this kind, which is valid for several classes of varieties and Lie subalgebras of $\mathrm{Der}(A_X)$. But, nevertheless, we are far from having a complete algebraic theory about Lie algebras of derivations — one main problem is the lack of a description of all integral subvarieties for an arbitrary Lie subalgebra of $\mathrm{Der}(A_X)$.

As a further main result we deduce:

An affine variety X with the ring of coordinates A_X is smooth if and only if the Lie algebra $\mathrm{Der}(A_X)$ is simple.

The fact that the smoothness of X implies the simplicity of $\mathrm{Der}(A_X)$ was proved by D.A. Jordan [10]. Here we introduce a trick which shows that the Lie subalgebra of Hamiltonian vector fields is always a Lie ideal, and the singular locus of the variety X must be an integral variety. From this we conclude that Lie algebras of derivations on singular varieties admit Lie ideals in any case.

The key idea of the whole approach is to relate Lie subalgebras to certain geometric subobjects of the associated variety or germ-like points, subvarieties or germs of subvarieties and to try and describe this class of Lie subalgebras by purely Lie theoretic properties. This was first developed by Purcell and Shanks ([18]) and then also successfully applied by J. Grabowski ([4]), I. Amemiya ([2]), K. Masuda ([14]), H. Omori ([16]) and H. Hauser and G. Müller ([8], [9]).

Furthermore, the Lie algebras of derivations are interesting from a computational aspect since the close interaction of the Lie structure and the commutative structure allow Gröbner algorithm-based programmes to calculate inside these Lie algebras. Especially in the case of Lie algebras of derivations on non-graded k-algebras no general structure theorem is known and therefore, Computeralgebra is a very powerful tool to study these kind of Lie algebras. Here we use the system SINGULAR to calculate the module structure for several examples of Lie algebras of derivations and for their Lie subalgebra (and submodule) of Hamiltonian derivations.

2 Integral varieties and Lie ideals in modular Lie algebras

In this section we aim at finding interactions between subobjects of the k–algebra A reflecting certain geometric properties of the variety associated to A and subobjects of a Lie algebra of derivations on A. The approach chosen here is mainly based on the study of the vanishing locus of Lie algebras of derivations (viewed as vector fields) and Lie algebras of derivations vanishing on certain loci of the variety. This seems to work quite well for the investigation of Lie algebras on abstract algebraic varieties. By comparison H. Hauser and G. Müller use, in their paper [8], the Lie subalgebras tangent to subvarieties of the given one for the study of Lie algebras of vector fields tangent to embedded varieties.

Let A denote a finitely generated k–algebra without zero divisors which is different from k. If we put more emphasis on the geometrical situation, we will also use the notation A_X, where X stands for the variety associated to A_X.

Definition 2.1 *A derivation of A is a k-linear map:*

$$\delta : A \longrightarrow A$$

which fulfils the Leibnitz equation $\delta(fg) = \delta(f)g + f\delta(g)$ for $f, g \in A$.

The set $\mathrm{Der}(A)$ of all derivations on A becomes a Lie algebra with the usual Lie bracket

$$[\delta, \theta] = \delta \circ \theta - \theta \circ \delta$$

and admits also naturally an A–module structure.

A Lie algebra is always understood to be a Lie subalgebra of some object $\mathrm{Der}(A)$ and so the reference to derivations is usually omitted.

First we fix all algebraic structures on $\mathrm{Der}(A)$ and their interactions for defining a category in which these structures and relations are universal. This yields the category of the modular Lie algebras of derivations.

Definition 2.2 *A modular Lie algebra is a Lie subalgebra of* $\mathrm{Der}(A)$ *where A is some commutative, finitely generated k-algebra with no zero divisors, which is a submodule of* $\mathrm{Der}(A)$ *simultaneously.*

Remark: A modular Lie algebra is always thought to be paired with its algebra A, but, if no confusion is possible, this algebra will not be mentioned explicitly.

Fixing a set of generators $x_1, \ldots, x_n$ of A one has a surjection $\phi : A_n \rightarrow A$ of the polynomial algebra A_n in n indeterminants onto A. Then all those derivations of A_n that stabilize the kernel of ϕ yield derivations on A. The embedding of $\mathrm{Der}_{ker(\phi)}(A_n) := \{\delta \in \mathrm{Der}(A_n) | \delta(ker(\phi)) \subset ker(\phi)\}$ into $\mathrm{Der}(A_n) = (A_n)^n$ induces now an injection

$$\mathrm{Der}_{ker(\phi)}(A_n) \otimes_{A_n} A \simeq \mathrm{Der}(A) \longrightarrow A^n.$$

Hence, modular Lie algebras are always torsion-free modules. The generators of the module A^n will also be denoted by the signs ∂_i of the partial derivations on A_n.

Definition 2.3 *A morphism of two modular Lie algebras $(A, \mathcal{L})$ and $(A', \mathcal{L}')$ is a pair of homomorphisms*

$$(\mu, \nu) : (A, \mathcal{L}) \longrightarrow (A', \mathcal{L}'),$$

where μ denotes a homomorphism of the underlying commutative algebras and ν a module homomorphism over μ that is a homomorphism of the Lie algebras simultaneously.

In every modular Lie algebra $\mathcal{L}$ over A there holds a universal equation describing the interaction between the different structures:

$$[\delta, f\vartheta] = \delta(f)\vartheta + f\,[\delta, \vartheta] \quad \forall f \in A, \forall \delta, \vartheta \in \mathcal{L}. \tag{1}$$

Some computation shows that this equation together with all necessary structures just determines the category of modular Lie algebras of derivations ([19] page 14). This means the property of being a derivation is included in the structure of equation (1).

Definition 2.4 *A Lie ideal of a modular Lie algebra which is also a submodule is called a modular Lie ideal.*

One very basic fact about the behaviour of modular Lie algebras is given by a lemma according to Amemiya:

Lemma 2.5 *(see [2] p. 547) Let $\mathcal{L}$ be a modular Lie algebra over A, $L \subset \mathcal{L}$ a Lie subalgebra, and denote by $L^{[1]} = \{\delta \in L | [\delta, \mathcal{L}] \subset L\}$ the set of all elements of L mapping $\mathcal{L}$ into L via the associated adjoint operator. Assume now that $\delta \in L^{[1]}$ and $f\delta \in L^{[1]}$ for some $f \in A$, then*

$$(\delta(f))^2 \in I_L := \{f \in A | f\mathcal{L} \subset L\},$$

which is the ideal of those elements of A multiplying $\mathcal{L}$ into L.

