Interner Bericht

A univariate method for plane elastic curves

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Abstract

The problem to interpolate Hermite-type data (i.e. two points with attached tangent vectors) with elastic curves of prescribed tension is known to have multiple solutions. A method is presented that finds all solutions of length not exceeding one period of its curvature function. The algorithm is based on algebraic relations between discrete curvature information which allow to transform the problem into a univariate one. The method operates with curves that by construction partially interpolate the given data. Hereby the objective function of the problem is drastically simplified. A bound on the maximum curvature value is established that provides an interval containing all solutions.

1 Introduction

An elastic curve is the idealized mathematical model of an elastic material that extends mainly in one dimension as e.g. a thin elastic beam. Based on D. Bernoulli's work plane elastic curves are defined as the extremals (critical points) of the functional

$$\int_0^L \kappa^2(s) + \sigma \, ds \tag{1}$$

where $\kappa = \kappa(s)$ denotes the curvature of the curve as a function of the arc length parameter s and where σ is constant. Applying the calculus of variation to this problem differential equations describing elastic curves have been derived by Euler and others (see [1], [10], [15], [16]).

In the context of CAGD this variational problem has gained widespread attention as a smoothing strategy in variational design and as the origin of the 'true' or nonlinear spline, see e.g. [3], [5], [6], [7], [11], [12], [13], [14], [18].

In this paper we address the basic interpolation problem to construct elastic curves of prescribed tension value σ and specified positions and tangents at the endpoints. This problem is known to have multiple solutions and we ask for a method to determine all solutions of length not exceeding one period of its curvature function. This set contains the shortest solution together with some alternatives with at most one loop. If curves that extend over several periods are wanted they can be produced by modifying the arc length formula used (see Theorem 7).

Existing methods are capable of computing particular solutions if suitable initial values are provided. But as the problem is treated in a multivariate setting it is a tedious procedure to search for initial values that produce alternative solutions.

In this paper it is shown that the problem can be tansformed into a zero location problem of a surprisingly simple univariate function. A bound on the maximum curvature values of the solution curves is established that provides an interval containing all zeros of the objective function. Thus, it becomes practical to determine all solutions to the problem.

Another important aspect of the univariate method is the reduction in the amount of data necessary to characterize particular solutions. It has been shown in [6] that fast computation of elastic splines interpolating sequences of points becomes possible if a table of initial values is available. The considerable amount of storage needed for such a table-based method could be drastically reduced by replacing the traditional approach that uses 3 reals to characterize one solution by the univariate method that needs only one real and one integer per curve.

The layout of this paper is as follows. After summarizing necessary preliminaries in section 2 we present the results fundamental to the new method in section 3. It is shown that the curvature information

$$\kappa(L), \kappa'(0), \kappa'(L)$$

of an interpolating elastica of length L can be computed from $\kappa(0)$ or – depending on a case distinction – from $\kappa'(0)$. Conditions are given that guarantee the existence of an elastic curve with a curvature function that interpolates all curvature information derived from an arbitrary value $\kappa(0)$ ($\kappa'(0)$ respectively). It is shown that this curve by constructing partially interpolates the given data.

A description of the algorithm is given in section 4. In particular it is shown how to derive a basic interval containing all solutions and how this search range can be restricted even further applying several conditions. Examples and empirical observations are given in section 5.

In the appendix the reader will find the elliptic integrals used in the notation of Burlisch [19].

2 Plane elastica

In this section we recall necessary preliminaries and the fundamentals of elastic curves as presented in [4].

A differentiable map $x : [0, L] \to \mathbf{R}^2$ is called an arc length parametrized curve if $|x'| = 1$. The curvature of an arc length parametrized curve is the function $\det(x', x'')$.

Any arc length parametrized plane curve of class C^2 has a representation of the form

$$x(s) = \int_0^s \begin{pmatrix} \cos(\Psi(\bar{s}) + \vartheta) \\ \sin(\Psi(\bar{s}) + \vartheta) \end{pmatrix} d\bar{s} + x(0) \tag{2}$$

where the function $\Psi : [0, L] \to \mathbf{R}$ is the turning angle of the tangent vector x' of x. Ψ is related to the curvature κ of x by

$$\Psi(s) = \int_0^s \kappa(\bar{s}) d\bar{s}.$$

The function

$$E(s) = \int_0^s \kappa^2(\bar{s}) d\bar{s}$$

is refered to as the energy function of x.

Definition 1 *An arc length parametrized plane curve $x \in C^2[0, L]$ is called elastic curve with tension parameter σ, if the curvature function κ of x satisfies the differential equation*

$$\kappa''(s) + \frac{1}{2}\kappa^3(s) - \frac{1}{2}\sigma\kappa(s) = 0 \tag{3}$$

for $s \in [0, L]$.

The solutions of differential equation (3) are analytic functions that can be expressed via Jacobi's elliptic functions.

Theorem 2 *For the curvature function κ of a plane elastic curve x with tension parameter σ the following statements hold:*

(i) The extension $\bar{\kappa}^2$ of κ^2 on $\mathbf{R}$ has a global maximum which will be denoted by κ_m^2.

(ii) $\kappa_m^2 \geq \sigma$

(iii) κ has a zero if and only $\kappa_m^2 > 2\sigma$. In this case κ is given by

$$\kappa(s) = \kappa_m \operatorname{cn}(\sqrt{(\kappa_m^2 - \sigma)/2}(s - s_m) \mid k^2) \tag{4}$$

with the parameter

$$k^2 = \frac{\kappa_m^2}{2(\kappa_m^2 - \sigma)}. \tag{5}$$

(iv) In the case $\kappa_m^2 \leq 2\sigma$ *the inequalities*

$$2\sigma - \kappa_m^2 \leq \kappa^2 \leq \kappa_m^2$$

hold. In this case κ *is given by*

$$\kappa(s) = \kappa_m \mathrm{dn}(\kappa_m(s - s_m)/2 \,|\, \frac{1}{k^2}). \tag{6}$$

Theorem 2 implies that the curvature function of an elastic curve extends to a periodic function on $\mathbf{R}$. In the case that $\kappa_m^2 > 2\sigma$ this periodic function is symmetric with respect to any zero of its derivative and antisymmetric with respect to any zero. According to Love [16] we will call this situation **inflectional** because the extension of x has turning points. In the case that $\kappa_m^2 \leq 2\sigma$ the extension of κ has no zeros but is still symmetric with respect to any zero of its derivative. The local extrema in this case are κ_m and $\pm\sqrt{2\sigma - \kappa_m^2}$. A classification of the different forms of elastica has been given by Euler (see [4], [10], [16]).

