Interner Bericht

The geometry of optimal degree reduction of Bezier curves

Guido Brunnett, Thomas Schreiber und Jörg Braun

Fachbereich Informatik

The geometry of optimal degree reduction
of Bezier curves

Guido Brunnett, Thomas Schreiber und Jörg Braun

The geometry of optimal degree reduction of Bezier curves

Guido Brunnett, Thomas Schreiber and Jörg Braun

Abstract

Optimal degree reductions, i.e. best approximations of n-th degree Bezier curves by Bezier curves of degree $n-1$, with respect to different norms are studied. It is shown that for any L_p-norm the euclidean degree reduction where the norm is applied to the euclidean distance function of two curves is identical to component-wise degree reduction. The Bezier points of the degree reductions are found to lie on parallel lines through the Bezier points of any Taylor expansion of degree $n-1$ of the original curve. This geometric situation is shown to hold also in the case of constrained degree reduction. The Bezier points of the degree reduction are explicitly given in the unconstrained case for $p=1$ and $p=2$ and in the constrained case for $p=2$.

1 Introduction

This paper addresses the problem of optimal approximation of Bezier curves of degree n by Bezier curves of reduced degree $n-1$ with respect to different norms on $C[0,1]$. We will consider the case where the norm is applied component-wise to the difference of two curves as well as the case where the norm is applied to the euclidean distance of two parametric curves.

In the following we will use Π_n^s to denote the space of parametric polynomials in $\mathbb{R}^s$ with degree at most n; $\Pi_n := \Pi_n^1$. The functions $e_n : e_n(t) = t^n$ will be used to denote the monomials. $|| \cdot ||^{[a,b]}$ denotes an arbitrary norm on $C[a,b]$. For $[a,b] = [0,1]$ we omit the interval specifier: $|| \cdot ||$ is a norm on $C[0,1]$. $| \cdot |$ will denote the euclidean vector norm: $|\mathbf{v}| = \sqrt{<\mathbf{v},\mathbf{v}>}$ and d is used for the euclidean distance: $d(\mathbf{v},\mathbf{w}) := |\mathbf{v}-\mathbf{w}|$.

Definition 1.1 *Let $\mathbf{x} \in \Pi_n^s$ with components $x^i (i = 1,\ldots,s)$.*

(i) $\bar{\mathbf{x}} \in \Pi_{n-1}^s$ with components $\bar{x}^i$ such that for each $i = 1,\ldots,s$

$$||x^i - \bar{x}^i|| \leq ||x^i - p|| \quad \text{for all} \quad p \in \Pi_{n-1}$$

is called an optimal component-wise degree reduction of $\mathbf{x}$ with respect to $|| \cdot ||$.

(ii) $\bar{\mathbf{x}} \in \Pi^s_{n-1}$ *with*

$$||d(\mathbf{x}, \bar{\mathbf{x}})|| \leq ||d(\mathbf{x}, \mathbf{y})|| \qquad \text{for all } \mathbf{y} \in \Pi^s_{n-1}$$

is called an optimal euclidean degree reduction of $\mathbf{x}$ *with respect to* $|| \cdot ||$.

Remark: Since for $s = 1$ the definitions (i) and (ii) are identical we will speak in this situation about the degree reduction without further specification.

It is intended to study optimal degree reductions in terms of their Bezier points with respect to different norms. Especially, we are interested in L_p-norms on $C[0,1]$

$$||f||_p = \left(\int_0^1 |f(t)|^p d\,t \right)^{1/p} \qquad 1 \leq p < \infty$$

and the uniform norm ($p = \infty$) on $C[0,1]$

$$||f||_\infty = \max_{t \in [0,1]} |f(t)|$$

for which both existence and uniqueness of the best approximation of $f \in C[0,1]$ in Π_m is guaranteed.

Clearly, if $\mathbf{x}$ is a degree elevated Bezier curve (i.e. the leading coefficient in monomial representation vanishes), then $\mathbf{x}$ is the optimal degree reduction of itself regardless what norm has been specified. Then the Bezier coefficients $\mathbf{b}_i$ ($i = 0, \ldots, n$) of $\mathbf{x}$ relate in the way

$$\mathbf{b}_i = \frac{1}{n}(i\bar{\mathbf{b}}_{i-1} + (n-i)\bar{\mathbf{b}}_i) \qquad i = 0, \ldots, n \tag{1}$$

to the coefficients $\bar{\mathbf{b}}_i$ ($i = 0, \ldots, n-1$) of $\bar{\mathbf{x}}$ which is the Bezier representation of degree $n-1$ of $\mathbf{x}$. In this situation the wanted Bezier points $\mathbf{b}_i$ can be computed from each of the following recursive formulas (see [Far83])

$$\mathbf{b}^I_0 = \mathbf{b}_0 \qquad \mathbf{b}^I_i = \frac{1}{n-i}(n \cdot \mathbf{b}_i - i\mathbf{b}^I_{i-1}) \qquad i = 1, \ldots, n-1 \tag{2}$$

or

$$\mathbf{b}^{II}_{n-1} = \mathbf{b}_n \qquad \mathbf{b}^{II}_{i-1} = \frac{1}{i}(n\mathbf{b}_i - (n-i)\mathbf{b}^{II}_i) \qquad i = n-1, \ldots, 1, \tag{3}$$

i.e.

$$\bar{\mathbf{b}}_i = \mathbf{b}^I_i = \mathbf{b}^{II}_i \qquad \text{for} \qquad i = 1, \ldots, n-1.$$

For an arbitrary Bezier curve $\mathbf{x}$ the points $\mathbf{b}^I_i$ and $\mathbf{b}^{II}_i$ defined by (2) and (3) or explicitly (cf [Eck93]) by

$$\mathbf{b}^I_i = \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=0}^i (-1)^j \binom{n}{j} \mathbf{b}_j \qquad i = 0, \ldots, n-1 \tag{4}$$

$$\mathbf{b}^{II}_i = -\frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=i+1}^n (-1)^j \binom{n}{j} \mathbf{b}_j \qquad i = 0, \ldots, n-1 \tag{5}$$

are of course in general not identical but still somehow related to approximations of degree $n-1$ of $\mathbf{x}$. (For the precise meaning of $\mathbf{b}_i^I$, $\mathbf{b}_i^{II}$ see section 2.)

