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Abstract

The quality of freeform surfaces is one of the major topics of CAD/CAM. Aesthetic and technical demands require the construction of high quality surfaces with strong shape conditions. Quality diminishing properties like dents or flat points have to be eliminated while approximation conditions must hold at the same time.
Our approach combines quality and approximation criteria to a nonlinear multicriteria optimization problem and achieves an automatic approximation and fitting process.

Keywords: Surface Fitting, Multicriteria Optimization, Partial Orders

1 Introduction

The quality of freeform surfaces is one of the major topics in CAD/CAM. Approximating surfaces have to fulfill quality criteria for aesthetic and technical demands. The construction of high quality surfaces implies the elimination of quality diminishing properties like dents, local violation of convexity, wavy shape etc. while approximation conditions must hold at the same time. In Multicriteria Optimization Problems more than one, even contradicting criteria - like in the regarded problem - can be used. The resulting not dominated solutions mostly are not optimal for any selected single criterion but make all the criteria as small as possible. In practice these solutions are the most wanted.

Due to the technical and aesthetic demands a *Decision Maker* must be integrated in the optimization process. The complexity of perceptions do not permit entirely the construction of a function catching the Decision Maker's purpose.
For achieving contenting solutions it is essential to get the Decision Maker's purpose before optimizing. Generally, the intention of the Decision Maker cannot be realized mathematically as a so called **Utility Function**. Even to ask the

Decision Maker for weightings concerning on problems with more than two criteria ends with misleading results when Pairwise Comparison or Scaling is used. Mostly problems arise when the Decision Maker is forced to claim a preference when he is just indifferent to objects. Here, partial orders are a promising facility.

In the following we use three criteria functions for approximation, smoothing and achieving elliptic points. Mark that there could be even more criteria functions used at the same time depending on the Decision Maker's issues.

W. Hohenberger and T.Reuding [8] combined the minimization of the rate of curvature change and minimization of weight differences of NURBS curves to minimize the curvature and shape variations of initial NURBS curves approximating the given data points in the more special case of parametric programming. We use a different approximation criterion employing the Minimum Square Approach. It is a powerful approach for approximating data points via parametric surfaces if the data is already smooth. However, inaccurate measurements usually cause dents and wavy shape of the surface. So a smoothing criterion is added supplementary to the approximation criteria. In this approach we use criteria based on linearization of strain energy using the coefficients of the fundamental forms which gave successful results in the work [7] of H. Hagen and P. Santarelli together with the least square constraint. Additionally we introduce a new criterion to avoid wavy shape to get rid of bumps and dents without flattening the surface and give a few academic examples . This criterion is strongly connected to our criterion we used in [3] where we used a criterion for convexity or concavity for curves and achieved very good results. This depends also on the strong connection of curve curvature and normal curvature.

2 Mathematical Fundamentals

The curvature of parametric curves and surfaces is essential for smoothing and convexity criteria. In the case of planar curves we distinguish between *signed* and *unsigned* curvature (see [3]). Regarding surfaces with respect to curvature we inspect normal curvatures especially the principal curvatures.

Definition 2.1 *Let F be a regular surface, φ a regular curve in F through the point $p \in F$, k the curvature of φ in p, n the normal vector of φ in p, N the normal vector of F in p, $\cos\Theta = < n, N >$. The number $k_n = k\cos\Theta$ is called* **Normal Curvature** *of $\varphi \subset F$ in p.*

Definition 2.2 *The maximal normal curvature k_1 and the minimal normal curvature k_2 of the surface at p are called* **principal curvature** *of the surface at p.*

Definition 2.3 *The* **Gaussian Curvature** K *and the* **Mean Curvature** H *are defined as follows:*

$$K = k_1 k_2$$
$$H = \frac{k_1+k_2}{2}$$

The curvatures mentioned above can be detected by using the coefficients of the first and the second fundamental form. The signs of the principal curvatures and even so the sign of the gaussian curvature are of special interest. If the signs of the principal curvature in a point is equal and therefore K > 0, the point is an **Elliptic Point** which means the dupin indicatrix is an ellipse.