The next theorem summarizes the fundamental equations of elastic curves. In order to avoid repetitions we use the following convention: Ψ denotes the turning angle of x', E is the energy function of x, φ is related to the curvature function by

$$\cos\varphi = (\kappa(0)^2 - \sigma)/(\kappa_m^2 - \sigma) \qquad \sin\varphi = 2\kappa'(0)/(\kappa_m^2 - \sigma). \tag{7}$$

For convenience we also use the notation $C(\alpha) = (\cos\alpha, \sin\alpha)$. ϑ is the angle in $[0, 2\pi[$ with $x'(0) = C(\vartheta)$.

Theorem 3 *For a plane elastic curve* x *with curvature function* κ *and tension parameter* σ *the following relations hold:*

(i)

$$4(\kappa'(s))^2 = (\kappa_m^2 - \sigma)^2 - (\kappa^2(s) - \sigma)^2 \tag{8}$$

(ii)

$$\kappa'(s) = -\frac{1}{2}(\kappa_m^2 - \sigma)\sin(\Psi(s) - \varphi) \tag{9}$$

(iii)

$$\kappa(s) - \kappa(0) = -\frac{1}{2}(\kappa_m^2 - \sigma) < C'(\varphi + \vartheta), x(s) - x(0) > \tag{10}$$

(iv)

$$\kappa^2(s) = (\kappa_m^2 - \sigma)\cos(\Psi(s) - \varphi) + \sigma \tag{11}$$

(v)

$$E(s) = (\kappa_m^2 - \sigma) < C(\varphi + \vartheta), x(s) - x(0) > +\sigma s \tag{12}$$

We complete our list of preliminaries with the following representation theorem of plane elastica.

Theorem 4 *The elastic curve x with curvature function κ and tension parameter $\sigma \neq \kappa_m^2$ has a representation of the form*

$$x(s) = \frac{1}{\kappa_m^2 - \sigma} \begin{pmatrix} \sin(\varphi + \vartheta) & \cos(\varphi + \vartheta) \\ -\cos(\varphi + \vartheta) & \sin(\varphi + \vartheta) \end{pmatrix} \begin{pmatrix} 2(\kappa(s) - \kappa(0)) \\ E(s) - \sigma s \end{pmatrix} + x(0). \tag{13}$$

Formula (13) provides an explicit representation of an elastic curve in terms of its curvature function if $\kappa_m^2 \neq \sigma$. Note, that in the case $\kappa_m^2 = \sigma$ the elastic curve is a circle of radius $1/|\kappa_m|$.

Remark 5 *It has been shown in [4] that for an arc length parametrized curve the representation formula (13) implies that x is an elastic curve even if κ is neither assumed to be the curvature of x nor to fulfill (3).*

3 Results fundamental to the univariate method

We begin this section by investigating the relations between the initial and terminal curvature for an interpolating elastic curve.

Let $x : [0, L] \to \mathbf{R}^2$ be an elastic curve with tension parameter σ and curvature function κ. Since elastic curves are invariant under rotation and translation we may assume that $x(0) = (0, 0)$ and $x'(0) = (0, 1)$. Evaluating the relations (9)-(11) at 0 and L we obtain three equations for the quantities $\kappa_0 := \kappa(0), \kappa_1 := \kappa(L), \kappa_0' := \kappa'(0), \kappa_1' := \kappa'(L), (P^1, P^2) := x(L)$ and $(V^1, V^2) := x'(L)$:

$$\kappa_1' = \kappa_0' V^2 + \frac{1}{2}(\kappa_0^2 - \sigma) V^1 \tag{14}$$

$$\kappa_1^2 = -2\kappa_0' V^1 + (\kappa_0^2 - \sigma) V^2 + \sigma \tag{15}$$

$$\kappa_1 = \kappa_0' P^2 + \frac{1}{2}(\kappa_0^2 - \sigma) P^1 + \kappa_0. \tag{16}$$

If an Hermite-type boundary value problem for x is given the quantities P^1, P^2, V^1, V^2 are known while $\kappa_0, \kappa_1, \kappa_0'$ and κ_1' are unknown. If one of the unknowns is treated as a parameter, it is possible to express the remaining unknowns in terms of this parameter. The equations are of most simple form if κ_0 is chosen to serve as the parameter. To further simplify notation we use the abbreviations:

$$A := (\kappa_0^2 - \sigma) V^2 + \sigma$$

$$B := \frac{1}{2}(\kappa_0^2 - \sigma) V^1$$

$$C := \frac{1}{2}(\kappa_0^2 - \sigma) P^1 + \kappa_0.$$

Theorem 6 *For an elastic curve* $x : [0, L] \to \mathbf{R}^2$ *with tension parameter* $\sigma, x(0) = (0,0), x'(0) = (0,1), x(L) = P = (P^1, P^2)$ *and* $x'(L) = V = (V^1, V^2)$ *the initial and terminal curvature values* κ_0, κ_1 *are related by the quadratic equation*

$$P^2\kappa_1^2 + 2V^1\kappa_1 - (\kappa_0^2 - \sigma)(V^1P^1 + V^2P^2) - 2V^1\kappa_0 - \sigma P^2 = 0. \tag{17}$$

Case 1: If $P^2 \neq 0$, then κ_0 satisfies the inequality

$$P^2 < P, V > \kappa_0^2 + P^2V^1\kappa_0 + \sigma P^2(P^2 - < P, V >) + (V^1)^2 \geq 0 \tag{18}$$

and

$$\kappa_0' = \frac{\kappa_1 - C}{P^2} \tag{19}$$

$$\kappa_1' = \kappa_0' V^2 + B \tag{20}$$

Case 2: If $P^2 = 0$ but $V^1 \neq 0$, then

$$\kappa_1 = C \tag{21}$$

$$\kappa_0' = \frac{A - C^2}{2V^1} \tag{22}$$

$$\kappa_1' = \kappa_0' V^2 + B \tag{23}$$

Case 3: If $P^2 = 0$ and $V^1 = 0$, then

$$\kappa_0^2 = \sigma \quad \text{or} \quad (P^1\kappa_0 + 2)^2 = \sigma(P^1)^2 + 4V^2 \tag{24}$$

$$\kappa_1 = C \tag{25}$$

$$\kappa_1' = \kappa_0' V^2 \tag{26}$$

Proof: Using (16) to eliminate κ_0' in (15) one obtains (17). This quadratic equation has real roots if and only if the discriminant is non-negative. This gives the inequality (18).
If $P^2 = 0$ and $V^1 = 0$, then we find from (14) - (16) that κ_0 satisfies the equation

$$\begin{aligned}(P^1)^2\kappa_0^4 &+ 4P^1\kappa_0^3 + [4 - 4V^2 - 2\sigma(P^1)^2]\kappa_0^2 \\ &- 4\sigma P^1\kappa_0 + \sigma^2(P^1)^2 + (4V^2 - 4)\sigma = 0\end{aligned}$$

that can be reduced to (24).
The rest of the proof is straightforward.