As a first consequence we conclude

Lemma 2.6 *Every non-vanishing Lie ideal of a modular Lie algebra contains a non-vanishing modular Lie ideal.*

Proof: Assume $L \subset \mathcal{L}$ to be a non-vanishing Lie ideal in the modular Lie algebra $\mathcal{L}$ over the algebra A and $\delta \in L$ an arbitrary element. Then we deduce:

$$\delta(f)\delta = [\delta, f\delta] \in L \quad \forall f \in A.$$

Now Amemiya's lemma states that $(\delta(\delta(f)))^2$ lies in the ideal I_L. Thus, if there is a non-vanishing $(\delta(\delta(f)))^2 \in A$, the Lie ideal L contains a non-trivial submodule of $\mathcal{L}$. Clearly, the maximal submodule L^m of a Lie ideal L is again a Lie ideal: assume $\delta \in L^m$. Then the structure equation 1 gives

$$f[\theta, \delta] = [\theta, f\delta] - \theta(f)\delta \in L \quad \forall f \in A, \forall\theta \in \mathcal{L},$$

which means that $[\theta, \delta] \in L^m \ \forall\theta \in \mathcal{L}$. Thus, L^m is a Lie ideal.

Hence, it remains to find a nonzero $(\delta(\delta(f)))^2$. Taking $f = x_i$, one of a fixed set of generators of the algebra A, we obtain

$$\delta^2(f) = \delta(g_i),$$

where g_i is the coefficient in a representation of δ in partial derivatives corresponding to these fixed generators of A. However, if δ annihilates g_i, we can take $f = x_i^2$ and obtain

$$\delta^2(f) = g_i^2,$$

which cannot vanish for all x_i.

Remark: The lemma implies that for a modular Lie algebra it is the same to be simple in the category of Lie algebras or in the category of modular Lie algebras. Furthermore, understanding the derivations as vector fields on the variety associated to A, any Lie ideal of a modular Lie algebra has to contain a vector field parallel to an arbitrarily chosen one of this modular Lie algebra.

Now we try to associate commutative objects, i.e. subobjects of the ring of coordinates A, to certain subobjects of our modular Lie algebra $\mathcal{L}$ over A.

Definition 2.7 *Let the k-algebra A be the ring of coordinates of some affine variety. Then an ideal i of A is said to be integral with respect to the Lie algebra $\mathcal{L}$ of derivations on A if it is stable under the action of $\mathcal{L}$ on A:*

$$\mathcal{L}(i) \subset i.$$

The factor representation of $\mathcal{L}$ on A/i is then, modulo the kernel, the usual representation of the Lie algebra restricted to the integral subset $V(i)$, i.e. the set of zeros of i. This justifies the notation of an integral ideal because one gets a one-to-one correspondence between integral ideals and closed, algebraic, integral (in the sense of differential geometry) subschemes of our variety.

Let us denote by $I_{\mathcal{L}}A$ the set of all integral ideals in A with respect to $\mathcal{L}$. One checks easily that this set is closed under addition, multiplication, taking intersections or radicals of ideals. A. Seidenberg has proved a more general statement:

Theorem 1 *([17] Theorem 1) Let $I \subset A$ be an integral ideal w.r.t. a modular Lie algebra $\mathcal{L}$ on A with associated primes $\mathbf{p}_1, \ldots, \mathbf{p}_s$. Then all these associated primes are integral and there are integral primary ideals $\mathbf{q}_1, \ldots, \mathbf{q}_s$ with*

$$I = \mathbf{q}_1 \cap \ldots \cap \mathbf{q}_s.$$

We have two maps describing the relationship between modular Lie algebras and integral ideals (see [10]). One of them sends the set of modular Lie ideals of a modular Lie algebra into the set of integral ideals by associating the image of its action on A to a modular Lie ideal L

$$int : L \longrightarrow L(A).$$

Reversely, the other one assigns the modular Lie ideal containing all those derivations that map A into i to an integral ideal i.

$$lid : i \longrightarrow \mathcal{L}_i := \mathcal{L} \cap \mathrm{Der}_k(A, i).$$

In both cases the general structure equation (1) ensures that the mappings are well-defined.

In general, these maps are neither the inverse of the other nor injective nor surjective.

There are some important examples of modular Lie algebras and of their substructures, which explain a little bit the relation between the Lie and the commutative structure:

1. The most important subobject of a modular Lie algebra on a variety is the modular Lie ideal of Hamiltonians. It is defined as follows:
   Write the k-algebra A_X of an r-dimensional, affine variety X as the factor $A_n/(f_1, \ldots, f_p)$ of a polynomial algebra in n indeterminants by an ideal generated by the polynomials $f_1, \ldots, f_p \in A_n$. Then, for any $q = n - r$ polynomials $f_{i_1}, \ldots, f_{i_q}$ and $q+1$ variables $x_{j_1}, \ldots, x_{j_{q+1}}$ the deteminant of the formal matrix

$$H_{\underline{i},\underline{j}} = \begin{vmatrix} \frac{\partial f_{i_1}}{\partial x_{j_1}} & \cdots & \frac{\partial f_{i_1}}{\partial x_{j_{q+1}}} \\ \vdots & & \vdots \\ \frac{\partial f_{i_q}}{\partial x_{j_1}} & \cdots & \frac{\partial f_{i_q}}{\partial x_{j_{q+1}}} \\ \frac{\partial}{\partial x_{j_1}} & \cdots & \frac{\partial}{\partial x_{j_{q+1}}} \end{vmatrix}$$

   defines a derivation on A_n for which $I(X) = (f_1, \ldots, f_p)$ is integral since every $q+1$-minor of the Jacobian matrix must vanish on X. Thus, the A_n-module generated by all these determinants stabilizes $I(X)$ and gives rise to an A_X-submodule Hamilton(X) of Der(A_X). By some computation it becomes obvious that the choice of other generators of A_X or other polynomials f_i splits the fixed Hamiltonians into sums of Hamiltonians w.r.t. the new variables and polynomials (see [19] or [10]). Hence, Hamilton(X) is independent of the choices of the variables and the f_i.

   Now we want to prove:

   Lemma 2.8 Hamilton(X) *is a Lie ideal of* Der(A_X).

   Proof: Let us fix some $H = H_{\underline{i},\underline{j}}$ and define $H_k(\ldots)$ (resp. $H^k(\ldots)$) to be the determinant of the matrix H, where the k-th row (resp. column) of H is replaced by the arguments in the brackets. It suffices to show that

$$[H, \delta] \in \mathrm{Hamilton}(X) \;\; \forall \delta \in \mathrm{Der}(A_X)$$

   for an arbitrary generator H because the structure equation (1) then ensures this relation for all elements from Hamilton(X). By fixing a set of generators of A_X we choose an embedding of X into some affine space and consider the generators H as derivations on A_n representing the corresponding generators of Hamilton(X). We will show that $[H, \delta]$ acts on a polynomial

$f \in A_n$ as a sum of Hamiltonian and trivial (on X) derivations. As usual, denote by ∂_i the partial derivations. If $\delta = \sum g_j \partial_j$, we obtain:

$$\delta(\partial_i(f)) = \partial_i(\delta(f)) - \partial_i(\delta)(f), \tag{2}$$

where $\partial_i(\delta)$ denotes the derivation $\sum \partial_i(g_j)\partial_j$ with derived coefficients.

