On the Number of Criteria Needed to Decide Pareto Optimality

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Abstract

In this paper we address the question of how many objective functions are really needed to decide whether a given point is Pareto optimal. We prove a reduction result for the case of quasi-convex objective functions and a convex feasible set. This result states that in order to decide whether a point x in the decision space is Pareto optimal it suffices to consider at most $n+1$ objectives at a time, where n is the dimension of the decision space. The main theorem is based on a geometric characterization of Pareto, strict Pareto and weak Pareto solutions and Helly's Theorem.

1 Introduction

In multiple criteria optimization, "optimal" decisions have to be found in the presence of several conflicting criteria. A decision is only considered optimal if an improvement with respect to one criterion implies a worse outcome with respect to at least one other criterion. The corresponding outcomes are called efficient points, the decisions Pareto optimal solutions.

One topic in the investigation of multiple criteria optimization problems is the determination of those objectives that determine the set of Pareto optimal solutions. Several authors have worked in that field: Gal and Leberling in [7],[12] and [9] introduced the notion of nonessential objectives and presented methods for their determination in the case of linear multiple criteria problems. Nonessential objectives can be dropped without changing the set of Pareto optimal solutions. Gal and Hanne in [8] investigated the consequences of dropping nonessential objectives in the search for a final compromise solution by MCDM methods. A more general concept of interdependent criteria has been discussed in [3], see also [2].

Our approach is related to this topic in the sense that we determine the number of objectives which are necessary to prove Pareto optimality for a given point. However, the theory presented in this paper is more general: the results also hold in the absence of nonessential criteria, as will be demonstrated by an example. The result that in fact only (at most) $n+1$ criteria have to be considered simultaneously leads to considerable advances for multiple criteria problems where the number of objectives Q is much larger than the dimension of the decision space (n). This situation arises in location theory, see [4] and [10]. Usually not one single person decides about the location of a new facility but a group of Q decision makers. Each of the Q decision makers gives his personal view of the location problems by means of a specific objective function. Typical objective functions in location theory are the weighted sum or the weighted maximum of the distances of existing facilities to the new one. Each decision maker may choose his individual set of weights as well as the type of the objective function. Therefore, we have a set of convex objective functions on the plane.

The paper is organized as follows. In Section 2 the multiple criteria optimization problem (MOP) is introduced. We present the definitions of (strict, weak) Pareto optimality and prove a geometrical characterization of all three optimality concepts based on level sets and level curves. This section is valid for general (MOP). The main part is Section 3, where we consider quasi-convex (MOP). We state Helly's representation theorem for the intersection of convex sets and deduce a reduction result for weak Pareto solutions. The main theorem generalizes this result for Pareto optimal solutions. Furthermore its proof provides a prototype polynomial time algorithm to check Pareto optimality for quasi-convex (MOP). Finally, an illustrative example is given in Section 4 and conclusions are stated in Section 5

2 Definitions and Basic Concepts

In this section we consider the general multiple criteria optimization problem

$$\min_{x \in X} f(x) \qquad \text{(MOP)}$$

where $X \subseteq \mathbb{R}^n$ is the feasible set and $f = (f^1, \ldots, f^Q) : \mathbb{R}^n \to \mathbb{R}^Q$ is the criterion mapping. The component functions $f^1, \ldots, f^Q$ are the criteria or objective functions. The index set of criteria will be denoted by $\mathcal{Q} := \{1, \ldots, Q\}$.

Optimizing the Q objective functions means minimization in $\mathbb{R}^Q$. Therefore, instead of the canonical order in $\mathbb{R}$, we consider three types of partial orders in $\mathbb{R}^Q$. Two vectors $z = (z_1, \ldots, z_Q)$ and $z' = (z'_1, \ldots, z'_Q) \in \mathbb{R}^Q$ are compared using three kinds of component-wise ordering

$$z \leqq z' :\Longleftrightarrow z_i \leq z'_i \;\; \forall\, i \in \mathcal{Q},$$

$$z \preceq z' :\Longleftrightarrow z_i \leq z'_i \;\; \forall\, i \in \mathcal{Q} \text{ and } \exists\, j \in \mathcal{Q} :\; z_j < z'_j,$$

$$z \prec z' :\Longleftrightarrow z_i < z'_i \;\; \forall\, i \in \mathcal{Q}.$$

Note that $\leqq$, $\preceq$ and $\prec$ define partial orders only, hence two vectors z and z' may not be comparable. Optimality is considered with respect to these three partial orders. A point $x \in X$ is called