□

The relevance of Theorem 6 can be summarized as follows:
given one additional curvature information (κ_0 in case 1 and 2, κ_0' in case 3) the initial and terminal curvature properties of x can be computed. Since the differential equation

for κ can be used to compute higher order derivatives, this statement extends to initial and terminal curvature derivatives of any degree.

We will now consider the following question: given P, V, σ and a set of values $\kappa_0, \kappa_1, \kappa_0', \kappa_1'$ such that the equations (14) - (16) hold, is there a curvature function κ with period T satisfying (3) and

$$\kappa(0) = \kappa_0, \kappa(L) = \kappa_1, \kappa'(0) = \kappa_0', \kappa'(L) = \kappa_1$$

where L is not bigger than T?

Theorem 7 *For given $\sigma, P^1, P^2, V^1, V^2 \in \mathbf{R}$ let $\kappa_0, \kappa_1, \kappa_0', \kappa_1'$ satisfy the system of equations (14) - (16). Let κ be the solution of the initial value problem*

$$\begin{aligned} \kappa'' + \frac{1}{2}\kappa^3 - \frac{1}{2}\sigma\kappa &= 0 \\ \kappa(0) = \kappa_0, \kappa'(0) &= \kappa_0'. \end{aligned} \tag{27}$$

Let T denote the period of κ.

(i) If $0 \neq \kappa_m^2 > 2\sigma$ or ($\kappa_m^2 \leq 2\sigma$ and $\kappa_0 \cdot \kappa_1 > 0$), there is a unique $L \in]0, T]$ such that

$$\kappa(L) = \kappa_1 \quad \textit{and} \quad \kappa'(L) = \kappa_1'. \tag{28}$$

(ii) If $0 \neq \kappa_m^2 \leq 2\sigma$ and $\kappa_0 \cdot \kappa_1 \leq 0$, there is no such L.

Proof: Let κ_m^2 denote the global maximum of κ^2 according to Theorem 2. Then (8) and (27) imply

$$4(\kappa_0')^2 = (\kappa_m^2 - \sigma)^2 - (\kappa_0^2 - \sigma)^2. \tag{29}$$

(14), (15) and (29) can be combined to yield

$$4(\kappa_1')^2 = (\kappa_m^2 - \sigma)^2 - (\kappa_1^2 - \sigma)^2. \tag{30}$$

Since $\kappa_m^2 \geq \sigma$ (30) implies that

$$\kappa_1^2 \leq \kappa_m^2. \tag{31}$$

Furthermore, (30) can be rewritten in the form

$$4(\kappa_1')^2 = (\kappa_m^2 - \kappa_1^2)(\kappa_1^2 + \kappa_m^2 - 2\sigma). \tag{32}$$

Thus, we obtain from (31) and (32)

$$\kappa_1^2 \geq 2\sigma - \kappa_m^2. \tag{33}$$

Now, we consider the inflectional case, i.e. $\kappa_m^2 > 2\sigma$, in which (33) is trivially fulfilled. Here

$$\kappa(s) = \kappa_m cn\left(\sqrt{(\kappa_m^2 - \sigma)/2}\, s_m/k^2\right)$$

and because of the cosine-like behaviour of this function for any $\kappa_1^2 < \kappa_m^2$ there are exactly two values $L_1, L_2 \in]0,T]$ with $\kappa(L_1) = \kappa(L_2) = \kappa_1$ and $\kappa'(L_1) = -\kappa'(L_2)$.

At each L_i (8) implies

$$4[\kappa'(L_i)]^2 = (\kappa_m^2 - \sigma)^2 - \left([\kappa(L_i)]^2 - \sigma\right)^2$$

Hence, with $\kappa(L_i) = \kappa_1$ and (30) one obtains

$$[\kappa'(L_i)]^2 = (\kappa_1')^2.$$

Since $\kappa'(L_1) = -\kappa'(L_2)$, $i \in \{1,2\}$ can be chosen such that

$$\kappa(L_i) = \kappa_1 \quad \text{and} \quad \kappa'(L_i) = \kappa_1'.$$

For $\kappa_1^2 = \kappa_m^2$ there is exactly one $L \in]0,T]$ where

$$\kappa(L) = \kappa_1.$$

In this case (8) and (30) imply

$$\kappa'(L) = \kappa_1 = 0.$$

In the non-inflectional case, i.e. $\kappa_m \leq 2\sigma$, κ is given by

$$\kappa(s) = \kappa_m dn\left(\kappa_m(s - s_m)/2|\frac{1}{k^2}\right).$$

As mentioned in section 2 the function dn is always positive and the range of κ^2 is

$$2\sigma - \kappa_m^2 \leq \kappa^2(s) \leq \kappa_m^2. \tag{34}$$

Therefore if $\kappa_0 \cdot \kappa_1 \leq 0$ there is no L with (28). However, if $\kappa_0 \cdot \kappa_1 > 0$ κ_1 lies in the range of κ because of (31) and (33). In analogy to the inflectional case the symmetry properties of dn give that for $\kappa_1^2 \neq \kappa_m^2$ and $\kappa_1^2 \neq 2\sigma - \kappa_m^2$ there are exactly two $L_i \in]0,T]$ with

$$\kappa(L_1) = \kappa(L_2) = \kappa_1 \quad \text{and} \quad \kappa'(L_1) = -\kappa'(L_2).$$

While for $\kappa_1^2 = \kappa_m^2$ or $\kappa_1^2 = 2\sigma - \kappa_m^2$ there is exactly one $L \in]0,T]$ with

$$\kappa(L) = \kappa_1 \quad \text{and} \quad \kappa'(L) = \kappa_1' = 0.$$

□

It is a straightforward procedure to compute the unique length L of Theorem 7. We state the result in the following supplement to Theorem 7 where for simplicity we assume that κ_0 is not positive. This situation can always be achieved by reflecting the interpolation data at the y-axis. Furthermore, we use the conventions that int_i with $i \in \{0,1\}$ denotes the integral

$$\text{int}_i := \int_{|\kappa_i|}^{|\kappa_m|} \frac{d\kappa}{\sqrt{(1/4)(\kappa_m^2 - \kappa^2)(\kappa^2 + \kappa_m^2 - 2\sigma)}}.$$

npos and dsign denote the functions

$$\text{npos}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\text{dsign}(x, y) = \begin{cases} -\text{sign}(y) & x > 0 \\ \text{sign}(y) & x \leq 0 \end{cases}$$

Supplement to Theorem 7:

If $\kappa_m^2 \neq \sigma$, and $\kappa_0 \leq 0$ the parameters κ_m, s_m of κ with (27) and (28) are given by:

$$\kappa_m^2 = \sqrt{4(\kappa_0')^2 + (\kappa_0^2 - \sigma)^2} + \sigma, \quad sign(\kappa_m) = -1;$$

$$s_m = -sign(\kappa_0')int_0.$$

For $\kappa_m^2 \neq 2\sigma$ the length parameter L can be obtained from

$$L = \begin{cases} L' & 0 < L' \leq T \\ L' + T & L' \leq 0 \\ L' - T & L' > 0 \end{cases}$$

where in the inflectional case

$$L' = s_m + T - npos(\kappa_1)\frac{T}{2} + dsign(\kappa_1, \kappa_1')int_1$$

and in the non-inflectional case

$$L' = s_m + T + sign(\kappa_1')int_1.$$

In the special case $\kappa_m^2 = 2\sigma$

$$L = s_m + sign(\kappa_1')int_1.$$

We will now show that the elastic curve x with curvature function κ according to Theorem 7 partially interpolates the given data.