In early papers on the issue of degree reduction [For72], [Far83] non-optimal (but easy to compute) component-wise degree reductions to arbitrary Bezier curves have been constructed using simple convex combinations of the points $\mathbf{b}_i^I$ and $\mathbf{b}_i^{II}$. Optimal component-wise degree reductions have been considered in [Eck93] for the case of the uniform norm. There it has been shown that the Bezier points $\bar{\mathbf{b}}_i$ of the optimal component-wise degree reduction of an arbitrary Bezier curve of degree n can be obtained as convex combinations of the points $\mathbf{b}_i^I$ and $\mathbf{b}_i^{II}(i=0,\ldots,n-1)$:

$$\bar{\mathbf{b}}_i = (1 - \alpha_{i,n})\mathbf{b}_i^I + \alpha_{i,n}\mathbf{b}_i^{II} \qquad (i=0,\ldots,n-1) \tag{6}$$

with

$$\alpha_{i,n} = \frac{1}{2^{2n-1}} \sum_{j=0}^{i} \binom{2n}{2j} \qquad (i=0,\ldots,n-1) \ . \tag{7}$$

We begin this paper with a new derivation of this result which is much shorter than the original one. Our proof is also direct in the sense that in contrast to [Eck93] we do not assume that a relation of the form (6) is valid. Furthermore, our derivation implies a geometric interpretation of the situation that has not yet been given: the Bezier points of the optimal component-wise degree reduction of $\mathbf{x}$ lie on parallel lines with the direction vector $\triangle^n \mathbf{b}_0$ through the Bezier points of a Taylor expansion of $\mathbf{x}$ of degree $n-1$. In section 3 we first show that this geometric statement holds true for arbitrary norms that are applied component-wise. Then we extend the range of this result even further by showing that for any L_p-norm the optimal component-wise degree reduction is in fact identical to the optimal euclidean degree reduction. In section 4 the Bezier coefficients of the optimal degree reduction are explicitly given for the cases of the L_1 and L_2-norm. In the last section we extend the results to the case of constrained degree reduction. We show for L_p-norms ($p > 1$) the identity of component-wise and euclidean degree reduction. The Bezier coefficients of the constrained degree reduction with respect to the L_2-norm are explicitly given.

2 Optimal component-wise degree reduction in the uniform norm

We start this section with a short and direct derivation of (6) and (7). Note, that in contrast to [Eck93] we do not assume that a relation of the form (6) holds.

Let $\mathbf{x}$ denote a Bezier curve of degree n with Bezier points $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$ and let $\bar{\mathbf{x}}$ be the optimal component-wise degree reduction of $\mathbf{x}$ with respect to the uniform norm on [0,1]. The Bezier points of $\bar{\mathbf{x}}$ are denoted by $\bar{\mathbf{b}}_0, \bar{\mathbf{b}}_1, \ldots, \bar{\mathbf{b}}_{n-1}$.

3

Since the Tschebyscheff polynomials of the first kind

$$T_n(t) := \cos(n \arccos t)$$

have the property

$$\frac{1}{2^{n-1}} \, ||T_n||_\infty^{[-1,1]} \leq ||p||_\infty^{[-1,1]}$$

for all polynomials $p \in \Pi_n$ with leading coefficient 1 it follows that

$$\mathbf{x}(t) - \bar{\mathbf{x}}(t) = \frac{\mathbf{a}_n}{2^{2n-1}} T_n(2t - 1) \tag{8}$$

where $\mathbf{a}_n$ denotes the leading coefficient of $\mathbf{x}$ in monomial form, i.e.

$$\mathbf{a}_n = \triangle^n \mathbf{b}_0.$$

Now, $\bar{\mathbf{x}}$ is degree elevated into a Bezier curve of degree n with Bezier coefficients $\tilde{\mathbf{b}}_0, \ldots, \tilde{\mathbf{b}}_n$ and T_n is expressed in Bezier form as

$$T_n(2t - 1) = \sum_{i=0}^{n} C_{i,n} B_i^n(t),$$

with (cf. [Sch81])

$$C_{i,n} = (-1)^{n+i} \frac{\binom{2n}{2i}}{\binom{n}{i}} \, .$$

Then comparing the coefficients in (8) leads to the equations

$$\tilde{\mathbf{b}}_i = \mathbf{b}_i - \frac{C_{i,n}}{2^{2n-1}} \triangle^n \mathbf{b}_0. \tag{9}$$

Using the explicit formula (4) for inverting the process of degree elevation we obtain

$$
\begin{aligned}
\bar{\mathbf{b}}_i &= \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=0}^{i} (-1)^j \binom{n}{j} \tilde{\mathbf{b}}_j \\
&= \frac{(-1)^i}{\binom{n-1}{i}} \left(\sum_{j=0}^{i} (-1)^j \binom{n}{j} \mathbf{b}_j - \sum_{j=0}^{i} (-1)^n \binom{2n}{2j} 2^{1-2n} \triangle^n \mathbf{b}_0 \right) \\
&= \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=0}^{i} (-1)^j \binom{n}{j} \mathbf{b}_j + \lambda_{i,n} \triangle^n \mathbf{b}_0
\end{aligned}
\tag{10}
$$

with

$$\lambda_{i,n} := \frac{(-1)^{n+i+1}}{\binom{n-1}{i}} \frac{1}{2^{2n-1}} \sum_{j=0}^{i} \binom{2n}{2j}.$$

4

We observe that the first part of the right side of (10) coincides with the explicit formula (4) for the points $\mathbf{b}_i^I$ obtained from reversing the process of degree elevation. Thus,

$$\bar{\mathbf{b}}_i = \mathbf{b}_i^I + \lambda_{i,n}\triangle^n \mathbf{b}_0 . \tag{11}$$

Formula (11) is equivalent to (6) because

$$\mathbf{b}_i^I - \mathbf{b}_i^{II} = \frac{(-1)^{n+i}}{\binom{n-1}{i}}\triangle^n \mathbf{b}_0 \tag{12}$$

but it reveals geometric information that has not been given (and incorrectly depicted in fig.1, p.242) in [Eck93]: The Bezier points of the optimal component-wise degree reduction with respect to the uniform norm lie on parallel lines through the point $\mathbf{b}_i^I$ with direction $\triangle^n \mathbf{b}_0$.