This gives:

$$[H,\delta](f) = H(\delta(f)) - \delta(H(f)) = H(\delta(f)) - \sum_{k=1}^{q+1} H^k(\delta(\partial_{j_k}(f_l))),$$

where $f_l = f_{i_l}$ for $l = 1, \dots, q$ and $f_{q+1} = f$. By (2) the second summand splits into

$$\sum_{k=1}^{q+1} H^k(\partial_{j_k}(\delta(f_l))) - \sum_{k=1}^{q+1} H^k(\partial_{j_k}(\delta)(f_l)).$$

Since $\partial_{j_k}(\delta) = \sum_m \partial_{j_k}(g_m)\partial_m$ is a constant derivation for every column k, the second summand here is a Hamiltonian derivation. The first sum can be written as

$$\sum_{k=1}^{q+1} H^k(\partial_{j_k}(\delta(f_l))) = \sum_{k'=1}^{q+1} H_{k'}(\partial_{j_k}(\delta(f_{k'})))$$

because the sums are taken over all columns and rows, respectively. Now δ represents a derivation of A_X and, therefore, $\delta(f_{k'}) = \sum r_s f_s \in I(X)$ is constant for $f_{k'} \in I(X)$ in every row. Thus, the summands for $k' = 1, \dots, q$ are either zero on X or Hamiltonians. Moreover, the summand for $k' = q+1$ is just $H(\delta(f))$ and, hence, deletes the first summand of the Lie bracket $[H, \delta]$. This completes the proof.

One can further prove that the modular Lie ideal Hamilton(X) spans the whole tangent space in every smooth point P of X, i.e. it contains $r = \dim X$ vector fields linearily independent in P (cf. [8] p. 315).

2. Let A_n be the algebra of polynomials in n indeterminants and $\mathcal{L}$ a modular Lie algebra over A_n. Let further $I = (f_1, \dots, f_p)$ be an integral ideal of A_n generated by the f_i. Then we can apply the same arguments as for the Hamiltonians to the ideal $\mathfrak{b}_r \subset A_n$ generated by all r–minors of the Jacobian and the f_i. We conclude that $\mathfrak{b}_r$ is again an integral ideal of A with respect to $\mathcal{L}$. This implies that after restricting to $X = V(I)$ the singular locus *Sing* X is defined by an integral ideal (see [19]).

3. Let A_X be the ring of coordinates of an affine variety as in the first example and assume that X is the quasicone over a smooth quasiprojective variety. In other words, X is the expansion with the exponential map of some Euler vector field $\delta_E = \sum_{i=1}^n w_i x_i \partial/\partial x_i$ of the germ of a quasihomogeneous isolated complete intersection singularity. A structure theorem proved by Aleksandov (cf. [1]) and Kersken (cf. [21]) independently states that, under these circumstances, the A_X–module Der(A_X) is generated by the Euler vector field δ_E and the Hamiltonians.

   In general, one can calculate the module of derivations of some finitely generated k–algebra as syzygies of the module generated by the columns of the Jacobian matrix over this algebra. For the easiest non–quasihomogeneous plane hypersurface singularity $f = x^5 + y^5 + x^3y^3$ in $\mathbf{C}^2$ this provides a submodule of $(\mathbf{C}[x,y])^2$ generated by the elements

$$\delta_1 = (6x^2y^2 - 25xy)\partial_x + (9xy^3 + 5x^3 - 25y^2)\partial_y$$

and

$$\delta_2 = (9x^3y + 5y^3 - 25x^2)\partial_x + (6x^2y^2 - 25xy)\partial_y.$$

In distinction to the quasihomogeneous case we observe that the single Hamiltonian

$$H = (5y^4 + 3x^3y^2)\partial_x - (5x^4 + 3x^2y^3)\partial_y$$

satisfies the relation $H = y\delta_2 - x\delta_1$. Hence, the Hamiltonians can be generated by non–Hamiltonians, which is definitively false for quasihomogeneous singularities.

4. If $\mathcal{L}$ is a modular Lie algebra over A and M a subset of $\mathcal{L}$, we set

$$I(M) := AM(A)$$

and call $V(M) := V(I(M))$ the vanishing locus of the set M. If A is the coordinate ring of some variety X, this is indeed the whole set of points where all vector fields of M vanish. The map *int* described above maps $\mathcal{L}$ to $I(\mathcal{L})$, which is therefore an integral ideal, and, further $I(L) = int(L)$ is an integral ideal for every modular Lie ideal L in $\mathcal{L}$.

From the discussion of Example 2 we deduce one of the main results:

Proposition 2.9 *An affine variety X with a ring of coordinates A_X is smooth if and only if the Lie algebra* $\mathrm{Der}(A_X)$ *is simple.*

Proof: To show that simplicity implies smoothness we take a sufficiently high power $I(Sing\ X)^n$ of the ideal defining the singular locus. Since this ideal is integral, $\mathrm{Der}(A_X, I(Sing\ X)^n)$, the set of all vector fields vanishing with order n on $Sing\ X$ defines a proper Lie ideal, which contradicts the simplicity of $\mathrm{Der}(A_X)$.

The other implication was proved by D.A. Jordan [10] for the first time. Here we give a slightly different proof, showing that this statement is a consequence of Lemma 2.6:
Let $\mathrm{Der}(A_X)$ contain a proper Lie ideal L. Then we can assume by Lemma 2.6 that L is a modular ideal and therefore A_X includes the integral ideal $L(A_X)$. It is easy to see that $L(A_X)$ must be a proper ideal, because otherwise there would exist derivations $\delta_i \in \mathrm{Der}(A_X)$ and regular functions $f_i \in A_X$ with $1 = \sum_i \delta_i(f_i)$. However, then

$$\theta = \sum_i \delta_i(f_i)\theta = \sum_i [\delta_i, f_i\theta] - f_i\,[\delta_i, \theta] \in L \quad \forall \theta \in \mathrm{Der}(A_X).$$

Hence, X admits an integral subvariety Y with respect to $\mathrm{Der}(A_X)$. But the discussion on Hamiltonians above shows that an integral subvariety of $\mathrm{Der}(A_X)$ in X cannot contain any smooth point $P \in X$, because $\mathrm{Der}(A_X)$ spans the whole dim X–dimensional tangent space of X at P. From the existence of Y we see that X has to be singular.

Now we introduce a further structure needed for the comparison of the commutative and Lie properties of our objects.

Definition 2.10 *Let L be a Lie subalgebra of a modular Lie algebra $\mathcal{L}$. Then we denote the maximal Lie ideal of $\mathcal{L}$ inside L by $L^{[\infty]}$.*

Remark: This maximal Lie ideal can be constructed by the so–called transporter series, which occur in the work of Guillemin and Sternberg [5], [6], [7] as well as in the articles of Omori [16], H. Hauser and G. Müller [8]: Define $L^{[0]} := L$ and, recursively,

$$L^{[i+1]} := \{\delta \in L^{[i]} \,|\, [\delta, \mathcal{L}] \subset L^{[i]}\}.$$

Then, by

$$L^{[\infty]} := \cap_{i \in \mathbf{N}} L^{[i]}$$

we obtain our maximal ideal of $\mathcal{L}$ inside L (see [5] Proposition 2.4). Moreover, if L is a modular Lie subalgebra, then, by the structure equation (1), all $L^{[i]}$ are A–modules, and so is $L^{[\infty]}$.

Furthermore, we introduce the notion of maximal integral ideals.