Theorem 8 *For given $\sigma, P^1, P^2, V^1, V^2 \in \mathbf{R}$ let $\kappa_0, \kappa_1, \kappa_0', \kappa_1'$ satisfy the system of equations (14) - (16). Furthermore, let $\kappa_0' \neq 0, \kappa_0^2 \neq \sigma$ and $\kappa_0 \cdot \kappa_1 > 0$ if $\kappa_m^2 \leq 2\sigma$. Then, the elastic curve x with $x(0) = (0,0), x'(0) = (0,1)$, curvature function κ according to (27) and (28) satisfies*

(i) $x'(L) = (V^1, V^2)$

(ii) $x^1(L) = P^1 \quad \Leftrightarrow \quad x^2(L) = P^2.$

Proof:

(i) Since x is an elastic curve (9) and (11) hold which under the assumptions made take the form

$$\begin{aligned} \kappa'(L) &= \kappa'(0)x^{2'}(L) + \frac{1}{2}((\kappa(0))^2 - \sigma)x^{1'}(L) \\ (\kappa(L))^2 &= -2\kappa'(0)x^{1'}(L) + ((\kappa(0))^2 - \sigma)x^{2'}(L) + \sigma. \end{aligned}$$

Since κ interpolates $\kappa_0, \kappa_1, \kappa_0'$ and κ_1' one solution of this uniquely solvable system is provided by (14) and (15): $x^{1'}(L) = V^1$ and $x^{2'}(L) = V^2$.

(ii) The linear curvature relation (10) for elastic curves takes the form

$$\kappa(L) = \kappa'(0)x^2(L) + \frac{1}{2}((\kappa(0))^2 - \sigma)x^1(L) + \kappa(0).$$

Again, since κ interpolates $\kappa_0, \kappa_1, \kappa_0'$ and κ_1' the comparison with (16) yields the claimed relation.

□

Obviously, Theorem 8 allows to simplify the objective function of the univariate method.

4 The algorithm

In this section a univariate method is presented to compute elastic curves with fixed tension parameter σ and prescribed initial and terminal positions $O, P \in \mathbf{R}^2$ and tangent vectors $U \in T_O\mathbf{R}^2$ and $V \in T_P\mathbf{R}^2$. We restrict ourself to compute elastic curves of length L not exceeding one period of their curvature function, i.e. we use the formulas for L given in Theorem 7. This restriction implies that the turning angle $\psi \in [-2\pi, 2\pi]$. Despite this restriction for each data set $\{O, P, U, V\}$ there may exist multiple solutions and it is the purpose of the algorithm to provide all of these. For convenience we assume that $O = (0, 0)$ and $U = (0, 1)$. Since elastic curves are invariant under uniform scaling we will further assume that $d = |P| = 1$. Note, that if d is scaled by λ, σ has to be scaled by $1/\lambda^2$.

The results of section 3 allow to transform the original problem into a zero location problem of a univariate function. In case 2 ($P^2 = 0, V^1 \neq 0$) we consider the objective function

$$f(\kappa_0) = x^1(L) - P^1 \tag{35}$$

which is evaluated in the following way:
For each κ_0 we compute the values of κ_1, κ_0' and κ_1' according to (21) - (23). Then

the parameters κ_m, s_m of the curvature function κ of x and the length L are computed according to the supplement to Theorem 7. Finally x^1 is evaluated at L by virtue of (13).

In case 1 ($P^2 \neq 0$) we have two objective functions of the form (35) to consider because (17), (19) and (20) provide two sets of values for κ_1, κ_0' and κ_1' for each κ_0. Both functions may contribute solutions to the problem.

In the very special case 3 ($P^2 = 0, V^1 = 0$) we have 4 objective functions of the form

$$f(\kappa_0') = x^1(L) - P^1 \tag{36}$$

to consider because there are four possible values of κ_0 according to (24).

We will now determine an interval containing all zeros of the objective functions that correspond to solutions of the problem.

Theorem 9 *The parameter κ_m^2 of an elastic curve $x : [0, L] \to \mathbf{R}$ with tension parameter $\sigma > 0, |x(L) - x(0)| = 1$ and period $T \geq L$ is bounded by*

$$\kappa_m^2 \leq \max\left\{2\sigma + \varepsilon, \left(4\sqrt{2}K(m)\right)^2 + \sigma\right\} \tag{37}$$

where K denotes the complete elliptic integral of the first kind, $\varepsilon > 0$ and

$$m := 1 - \frac{1}{2}\frac{\varepsilon}{\sigma + \varepsilon}.$$

Proof: If $\kappa_m^2 \geq 2\sigma + \varepsilon$ the modulus k^2 of x is given by

$$k^2 = \frac{\kappa_m^2}{2(\kappa_m^2 - \sigma)}.$$

For positive σ k^2 is a monotone decreasing function of κ_m^2, thus

$$k^2 \leq \frac{2\sigma + \varepsilon}{2(\sigma + \varepsilon)} = 1 - \frac{1}{2}\frac{\varepsilon}{\sigma + \varepsilon} = m.$$

Since K is monotone increasing with k^2 we have

$$K(k^2) \leq K(m).$$

The period T of x is in this case given by

$$T = \frac{4}{\sqrt{\frac{1}{2}(\kappa_m^2 - \sigma)}} K(k^2).$$

Thus the assumption $T \geq L \geq 1$ gives

$$\kappa_m^2 \leq \left(4\sqrt{2}\, K(m)\right)^2 + \sigma.$$

□

Theorem 9 implies

Corollary 10 *If in the situation of Theorem 9 ε is chosen such that*

$$\sigma < \left[4\sqrt{2}\, K(m)\right]^2 - \varepsilon \tag{38}$$

then

$$\kappa_m^2 \leq \left[4\sqrt{2}\, K(m)\right]^2 + \sigma \tag{39}$$

Proof: Follows directly from (37)

□

Corollary 10 suggests the following procedure to obtain the upper bound for $|\kappa_m|$: for each σ given ε should be chosen such that (38) holds and $\sigma \approx \left[4\sqrt{2}\, K(m)\right]^2 - \varepsilon$. However, in practice it is sufficient to determine upper bounds for ranges of tension values.