Although the points $\mathbf{b}_i^I, \mathbf{b}_i^{II}$ have been used for a long time it has not been noted that these are the Bezier control points of the Taylor expansion of $\mathbf{x}$ of degree $n-1$ at $0,1$ respectively. This can be verified for $\mathbf{b}_i^I$ in the following way.

Let $T_m \mathbf{x}(a,t)$ denote the Taylor expansion of $\mathbf{x}$ of degree m with expansion point a. Then, using standard formulas for expressing the monomials into the Bernstein bases we obtain

$$
\begin{aligned}
T_{n-1}\mathbf{x}(0,t) &= \sum_{k=0}^{n-1} \frac{\mathbf{x}^{(k)}(0)}{k!} t^k \\
&= \sum_{k=0}^{n-1}\sum_{i=k}^{n-1} \frac{\mathbf{x}^{(k)}(0)}{k!} \frac{\binom{i}{k}}{\binom{n-1}{k}} B_i^{n-1}(t) \\
&= \sum_{i=0}^{n-1}\sum_{k=0}^{i} \frac{\mathbf{x}^{(k)}(0)}{k!} \frac{\binom{i}{k}}{\binom{n-1}{k}} B_i^{n-1}(t) .
\end{aligned}
$$

Thus, with $\mathbf{x}^{(k)}(0) = \frac{n!}{(n-k)!}\triangle^n \mathbf{b}_0$ the i-th coefficient $\mathbf{b}_i^T(0)$ of $T_{n-1}\mathbf{x}(0,t)$ in the Bernstein bases is

$$
\begin{aligned}
\mathbf{b}_i^T(0) &= \sum_{k=0}^{i} \frac{\binom{n}{k}\binom{i}{k}}{\binom{n-1}{k}} \triangle^k \mathbf{b}_0 \\
&= \sum_{k=0}^{i}\sum_{j=0}^{k}(-1)^{j+k} \frac{\binom{n}{k}\binom{i}{k}\binom{k}{j}}{\binom{n-1}{k}} \mathbf{b}_j \\
&= \sum_{j=0}^{i}\sum_{k=j}^{i}(-1)^{j+k} \frac{\binom{n}{k}\binom{i}{k}\binom{k}{j}}{\binom{n-1}{k}} \mathbf{b}_j \\
&= \sum_{j=0}^{i}(-1)^{i+j} \frac{\binom{n}{j}}{\binom{n-1}{i}} \mathbf{b}_j = \mathbf{b}_i^I .
\end{aligned}
$$

The Bezier control points of $T_{n-1}\,\mathbf{x}(a,t)$ for arbitrary expansion point a can be derived as follows. Since $\mathbf{x}^{(n)}(t) = \triangle^n \mathbf{b}_0$

$$\mathbf{x}(t) = T_{n-1}\,\mathbf{x}(a,t) + (t-a)^n \triangle^n \mathbf{b}_0$$

for any a. Thus,

$$T_{n-1}\,\mathbf{x}(a,t) = T_{n-1}\,\mathbf{x}(0,t) + (t^n - (t-a)^n)\triangle^n \mathbf{b}_0.$$

Now, we transform $t^n - (t-a)^n$ into the Bernstein bases in order to derive its Bezier coefficients

$$
\begin{aligned}
t^n - (t-a)^n &= t^n - \sum_{k=0}^{n} \binom{n}{k} t^k (-a)^{n-k} \\
&= -\sum_{k=0}^{n-1} \sum_{i=k}^{n-1} (-a)^{n-k} \frac{\binom{n}{k}\binom{i}{k}}{\binom{n-1}{k}} B_i^{n-1}(t) \\
&= -\sum_{i=0}^{n-1} \sum_{k=0}^{i} (-a)^{n-k} \frac{n}{n-k} \binom{i}{k} B_i^{n-1}(t) .
\end{aligned}
$$

Hence, we obtain for the Bezier control points $\mathbf{b}_i^T(a)$ of $T_{n-1}\,\mathbf{x}(a,t)$:

$$\mathbf{b}_i^T(a) = \mathbf{b}_i^I - \mu_{i,n}(a)\triangle^n \mathbf{b}_0 \tag{13}$$

with

$$\mu_{i,n}(a) = \sum_{k=0}^{i} \binom{i}{k} \frac{n}{n-k} (-a)^{n-k}$$

Especially, we verify

$$
\begin{aligned}
\mathbf{b}_i^T(1) &= \mathbf{b}_i^I - \sum_{k=0}^{i} (-1)^{n-k} \frac{\binom{n}{k}\binom{i}{k}}{\binom{n-1}{k}} \triangle^n \mathbf{b}_0 \\
&= \mathbf{b}_i^I - \frac{(-1)^{n+i}}{\binom{n-1}{i}} \triangle^n \mathbf{b}_0 = \mathbf{b}_i^{II}.
\end{aligned}
$$

With (13) the generalized form of (11) becomes

$$\bar{\mathbf{b}}_i = \mathbf{b}_i^T(a) + (\mu_{i,n}(a) + \lambda_{i,n})\triangle^n \mathbf{b}_0 .$$

3 The geometry of optimal degree reduction

In this section we will first show that the geometric interpretation given in the previous section holds for the component-wise approach to best approximation of parametric curves regardless what norm has been specified. Then we will extend the range of this result even further by showing that for arbitrary L_p-norms the optimal euclidean degree reduction is in fact identical to the optimal component-wise degree reduction.

