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On the Dependence of the Solution of Generalized Boltzmann Equations on the Scattering Cross Section: The Direct Problem

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Abstract
In this paper we prove the smooth dependence of the solution of a phenomenological generalized Boltzmann equation on the scattering cross section. We get an upper bound for the norm of the difference of two solutions.

Key words: Generalized Boltzmann Equation, Differential Equations in Banach Spaces

1) Introduction: The Boltzmann equation

The evolution of the distribution function of a spatially homogeneous gas consisting of molecules with internal energy is given by:

$$\frac{\partial}{\partial t} f(t,v,\varepsilon_1) = J(\sigma,f,f)(t,v,\varepsilon_1) \quad \text{with initial value } f_o \in L_1(\mathbb{R}^3 x \mathbb{R}_+). \tag{1.1}$$

In (1.1) we have used the following notations:

$$J(\sigma,f,g) = \frac{1}{2}\int_{\Pi'} \sqrt{1 - e_1' - e_2'}\ \sigma(E,e_1,e_2,e_1',e_2',\eta\cdot\eta')\,[f'g'_* + f'_*g' - fg_* - f_*g]\ d\Omega(\eta')\,de_1'de_2'\ d\varepsilon_2 dw, \tag{1.2}$$

$\Pi' = \mathbb{R}^3 x \mathbb{R}_+ x \Delta_1 x S_2$ with $\Delta_1 = \{(e_1',e_2') : 0 \le e_1',\ 0 \le e_2' \text{ and } e_1' + e_2' \le 1\}$,

$E = \frac{1}{2}|v-w|^2 + \varepsilon_1 + \varepsilon_2$, $c' = \sqrt{2E(1 - e_1' - e_2')}$ and $e_i = \varepsilon_i/E$, i=1,2.

$v' = \frac{1}{2}(v + w + \eta'c')$, $\varepsilon_1' = e_1'E$,

$w' = \frac{1}{2}(v + w - \eta'c')$, $\varepsilon_2' = e_2'E$,

$f' = f(t,v',\varepsilon_1')$, $f'_* = f(t,w',\varepsilon_2')$, $f_* = f(t,w,\varepsilon_2)$.

The aim of this paper is to study the behavior of the solutions of (1.1) in $C([0,t_o]\to L_1)$. As a first step in this direction we note a scaling property of such solutions. Suppose we have found a solution $f(\cdot,\sigma)$ of (1.1) in $C([0,t_o]\to L_1)$ for some $t_o > 0$. We define for $\lambda,\mu > 0$

$$g(t) = \lambda f(\lambda\mu t,\sigma)$$

and we get

$$g(t) = \lambda f(\lambda\mu t,\sigma) = \lambda f_o + \lambda\int_0^{\lambda\mu t} J(\sigma,f(s),f(s))\,ds$$
$$= \lambda f_o + \int_0^t J(\mu\sigma,\lambda f(\lambda\mu s),\lambda f(\lambda\mu s))\,ds = g_o + \int_0^t J(\mu\sigma,g(s),g(s))\,ds$$

which shows, that $g(\cdot)$ solves (1.1) with data λf_o and $\mu\sigma$ in $C([0,t_o/\mu\lambda]\to L_1)$. Because of this property we assume in the following:

$$\|f_o\| = 1.$$

2) The space of scattering cross sections and basic properties

In this section we introduce the function space for the scattering cross sections and note some results which have been proved in refs. [1] - [3]. As usual we denote by $C(X \to Y)$ the space of continous functions from a metric space X into a metric space Y.

<u>Definition</u> <u>2.1:</u> The set S of scattering cross sections is the set of all measurable real valued functions k defined on $\mathbb{R}_+ x \Delta_1 x \Delta_1 x S_2$ which have the properties:

(i) $k \in C(\mathbb{R}_+ x \Delta_1 \to L_1(\Delta_1 x[-1,1]))$

(ii) $k(E,e,e',x) = k(E,e',e,x)$ a.e (2.1)

(iii) $e_1 + e_2 = 1 \Rightarrow k(E,e,e',x) = 0$ a.e.