The following examples of upper bounds have been taken from the list of bounds used in our implementation:

$\sigma <$	$\lvert\kappa_m\rvert <$
7	11
22.8	12
40	13
55.1	14
74.4	15
1000	45

In case 1 and 2 all possible solutions are contained in the basic interval $I = [-|\kappa_m|, |\kappa_m|]$ because $\kappa_0^2 \leq \kappa_m^2$. In case 3 the basic interval is obtained from (39) and the inequality

$$4(\kappa_0')^2 \leq (\kappa_m^2 - \sigma)^2$$

which follows from (8).

In the most general case 1 the basic search interval can be further reduced using condition (18). Obviously, solutions can only exist in the intersection $\hat{I}$ of I with the solution set of the quadratic inequality (18). Note, that $\hat{I}$ may consist of 0, 1 or two intervals .

As a composition of continuous functions the objective functions (35) and (36) are continuous but these functions may not be differentiable at values $\kappa_0(\kappa_0'$ resp.) that separate the inflectional and non-inflectional case, i.e. where $\kappa_m^2 = 2\sigma$. The cusps in the objective functions shown in figure 1-3 correspond to such values. Although our zero finder does not require differentiability the performance is improved if $\hat{I}$ is subdivided at those points.

Theorem 11 *Let $x : [0, L] \to \mathbf{R}^2$ be an elastic curve with parameters σ and κ_m related by $2\sigma = \kappa_m^2$. Let $x(0) = (0,0), x'(0) = (0,1), x(L) = (P^1, P^2)$ and $x'(L) = (V^1, V^2)$.*

(i) If $P^2 \neq 0$ then the initial curvature κ_0 of x is a zero of the polynomial

$$\begin{aligned}
& \kappa_0^8\left((P^1)^2+(P^2)^2\right)^2 \qquad (40)\\
+ & \; 8P^1\kappa_0^7\left((P^1)^2+(P^2)^2\right)\\
- & \; 4\kappa_0^6\left(-4V^1P^1P^2+2V^2\left((P^1)^2-(P^2)^2\right)+\sigma\left((P^1)^2+(P^2)^2\right)^2-2\left(3(P^1)^2+(P^2)^2\right)\right)\\
- & \; 8\kappa_0^5\left(-4V^1P^2+P^1\left(4V^2+3\sigma\left((P^1)^2+(P^2)^2\right)-4\right)\right)\\
+ & \; 2\kappa_0^4\left(-24V^1\sigma P^1P^2+4V^2\left(3\sigma\left((P^1)^2-(P^2)^2\right)-4\right)\right.\\
& \qquad \left.+\sigma^2\left((P^1)^2+(P^2)^2\right)\left(3(P^1)^2+2(P^2)^2\right)-4\sigma\left(7(P^1)^2+(P^2)^2\right)+16\right)\\
+ & \; 8\sigma\kappa_0^3\left(-8V^1P^2+P^1\left(8V^2+\sigma\left(3(P^1)^2+2(P^2)^2\right)-8\right)\right)\\
- & \; 4\sigma\kappa_0^2\left(-8V^1\sigma P^1P^2+2V^2\left(\sigma\left(3(P^1)^2-2(P^2)^2\right)-8\right)\right.\\
& \qquad \left.+\sigma^2(P^1)^2\left((P^1)^2+(P^2)^2\right)-2\sigma\left(5(P^1)^2-2(P^2)^2\right)+16\right)\\
- & \; 8\sigma^2P^1\kappa_0\left(4V^2+\sigma(P^1)^2-4\right)\\
+ & \; \sigma^2\left(4V^2+\sigma(P^1)^2-4\right)^2
\end{aligned}$$

(ii) If $P^2 = 0$ and $V^1 \neq 0$ κ_0 is a zero of the polynominal that is obtained by setting $P^2 = 0$ in (40).

(iii) If $P^2 = 0$ and $V^1 = 0$ the initial curvature derivative κ_0' is given by

$$4(\kappa_0')^2 = \sigma^2 \quad or \quad 4(\kappa_0')^2 = \sigma^2 - \left[\left(\frac{-2 \pm \sqrt{\sigma(P^1)^2+4V^2}}{P^1}\right)^2 - \sigma\right]^2 \qquad (41)$$

Proof:

(i) Let $P^2 \neq 0$.
(29) together with $\kappa_m^2 = 2\sigma$ implies

$$4(\kappa_0')^2 = \sigma^2 - (\kappa_0^2 - \sigma)^2. \qquad (42)$$

Replacing κ_0' in (42) using (16) we obtain

$$4\left[\kappa_1 - \kappa_0 - \frac{1}{2}(\kappa_0^2-\sigma)P^1\right]^2 + (P^2)^2((\kappa_0^2-\sigma)^2-\sigma^2) = 0. \qquad (43)$$

Replacing κ_1^2 by (17) a linear equation in κ_1 is obtained. Solving this linear equation and substituting the result into (43) yields (40).

(ii) Let $P^2 = 0$, $V^1 \neq 0$.
Replacing κ_0' in (42) using (15) and substituting κ_1 by (21) yields (40) with $P^2 = 0$.

(iii) Let $P^2 = 0,\ V^1 = 0$.
Substituting (24) into (42) yields (41).

□

In case 1 and 2 the zeros of the polynomial (40) that belong to $\hat{I}$ are computed; double roots are discarded. In case 2 the sequence of roots provides a subdivision of $\hat{I}$ into subintervals such that each interval corresponds to only one type of curves: inflectional or non-inflectional.

In case 1 such a sequence of classified subintervals is constructed for each objective function. This is done by checking for each zero of (40) which of the two κ_1-values according to (17) in fact provides $\kappa_m^2 = 2\sigma$. Thus the roots are divided in two groups one for each objective function.

Following Theorem 7(ii) the non-inflectional subintervals have to be further processed according to the condition

$$\kappa_0\,\kappa_1 \geq 0. \tag{44}$$

Using (17) in case 1 and (21) in case 2 the region where (44) holds is explicitely determined and intersected with the non-inflectional subintervals.

Finally, we have to exclude the special cases mentioned in Theorem 8:
$\kappa_0^2 = \sigma$ and $\kappa_0' = 0$. Since

$$\kappa_0' = 0 \quad \Leftrightarrow \quad (P^1\kappa_0 + 2)^2 = \sigma(P^1)^2 + 4V^2 \quad \vee \quad \kappa_0^2 = \sigma \tag{45}$$

there are four critical values of κ_0 that have to be observed. At each of these values the objective function according to (35) is evaluated. If f is non-zero the corresponding curvature value is simply canceled from the list of critical values. However, if f is zero at $\bar{\kappa_0}$ it has to be checked if $\bar{\kappa_0}$ provides a solution to the problem, i.e. if $x^2(L) = P^2$ (for $\kappa_0^2 = \sigma$ it is also necessary to check if $x'(L) = V$). If this is not the case the corresponding subinterval is subdivided at $\bar{\kappa_0}$ in order to avoid the zero finder to run into this unwanted zero of the objective function.