6

Theorem 3.1 *Let* $\mathbf{x}$ *be a Bezier curve of degree* n *with Bezier points* $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$ *and* $|| \cdot ||$ *be an arbitrary norm on* $C[0,1]$ *for which uniqueness of the best approximation in* Π_{n-1} *is guaranteed.*

The optimal component-wise degree reduction $\bar{\mathbf{x}}$ *of* $\mathbf{x}$ *with respect to* $|| \cdot ||$ *is of the form*

$$\bar{\mathbf{x}}(t) = T_{n-1}\mathbf{x}(0,t) + z_n(t)\triangle^n \mathbf{b}_0 \tag{14}$$

where z_n *denotes the optimal degree reduction of the monomial* e_n *with respect to* $|| \cdot ||$.

Proof: We show that (14) holds for every vector component.
Let x^i be a component of $\mathbf{x}$ with coefficients $b_0^i, \ldots, b_n^i$ and let y be an arbitrary polynomial of degree $n-1$. Then

$$
\begin{aligned}
||x^i - y|| &= ||x^i - T_{n-1}x^i(0,\cdot) - (y - T_{n-1}x^i(0,\cdot))|| \\
&= |\triangle_n b_0^i| \;\; ||e_n - \frac{1}{\triangle_n b_0^i}(y - T_{n-1}x^i(0,\cdot))|| \\
&\geq |\triangle_n b_0^i| \;\; ||e_n - z||
\end{aligned}
$$

from the definition of z as the best approximation to the monomial e_n with respect to $|| \cdot ||$.
Since for the i-th component $\bar{x}^i$ of $\bar{x}$ according to (14)

$$||x^i - \bar{x}^i|| = |\triangle_n b_0^i| \;\; ||e_n - z||$$

it follows from the uniqueness of the best approximation that $\bar{x}^i$ is the best approximation to x^i.

$\square$

If we denote with $\sum_{i=0}^{n-1} \lambda_{i,n} B_i^{n-1}$ the Bezier representation of the unique polynomial $z \in \Pi_{n-1}$, comparing the coefficients in (14) yields

$$\bar{\mathbf{b}}_i = \mathbf{b}_i^I + \lambda_{i,n}\triangle^n \mathbf{b}_0, \quad i = 0, \ldots, n-1,$$

i.e. the Bezier points of the optimal component-wise degree reduced Bezier curve lie on lines parallel to $\triangle^n \mathbf{b}_0$ through the Bezier points of the Taylor expansion of $\mathbf{x}$ at 0.

In [Lac91] the following example has been given which shows that the euclidean best approximation to a parametric polynomial is in general different from the component-wise obtained best approximation.

Example: The best euclidean constant approximation to $\mathbf{x}(t) = (t, t^2)$ on $[-1, 1]$ with respect to the uniform norm is $\bar{\mathbf{x}}_e(t) = (0, 1)$, with maximum error 1. The best component-wise constant approximation to $\mathbf{x}$ with respect to the uniform norm is $\bar{\mathbf{x}}_c(t) = (0, 0.5)$, the euclidean norm of the error vector is $\frac{\sqrt{5}}{2}$.

However, in [Lac88] it has been shown for the uniform norm that in the case of approximation of parametric polynomials of degree n with polynomials of degree $n-1$ the euclidean best approximation can be determined component-wise. We will now generalize this result to arbitrary L_p-norms.

Theorem 3.2 *The optimal euclidean degree reduction $\bar{\mathbf{x}}_e$ of a parametric polynomial $\mathbf{x}$ of degree n with respect to any L_p-norm $(p \geq 1)$ is identical to the optimal component-wise degree reduction $\bar{\mathbf{x}}$ of $\mathbf{x}$.*

Proof: We will show for $p \leq 1 < \infty$ that $\bar{\mathbf{x}}_e$ is of the form (14). The case $p = \infty$ is in [Lac88].

As usual, we denote the control points of the Bezier representation of $\mathbf{x}$ with $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$. Since the property of best approximation of parametric polynomials is independent of the coordinate system, we may specify the coordinate system such that

$$\triangle^n \mathbf{b}_0 = |\triangle^n \mathbf{b}_0| \, \mathbf{v} = |\triangle^n \mathbf{b}_0| \, (1, 0, \ldots, 0).$$

Then, for an arbitrary curve $\mathbf{y} \in \Pi_{n-1}^s$ one obtains

$$
\begin{aligned}
d(\mathbf{x}(t), \mathbf{y}(t)) &= |\triangle^n \mathbf{b}_0| \, d(t^n \, \mathbf{v}, \frac{1}{|\triangle^n \mathbf{b}_0|} (\mathbf{y}(t) - T_{n-1}\mathbf{x}(0, t))) \\
&= |\triangle^n \mathbf{b}_0| \sqrt{[t^n - q^1(t)]^2 + \sum_{i=2}^{s} [q^i(t)]^2} \\
&\geq |\triangle^n \mathbf{b}_0| \, |t^n - q^1(t)|
\end{aligned}
$$

with q^i being the components of $\mathbf{q}(t) = \frac{1}{|\triangle^n \mathbf{b}_0|} \, (\mathbf{y}(t) - T_{n-1} \, \mathbf{x}(0, t))$.

Thus,

$$
\begin{aligned}
(||d(\mathbf{x}, \mathbf{y})||_p)^p &= \int_0^1 (d(\mathbf{x}(t), \mathbf{y}(t)))^p dt \\
&\geq |\triangle^n \mathbf{b}_0|^p \int_0^1 |t^n - q^1(t)|^p dt \\
&\geq |\triangle^n \mathbf{b}_0|^p \int_0^1 |t^n - z_{p,n}(t)|^p dt
\end{aligned}
$$

where $z_{p,n}$ denotes the L_p-best approximation of e_n in Π_{n-1}.