3 Multicriteria Optimization

Concerning our problem we find different, contradicting criteria which have to be optimized at once. This leads to a Multicriteria Optimization Problem. Due to the technical and aesthetic demands a *Decision Maker* must be integrated in the optimization process. The complexity of perceptions do not permit entirely the construction of a function catching the Decision Maker's purpose.

Definition 3.1 *A* **Multicriteria Optimization Problem (MOP)** *is given by*

$$\begin{array}{l} \min\ f_1(x) \\ \vdots \\ \min\ f_p(x) \\ h(x) = 0, \\ g(x) \leq z \\ x \geq 0, x \in \mathbb{R}^n, z \in \mathbb{R}^n \end{array}$$

where $f_1, \ldots, f_p$ *are criteria functions,* p, $n \in \mathbb{N}$, *and* h *and* g *are restriction functions.*
The solutions of the MOP are called **Efficient Solutions**.

In Multicriteria Optimization Problems more than one, even contradicting criteria functions can be used. In contrary to Single Objective Optimization Problems there could be more than one facet of non dominated solutions.

Also an efficient solution does not need to be optimal for any of the regarded criteria functions. Such efficient solutions are so called **Compromise Solutions** which in practice usually are the most wanted. In our example a suitable surface need neither to be the best approximation nor very flat nor convex (for example with extrem high curvature) than must be a contenting federation of those entities.

To get such efficient solutions we use the approach of weighted sums which reduce the original Multicriteria Optimization Problem to an optimization problem with only one criteria function.

By using the weighting sum approach every criterion is multiplied with a strictly positive scalar and then summed to a weighted sum criterion. The **Weighting Vector** consisting of all the weights as its parameters is normalized such that it sums to one. Every solution that minimizes the weighted sum program is an efficient solution. On the other hand for every efficient solution exists at least one weighting vector that leads to that initial efficient solution as an optimal solution in the Weighted Sum Program.

For achieving contenting solutions it is essential to get the Decision Makers purpose before optimizing. Generally, the intention of the Decision Maker cannot be realized mathematically as a so called **Utility Function**. Even to ask the Decision Maker for weightings concerning on problems with more than two criteria ends with misleading results when Pairwise Comparison or Scaling is used. Mostly problems arise when the Decision Maker is forced to claim a preference when he is just indifferent to objects. Here, partial orders are a promising facility.

Definition 3.2 *Let $\leq$ be a binary relation on a set M. $\leq$ is called* **Partial Order** *if the following holds:*
(i) $a \leq a$ for all $a \in M$
(Reflexivity)
(ii) For all a, b, $c \in M$ with $a \leq b$ and $b \leq c$ follows $a \leq c$
(Transitivity)
(iii) For all a, b, $c \in M$ with $a \leq b$ and $b \leq a$ follows $a = b$
(Antisymmetry)

Condition (iii) is the one which differs partial orders from total orders which are usually used. This condition implies that two elements of a partially ordered set even need not to be comparable with respect to the partial order.

Definition 3.3 *Let $\leq$ be a partial order on a set M. Elements a, $b \in M$ with neither $a \leq b$ nor $b \leq a$ are called* **not comparable**.

This entity of noncomparability offers the possibility to express indifference to two elements to the Decision Maker and therefore avoids the most serious faults in weight chosen by Scaling or Pairwise Comparison. Regarding the componentwise order we are able to integrate this powerful decision facility in the Weighted Sum Approach.

Definition 3.4 *Let $\leq$ be a binary relation on a set $M \subset \mathbb{R}^n, n \in \mathbb{N}$. $\leq$ is called* **Componentwise Order** *if the following holds:*
Let $v, w \in \mathbb{R}^n$, then $v \leq w$ if and only if $v_i \leq w_i$ for all $i = 0, \ldots, n$.
We say $v < w$ if and only if $v \leq w$ and there exist a number $j \in \mathbb{N}$ with $v_i < w_i$.