After these preparations a two step root finding procedure consisting of zero bracketing and subsequent application of Brent's methods is started on each subinterval of $\hat{I}$ (see [2], [19]). For the bracketing process empirical observations on the behaviour of the objective function can be used (see next section).

In case 3 we procede in a similar way. $\hat{I}$ is subdivided at values according to (41) and non-inflectional subintervals are intersected with the solution set of (44). However, since (24) is valid any zero provided by the zero finder working on the function (36) has to be verified to be a solution of the problem.

5 Examples and observations

The following figures have been produced with an implementation of the algorithm described in the previous section. In each figure all interpolating elastic curves of length not exceeding one period have been shown together with the graphs of the objective functions. The zeros of of the objective functions have been marked in the color of the corresponding solution curve.

Figure 1 shows a case 1 example, thus, two objective functions are shown. The normalized interpolation data is $P^1 = 0.868, P^2 = -0.497, V^1 = -0.917, V^2 = -0.398$ and $\sigma = 10.99$.

Using our list of upper bounds derived from Corollary 10 we obtain that all solutions are contained in $I = [-11.5, 11.5]$. The intersection of I with the solutions set of (18) yields $\hat{I} = [-11.5, -2.71] \cup [-0.356, 11.5]$.

In this example all roots of the polynomial (40) are real and contained in $\hat{I}$. For the first objective function f_1 the zeros

$$R_1 = \{-4.006, 0.478, 1.079\}$$

are relevant, while for the second objective function f_2

$$R_2 = \{-4.375, -3.777, -2.718, 2.299, 4.076\}$$

is the set of zeros to be considered.

We take a closer look at the second objective function which is slightly more interesting. In this case the subintervals of $\hat{I}$ given by R_2 correspond to the following curve types:

$[-11.5, -4.375]$	inflectional;	$[-4.375, -3.777]$	non-inflectional;
$[-3.777, -2.718]$	inflectional;	$[-2.718, -2.71]$	non-inflectional;
$[-0.356, 2.299]$	inflectional;	$[2.299, 4.076]$	non-inflectional;
$[4.076, 11.5]$	inflectional.		

The non-inflectional subintervals have to be intersected with the solution set of (44) which is [-3587, 11.5].

Furthermore, the four special values according to (45) are

$$-3.315, 3.315, 0.675 \text{ and } -5.283.$$

Since f_2 is zero at -5.283 this value serves as a new intersection point. Note, that this zero does not correspond to a solution of the problem. Thus, we obtain the following sequence of intervals

$$\begin{aligned} I_1 &= [-11.5, -5.283] \quad , \quad I_2 = [-5.283, -4.375], \\ I_3 &= [-3.777, -2.718] \quad , \quad I_4 = [-2.718, -2.71], \\ I_5 &= [-0.356, 2.299] \quad , \quad I_6 = [2.299, 4.076], \\ I_7 &= [4.076, 11.5] \quad . \end{aligned}$$

One implemented variant of our algorithm checks if adjacent intervals may be joined because the objective function is (numerically) differentiable at the junction point. In this example I_3 may be joined with I_4 and I_6 may be joined with I_7. The resulting five intervals are shown in figure 1; interval endpoints are marked by vertical lines. The only solution found is $\kappa_0 = -4.852$.

Figure 2 shows a second case 1 example while in figure 3 a case 3 example is drawn.

Observing the objective functions for these examples we find that

(a) on each subinterval there are at most 2 zeros

(b) if two zeros occur the objective function is convex on this subinterval.

These observations are not specific to these examples but have been verified for all examples considered. Taking these properties of the objective functions for granted a bracketing procedure has been designed that uses only a few function evaluations per interval, thus, making the algorithm extremly fast.

To give the reader an impression of the performance we mention that on an SGI Indigo XS R4400 in one second real time all solutions of about 10 different interpolation problems can be computed.

A Appendix

In the inflectional case $\kappa_m^2 > 2\sigma$

$$\begin{aligned} \text{int}_i &= \frac{1}{\sqrt{(1/2)(\kappa_m^2 - \sigma)}} \int_0^{z_i} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \\ &= \frac{1}{\sqrt{(1/2)(\kappa_m^2 - \sigma)}} \int_0^{x_i} \frac{dx}{\sqrt{(1+x^2)(1+k_c^2x^2)}} \\ &= \frac{1}{\sqrt{(1/2)(\kappa_m^2 - \sigma)}} el2(x_i, k_c, 1, 1) \end{aligned}$$

with

$$z_i = \frac{\sqrt{\kappa_m^2 - \kappa_i^2}}{|\kappa_m|}, x_i = \frac{z_i}{\sqrt{1 - z_i^2}}$$

and where k^2 is defined through (5) and $k_c^2 = 1 - k^2$ denotes the complementary modul. The function el2 is the general elliptic integral of the second kind in the notation of Burlisch as available in [8].

The period T is given by

$$T = \frac{4}{\sqrt{(1/2)(\kappa_m^2 - \sigma)}} cel(k_c, 1, 1, 1)$$

where $K := cel(k_c, 1, 1, 1)$ is the notation of Burlisch for the complete elliptic integral of the first kind.

In the non-inflectional case $\kappa_m^2 \leq 2\sigma$

$$\begin{aligned} \text{int}_i &= \frac{2}{|\kappa_m|}\int_0^{z_i} \frac{dz}{\sqrt{(1-z^2)(1-l^2z^2)}} \\ &= \frac{2}{|\kappa_m|}\int_0^{x_i} \frac{dx}{\sqrt{(1+x^2)(1+l_c^2x^2)}} \\ &= \frac{2}{|\kappa_m|} el2(x_i, l_c, 1, 1). \end{aligned}$$

with

$$z_i = \sqrt{\frac{\kappa_m^2 - \kappa_i^2}{2(\kappa_m^2 - \sigma)}}, x_i = \frac{z_i}{\sqrt{1 - z_i^2}}$$

and where $l^2 = 1/k^2$ and $l_c^2 = 1 - l^2$.