Now, we conclude that $\bar{\mathbf{x}}_e$ is of the form (14) since for $\bar{\mathbf{x}}(t) = T_{n-1}\mathbf{x}(0, t) + \triangle^n \mathbf{b}_0 z(t)$ the minimum error is obtained:

$$(||d(\mathbf{x}, \bar{\mathbf{x}})||_p)^p = \int_0^1 |\triangle^n \mathbf{b}_0|^p \, |t^n - z_{p,n}(t)|^p dt \; .$$

$\square$

4 Polynomials of least deviation from zero

Formula (14) of the previous section implies that the optimal degree reduced Bezier curve of degree $n - 1$ is determined, if the best approximation $z \in \Pi_{n-1}$ of the monomial e_n

8

with respect to the given norm is known. Since $e_n - z$ is a polynomial of degree n with leading coefficient one, it is an equivalent problem to determine the polynomial of degree n with leading coefficient 1 and least deviation from zero.

For L_p-norms the notation $Q_{p,n}$ is used to denote the polynomial of degree n with

$$\|Q_{p,n}\|_p^{[-1,1]} = \inf_{t_i} \|(t - t_1) \cdots (t - t_n)\|_p^{[-1,1]}.$$

Besides the case of the uniform norm ($p = \infty, Q_{\infty,n}(t) = \frac{1}{2^{n-1}} T_n(t)$), there are two other cases were these polynomials are explicitly known. For $p = 1$,

$$Q_{1,n}(t) = \frac{1}{2^n} U_n(t)$$

where

$$U_n(t) := \frac{\sin[(n+1)\arccos t]}{\sin[\arccos t]}$$

are the Tschebyscheff polynomials of the 2. kind.

For $p = 2$,

$$Q_{2,n}(t) = \frac{2^n}{\binom{2n}{n}} P_n(t)$$

where

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\{(x^2 - 1)^n\}$$

are the Legendre polynomials.

The three polynomials $Q_{1,n}, Q_{2,n}$ and $Q_{\infty,n}$ are special cases of the Jacobi polynomials $P_n^{(a,b)}$

$$P_n^{(a,b)}(t) = \sum_{i=0}^{n} \binom{n+a}{i}\binom{n+b}{n-i}\left(\frac{t-1}{2}\right)^{n-i}\left(\frac{t+1}{2}\right)^{i} \tag{15}$$

(see [Sze33], p.38) which are orthogonal with respect to the weight function $(1-t)^a(1+t)^b$:

$$Q_{1,n} = \frac{1}{2^n} P_n^{(-\frac{1}{2},-\frac{1}{2})}, Q_{2,n} = \frac{2^n}{\binom{2n}{n}} P_n^{(0,0)} \text{ and } Q_{\infty,n} = \frac{1}{2^{n-1}} P_n^{(\frac{1}{2},\frac{1}{2})}.$$

However, no such statement is known for the other cases. By direct calculation, [Bur67] shows by a counterexample that, in general, the weight function is not $(1-t^2)^{\frac{1}{p}-\frac{1}{2}}$. He also calculates the zeros of $Q_{p,n}$ for the cases $p = 2, 3, \ldots, 7$ and $n = 1, 2, \ldots, 6$. Thus, in these cases an approximation formula for $Q_{p,n}$ is available.

Assume that a Bezier representation of $Q_{p,n}$ has been established, i.e.

$$Q_{p,n}(2t - 1) = \sum_{i=0}^{n} c_{p,n}^{(i)} B_i^n(t), \quad t \in [0, 1],$$

9

also denote with $\tilde{\lambda}_{p,n}^{(i)}$ $(i = 0, \ldots, n)$ the Bezier coefficients of the degree elevation of $z_{p,n}$. Then from

$$z_{p,n}(t) = t^n - \frac{1}{2^n} Q_{p,n}(2t - 1)$$

one obtains

$$\tilde{\lambda}_{p,n}^{(i)} = \frac{-c_{p,n}^{(i)}}{2^n} \qquad i = 0, \ldots, n-1$$

and with (4)

$$\lambda_{p,n}^{(i)} = \frac{(-1)^{i+1}}{\binom{n-i}{i}} \frac{1}{2^n} \sum_{j=0}^{i} (-1)^j \binom{n}{j} c_{p,n}^{(i)}. \tag{16}$$

Since the coefficients of $P_n^{(a,b)}$ in Bezier representation can be obtained directly from (15) to be

$$(-1)^{n+i} \frac{\binom{n+a}{i} \binom{n+b}{n-i}}{\binom{n}{i}}$$

one obtains

$$c_{1,n}^{(i)} = \frac{(-1)^{n+i}}{2^{n+1}} \frac{\binom{2n+2}{2i+1}}{\binom{n}{i}} ,$$

$$c_{2,n}^{(i)} = \frac{(-1)^{n+i}}{\binom{2n}{n}} 2^n \binom{n}{i}$$

and again

$$c_{\infty,n}^{(i)} = \frac{(-1)^{n+i}}{2^{n-1}} \frac{\binom{2n}{2i}}{\binom{n}{i}} .$$

Thus (16) yields

$$\lambda_{1,n}^{(i)} = \frac{(-1)^{n+i+1}}{\binom{n-i}{i}} \frac{1}{2^{2n+1}} \sum_{j=0}^{i} \binom{2n+2}{2j+1} ,$$

$$\lambda_{2,n}^{(i)} = \frac{(-1)^{n+i+1}}{\binom{n-i}{i}} \frac{1}{\binom{2n}{n}} \sum_{j=0}^{i} \binom{n}{j}^2$$

and again

$$\lambda_{\infty,n}^{(i)} = \frac{(-1)^{n+i+1}}{\binom{n-i}{i}} \frac{1}{2^{2n-1}} \sum_{j=0}^{i} \binom{2n}{2j} .$$