Definition 2.11 *Let a prime ideal* $\mathbf{p} \subset A$ *define a subvariety* $Y \subset X$ *and* $\mathcal{L}$ *be a modular Lie algebra over* A*, then*

$$\mathbf{i}_{\mathbf{p}}^{\mathcal{L}} := \max_{\mathbf{i} \in I_{\mathcal{L}} A} \{\mathbf{i} \subset \mathbf{p}\}$$

is the maximal integral ideal in $\mathbf{p}$ *and, thus, defines the minimal integral subvariety* Z *containing* Y*.*

Remark: The ideal $\mathbf{i}_{\mathbf{p}}^{\mathcal{L}}$ can be constructed from $\mathbf{p}$ by

$$\mathbf{i}_{\mathbf{p}}^{\mathcal{L}} = \{f \in \mathbf{p} \mid \delta_1 \circ \ldots \circ \delta_t(f) \in \mathbf{p};\ \forall \delta_i \in \mathcal{L},\ \forall t \in \mathbf{N}\}.$$

It is obvious that this set contains $\mathbf{i}_{\mathbf{p}}^{\mathcal{L}}$, but it is also an ideal by the Leibnitz rule. Further one checks immediately that $\mathbf{i}_{\mathbf{p}}^{\mathcal{L}}$ is again a prime ideal. This means that the minimal integral subvariety containing an irreducible variety is itself irreducible.

Let us denote by $\mathcal{L}_{\mathbf{p}}$ the intersection of the two modular Lie algebras $\mathcal{L}$ and $\mathrm{Der}_k(A, \mathbf{p})$ over A, which is exactly the modular Lie subalgebra of $\mathcal{L}$ of vector fields vanishing on the subvariety $V(\mathbf{p})$. Then we obtain the following relationship of maximal Lie ideals in a Lie subalgebra of type $\mathcal{L}_{\mathbf{p}}$ to minimal integral subvarieties over given subvarieties in the following way:

Proposition 2.12 *Let* $Y \subset X$ *be an arbitrary subvariety defined by a prime ideal* $\mathbf{p} \subset A_X$*. Then the maximal Lie ideal* $(\mathcal{L}_{\mathbf{p}})^{[\infty]}$ *of* $\mathcal{L}$ *in the Lie subalgebra* $\mathcal{L}_{\mathbf{p}}$ *is exactly the Lie ideal related to the minimal integral subvariety* Z *containing* Y*:*

$$(\mathcal{L}_{\mathbf{p}})^{[\infty]} = \mathcal{L}_{\mathbf{i}_{\mathbf{p}}^{\mathcal{L}}} = \mathcal{L}_{I(Z)}.$$

Proof: Set $\mathbf{M} = \mathcal{L}_{\mathbf{i}_{\mathbf{p}}^{\mathcal{L}}}$. Then, clearly, $\mathbf{M} \subset (\mathcal{L}_{\mathbf{p}})^{[\infty]}$ because it is a Lie ideal.

Now we take a $\delta \in (\mathcal{L}_{\mathbf{p}})^{[\infty]}$ with a representation $\delta = \sum_{i=1}^{n} h_i \partial_i$ induced by an embedding of X into a suitable affine space and want to show that $\delta \in \mathbf{M}$. We see easily that δ is in $\mathbf{M}$ if and only if $h_i \in \mathbf{i}_{\mathbf{p}}^{\mathcal{L}}$ for all $i = 1, \ldots, n$. We will show by induction on the number t of derivations that:

- $\forall h_i :\ \delta_1 \circ \ldots \circ \delta_t(h_i) \in \mathbf{p}\ \ \forall t \in \mathbf{N}\ \ \forall \delta_j \in \mathcal{L}$
- If $\theta = [\delta_1 [\ldots [\delta_t, \delta] \ldots]] = \sum_i g_i \partial_i$, then the g_i are sums of the type:

$$g_i = \delta_1 \circ \ldots \circ \delta_t(h_i) + \sum_{\substack{\tau \subset \{1,\ldots,t\} \\ \tau \neq \{1,\ldots,t\} \\ j = 1,\ldots,n}} f_\tau \delta_\tau(h_j),$$

  where the sum runs over all proper, ordered subsets τ of $\{1, \ldots, t\}$ and all h_j. The f_τ are some elements from A_X, and δ_τ denotes the successive application of the δ_s to h_j given by the ordered set τ.

The main component is the following formula:
If two vector fields θ, θ' are represented by $\theta = \sum_{i=1}^{n} f_i \partial_i$ and $\theta' = \sum_{i=1}^{n} f_i' \partial_i$, then their bracket is given by

$$[\theta, \theta'] = \sum_{i=1}^{n} (\theta(f_i') - \theta'(f_i)) \partial_i. \tag{3}$$

This follows from the corresponding equation, which is valid for derivations on A_n.

Replacing θ' by δ in equation (3) we directly obtain the second statement for the start of our induction. The first can be deduced from the facts that $[\theta, \delta] \in \mathcal{L}_{\mathbf{p}}\ \forall \theta \in \mathcal{L}$ and $\delta(f_i) \in \mathbf{p}\ \forall f_i \in A_X$.

Now set $\theta = [\delta_1 [\ldots [\delta_t, \delta] \ldots]] = \sum_i g_i \partial_i$ and $\delta_0 = \sum_i f_i \partial_i$. Then (3) yields

$$[\delta_0, \theta] = \sum_i (\delta_0(g_i) - \theta(f_i)) \partial_i.$$

By the assumption on the g_i we see

$$\delta_0(g_i) = \delta_0 \circ \ldots \circ \delta_t(h_i) + \sum_{\substack{\tau \subset \{0,\ldots,t\} \\ \tau \neq \{0,\ldots,t\} \\ j=1,\ldots,n}} f_\tau \delta_\tau(h_j).$$

On the other hand, the action of θ on f_i does not increase the number of derivations applied to the h_i in the coefficients and, hence, the part of the coefficients of $[\delta_0, \theta]$ generated by $\theta(f_i)$ can be written in the form of the sum

$$\sum_{\substack{\tau \subset \{0,\ldots,t\} \\ \tau \neq \{0,\ldots,t\} \\ j=1,\ldots,n}} f_\tau \delta_\tau(h_j)$$

since they do not include the action of δ_0. This means, we have proved the statement on the coefficients of $\theta' = [\delta_0 [\ldots [\delta_t, \delta] \ldots]] = \sum_i g_i' \partial_i$.

By the first statement of our induction it follows now that

$$\sum_{\substack{\tau \subset \{0,\ldots,t\} \\ \tau \neq \{0,\ldots,t\} \\ j=1,\ldots,n}} f_\tau \delta_\tau(h_j) \in \mathbf{p}$$

because the sets τ include at most t elements. However, $\theta' \in \mathcal{L}_\mathbf{p}$ and, hence, we have

$$\delta_0 \circ \ldots \circ \delta_t(h_i) \in \mathbf{p}.$$

Our proposition is proved.