(iv) $\|k\| = \sup_{(E,e)} \int_{\Delta_1 x S_2} |k(E,e,e',x)| \sqrt{1 - e_1' - e_2'}\, de'dx < \infty$

The set of all nonnegative functions in S will be denoted by S_+. We denote the closed unit ball of S by B_1 and the open unit ball by B_1'.

By inspection of (1.2) the collision operator can be split into a gain and a loss term

$$J(\sigma,f,g) = G(\sigma,f,g) - V(\sigma,f,g) , \tag{2.2}$$

where the functions G and V are nonnegative if the scattering cross section and the functions f and g are nonnegative. We summarize some known results, see [1] - [3], in the following propositions.

<u>Proposition</u> <u>2.1:</u> Let σ be in S. Then both $G(\sigma,\cdot,\cdot)$ and $V(\sigma,\cdot,\cdot)$ are mappings from $L_1 x L_1$ into L_1 and there hold the estimates:

$$\|V(\sigma,f,g)\| \le 2\pi \|\sigma\| \|f\| \|g\| \quad \text{and} \quad \|G(\sigma,f,g)\| \le 2\pi \|\sigma\| \|f\| \|g\| . \tag{2.3}$$

For any σ in B_1' and $t_o \le [4\pi]^{-1}$ there is a unique function $f(\cdot,\sigma) \in C([0,t_o] \to L_1)$ which solves (1.1) with data f_o and σ. If σ and f_o are non-negative then there is for any $t_o > 0$ a unique solution $f(\cdot,\sigma) \in C([0,t_o] \to L_1)$ of (1.1) which has the property

$$\forall t \ge 0 : \|f(t,\sigma)\| = \|f_o\|. \tag{2.4}$$

<u>Definition</u> <u>2.2:</u> Let σ and f_o be a nonnegative functions in S and $L_1(\mathbb{R}^3 x \mathbb{R}_+)$ respectiveley. We define for $h \ge 2\pi \|\sigma\|$ the operator

$$Q_h(\sigma,f,g) = J(\sigma,f,g) + \frac{h}{2} \{ f \int_{\mathbb{R}^3 x \mathbb{R}_+} g(w,\varepsilon_2)\, d\varepsilon_2 dw + g \int_{\mathbb{R}^3 x \mathbb{R}_+} f(w,\varepsilon_2)\, d\varepsilon_2 dw \}, \tag{2.5}$$

and the following sequence $\{H_n(\sigma)\}$ of functions

$$H_o(\sigma) = f_o$$
$$H_n(\sigma) = \frac{1}{nh} \sum_{\mu=0}^{n-1} Q_h(\sigma, H_{n-1-\mu}(\sigma), H_\mu(\sigma)) \quad , \text{ if } n \geq 1 . \tag{2.6}$$

Proposition 2.2: Let $\sigma \in S$ and f_o be nonnegative functions. Then each of the functions $H_n(\sigma)$ in (2.6) is nonnegative and we have $\|H_n(\sigma)\| = \|f_o\| = 1$ for any $n \in \mathbb{N}$. The unique solution (1.1) can be represented as

$$f(t,\sigma) = \sum_{n=0}^{\infty} e^{-ht} (1 - e^{-ht})^n \; H_n(\sigma) , \tag{2.7}$$

where we can choose any $h \geq 2\pi \|\sigma\|$.

3) Frechet differentiability of the solution for sufficiently short times

In the following we assume the initial condition f_o to be a nonnegative function. We recall the standard notations of Frechet differentiability, see e.g [4].