Remark: The approximation error $E_{p,n}(\mathbf{x})$ for the optimal euclidean degree reduction of $\mathbf{x}$ with respect to $|| \cdot ||_p$ is given by

$$\begin{aligned} E_{p,n}(\mathbf{x}) &= |\triangle^n \mathbf{b}_0| \, ||e_n - z_{p,n}||_p \\ &= \frac{|\triangle^n \mathbf{b}_0|}{2^n} \, ||Q_{p,n}||_p^{[-1,1]}, \end{aligned}$$

10

i.e. for the three cases considered above

$$E_{1,n}(\mathbf{x}) = \frac{|\triangle^n \mathbf{b}_0|}{2^{2n}},$$

$$E_{2,n}(\mathbf{x}) = \frac{1}{\sqrt{2n+1}} \frac{|\triangle^n \mathbf{b}_0|}{\binom{2n}{n}}$$

and

$$E_{\infty,n}(\mathbf{x}) = \frac{|\triangle^n \mathbf{b}_0|}{2^{2n-1}}.$$

The following figure shows the optimal L_p degree reductions of a quintic Bezier curve for the cases $p = 1, 2$ and ∞. The dotted line corresponds to the L_1 approximation, the solid (thin) line is the L_2 approximation and the dashed line is the L_∞ approximation. The corresponding error functions are drawn in the same line style.
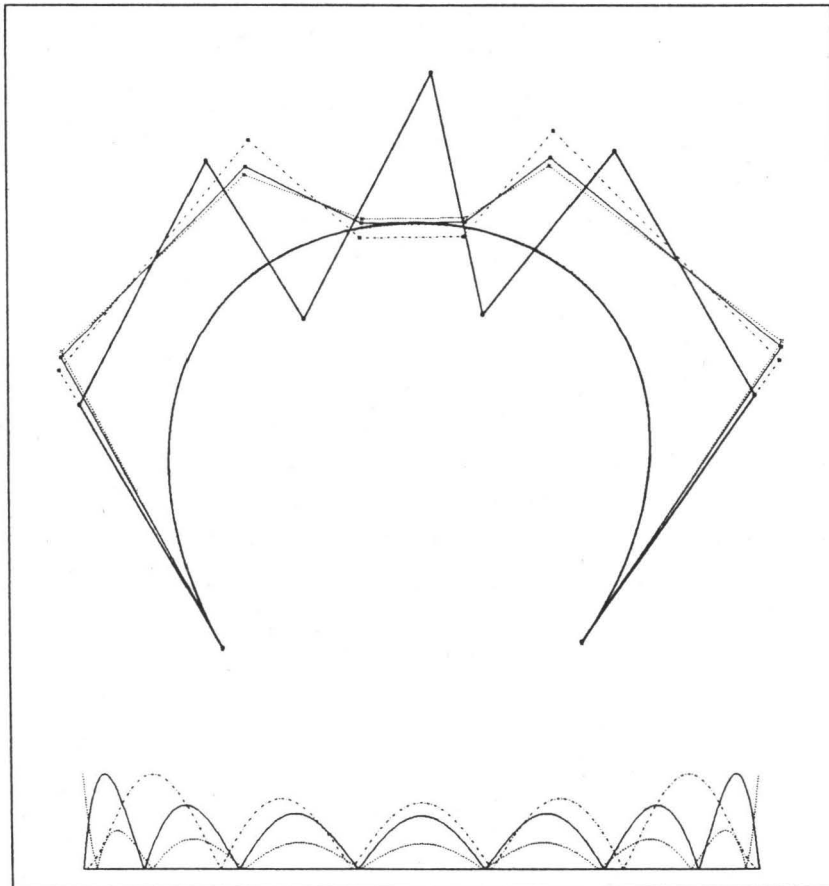
5 Constrained degree reduction

Curves considered in CAD are frequently required to match boundary constraints that guarantee a prescribed order of continuity. This motivates

Definition 5.1 *Let* $\mathbf{x} \in \Pi_n^s, \alpha \in \mathbf{N}, ||\cdot||$ *denote a norm on* $C[0,1]$.

   (i) $\Lambda^\alpha(\mathbf{x}) := \{\mathbf{y} \in \Pi_{n-1}^s \mid \mathbf{y}^\nu(j) = \mathbf{x}^\nu(j) \text{ for } \nu = 0,1,\ldots,\alpha-1 \text{ and } j = 0,1\}$

  (ii) $\bar{\mathbf{x}} \in \Lambda^\alpha(\mathbf{x})$ *such that for each* $i \in \{1,\ldots,s\}$

$$||x^i - \bar{x}^i|| \leq ||x^i - q|| \text{ for all } q \in \Lambda^\alpha(x^i)$$

     is called an optimal component-wise $C^{\alpha-1}$-*degree reduction of* $\mathbf{x}$ *with respect to* $||\cdot||$.

 (iii) $\bar{\mathbf{x}} \in \Lambda^\alpha(\mathbf{x})$ *with*

$$||d(\mathbf{x},\bar{\mathbf{x}})|| \leq ||d(\mathbf{x},\mathbf{q})|| \text{ for all } \mathbf{q} \in \Lambda^\alpha(\mathbf{x})$$

     is called an optimal euclidean $C^{\alpha-1}$-*degree reduction of* $\mathbf{x}$ *with respect to* $||\cdot||$.

In the following we will generalize the results of the preceeding sections to the more general situation of optimal $C^{\alpha-1}$-degree reduction. We will make use of the following statements

Lemma 5.1 *If* $||\cdot||$ *is a strictly convex norm on* $C[0,1], h \in C[0,1]$ *with a finite number of zeros in* $[0,1]$, *then the functional* $N_h : C[0,1] \to \mathbb{R}$ *with* $N_h(f) := ||h \cdot f||$ *is a strictly convex norm on* $C[0,1]$.