3 Characterizing points on the variety

Here we try to find a correspondence between certain maximal (w.r.t. the inclusion) Lie subalgebras of a modular Lie algebra and points of our variety. This is the usual approach to construct finally an isomorphism of the varieties when their Lie algebras of derivations are isomorphic (see [18], [16], [4], [8] and [9]). The most recent paper [8] of H. Hauser and G. Müller actually presents a Lie theoretic description of all Lie subalgebras of $\mathrm{Der}(O_n)$ tangent to subvarieties not included in the singular locus in the case of embedded germs of analytic varieties. These are characterized as balanced Lie subalgebras, which means for a Lie subalgebra $L \subset \mathrm{Der}(O_n)$ to fulfil $L^{[\infty]} = 0$ and $L^{[2]} \neq 0$. Moreover, it is mentioned there that the whole approach holds true for algebraic varieties.

Nevertheless, from the viewpoint of a general algebraic theory of modular Lie algebras it seems to be a little bit unsatisfactory to be restricted to modular Lie algebras of all derivations of a k–algebra A. For that reason we give a different proof of the reconstruction of points based on the article [4] of J. Grabowski, which allows us to apply the whole approach to a wider class of modular Lie algebras. Especially the modular Lie algebras with 0–dimensional vanishing locus are interesting for the study of quasihomogeneous isolated complete intersection singularities as they arise naturally there.

Definition 3.1 *Let $P \in X \setminus V(\mathcal{L})$ be a point of the variety X with the maximal ideal $\mathbf{m}_P$ not lying in the vanishing locus of the modular Lie algebra $\mathcal{L}$. Then we denote by*

$$\mathcal{L}_P := \mathcal{L}_{\mathbf{m}_P}$$

the modular Lie subalgebra of all those vector fields vanishing in P.

For this type of modular Lie subalgebras we can show the following properties:

Lemma 3.2 *$\mathcal{L}_P$ is a proper, finite codimensional and maximal (w.r.t. the inclusion) Lie subalgebra.*

Proof: $\mathcal{L}_P$ is a proper and finite codimensional Lie subalgebra by definition. To prove the maximality let $\delta \in \mathcal{L} \setminus \mathcal{L}_P$ and set $L = \langle \delta, \mathcal{L}_P \rangle$ to be the Lie algebra generated by δ over $\mathcal{L}_P$. Not all coefficients of $\delta = \sum_i f_i \partial_i$ vanish at P and we can assume $f_1(P) = 1$. Hence, $\delta(f)|_P = 1$, where $f = x_1 - x_1(P)$ is the coordinate function corresponding to x_1 shifted by its absolute value at P. From this it follows:

$$\delta(f)\theta = [\delta, f\theta] - f\,[\delta, \theta] \in L \quad \forall \theta \in \mathcal{L}$$

and because of

$$\theta - \delta(f)\theta \in \mathcal{L}_P \subset L$$

we obtain $\theta \in L$ for all $\theta \in \mathcal{L}$.

Next we want to characterize the Lie subalgebras of the type $\mathcal{L}_P$ by algebraic properties of the maximal Lie ideals $\mathcal{L}_P^{[\infty]}$. The main problem is to find conditions that distinguish between Lie subalgebras of the type $\mathcal{L}_P$ and those maximal Lie subalgebras vanishing at the same locus as the whole Lie algebra $\mathcal{L}$.

Lemma 3.3 *For every $P \in X \setminus V(\mathcal{L})$ the Lie ideal $\mathcal{L}_P^{[\infty]}$ is infinite codimensional in $\mathcal{L}$ and contains all vector fields vanishing along the minimal integral subvariety Y_P including P.*

If, moreover, $P \in X_{reg}$ is a regular point and $\mathcal{L}$ generates the whole tangent space at P, then $\mathcal{L}_P^{[\infty]} = 0$.

Proof: The minimal integral subvariety Y_P containing P is clearly not identical with P since otherwise P would lie in the vanishing locus of $\mathcal{L}$. Therefore, the first statement is a direct consequence of Proposition 2.12 of the preceding section. The infinite codimensionality follows from the fact that the factor of $\mathcal{L}$ by $\mathcal{L}_P^{[\infty]}$ yields a modular Lie algebra on Y_P, which is clearly an infinite–dimensional Lie algebra.

The second statement is now an easy consequence of the first one:
Since the tangent space at P is $\dim X$–dimensional, the minimal integral subvariety including P is X itself.

Corollary 3.4 *Let P be a regular point of X. Then, for the Lie subalgebra* $\mathrm{Der}(A_X, \mathbf{m}_P)$ *of the Lie algebra of all derivations* $\mathrm{Der}(A_X)$ *we see*

$$(\mathrm{Der}(A_X, \mathbf{m}_P))^{[\infty]} = 0.$$

The key for the characterization of finite codimensional Lie subalgebras is the fundamental lemma due to J. Grabowski and its specialization to our case below:

Theorem 2 *(J. Grabowski [4] p. 20) Let $\mathcal{L}$ be a modular Lie algebra over a k-algebra A and, further, let L be a finite codimensional, proper Lie subalgebra of L. Then there exists a proper ideal I_0 of A such that $L \subset \mathcal{L}_{\mathbf{p}}$ for all prime ideals $\mathbf{p}$ satisfying $I_0 \subset \mathbf{p}$.*

Geometrically this means that the zeros of I_0 describe a sublocus of the vanishing locus of L and, hence, this vanishing locus cannot be empty. However, for our purpose this result is not strong enough because the singular locus itself may be a set of common zeros of all vector fields of $\mathcal{L}$. But we can prove the existence of some I_0 with additional properties, which helps to solve the problem in the singular case, too.

First we introduce a new notation:

Definition 3.5 *An ideal I that fulfils the property of the preceding theorem is called a characterizing ideal for a modular Lie subalgebra L in $\mathcal{L}$. This means that $L \subset \mathcal{L}_{\mathbf{p}}$ whenever $I \subset \mathbf{p}$ for some prime ideal $p \subset A$.*

Remark: Every ideal with a set of common zeros in the vanishing locus of L is also characterizing for L, indeed. This gives us a geometrical interpretation of being characterized and we can deduce that the set of all characterizing ideals is closed under addition, multiplication, taking intersections and radicals. As the main conclusion we obtain: if the ideal $I(L)$ is proper, it is characterizing for L and the radical $\sqrt{I(L)}$ is contained in every reduced, characterizing ideal. Moreover, $I(\mathcal{L})$ is characterizing for every Lie subalgebra if it is proper.

We can sharpen the fundamental lemma for finitely generated k-algebras:

Lemma 3.6 *Let $\mathcal{L}$ be a modular Lie algebra over a finitely generated k-algebra A_X without zero divisors (which implies $\mathcal{L}$ to be finitely generated as an A-module) and L be a finite codimensional, proper and maximal Lie subalgebra. Then there is a characterizing ideal I_0 with:*

$$I_0\mathcal{L} \subset L.$$

Proof: This proof is based on the constructions given by J. Grabowski in the proof of his fundamental lemma (see [4] p. 23). He derives characterizing ideals in the following way:

1. Assume that L is an A_X-module. Then we choose elements $\delta_1, \ldots, \delta_r \in \mathcal{L} \setminus L$ representing a basis of $\mathcal{L}/L$ and set for δ_i
$$I_i = \{f \in A_X \,|\, f\delta_i \in L\}.$$
These are finite codimensional ideals and by
$$I_0 = \cap_{i=1}^r I_i$$
we obtain a characterizing ideal for L.