Theorem 3.1: Suppose $t_o = [4\pi]^{-1}$. Let $f(\cdot,\sigma)$ be solution of (1.1) with data f_o and σ. The mappings

$$B_1' \ni \sigma \;\rightarrow\; f(\cdot,\sigma) \in C([0,t_o] \rightarrow L_1) ,$$

$$[0,t_o] x B_1' \ni (t,\sigma) \;\rightarrow\; f(t,\sigma) \in L_1$$

are Frechet differentiable. If σ is a nonnegative function in B_1' then we have

$$\|D_2 f(t,\sigma)\| \leq \frac{(\exp[8\pi\|\sigma\| \; t] - 1)}{2\|\sigma\|} , \quad t \in [0,t_o] \tag{3.1}$$

Suppose we have two nonnegative scattering cross sections σ and σ_1 in B_1' with $\|\sigma\| \geq \|\sigma_1\| > 0$. Then we have the estimate

$$\|f(t,\sigma) - f(t,\sigma_1)\| \leq \frac{(\exp[8\pi\|\sigma\| \; t] - 1)}{2\|\sigma_1\|} \|\sigma - \sigma_1\| \quad , \; t \in [0,t_o] \tag{3.2}$$

Proof: The Frechet differentiability of the solution of (1.1) follows from standard theorems on the parameter dependence of the solution of differential equations in Banach spaces, see e.g. [4], theorem VI,4.3, provided we can show that the mapping

$$S x L_1(\mathbb{R}^3 x \mathbb{R}_+) \ni (\sigma,f) \;\rightarrow\; J(\sigma,f,f) \in L_1(\mathbb{R}^3 x \mathbb{R}_+) \tag{3.3}$$

is Frechet differentiable. But this is an easy consequence of (2.2) and (2.3) and the linearity of $J(\cdot,\cdot,\cdot)$ in all of its arguments. Moreover we obtain:

$$\|D_1J(\sigma,f,g)\| \leq 4\pi\|f\|\,\|g\| \quad \text{and} \quad \|D_2J(\sigma,f,g)\| \leq 4\pi\|\sigma\|\,\|g\| \tag{3.4}$$

To prove (3.1) and (3.2) we start with the evolution equation for the partial derivative $D_2f(t,\sigma)$ of $f(t,\sigma)$ with respect to σ. We get form the above mentioned theorem:

$$\begin{aligned} \frac{\partial}{\partial t} D_2f(t,\sigma) &= D_1J(\sigma,f(t,\sigma),f(t,\sigma)) + \langle T(\sigma,f(t,\sigma)), D_2f(t,\sigma)\rangle \\ D_2f(0,\sigma) &= 0, \end{aligned} \tag{3.5}$$

where $D_1J(\sigma,f(t,\sigma),f(t,\sigma))$ is the partial derivative of the collision operator (1.2) with respect to σ evaluated at $(\sigma,f(t,\sigma),f(t,\sigma))$. The two parameter family $T(\sigma,f)$, $\sigma \in S$, $f \in L_1(\mathbb{R}^3\mathrm{x}\mathbb{R}_+)$, of operators acts on the space $\mathcal{L}(S,L_1(\mathbb{R}^3\mathrm{x}\mathbb{R}_+))$ of linear operators from S into $L_1(\mathbb{R}^3\mathrm{x}\mathbb{R}_+)$ as follows: for any $A \in \mathcal{L}(S,L_1(\mathbb{R}^3\mathrm{x}\mathbb{R}_+))$ we have

$$S \ni k \to \langle\langle T(\sigma,f),A\rangle,k\rangle = 2J(\sigma,f,\langle A,k\rangle) \in L_1(\mathbb{R}^3\mathrm{x}\mathbb{R}_+).$$