- **strict Pareto solution** (of (MOP)) or **strictly Pareto optimal** if there is no $\bar{x} \in X \setminus \{x\}$ **weakly dominating** x, i.e. satisfying

$$f(\bar{x}) \leqq f(x),$$

- **Pareto solution** (of (MOP)) or **Pareto optimal** if there is no $\bar{x} \in X$ **dominating** x, i.e. satisfying

$$f(\bar{x}) \preceq f(x),$$

- **weak Pareto solution** (of (MOP)) or **weakly Pareto optimal** if there is no $\bar{x} \in X$ **strictly dominating** x, i.e. satisfying

$$f(\bar{x}) \prec f(x).$$

The sets of all strict Pareto, Pareto and weak Pareto solutions are denoted by $\mathcal{X}_{s-Par}$, $\mathcal{X}_{Par}$, and $\mathcal{X}_{w-Par}$, respectively. For $q_1, \ldots, q_P \in \{1, \ldots, Q\}$ and $P \leq Q$, we will also use the notations $\mathcal{X}_{s-Par}\left(f^{q_1}, \ldots, f^{q_P}\right)$, $\mathcal{X}_{w-Par}\left(f^{q_1}, \ldots, f^{q_P}\right)$ and $\mathcal{X}_{Par}\left(f^{q_1}, \ldots, f^{q_P}\right)$ if (strict, weak) Pareto solutions for the criterion mapping $\left(f^{q_1}, \ldots, f^{q_P}\right)$ with range space in $\mathbb{R}^P$ are considered.

Geometrically, the optimality definitions presented above can be characterized using level curves

$$L^q_=(z) := \{x \in X :\; f^q(x) = z\},$$

level sets

$$L^q_\leq(z) := \{x \in X :\; f^q(x) \leq z\},$$

and strict level sets

$$L^q_<(z) := \{x \in X :\; f^q(x) < z\}.$$

The following Theorem can be found in [6]. However, we include the proof for completeness.

Theorem 1 *Let $x \in X$ and $z_q := f^q(x)$ for $q \in \mathcal{Q}$. Then the following hold:*

1. *x is a strict Pareto solution if and only if*

$$\bigcap_{q \in \mathcal{Q}} L^q_{\leq}(z_q) = \{x\}.$$

2. *x is a Pareto solution if and only if*

$$\bigcap_{q \in \mathcal{Q}} L^q_{\leq}(z_q) = \bigcap_{q \in \mathcal{Q}} L^q_{=}(z_q).$$

3. *x is a weak Pareto solution if and only if*

$$\bigcap_{q \in \mathcal{Q}} (L^q_{<}(z_q)) = \emptyset.$$

Proof.

1. x is a strict Pareto solution

$$\Longleftrightarrow \nexists\ \bar{x} \neq x \in X \text{ such that } f(\bar{x}) \leqq f(x)$$
$$\Longleftrightarrow \nexists\ \bar{x} \neq x \in X \text{ such that } f^q(\bar{x}) \leq f^q(x)\ \forall\, q \in \mathcal{Q}$$
$$\Longleftrightarrow \nexists\ \bar{x} \neq x \in X \text{ such that } \bar{x} \in \bigcap_{q=1}^{Q} L^q_{\leq}(z_q)$$
$$\Longleftrightarrow \bigcap_{q=1}^{Q} L^q_{\leq}(z_q) = \{x\}.$$

2. x is a Pareto solution

$$\Longleftrightarrow \nexists\ \bar{x} \in X \text{ such that } f(\bar{x}) \preceq f(x)$$
$$\Longleftrightarrow \nexists\ \bar{x} \in X \text{ such that } (f^q(\bar{x}) \leq f^q(x)\ \forall\, q \in \mathcal{Q}\ \wedge\ \exists j \in \mathcal{Q}:\ f^j(\bar{x}) < f^j(x))$$
$$\Longleftrightarrow \nexists\ \bar{x} \in X \text{ such that } \left(\bar{x} \in \bigcap_{q=1}^{Q} L^q_{\leq}(z_q)\ \wedge\ \exists j \in \mathcal{Q}:\ \bar{x} \in L^j_{<}(z_j)\right)$$
$$\Longleftrightarrow \bigcap_{q=1}^{Q} L^q_{\leq}(z_q) = \bigcap_{q=1}^{Q} L^q_{=}(z_q).$$

3. x is a weak Pareto solution

$$\Longleftrightarrow \nexists\ \bar{x} \in X \text{ such that } f(\bar{x}) \prec f(x)$$
$$\Longleftrightarrow \nexists\ \bar{x} \in X \text{ such that } f^q(\bar{x}) < f^q(x)\ \forall\, q \in \mathcal{Q}$$
$$\Longleftrightarrow \nexists\ \bar{x} \in X \text{ such that } \bar{x} \in \bigcap_{q=1}^{Q} L^q_{<}(z_q)$$
$$\Longleftrightarrow \bigcap_{q=1}^{Q} L^q_{<}(z_q) = \emptyset.$$

□

An immediate consequence of Theorem 1 is the following result, which states that if a point x is a weak or a strict Pareto solution with respect to at least two of the objectives it is so with respect to all Q criteria.