Example 3.5 *In the sense of componentwise ordering (6,0,0), (0,0,4) and (1,1,1) are incomparable to each other, but (1,1,1) $\leq$ (6,1,1), (6,0,0) $\leq$ (6,1,1) and (6,0,0) $\leq$ (6,0,4), (0,0,4) $\leq$ (6,0,4). This can be shown graphical with a hasse diagram (see figure 1.). Look at the hasse diagram as a directed graph. If two points a, b where a is lower in the picture than b are connected say there is an arc from a to b, then the corresponding numbers to points a2, b2 are said a2 $\leq$ b2 when there is a directed way from a2 to b2.*

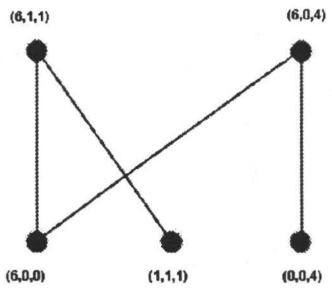


Figure 1: Numbers ordered by Componentwise Ordering illustrated by a Hasse Diagram

The componentwise order is a partial order. Therefore the weightings of the Weighted Sum Approach can be partially ordered. As pointed out a representation of criteria function weights as a partial order helps the Decision Maker to select a weighting corresponding to his purpose and therefore is an important step before optimizing. The partitioning of the parameter space of weightings getting representatives corresponding to efficient solutions is just ongoing research and will reduce complexity a lot.

4 Formulation of the Multicriteria Optimization Problem

In this section we formulate a Multicriteria Optimization Problem using different, partially contradicting criteria for surface fitting. In the following we elaborate three criteria functions for approximation, smoothing and achieving elliptic points. The minimum square approach is a powerful approach for approximating data points via parametric surfaces if the data is already smooth. A new criterion to avoid wavy shape introduced and examples are given. Mark that there could be even more criteria functions used at the same time depending on the Decision Maker's issues. We assume that the data is already reduced. Successful techniques for data reducing are Clustering or Multiresolution Methods.

For approximation of given data points we employ biquintic Tensor Product Bézier Spline Surfaces on the parameter interval [0,1]x[0,1]. A **Tensor Product Bézier Spline Surface** B:[0,1]x[0,1] $\rightarrow$ $\mathbb{R}$ is called a segmented surface of degree (m,n) (m,n $\in$ $\mathbb{N}$) with parametrization $(u_0 = 0, \ldots, u_m = 1; w_0 = 0, \ldots, w_n = 1)$. The **Bézier Segments** $B_{k1,k2}$ $(k1 = 0, \ldots, m-1; k2 = 0, \ldots, n-1)$ are given as $B_{k1,k2} = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{mk1+i,nk2+j} B_i^m(\frac{u-u_{k1}}{u_{k1+1}-u_{k1}}) B_j^n(\frac{w-w_{k2}}{w_{k2+1}-w_{k2}})$
for all $u \in [u_{k1}, u_{k1+1}]$, $w \in [w_{k2}, w_{k2+1}]$, $b_j \in \mathbb{R}^3$
and $B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$ are Bernstein Polynomials.
We manipulate the surface by moving control points $b_{mk1+i,nk2+j}$ automatically with respect to our approximation and smoothing criteria. As mentioned above smoothing criteria and criteria to achieve elliptic points have to be added by technical and aesthetic demands.

We formulate our MOP in the following way:
$min\ c_1(b)$
$\vdots$
$min\ c_p(b)$
Ab $= 0$
$b_j \geq 0,$
where $c_1, \ldots, c_p$ are the criteria functions, p $\in$ $\mathbb{N}$, A is the C^3 restriction matrix, b $= (b_{0,0_{(0)}}, \ldots, b_{mk1+i,nk2+j_{(3)}})$ where $b_j \in \mathbb{R}^3$ are the control points of our Tensor Product Bézier Spline Surface.

Note that the restrictions are linear whereas the whole MOP is nonlinear.