This equation shows that $T(\sigma,f)$ is a linear operator and we get from (2.2) and (2.3):

$$\|T(\sigma,f)\| \leq 8\pi\|\sigma\|\,\|f\|. \tag{3.6}$$

Moreover we have: $T(\sigma,f_1+f_2) = T(\sigma,f_1)+T(\sigma,f_2)$. With the help of (3.6) and proposition 2.1 we see that the mapping

$$[0,t_0] \ni t \to T(\sigma,f(t,\sigma)),$$

$f(\cdot,\sigma)$ being the solution of (1.1) with data f_0 and σ, is in $C([0,t_0] \to \mathcal{L}(\mathcal{L}(S,L_1(\mathbb{R}^3\mathrm{x}\mathbb{R}_+)), \mathcal{L}(S,L_1(\mathbb{R}^3\mathrm{x}\mathbb{R}_+)))$. From the theory of evolution equations, see e.g.[5], we now get with the help of (3.4), (3.5) and (3.6):

$$\|D_2f(t,\sigma)\| \leq \int_0^t 4\pi\|f(s,\sigma)\|^2 \exp\left[\int_s^t 8\pi\|\sigma\|\,\|f(s',\sigma)\|\,ds'\right]ds, \quad 0 \leq t \leq t_0. \tag{3.7}$$

Now (3.1) is a direct consequence of (2.4). To prove (3.2) we use the mean value theorem, see e.g [4], to get

$$\|f(t,\sigma_1)-f(t,\sigma_1)\| \leq \|\sigma-\sigma_1\|\,[\sup\|D_2f(t,\sigma')],$$

where the supremum has to be taken over all σ' on the line between σ and σ_1. Due to the nonnegativity of σ and σ_1 we have:

$$\forall\ \lambda\in[0,1] : \lambda\sigma + (1-\lambda)\sigma_1 \geq 0 \quad \text{and} \quad \|\sigma_1\| \leq \|\lambda\sigma + (1-\lambda)\sigma_1\| \leq \|\sigma\|.$$

Now (3.2) is a direct consequence of (3.1).

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4) Differentiability of the solution for large times

In section 3 we had to limit ourselves on sufficiently small times because we wanted to use the calculus of Frechet derivatives. In this section we consider smoothness properties on arbitrary time intervalls of the solution of (1.1) in the case of nonnegative scattering cross sections. To this end we have to weaken the property of differentiability. Our definition is mainly motivated by the fact, that the set of all nonnegative scattering cross sections is of first category in S but forms a positive cone.

Definition 4.1: Let X and Y be Banach spaces and $D \subset X$ be a convex set. We call a map $f: D \to Y$ differentiable at x in the direction of y, if the map

$$[0,1] \ni \lambda \to f(\lambda x + (1-\lambda)y) \in Y$$

is differentiable. This derivative will be denoted by $\delta_{x,y} f(\lambda)$.

Theorem 4.1: Let f_0 be a nonnegative function in $L_1(\mathbb{R}^3 x \mathbb{R}_+)$ and let t_0 be any positive time. The map

$$S_+ \ni \sigma \to f(\cdot,\sigma) \in C([0,t_0] \to L_1),$$

$f(\cdot,\sigma)$ being the solution of (1.1) with data f_0 and σ, is differentiable at any $\sigma \in S_+$ in any direction $\sigma_1 \in S_+$. Suppose we have to scattering cross sections σ_1 and σ_2 with $\|\sigma_1\| \geq \|\sigma_2\|$. Then, for any $t > 0$, there holds the estimate

$$\begin{aligned} \|f(t,\sigma_1)-f(t,\sigma_2)\| \leq \; & \frac{\|\sigma_1-\sigma_2\|}{2\|\sigma_1\|} \left(\exp[8\pi\|\sigma_1\|t] - 1\right) \\ & \cdot \left(1 - (N_0+1)(1 - \exp[-8\pi\|\sigma_1\|t])^{N_0} \right. \\ & \left. + N_0(1 - \exp[-8\pi\|\sigma_1\|t])^{N_0+1} \right) \\ & + 2\left(1 - \exp[-8\pi\|\sigma_1\|t]\right)^{N_0+1} , \end{aligned} \tag{4.1}$$

where N_0 is given by

$$N_0 = \begin{cases} \left[\dfrac{4\|\sigma_1\|}{\|\sigma_1-\sigma_2\|} \right] , & \|\sigma_1-\sigma_2\| > 0 \\ \infty , & \sigma_1 = \sigma_2 , \end{cases} \tag{4.2}$$

and [x] is the Gauss bracket of a real number x.