2. Unless L is a module, we define for an arbitrary $\delta \in L$
$$I_\delta = \{f \in A_X \,|\, \forall g \in A_X : \; gf\delta \in L\}$$
to be the maximal ideal mapping δ into L by multiplication. There must exist a $\delta \in L$ for which $I_\delta \neq A_X$ because L is not a module. The I_δ for such an element is then a characterizing ideal.

We see that in the module case the constructed ideal I_0 just fulfils the additional inclusion. So we are left with the non-modular case:

Here one observes that L must contain a set of generators of $\mathcal{L}$ as an A_X-module since otherwise the module $A_X L$ generated by L would be a greater proper Lie subalgebra. Now, fixing such a set $\delta_1, \ldots, \delta_s$, it follows that for each δ_i the ideal I_{δ_i} is either the whole ring A_X or a charcterizing ideal for L. However, it cannot be equal to A_X for all δ_i and, hence, with

$$I_0 = \cap_{i=1}^{s} I_{\delta_i}$$

we obtain our desired ideal.

Remark: In [16], Corollary 3.5, H. Omori proves that, with the additional assumption of $L^{[\infty]} = 0$, a maximal, finite codimensional Lie subalgebra L of $\mathcal{L}$ must be modular. This means that the second case of our proof does not occur under these circumstances.

Now we can classify Lie subalgebras associated to points of certain Zariski–open sets in X:

Theorem 3 *Let $\mathcal{L}$ be a modular Lie algebra over the k-algebra A_X corresponding to the affine variety X. Then:*

1. *If $\mathcal{L}$ is "strongly nowhere vanishing" (s. J. Grabowski [4]), i.e. $\mathcal{L}(A_X) = A_X$, then the set of Lie subalgebras*
$$\mathcal{M}_1 = \{L \subset \mathcal{L} | \dim \mathcal{L}/L < \infty, \ L \text{ is maximal}\}$$
is bijective to the set of points of X.

2. *If $\mathcal{L}$ generates the whole tangent space to X in every regular point $P \in X$, then the set of Lie subalgebras*
$$\mathcal{M}_2 = \{L \subset \mathcal{L} | \dim \mathcal{L}/L < \infty, \ L \text{ is maximal}, \ L^{[\infty]} = 0\}$$
is bijective to the set of regular points of X.

3. *If $\mathcal{L}$ has a 0-dimensional vanishing locus, which means that $\mathcal{L}(A_X)$ is a finite codimensional ideal, then the set of Lie subalgebras*
$$\mathcal{M}_3 = \{L \subset \mathcal{L} | \dim \mathcal{L}/L < \infty, \ L \text{ is maximal}, \ \operatorname{codim} L^{[\infty]} = \infty\}$$
is bijective to the set of points of $X \setminus V(\mathcal{L})$.

Remark: In the sequel we will refer to these different settings as modular Lie algebras of type 1, 2 or 3. Proof: The first result is the classical one by J. Grabowski ([4] Theorem 5.1).

Consider now the second type:

We assume that L is a maximal, finite codimensional Lie algebra and will show that $L^{[\infty]} = 0$ only if $L = \mathcal{L}_P$ for some regular point $P \in X$.

Set $L = \mathcal{L}_P$ for a singular point $P \in X$. Then L contains the Lie ideal $\mathcal{L}_{Sing\ X}$, which is proper since there exists a non-vanishing vector field in P. Thus $L^{[\infty]} \neq 0$ in this case.

If L is not of the form $L = \mathcal{L}_P$ for any $P \in X$, then L admits the same vanishing locus as $\mathcal{L}$ by maximality. Furthermore, there exists a characterizing ideal I_0 with $I_0\mathcal{L} \subset L$. Let $I = \sqrt{\mathcal{L}(A_X)}$ denote the reduced ideal describing the zero set of $\mathcal{L}$. Then, for some power I^k, it holds that $I^k \subset I_0$ and, hence, $I^k\mathcal{L} \subset L$. However, $I^k\mathcal{L}$ is easily shown to be a Lie ideal of $\mathcal{L}$ since $\mathcal{L}(A_X) \subset I$. So we conclude $L^{[\infty]} \neq 0$ for such L, too.

Next we consider the third type:

Let $P \in X \setminus V(\mathcal{L})$ be a point outside the zeros of $\mathcal{L}$ and consider $L = \mathcal{L}_P$. Then P itself is clearly not an integral subvariety. Thus the minimal integral subvariety over P has positive dimension. But then $L^{[\infty]}$ is infinite–codimensional because, by Lemma 2.12, the factor $\mathcal{L}/L^{[\infty]}$ defines a modular Lie algebra on a variety of positive dimension.

If, otherwise, L is not of the form $\mathcal{L}_P$ for some $P \in X \setminus V(\mathcal{L})$, we deduce $I^k\mathcal{L} \subset L$ as in the second case, where I is the reduced ideal describing the common vanshing locus of $\mathcal{L}$ and L. Here, however, because of the assumptions on the vanishing locus of $\mathcal{L}$, the ideal I is finite–codimensional and so is $I^k\mathcal{L}$ as $\mathcal{L}$ is finitely generated. Hence, L contains a finite–codimensional Lie ideal of $\mathcal{L}$ and we are finished.

Note that all three settings are not determined by purely Lie theoretic conditions since they use either the action of the Lie algebra of derivations on the k–algebra or a special assumption on the variety. For a general approach it is still open to find a condition that excludes all maximal Lie subalgebras not related to geometric subobjects.

At this point we illustrate the behaviour of the approach by two additional examples:

1. First we take the intersection X of two hyperplanes in the space given by the equation $f = xy$ in the ring $\mathbf{C}[x, y, z]$. Here we have singularities along the z–axis as X is the product of two intersecting lines in the plane with an affine line $\mathbf{C}^1$. Hence, the module $\mathrm{Der}(A_X) \subset (\mathbf{C}[x, y, z])^3$ of all derivations is generated by $\delta_1 = \partial_z, \delta_2 = y\partial_y$ and $\delta_3 = x\partial_x$. This means, $\mathrm{Der}(A_X)$ is a strongly nowhere vanishing Lie algebra and, thus, the set of Lie subalgebras

$$\mathcal{M}_1 = \{L \subset \mathcal{L} | \dim \mathcal{L}/L < \infty,\ L \text{ is maximal}\}$$

includes all points of X whereas the additional assumption $L^{[\infty]} = 0$ distinguishes the regular points from those on the z–axis.