For approximation of data points the Minimum Square Method is an approved approach which gives good results when data is already smooth. Therefore we use it as a first criteria function in the following way:
$c_1(b) = \sum_{i=0}^{n_1} (F(\tilde{u_i}, \tilde{w_i}) - P_i)^2$
where P_i are approximation points, $\tilde{u_0} = 0, \tilde{w_0} = 0, \tilde{u_{n_1}} = 1, \tilde{w_{n_1}} = 1, \tilde{u}_{i-1} \leq \tilde{u}_i \leq \tilde{u}_{i+1}$, $\tilde{w}_{i-1} \leq \tilde{w}_i \leq \tilde{w}_{i+1}$ and $i = 1, \ldots, n_1 - 1, n_1 \in \mathbb{N}$.

However, inaccurate measurements usually cause dents and wavy shape which leads to the utilization of a smoothing criterion. We use the linearization of the strain energy. For computation we use the trapezoid rule for double integrals of the following formula.

$$c_2(b) = \int_0^1 \int_0^1 k_1(u,w)^2 + k_2(u,w)^2 du dw$$

With the aid of this criterion we achieve fairly smooth but flat surfaces. In contrary one may want only to get rid of a bump or dent without minimizing the curvature in such a strong way. To handle that kind of situation we construct a criterion for achieving elliptic points.

We stated in the previous section that the signs of the principal curvatures and therefore the gaussian curvature is an accurate detector for elliptic points. If one of the principal curvatures is equal to zero but the surface also has a wavy shape the sign of the other principal curvature gives also worthful information whereas gaussian curvature fails to give any information than that there is a flat point. So we look at the sum of squared principal curvatures multiplied with sign. We normalize the integral

$$\int_0^1\int_0^1 sign(k_1(u,w))k_1(u,w)^2 + sign(k_2(u,w))k_2(u,w)^2 dudw$$

with the integral

$$(\int_0^1\int_0^1 k_1(u,w)^2 + k_2(u,w)^2 dudw)^2$$

to one. To avoid denominator equal to 0 we add the constant 1 and get

$$C(b) = \frac{(\int_0^1\int_0^1 sign(k_1(u,w))k_1(u,w)^2 + sign(k_2(u,w))k_2(u,w)^2 dudw)^2 + 1}{(\int_0^1\int_0^1 k_1(u,w)^2 + k_2(u,w)^2 dudw)^2 + 1}$$

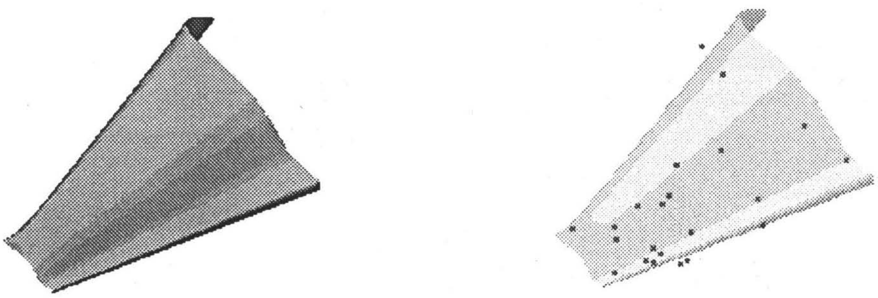
C(b) equals to one if and only if the sign of the principal curvatures are equal and do not change the sign. Therefore the function $c_3(b) = 1 - C^2(b)$ is minimized exactly when the signs of the principal curvatures equal and do not change the sign which means that there are only elliptic or flat points. We use c_3 as a third criteria function and use the trapezoid rule for double integrals for computation.

This criterion is strongly connected to our criterion we used in [3], where we used a criterion for convexity or concavity for curves and achieved very good results. This depends also on the strong connection of curve curvature and normal curvature. We could also use any curvature like gaussian or mean curvature, but as pointed out before, for detecting elliptic points and avoiding wavy shape the sum of the squares of principal curvatures with respect to sign (which means multiplication with its sign) is a promising approach and the results confirm that. In that context gaussian curvature does work for detecting elliptic points for the mentioned reason but does fail for wavy shape when one of the principal curvatures is zero.