For the proof of theorem 4.1 we need a technical lemma

Lemma 4.1: Let c and d be nonnegative real numbers with $c \le 2$. Suppose we have a sequence $\{a_n\}$ of numbers satisfying

(i) $a_0 \ge 0$

(ii) $a_n \le \frac{c}{n} \sum_{\mu=0}^{n-1} a_\mu + d$, $n \ge 1$.

Then there holds the estimate: $a_n \le (n+1)a_0 + nd$ for $n \ge 0$.

Proof of the lemma: Obviously the above inequality is true for $n = 0$. Now suppose it is true for all $k \in \{0,1,..,n-1\}$, $n \ge 1$. Then we have

$$a_n \le \frac{c}{n} \sum_{\mu=0}^{n-1} [(\mu+1)a_0 + \mu d] + d$$

$$= \frac{c}{n}\frac{n+1}{2} n a_0 + \frac{c}{n}\frac{n}{2}(n-1) d + d$$

$$\le (n+1)a_0 + (n-1)d + d \quad , \quad \text{because we assumed : } c \le 2,$$

which implies that the above estimate is true for all $k \in \{0,1,...,n\}$

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Proof of theorem 4.1: We first prove (4.1). To this end we use the series representation (2.7) of the solution of (1.1). This means that we have to consider the difference $H_n(\sigma_1) - H_n(\sigma_2)$. For any $h \ge 2\pi \|\sigma_1\|$ we have due to (2.6):

$$H_n(\sigma_1) - H_n(\sigma_2) = \frac{1}{nh} \sum_{\mu=0}^{n-1} [Q_h(\sigma_1,H_{n-1-\mu}(\sigma_1),H_\mu(\sigma_1)) - Q_h(\sigma_2,H_{n-1-\mu}(\sigma_1),H_\mu(\sigma_2)] \text{ , } n \ge 1,$$

where $Q_h(\sigma,\cdot,\cdot)$ is given by (2.5). In addition, due to proposition 2.2 we have

$$H_n(\sigma_1) - H_n(\sigma_2) = \frac{1}{nh} \sum_{\mu=0}^{n-1} J(\sigma_1,H_{n-1-\mu}(\sigma_1),H_\mu(\sigma_1)) - J(\sigma_2,H_{n-1-\mu}(\sigma_2),H_\mu(\sigma_2)) + h[H_\mu(\sigma_1) - H_\mu(\sigma_2)] . \tag{4.3}$$

Let us consider the first term on the right hand side of (4.3) in greater detail. Due to the symmetry properties of $J(\cdot,\cdot,\cdot)$, see (1.2), we get:

$$\sum_{\mu=0}^{n-1} J(\sigma_1,H_{n-1-\mu}(\sigma_1),H_\mu(\sigma_1)) - J(\sigma_2,H_{n-1-\mu}(\sigma_2),H_\mu(\sigma_2)) =$$
$$\sum_{\mu=0}^{n-1} J(\sigma_1,H_{n-1-\mu}(\sigma_1),H_\mu(\sigma_1)) - J(\sigma_1,H_{n-1-\mu}(\sigma_2),H_\mu(\sigma_2)) + J(\sigma_1-\sigma_2,H_{n-1-\mu}(\sigma_2),H_\mu(\sigma_2)) . \tag{4.4}$$

and

$$\sum_{\mu=0}^{n-1} J(\sigma_1,H_{n-1-\mu}(\sigma_1),H_\mu(\sigma_1)) - J(\sigma_2,H_{n-1-\mu}(\sigma_2),H_\mu(\sigma_2)) = \sum_{\mu=0}^{n-1} J(\sigma_1,H_{n-1-\mu}(\sigma_1)+H_{n-1-\mu}(\sigma_2),H_\mu(\sigma_1)-H_\mu(\sigma_2)) . \tag{4.5}$$