2. Next we consider the Whitney umbrella X given by the polynomial $f = x^2 - zy^2$ in the ring $\mathbf{C}[x, y, z]$ as an affine variety. Again we have singularities along the z–axis, but now these singularities are all analytically isomorphic except for the point 0. Hence, $\mathrm{Der}(A_X)$ is generated by the vector fields

$$\delta_1 = y\partial_y - 2z\partial_z,$$
$$\delta_2 = x\partial_x + 2z\partial_z,$$
$$\delta_3 = yz\partial_x + x\partial_y$$

and

$$\delta_4 = y^2\partial_x + 2x\partial_z,$$

which all vanish at the origin. Therefore, the set of Lie subalgebras

$$\mathcal{M}_3 = \{L \subset \mathcal{L} | \dim \mathcal{L}/L < \infty,\ L \text{ is maximal},\ \operatorname{codim} L^{[\infty]} = \infty\}$$

consists of all points of $X \setminus \{0\}$, and the set

$$\mathcal{M}_2 = \{L \subset \mathcal{L} | \dim \mathcal{L}/L < \infty,\ L \text{ is maximal},\ L^{[\infty]} = 0\}$$

describes all point outside the z–axis.

Concerning the structure of the set $\mathcal{M}_i$ we can show:

Corollary 3.7 *Let $\mathcal{L}$ be a modular Lie algebra of type 1, 2 or 3 over A_X. By the data of the Lie algebra $\mathcal{L}$ the set $\mathcal{M}_i$ is not only determined as a set of points, but also as a topological space w.r.t. the Zariski–topology.*

Proof: One has to observe that a Lie subalgebra L in $\mathcal{M}_i$ contains just the vector fields vanishing at the point represented by L. Thus, the zeros of a vector field δ can be found by deciding the membership in the Lie subalgebras contained in $\mathcal{M}_i$. Moreover, for every $P \in \mathcal{M}_i$ there exists a vector field that is not zero at P. Hence, the closed sets given by zeros of single vector fields define a subbasis for the Zariski–topology on $\mathcal{M}_i$.

We see that all purely topological properties determined by these Zariski–open sets are indeed Lie invariants. This includes the determination of algebraic subsets in $\mathcal{M}_i$, the irreducibility of closed sets and dimensions of subvarieties. Furthermore, we obtain the modular Lie subalgebra of vector fields vanishing on an arbitrary algebraically closed set as an intersection of the Lie subalgebras corresponding to its points.

4 Construction of the ring of regular functions on the varieties

Now we are left with the task of using our data to construct isomorphisms of the variety from a given isomorphism of some modular Lie algebras of type 1, 2 or 3. The usual idea (see [16], [4] or [8]) is to study functions of an appropriate class on the manifold, embedded germ or variety and then to show that the pull–back with the bijection of points induced by an isomorphism of modular Lie algebras yields a function of the same class.

Here we use another idea: first we try to construct a k–algebra of certain rational functions purely algebraically from the data of the given Lie algebra. This would imply that an isomorphism of the Lie algebras a priori gives rise to an isomorphism of the constructed k–algebras. But then we show that, under certain conditions, these k–algebras are just the algebras of regular functions on the varieties.

We start with the following information:
Let A_X and $A_{X'}$ be two finitely generated k–algebras with associated affine varieties X and X', and

$$\Phi : \mathcal{L} \longrightarrow \mathcal{L}'$$

the given isomorphism of some modular Lie algebras of the same type over these k–algebras.

- We know the Zariski–open sets $\mathcal{M}_i$ and $\mathcal{M}'_i$ of both varieties, including all regular points of them, by the construction above. Moreover, Φ induces a bijection between them.
- We have a module on each variety — the given modular Lie algebra.
- As we have proved in the last theorem of the preceding section: each element of the sets $\mathcal{M}_i$ and $\mathcal{M}'_i$ represents not only a point of X or X', but includes exactly all those vector fields that vanish at this point.

Now assume $f \in A_X$ to be a regular function on X. Then we can find a $\theta \in \mathcal{L}$ for each $\delta \in \mathcal{L}$ with

$$f(P) * \delta - \theta|_P = 0 \ \forall P \in X.$$

Of course, θ is nothing else but $f * \delta \in \mathcal{L}$. The advantage of this description is that the vanishing at a point $P \in X_{reg}$ can be expressed directly in terms of our modular Lie algebra:

Definition 4.1 *Let $\mathcal{L}$ be a modular Lie algebra of type 1, 2 or 3, and $\mathcal{M}_i$ the corresponding set of maximal, finite–codimensional Lie subalgebras L. Then we define a k–algebra of functions on $\mathcal{M}_i$ by:*

$$B(\mathcal{L}) := \{f \in k^{\mathcal{M}_i} | \forall \delta \in \mathcal{L} \ \exists \theta \in \mathcal{L} : \ f(L) * \delta - \theta \in L \ \forall L \in \mathcal{M}_i\}.$$

Remark: This definition is clearly a copy of the above condition restricted to the Zariski–open set $\mathcal{M}_i$ of X. Moreover, by the definition we have a $B(\mathcal{L})$–module structure on $\mathcal{L}$: For $\delta \in \mathcal{L}$ and $f \in B(\mathcal{L})$ we set

$$f * \delta := \theta,$$

where θ is exactly the fixed element occuring in Definition 4.1. From the construction and the interpretation of the $L \in \mathcal{M}_i$ we deduce the uniqueness of θ and all algebraic properties of a module structure.

Lemma 4.2 *Let $\mathcal{L}$ be a modular Lie algebra of type 1, 2 or 3. Then $B(\mathcal{L})$ is an integral extension of the k–algebra A_X in its quotient field. If, moreover, either A_X is normal or $\mathcal{L}$ is of rank 1, then $B(\mathcal{L})$ is isomorphic to A_X. This holds also for every "strongly nowhere vanishing" modular Lie algebra $\mathcal{L}$ on an arbitrary k–algebra A_X if $B(\mathcal{L})$ is constructed on the set $\mathcal{M}_1$.*

Proof: From the remark to Definition 4.1 it follows that every regular function on X fulfils the conditions determining $B(\mathcal{L})$. Thus, we see that $A_X \subset B(\mathcal{L})$. Let f be an arbitrary element of $B(\mathcal{L})$

and take a pair δ, θ for which the defining relation holds. We choose representations $\delta = \sum_j g_j \partial_j$ and $\theta = \sum_j h_j \partial_j$ w.r.t. a fixed set of generators of A_X. Then the definition of $B(\mathcal{L})$ gives

$$f(P) g_j(P) = h_j(P) \quad \forall j, \forall P \in \mathcal{M}_i.$$

But this means that f is a rational function on X and that it must be regular on $\mathcal{M}_i$ because one finds a non-vanishing coefficient g_j at each point $P \in \mathcal{M}_i$. We conclude

$$B(\mathcal{L}) \subset K(A_X).$$

Further, if $\mathcal{L}$ is "strongly nowhere vanishing", and we use $\mathcal{M}_1$ for the construction, $A_X \cong B(\mathcal{L})$ since $\mathcal{M}_1$ is then bijective to X.

If $\mathcal{L}$ is of rank 1, we can choose δ to be the generator of $\mathcal{L}$. Then $\theta = f'\delta$ holds a priori for some $f' \in A_X$ and we deduce

$$f(P) = f'(P) \quad \forall P \in \mathcal{M}_i.$$

This gives $A_X \cong B(\mathcal{L})$ as desired.