Proof: It is straightforward to verify that N_h fullfils the norm axioms. To show definiteness note that $hf = 0$ implies $f = 0$ because h has only a finite number of zeros in $[0,1]$.

Assume that N_h is not strictly convex then there exist $f, g \in C[0,1], f \neq g$, with $||hf|| \leq r, ||hg|| \leq r$ but $||hf|| + ||hg|| = 2r$. Thus, $||\cdot||$ is not strictly convex. $\qquad\square$

Lemma 5.2 *For* $\alpha \in \mathbf{N}, 1 < p \leq \infty$ *there is a unique best approximation* $\check{z}_{p,n}^{(\alpha)} \in \Pi_{n-1}$ *to* e_n *with respect to* $N_p^\alpha : N_p^\alpha(f) := ||[e_1(e_1 - 1)]^\alpha f||_p$.

Proof: For $1 < p < \infty$ N_p^α is a strictly convex norm. Thus, the best approximation to any $f \in C[0,1]$ in Π_{n-1} is unique. For $p = \infty$, we note that for a minimization problem equivalent to the one considered here unique existence has been proven in [LSV79]. $\qquad\square$

Theorem 5.1 *Let* $\mathbf{x}$ *be a Bezier curve of degree* n *with Bezier coefficients* $\mathbf{b}_0, \ldots, \mathbf{b}_n$ *and* $\alpha \in \mathbf{N}, 1 < p \leq \infty$.
The optimal component-wise $C^{\alpha-1}$-*degree reduction* $\bar{\mathbf{x}} \in \Lambda^\alpha(\mathbf{x})$ *of* x *with respect to* $|| \cdot ||_p$
exists uniquely and is given by

$$\bar{\mathbf{x}}(t) = T_{n-1}\mathbf{x}(0,t) + z_{p,n}^{(\alpha)}(t)\triangle^n\mathbf{b}_0 \tag{17}$$

where

$$z_{p,n}^{(\alpha)} := t^n - [t(t-1)]^\alpha \left(t^{n-2\alpha} - \tilde{z}_{p,n-2\alpha}^{(\alpha)} \right) \tag{18}$$

is the unique $C^{\alpha-1}$ *degree reduction of* e_n *with respect to* $|| \cdot ||_p$.

Proof: Let $\mathbf{y} \in \Lambda^\alpha(\mathbf{x})$ be arbitrary. Then

$$x^i(t) - y^i(t) = \triangle^n\mathbf{b}_0^i \, [t(t-1)]^\alpha \, (t^{n-2\alpha} - q^i(t)) \tag{19}$$

with $q^i \in \Pi_{n-(2\alpha+1)}$. Hence, the problem to find the optimal $C^{\alpha-1}$ degree reduction of x^i with respect to $|| \cdot ||_p$ is equivalent to determine the best approximation in $\Pi_{n-(2\alpha+1)}$ of $e_{n-2\alpha}$ with respect to N_p^α. Especially, the optimal $C^{\alpha-1}$ degree reduction of x^i with respect to $|| \cdot ||_p$ exists uniquely and the minimal error is

$$\inf_{y^i \in \Lambda^\alpha(x^i)} ||x^i - y^i||_p = |\triangle^n\mathbf{b}_0^i| \, N_p^\alpha(e_{n-2\alpha} - \tilde{z}_{p,n-2\alpha}^{(\alpha)}).$$

Since for $\bar{x}^i$ according to (17) and (18)

$$x^i(t) - \bar{x}^i(t) = \triangle^n\mathbf{b}_0^i \, [t(t-1)]^\alpha \, (t^{n-2\alpha} - \tilde{z}_{p,n-2\alpha}^{(\alpha)}),$$

$\bar{\mathbf{x}}$ is the optimal component-wise $C^{\alpha-1}$ degree reduction of $\mathbf{x}$.
Applying (17) to the special case where $x(t) = t^n$ one verifies that $z_{p,n}^{(\alpha)}$ is the unique $C^{\alpha-1}$ degree reduction of e_n. $\qquad\square$

We will now generalize Theorem 3.2 to the case of constrained best approximation.

Theorem 5.2 *The optimal euclidean* $C^{\alpha-1}$-*degree reduction* $\bar{\mathbf{x}}_e$ *of a parametric polynomial* $\mathbf{x}$ *of degree* n *with respect to any* L_p-*norm* $(p > 1)$ *exists uniquely and is identical to the optimal component-wise* $C^{\alpha-1}$ *degree reduction* $\bar{\mathbf{x}}$ *of* $\mathbf{x}$.

Proof: Let $\mathbf{y} \in \Lambda^\alpha(\mathbf{x})$ be arbitrary. Then from (19)

$$\sum_{i=1}^{s} (x^i(t) - y^i(t))^2 = [t(t-1)]^{2\alpha} \sum_{i=1}^{s} (\triangle^n\mathbf{b}_0^i)^2 (t^{n-2\alpha} - q^i(t))^2$$

which implies the unique existence of $\bar{\mathbf{x}}_e$.

13

If the coordinate system is chosen such that

$$\triangle^n \mathbf{b}_0 = |\triangle^n \mathbf{b}_0| \, (1, 0, \ldots, 0) = |\triangle^n \mathbf{b}_0| \mathbf{v}$$

$$
\begin{aligned}
d(\mathbf{x}(t), \mathbf{y}(t)) &= |\triangle^n \mathbf{b}_0| \, [t(t-1)]^\alpha d(t^{n-2\alpha}\mathbf{v}, \mathbf{q}(t)) \\
&\geq |\triangle^n \mathbf{b}_0| \, [t(t-1)]^\alpha |(t^{n-2\alpha} - q^1(t)| \, .
\end{aligned}
$$

Hence,

$$\int_0^1 d(\mathbf{x}(t), \mathbf{y}(t))^p dt \geq |\triangle^n \mathbf{b}_0|^p \int_0^1 [t(t-1)]^{\alpha p} |t^{n-2\alpha} - q^1(t)|^p dt \, .$$

Since with $\bar{\mathbf{x}}$ according to (17)

$$d(\mathbf{x}(t), \bar{\mathbf{x}}(t)) = |\triangle^n \mathbf{b}_0| [t(t-1)]^\alpha |t^{n-2\alpha} - \tilde{z}^{(\alpha)}_{p,n-2\alpha}(t)|$$

one verifies that $\bar{\mathbf{x}}_e = \bar{\mathbf{x}}$.