Equations (4.4) and (4.5) together yield:

$$H_n(\sigma_1) - H_n(\sigma_2) = \frac{1}{nh} \sum_{\mu=0}^{n-1} \Big(J(\sigma_1-\sigma_2,H_{n-1-\mu}(\sigma_2),H_\mu(\sigma_2)) + J(\sigma_1,H_{n-1-\mu}(\sigma_1)+H_{n-1-\mu}(\sigma_2),H_\mu(\sigma_1)-H_\mu(\sigma_2)) + h\,[H_\mu(\sigma_1)-H_\mu(\sigma_2)] \Big) . \tag{4.6}$$

Using (2.3) and proposition 2.2 we obtain now a recursively defined sequence of upper bounds for the norm of the difference $H_n(\sigma_1) - H_n(\sigma_2)$:

$$\|H_n(\sigma_1)-H_n(\sigma_2)\| \le \frac{1}{nh} \sum_{\mu=0}^{n-1} 4\pi \,\|\sigma_1-\sigma_2\| + [8\pi\|\sigma_1\|+h]\, \|H_\mu(\sigma_1)-H_\mu(\sigma_2)\| \tag{4.7}$$

If we use in (4.7) $h = 8\pi\|\sigma_1\|$ then the hierarchy (4.7) is of the same type as that discussed in lemma 4.1. Therefore we get for this h :

$$\|H_n(\sigma_1)-H_n(\sigma_2)\| \le \frac{n}{2} \frac{\|\sigma_1-\sigma_2\|}{\|\sigma_1\|} \tag{4.8}$$

Using (4.8) and (2.7) we can estimate the difference of to solutions of (1.1):

$$\|f(t,\sigma_1)-f(t,\sigma_2)\| \le \sum_{n=0}^{\infty} e^{-ht} (1 - e^{-ht})^n \frac{n}{2} \frac{\|\sigma_1-\sigma_2\|}{\|\sigma_1\|} , \quad h = 8\pi\|\sigma_1\|. \tag{4.9}$$

With the help of

$$\sum_{n=0}^{\infty} nq^n = \frac{q}{(1-q)^2}$$
$$\sum_{i=0}^{N} iq^i = \frac{q - (N+1)q^{N+1} + Nq^{N+2}}{(1-q)^2} , \quad q \in [0,1[\tag{4.10}$$

we obtain:

$$\|f(t,\sigma_1)-f(t,\sigma_2)\| \le \frac{\|\sigma_1-\sigma_2\|}{2\|\sigma_1\|} \Big(\exp[8\pi\|\sigma_1\|t] -1 \Big). \tag{4.11}$$

Notice that we have in (4.11): $\|\sigma_1\| = \max(\|\sigma_1\|,\|\sigma_2\|)$. To get (4.1) we use the fact that we have, due to proposition 2.2, $\|H_n(\sigma_1) - H_n(\sigma_2)\| \le 2$. With the help of this estimate and of (4.8) we get from (4.9):

$$\|f(t,\sigma_1)-f(t,\sigma_2)\| \le \sum_{n=0}^{N_0} e^{-ht}(1-e^{-ht})^n \frac{n}{2}\frac{\|\sigma_1-\sigma_2\|}{\|\sigma_1\|} + \sum_{n=N_0+1}^{\infty} 2\, e^{-ht}(1-e^{-ht})^n \;,\; h = 8\pi\|\sigma_1\|, \tag{4.12}$$

where N_0 is given by (4.2). If we use now (4.10), (4.2) is an easy consequence of (4.12).

In a second step we prove the directional differentiability of the solution of (1.1) in the convex cone S_+. To this end we first note that we can identify the space of linear operators from $\mathbb{R}$ into $L_1(\mathbb{R}^3 x \mathbb{R}_+)$ with the space $L_1(\mathbb{R}^3 x \mathbb{R}_+)$. via the isometric isomorphism $\langle\lambda,\cdot\rangle \to \lambda(1)$.