Corollary 1 *Let $f^1, \ldots, f^Q$ be Q functions, $f^q : \mathbb{R}^n \to \mathbb{R}$, $x \in X$ and $\{q_1, \ldots, q_P\} \subseteq \mathcal{Q}$ with $P \geq 2$. Then the following hold:*

1. $x \in \mathcal{X}_{w-Par}\left(f^{q_1}, \ldots, f^{q_P}\right) \Rightarrow x \in \mathcal{X}_{w-Par}\left(f^1, \ldots, f^Q\right)$.

2. $x \in \mathcal{X}_{s-Par}\left(f^{q_1}, \ldots, f^{q_P}\right) \Rightarrow x \in \mathcal{X}_{s-Par}\left(f^1, \ldots, f^Q\right)$.

3 The Reduction Result

In the main part of the paper we consider quasi-convex (MOP), i.e. $f^q : \mathbb{R}^n \to \mathbb{R}$ are quasi-convex functions for all $q \in \mathcal{Q}$ and $X = \mathcal{C}$ is a convex subset of $\mathbb{R}^n$. A function f^q is quasi-convex if for each x and $y \in \mathbb{R}^n$

$$f^q\left(\lambda x + (1-\lambda)y\right) \leq \max\{f^q(x), f^q(y)\}$$

for all $\lambda \in (0,1)$. To avoid technicalities we assume that all functions are defined on the whole space $\mathbb{R}^n$. An appealing feature of quasi-convex functions is that they can be characterized in terms of level sets. The following Lemma is well known, see e.g. [1].

Lemma 1 $f^q : \mathbb{R}^n \to \mathbb{R}$ *is quasi-convex if and only if $L^q_{\leq}(z)$ is convex for all $z \in \mathbb{R}$.*

We will also use the notion of the dimension of a convex set $\mathcal{C}$ which is defined as the dimension of the affine subspace spanned by $\mathcal{C}$, i.e.

$$\dim(\mathcal{C}) := \dim\,(\mathrm{span}(\mathcal{C})).$$

The main result of the paper relies on the following result from convex analysis, known as Helly's Theorem, see [16] and [11].

Lemma 2 *Let $C_1, \ldots, C_Q$, $Q \geq n+1$ be convex sets in $\mathbb{R}^n$. Then*

$$\bigcap_{q=1}^{Q} C_q \neq \emptyset \Leftrightarrow \forall q_1, \ldots, q_{n+1} \in \mathcal{Q}: \; C_{q_1} \cap \ldots \cap C_{q_{n+1}} \neq \emptyset,$$

or equivalently

$$\bigcap_{q=1}^{Q} C_q = \emptyset \Leftrightarrow \exists q_1, \ldots, q_{n+1} \in \mathcal{Q}: \; C_{q_1} \cap \ldots \cap C_{q_{n+1}} = \emptyset.$$

In this section we only deal with quasi-convex objectives and hence all level sets will be convex. As we already know strict and weak Pareto optimal solutions as well as Pareto optimal ones can be characterized geometrically by investigating intersections of level sets as shown in Theorem 1. To do so, we combine the results of Section 2 and Helly's Theorem. An immediate consequence of Corollary 1, Lemma 2 and Theorem 1 is the following result.

Corollary 2 *Let $f^1, \ldots, f^Q$ be quasi-convex functions, $f^q : \mathbb{R}^n \to \mathbb{R}$. Then a point $x \in \mathcal{C}$ is in $\mathcal{X}_{w-Par}\left(f^1, \ldots, f^Q\right)$ if and only if there exists a subset $\{f^{q_1}, \ldots, f^{q_P}\}$ of objective functions, $P \leq n+1$, such that $x \in \mathcal{X}_{w-Par}\left(f^{q_1}, \ldots, f^{q_P}\right)$.*