The period is given by

$$T = \frac{4}{|\kappa_m|} cel(l_c, 1, 1, 1).$$

In the special case $\kappa_m^2 = \sigma$

$$\text{int}_i = \frac{2}{|\kappa_m|}\text{atanh}(z_i).$$

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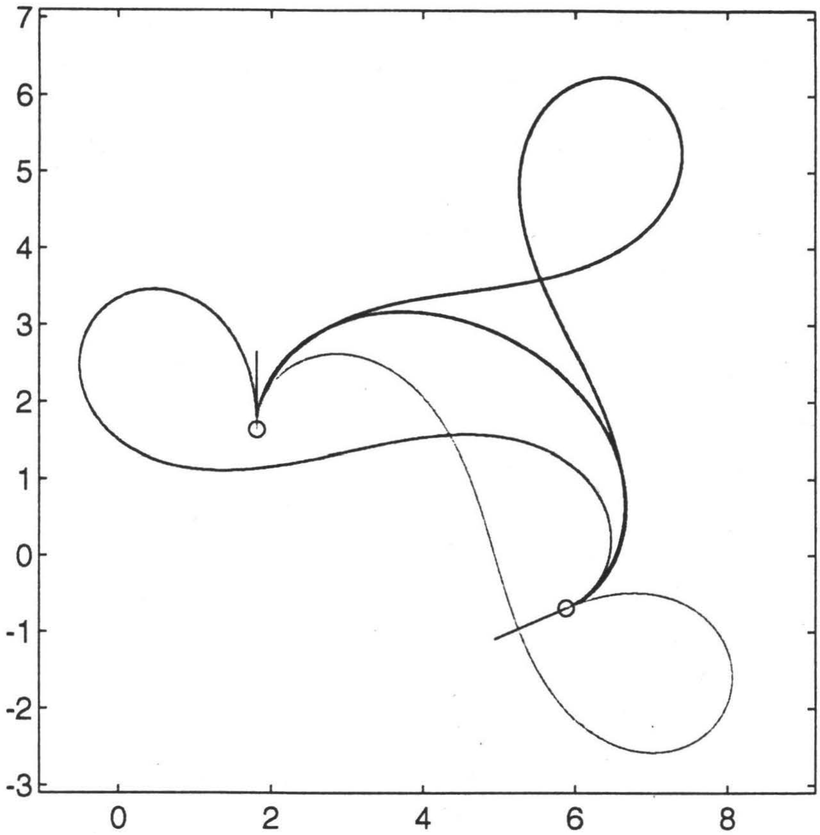
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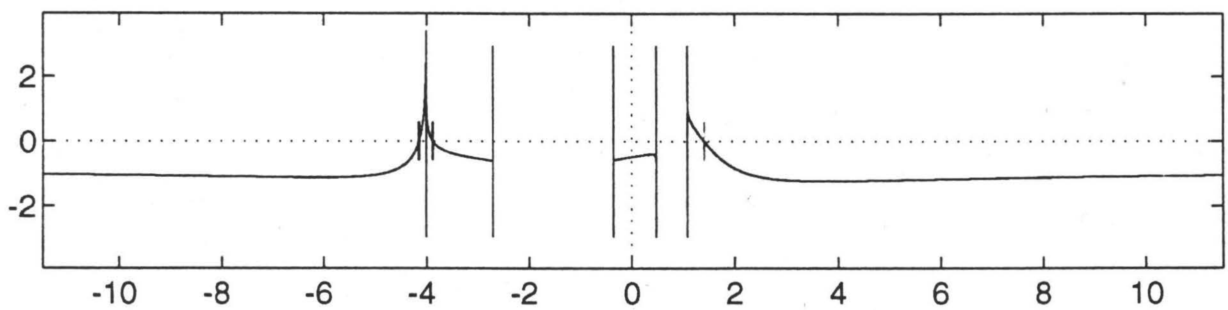
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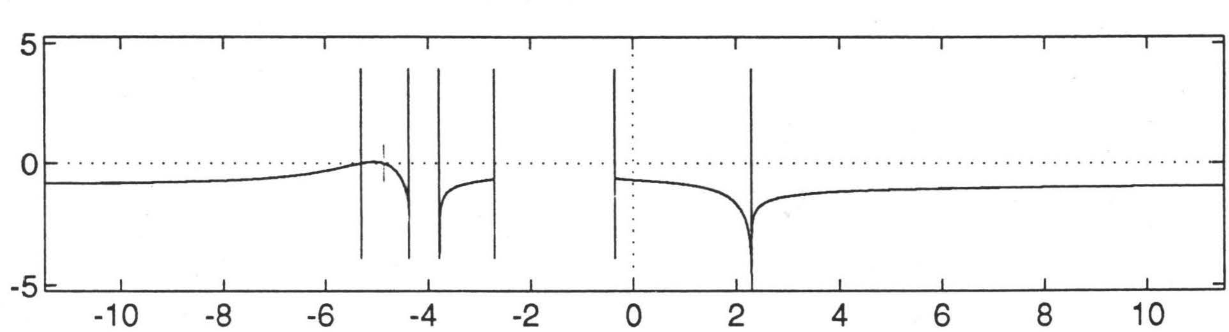
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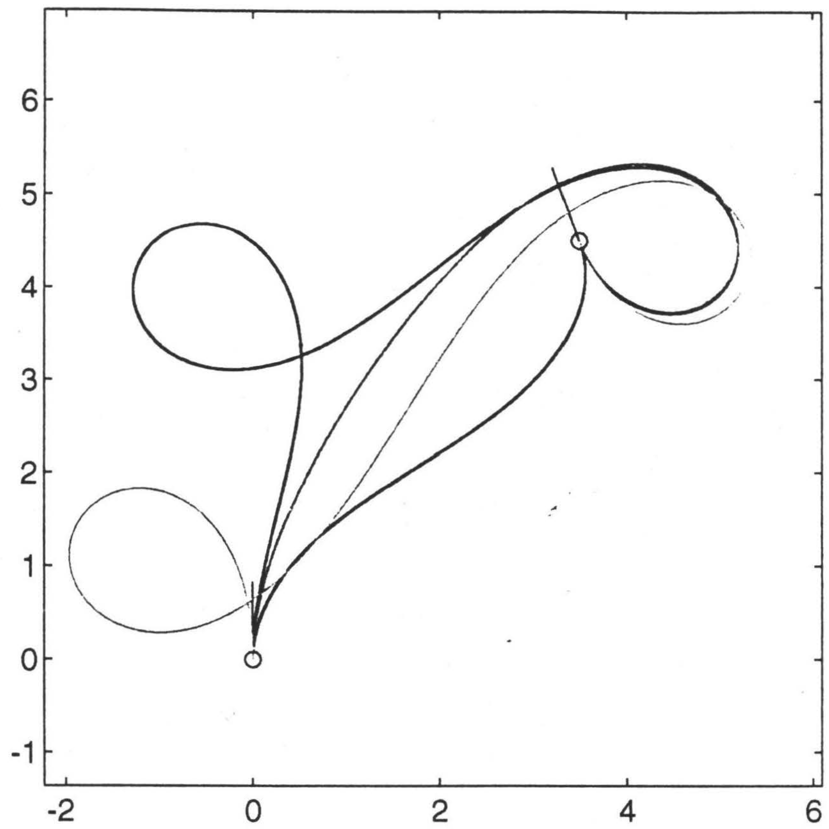