Let k and σ be two scattering cross sections in S_+. Without loss of generality we assume: max $(\|\sigma\|,\|k\|) > 0$. To prove our claim we use the series representation (2.7) of the solution of (1.1) with $h = 8\pi \max(\|\sigma\|,\|k\|)$. To keep the notation in following formulas as simple as possible we introduce

$$H_n(\lambda) = H_n(\sigma(\lambda)),\ n \ge 1, \quad \text{where we have } \sigma(\lambda) = \lambda\sigma + (1-\lambda)k. \tag{4.13}$$

Recall that H_0 is independent of λ. Because of (4.13) and the Frechet differentiability of the collision operator $J(\cdot,\cdot,\cdot)$ we see that $\delta_{\sigma,k}H_n(\lambda)$ exists for all $n \ge 1$. Furthermore we get from L_1-continuity of $J(\cdot,\cdot,\cdot)$, see (2.2) and (2.3):

$$\forall n\ge 0 : \quad \delta_{\sigma,k}H_n(\,.\,) \in C([0,1] \to L_1) \tag{4.14}$$

The directional derivatives of $\delta_{\sigma,k}H_n(\lambda)$ can be calculated recursively:

$$\delta_{\sigma,k}\, H_n(\lambda) = \frac{1}{nh}\sum_{\mu=0}^{n-1} J(\sigma-k,H_{n-1-\mu}(\lambda),H_\mu(\lambda)) + \frac{1}{nh}\sum_{\mu=0}^{n-1} [\, 2J(\sigma(\lambda),H_{n-1-\mu}(\lambda),\delta_{\sigma,k}H_\mu(\lambda)) + h\, \delta_{\sigma,k}H_\mu(\lambda)\,] .$$

Using now the fact that (4.13) yields: $\|\sigma(\lambda)\| \le \max(\|\sigma\|,\|k\|)$ we get a hierarchy of upper bounds for $\|\delta_{\sigma,k}H_n(\lambda)\|$:

$$\|\delta_{\sigma,k}\, H_n(\lambda)\| \le \frac{1}{2}\frac{\|\sigma-k\|}{h'} + \frac{1}{n}\sum_{\mu=0}^{n-1} 2\|\delta_{\sigma,k}H_\mu(\lambda)\| \quad , h' = \max(\|\sigma\|,\|k\|) . \tag{4.15}$$

Notice that the left hand side of this inequality is independent of λ. The above hierarchy fits to the situation discussed in lemma 4.1. Therefore (4.15) yields:

$$\forall\, \lambda \in [0,1] \;:\; \|\delta_{\sigma,k}H_n(\lambda)\|_0 \le \frac{n}{2}\frac{\|\sigma-k\|}{h'} \quad , \; h' = \max(\|\sigma\|,\|k\|) . \tag{4.16}$$

For a given $t_0 > 0$ we consider now the directional derivatives of the partial sums of the series representation (2.7). We introduce the following sequence of mappings:

$[0,1] \ni \lambda \rightarrow f_N'(\cdot,\lambda) \in C([0,t_0] \rightarrow L_1)$ where $f_N'(\cdot,\lambda)$ is given by

$$f_N'(t,\lambda) = \sum_{n=0}^{N} e^{-ht} (1 - e^{-ht})^n \, \delta_{\sigma,k} H_n(\lambda) \quad , \; t \in [0,t_0] \tag{4.17}$$

Recall that we have in (4.17): $h = 8\pi \max(\|\sigma\|,\|k\|)$. Due to (4.16) we have uniform convergence of $f_N'(\cdot,\lambda)$ in $C([0,t_0] \rightarrow L_1)$. Now, as a consequence of (4.14) the limit function

$$f'(\cdot,\lambda) \; : \; t \rightarrow \sum_{n=0}^{\infty} e^{-ht} (1 - e^{-ht})^n \, \delta_{\sigma,k} H_n(\lambda) \; , \; h = 2\pi \max(\|\sigma\|,\|k\|)$$

is an element of $C([0,t_0] \rightarrow L_1)$. This proves our claim.

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