Up to now we know that, in general, $B(\mathcal{L})$ is an extension of A_X for which $\mathcal{L}$ admits a finitely generated module structure by Definition 4.1. Moreover, it follows from Definition 4.1 that the annihilator in $B(\mathcal{L})$ of $\mathcal{L}$ is 0. So we can apply a standard theorem of commutative algbra (see [15] Theorem 2.1), the so-called "determinant trick", which yields, under these assumptions, that every element f of $B(\mathcal{L})$ is integral over A_X. This means $B(\mathcal{L})$ is an integral extension of A_X and if A_X is normal, both k-algebras have to be isomorphic.

Throughout this and the preceding section we have established the following "summary" statement:

Theorem 4 *Let $\mathcal{L}$ be a modular Lie algebra of derivations on A_X and $\mathcal{L}'$ a modular Lie algebra of derivations on $A_{X'}$ both satisfying one of the following points:*

1. *$\mathcal{L}$ resp. $\mathcal{L}'$ are "strongly nowhere vanishing" modular Lie algebras.*

2. *$\mathcal{L}$ resp. $\mathcal{L}'$ are modular Lie algebras of rank 1 as modules with 0-dimensional vanishing locus.*

3. *$\mathcal{L}$ and $\mathcal{L}'$ are modular Lie algebras with 0-dimensional vanishing locus over normal k-algebras A_X and $A_{X'}$, respectively.*

4. *$\mathcal{L}$ and $\mathcal{L}'$ are modular Lie algebras spanning the whole tangent space in every regular point over normal k-algebras A_X and $A_{X'}$ respectively.*

If the Lie algebras $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic, then A_X and $A_{X'}$ are so, too.

Proof: These are just all possible cases for $\mathcal{L}$ and A_X (resp. $\mathcal{L}'$ and $A_{X'}$) for which Lemma 4.2 gives rise to an isomorphism of $B(\mathcal{L})$ and A_X (resp. $B(\mathcal{L}')$ and $A_{X'}$). Hence, for each of these cases we can construct the ring of coordinates on X and X' purely algebraically using only the data of the Lie algebras $\mathcal{L}$ and $\mathcal{L}'$. Thus, an isomorphism of them will induce an isomorphism of the rings of coordinates A_X and $A_{X'}$.

Next we note some corollaries to emphasize important special cases:

Corollary 4.3 *Let X and X' be two normal, affine varieties over k with k-algebras A and A' as rings of coordinates. They are isomorphic if and only if the Lie algebras* Der(A) *and* Der(A') *are so.*

This is one of the main results claimed in the introduction.

Corollary 4.4 *Let A_X and $A_{X'}$ admit a non-vanishing derivation at every point. Then they are isomorphic if and only if the Lie algebras* Der(A) *and* Der(A') *are isomorphic.*

This is a special case of a theorem according to S.M. Skryabin (see [20] theorem 2) which proves such a statement for every pair of a "strongly nowhere vanishing" modular Lie algebra $\mathcal{L}$ over a k-algebra R with $R = 6R$ inducing, in R, an ideal

$$\delta(f)\delta'(g) - \delta(g)\delta'(f) \quad \forall f, g \in R$$

which either vanishes or contains at least one non-zero divisor of R.

Corollary 4.5 *Let A_X and $A_{X'}$ be non–negative graded k–algebras with zero graded parts equal to the constant functions. Then we have Euler derivations (possibly more than one)*

$$\delta_E : f \longrightarrow deg(f)f \quad \forall f \text{ homogeneous}$$

on both varieties. If now the modular Lie algebras $\mathcal{L} = A_X\delta_E$ and $\mathcal{L}' = A_{X'}\delta_E$ are isomorphic Lie algebras, then A_X and $A_{X'}$ are isomorphic.

Proof: This follows from the fact that the vanishing locus of an Euler vector field associated to a complete graduation is only the single point $\{0\}$. By complete graduation we mean that only the constant functions are zero–graded.

These objects seem to be a little bit exotic, but they come out very naturally during the investigation of quasihomogeneous isolated complete intersection singularities. This connection will be part of a joint work with B. Martin [13].

References

[1] A.G. Aleksandrov, Cohomology of a quasihomogeneous complete intersection (russ), Isv. Akad. Nauk UdSSR 49 (1985) No 3, 467–509.

[2] I. Amemiya, Lie algebras of vector fields and complex structures, J. Math. Soc., Japan 27 (1975), 545–549

[3] H. Bjahr, O.A. Laudal, Deformations of Lie–algebras and Lie–algebras of deformations. Application to the study of hypersurface singularities, Preprint No 3 (1987), University of Oslo.

[4] J. Grabowski, Isomorphisms and ideals of the Lie–algebra of vector fields, Inv. math. 50 (1978), 13–33.

[5] V.W. Guillemin, A Jordan–Hölder decomposition for a certain class of infinite–dimensional Lie algebras, J. Diff. Geom. 2 (1968), 313–345

[6] V.W. Guillemin, Infinite dimensional primitive Lie algebras, J. Diff. Geom. 4 (1970), 257–282

[7] V.W. Guillemin, S. Sternberg, An algebraic model of transitive differential geometry, Bull. Amer. Math. Soc. 70 (1964), 16–47

[8] H. Hauser, G. Müller, Affine varieties and Lie algebras of vector fields, Manuscr. math. 80 (1993), 309–337

[9] H. Hauser, G. Müller, On the Lie algebra $\Theta(X)$ of vector fields on a singularity, Preprint 4 (1993), Universität Mainz

[10] D.A. Jordan, On the ideals of a Lie algebra of derivations, J. London Math. Soc. (2) 33 (1986), 33–39.

[11] V.G. Kac, Infinite dimensional Lie algebras, Progress in Mathematics Vol. 44, Birkäuser, Boston ..., 1983

[12] B. Martin, Th. Siebert, The tangent cohomology of a quasihomogeneous isolated complete intersection singularity, Preprint 251 (1990), Humboldt–Universität Berlin

[13] B. Martin, Th. Siebert, On the zero–th tangent cohomology of a quasihomogeneous isolated complete intersection singularity, In preparation

[14] K. Masuda, Homomorphism of the Lie algebras of vector fields, J. Math. Soc. Japan 28 (1976) 506–528

[15] H. Matsumura, Commutative ring theory, Cambridge University Press, Cambridge, 1986.

[16] H. Omori, A method of classifying expansive singularities, J. Diff. Geom. 15 (1980), 493–512.

[17] A. Seidenberg, Differential ideals in rings of finitely generated type, Am. J. Math. 89 (1967), 22–42.

[18] L.E. Shanks, M.E. Purcell, The Lie algebra of a smooth manifold, Proc. Amer. Math. Soc. 5 (1954), 468–472

[19] Th. Siebert, Lie–Algebren von Derivationen und affine algebraische Geometrie über Körpern der Charakteristik 0, Dissertation Humboldt–Universität zu Berlin, Berlin, 1992

[20] S.M. Skrjabin, Regular Lie rings of derivations (Russ.), Vestn. Mosk. Univ. Ser. I (1988) No. 3, 59–62.

[21] J. Wahl, Derivations, automorphisms and deformations of quasi–homogeneous singularities, Proc. Symp. Pure Math. 40, part 2 (1983), 613–624.