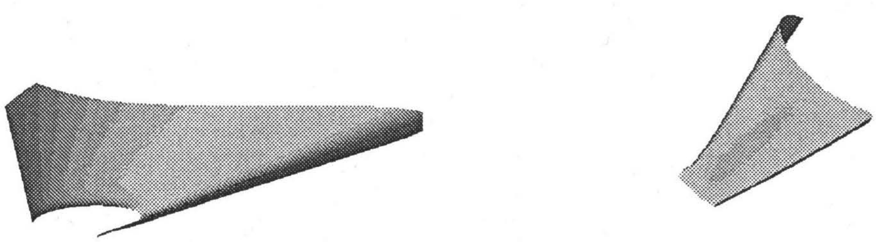
For numerical integration we use the trapezoid rule for double integrals. We use a Conjugate Gradient Method solving the constructed optimization problem. The gradient of the regarded weighted criteria vector

$$\text{grad}\ (\sum_{j=1}^{p} w_j c_j(b)) = (\sum_{j=1}^{p} w_j \frac{\partial}{\partial b_{0,0_{(0)}}} c_j(b), \ldots, \sum_{j=1}^{p} w_j \frac{\partial}{\partial b_{k1m,k2n_{(3)}}} c_j(b))$$

is itself a weighted sum of gradients of each criteria. The weights $w_j (j = 1, \ldots, p)$ are non negative real numbers which sums to one. The weights are chosen by the Decision Maker as described in the previous section. We add a few academic examples (see figure 2) and 3). The surface at the left hand side of figure 2) is the initial surface with a unwanted wavy shape. To illustrate the approximation quality at the right hand side the approximation points are drawn at the surface. At figure 3) the surface at the left hand side is the surface optimized only with criterion c_3. It has a convex shape but a great variation to the approximation points. Therefore you see the great effort in using the other to criteria contradicting to c_3 in combination with it.



Left Side: Initial Surface.
Right Side: Approximation points drawn on initial surface



Left Side: Resulting surface optimized with criterion c_3.
Right Side: Resulting surface optimized with criteria $c_1 - c_3$.
(Weight (0.05,0.475,0.475))

5 Conclusions and Further Research

A method for automatically smoothing of surfaces using Multicriteria Optimization Techniques was presented. Different, partially contradicting criteria for surface fitting were used. A new criterion for achieving elliptic points was introduced. In the present we are dividing the parameter space of the weightings to get representatives for the efficient solutions and reduce computing time. Further research in segmentation and parametrization has to be done.

References

[1] U. Bossong, *Reflecting partial orders and weighted sums*, to appear

[2] U. Bossong, D. Schweigert, *Minimal paths on ordered graphs*, to appear

[3] U. Bossong, H. Hagen, *Approximations and Smoothing with Spline Curves using Multicriteria Optimization Techniques*, Proceedings of the Spring Conference on Computer Graphics, Budmerice Castle, April 23 to 25, 1998

[4] G. Brunnett, H. Hagen, P. Santarelli, *Variational design of curves and surfaces*, Surv. Math. Ind. (1993) 3:, pp. 1–27

[5] M. P. Do Camo, *Differential geometry of curves and surfaces* Englewood Cliffs, NJ : Prentice-Hall (1976)

[6] H. Hagen, St. Hahmann, Th. Schreiber, *Surface Interrogation Algorithms*, IEEE Computer Graphics and Application, Vol.12, No.5 (1992)pp. 53–60

[7] H. Hagen, P. Santarelli, *Variational Design of Smooth B-splines surfaces*, In: Topics in surface modeling (Hagen, H., ed.), Philadelphia: SIAM 1992, pp. 85–92

[8] W. Hohenberger, T. Reuding, *Smoothing rational B–Spline curves using the weights in an optimization procedure*, CAGD12 (1995) , pp. 837–848

[9] N. Sapides, G. Farin, *Automatic fairing algorithm for B-spline curves*, CAD Vol. 22, No. 2, Mai 1990, pp. 120–129

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