$\square$

In analogy to section 4 we denote with $Q^{(\alpha)}_{p,n} \in \Pi_n$ the constrained polynomial of least deviation from zero, i.e.

$$||Q^{(\alpha)}_{p,n}||_p^{[-1,1]} = \inf_{t_i} ||(t-1)^\alpha (t+1)^\alpha (t - t_1) \ldots (t - t_{n-2\alpha})||_p.$$

If a Bezier representation of $Q^{(\alpha)}_{p,n}$ is available

$$Q^{(\alpha)}_{p,n}(2t - 1) = \sum_{i=0}^n c^{(i)}_{p,n,\alpha} B^{(n)}_i(t) \, ,$$

one obtains from

$$z^{(\alpha)}_{p,n}(t) = t^n - \frac{1}{2^n} Q^{(\alpha)}_{p,n}(2t - 1)$$

the Bezier coefficients $\lambda^{(i)}_{p,n,\alpha}$ ($i = 0, \ldots, n-1$) of $z^{(\alpha)}_{p,n}$:

$$\lambda^{(i)}_{p,n,\alpha} = \frac{(-1)^{i+1}}{\binom{n-i}{i}} \frac{1}{2^n} \sum_{j=0}^i (-1)^j \binom{n}{j} c^{(i)}_{p,n,\alpha} \, .$$

In the case of the L_2-norm the polynomials $Q^{(\alpha)}_{p,n}$ are related to the Jacobi polynomials. According to [LSV79]

$$t^m - \tilde{z}^{(\alpha)}_{2,m}(t) = \frac{1}{\binom{2m+2\alpha}{m}} P^{(2\alpha, 2\alpha)}_m (2t - 1) \, .$$

Hence,

$$
\begin{aligned}
Q^{(\alpha)}_{2,n}(t) &= (t-1)^\alpha (t+1)^\alpha \frac{2^{n-2\alpha}}{\binom{2n}{n-2\alpha}} P^{(2\alpha, 2\alpha)}_{n-2\alpha}(t) \\
&= \frac{2^n}{\binom{2n}{n}} \sum_{i=0}^n \binom{n+2\alpha}{i+\alpha} \binom{n-2\alpha}{i-\alpha} \left(\frac{t-1}{2}\right)^{n-i} \left(\frac{t+1}{2}\right)^i
\end{aligned}
$$

14

Thus, the Bezier coefficients of $Q_{2,n}^{(\alpha)}$ are given by

$$c_{2,n,\alpha}^{(i)} = (-1)^{n+i} \frac{2^n}{\binom{2n}{n}} \frac{\binom{n+2\alpha}{i+\alpha}\binom{n-2\alpha}{i-\alpha}}{\binom{n}{i}},$$

and one obtains

$$\lambda_{2,n,\alpha}^{(i)} = \frac{(-1)^{n+i+1}}{\binom{n-i}{i}} \frac{1}{\binom{2n}{n}} \sum_{j=0}^{i} \binom{n+2\alpha}{j+\alpha}\binom{n-2\alpha}{j-\alpha}.$$

Remark: The approximation error $E_{p,n}^{\alpha}(\mathbf{x})$ for the optimal euclidean $C^{\alpha-1}$ degree reduction of $\mathbf{x}$ with respect to $||\cdot||_p$ is

$$E_{p,n}^{\alpha}(\mathbf{x}) = \frac{|\triangle^n \mathbf{b}_0|}{2^n} \, ||Q_{p,n}^{(\alpha)}||_p^{[-1,1]}.$$

In the case $p = 2$ considered above one obtains

$$E_{2,n}^{\alpha}(\mathbf{x}) = \left(\frac{1}{2n+1} \frac{\binom{n+2\alpha}{2\alpha}}{\binom{n}{2\alpha}} \right)^{\frac{1}{2}} \frac{|\triangle^n \mathbf{b}_0|}{\binom{2n}{n}}.$$

References

[Bur67]   F.D. Burgoyne. Practical L^p Polynomial Approximation. *Math. Comp.*, 21:113–115, 1967.

[Eck93]   Matthias Eck. Degree reduction of Bézier curves. *CAGD*, 10:237–251, 1993.

[Far83]   Gerald Farin. Algorithms for rational Bézier curves. *CAD*, 15(2):73–77, 1983.

[For72]   A.R. Forrest. Interactive interpolation and approximation by Bézier polynomials. *The Computer Journal*, 15(1):71–79, 1972.

[Lac88]   Michael A. Lachance. Chebyshev economization for parametric surfaces. *CAGD*, 5:195–205, 1988.

[Lac91]   Michael A. Lachance. Approximation by constrained parametric polynomials. *Rocky Mountain Journal of Mathematics*, 21(1):473–488, 1991.

[LSV79]   Michael A. Lachance, E.B. Saff, and R.S. Varga. Bounds for Incomplete Polynomials Vanishing at Both Endpoints of an interval. In C.V. Coffman and G.J. Fix, editors, *Constructive Approaches to Mathematical Models*, pages 421–437. Academic Press, 1979.

[Sch81]   L.L. Schumaker. *Spline Functions: Basic Theory*. Wiley, New York, 1981.

[Sze33]   G. Szegö. *Asymptotische Entwicklungen der Jacobischen Polynome*. Niemeyer, Halle(Saale), 1933.