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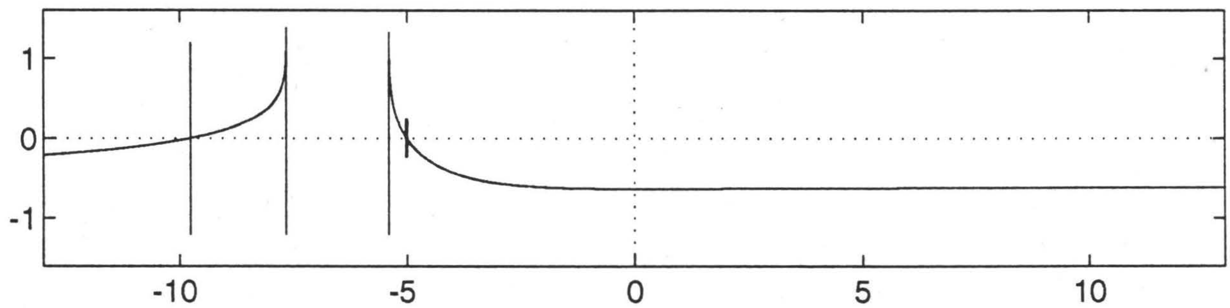

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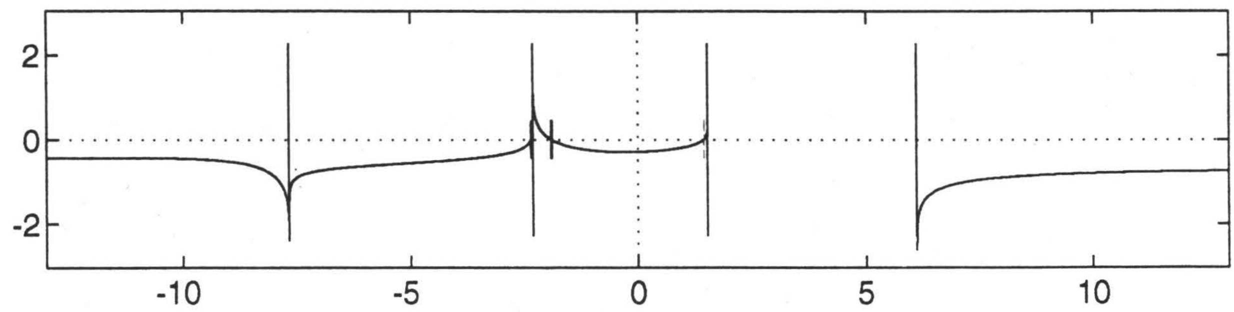

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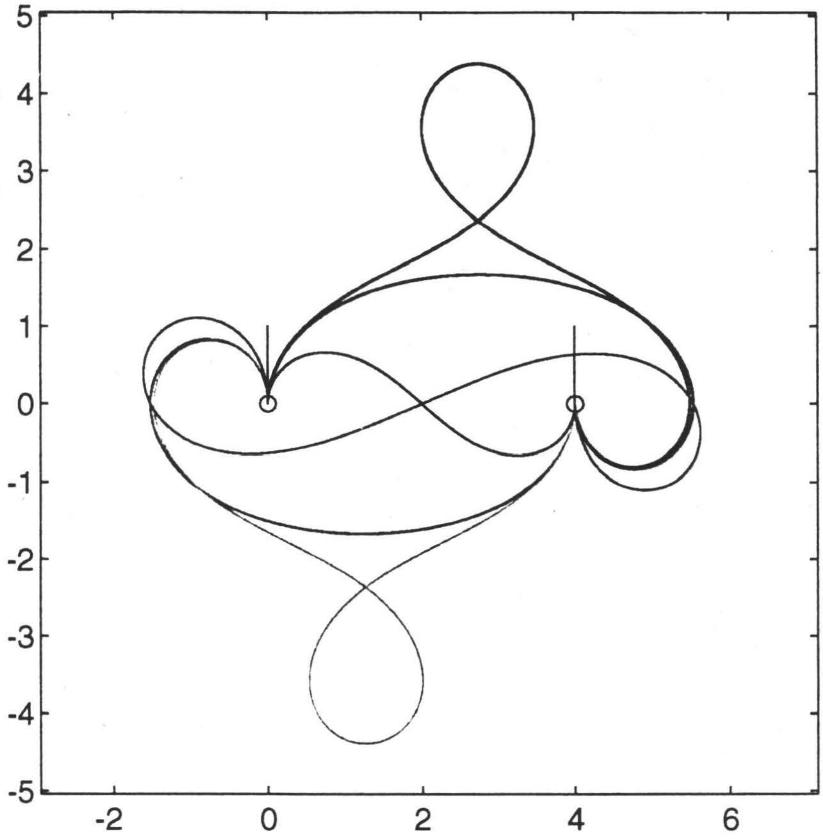
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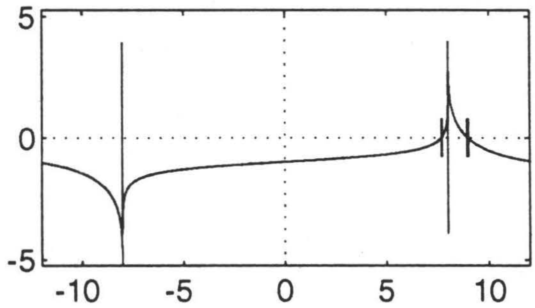
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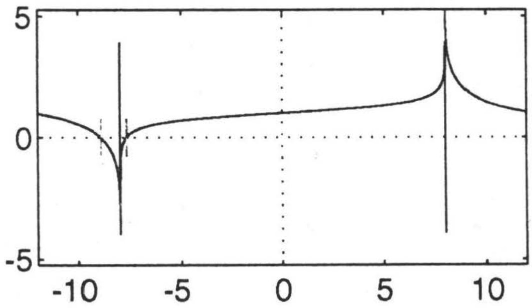
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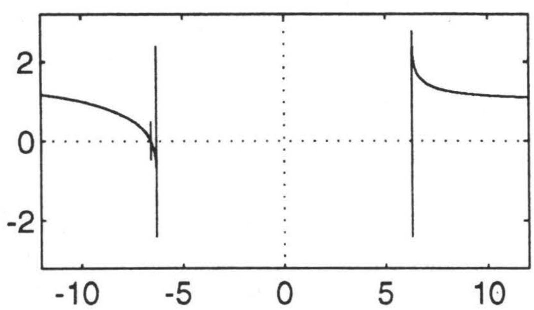
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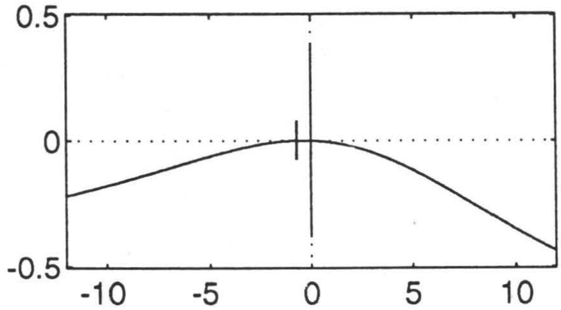
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