Corollary 2 implies that, in order to check whether x is a weak Pareto solution, it is sufficient to check (at most) all subsets of no more than three level sets through the point x for an empty intersection. When $n = 2$ this means that in the worst case $O(Q^3)$ sets of three convex sets have to be checked for an empty intersection. The practical relevance for (MOP) with $Q \gg 2$, as is the case in location theory, is immediate.
The main purpose of this paper is to generalize the rather straightforward result of Corollary 2 for Pareto optimal solutions $\mathcal{X}_{Par}\left(f^1, \ldots, f^Q\right)$. Although this leap does not seem to be that big, the different characterization in terms of level sets prohibits an easy solution. Checking for equality of the intersections in part 2 of Theorem 1 is more difficult than simply checking for empty intersection as in part 3.
First results in this regard for special cases have been proved earlier by the authors. We refer to [13] for multiple criteria location problems. In [14], finite convex functions on $\mathbb{R}^2$ have been considered. The following theorem deals with the most general case of quasi-convex functions on $\mathbb{R}^n$, see also [5].
Theorem 2 does not only show that checking intersections of at most $n+1$ level sets is sufficient to check Pareto optimality of a point $x \in \mathcal{C}$. It also gives a constructive proof applicable to find Pareto optimal solutions. This has been done for the special case of location problems in [13]. As mentioned before, the proof relies on Lemma 2 and on the use of level sets and level curves. Note that due to Theorem 1 checking intersections of level curves is equivalent to checking Pareto optimality for a subset of criteria.

Theorem 2 *Let $f^1, \ldots, f^Q$ be quasi-convex functions and let $\mathcal{C} \subset \mathbb{R}^n$ be a convex set. For fixed $n \in \mathbb{N}$ and arbitrary $Q > n+1$ the problem $x \in \mathcal{X}_{Par}\left(f^1, \ldots, f^Q\right)$ can be decided in polynomial time, if $x \in \mathcal{X}_{Par}\left(f^{q_1}, \ldots, f^{q_P}\right)$ can be decided in polynomial time, for all $P \leq n+1$.*

The important consequence of Theorem 2 is that in order to decide whether a given point x is Pareto optimal, it suffices to check Pareto optimality with respect to subsets of at most $n+1$ of the Q objective functions. The number of such subsets is bounded by $O(Q^{n+1})$ and is polynomial in Q.
Before we present the proof we will introduce some notation to facilitate readability. We denote the set of all subsets of $\mathcal{Q}$ with no more than $n+1$ elements by $\mathcal{Q}(n+1)$

$$\mathcal{Q}(n+1) := \{\mathcal{J} \subset \mathcal{Q} : |\mathcal{J}| \leq n+1\}$$

and let $\mathcal{J} = \{q_1, \ldots, q_m\}$ with $2 \leq m \leq n$ be an element of $\mathcal{Q}(n+1)$. Note that $m \leq n < n+1 \leq Q$. Let $L^{q_i}_{\leqq}(f^{q_i}(x))$, $i = 1, \ldots, m$ be the corresponding level curves through x. For $q_i \in \mathcal{J}$ we use the following abbreviations:

$$\begin{aligned} L^{q_i}_{\leq}(f^{q_i}(x)) &=: L^{q_i}_{\leq} \\ \bigcap_{q_i \in \mathcal{J}} L^{q_i}_{\leq} = \bigcap_{i=1}^{m} L^{q_i}_{\leq}(f^{q_i}(x)) &=: L_{\leq} \\ dim(L^{q_i}_{\leq}) &=: d_i \\ dim(L_{\leq}) &=: d. \end{aligned}$$

Analogously we use $L^{q_i}_{<}$ for $L^{q_i}_{<}(f_{q_i}(x))$ and $L_{<}$ for $\bigcap_{i=1}^{m} L^{q_i}_{<}$. Note that by the definition of level sets $x \in L_{\leq}$ and therefore $L_{\leq} \neq \emptyset$. Lemma 3 provides a necessary condition for x to be Pareto optimal.

Lemma 3 *If x is Pareto optimal there exists a set $\{q_1, \ldots, q_{n+1}\} \subseteq \mathcal{Q}$ and a number $m < n+1$ such that*

$$L_{<} \cap \bigcap_{i=m+1}^{n+1} L^{q_i}_{<} = \emptyset,$$

where $L_{<}$ is defined as above.

Proof.
If x is a Pareto solution x is also weakly Pareto optimal, i.e. $x \in \mathcal{X}_{w-Par}\left(f^1, \ldots, f^Q\right)$. By Theorem 1 this implies $\bigcap_{i=1}^{Q} L^{i}_{<} = \emptyset$. The result now follows from Corollary 2. □

An outline of the proof of Theorem 2 is as follows. Let $x \in \mathcal{C}$ be a feasible point. Using the notations introduced above we select index sets from $\mathcal{Q}(n+1)$ and check the intersection of the corresponding level sets. We prove that, after having checked at most all members of $\mathcal{Q}(n+1)$, we can decide whether x is Pareto optimal or not. We will start with an index set $\mathcal{J} = \{q_1, \ldots, q_m\} \in \mathcal{Q}(n+1)$ such that, as noted above $|\mathcal{J}| = m \leq n$. Index sets already checked will always be deleted from $\mathcal{Q}(n+1)$.

Proof of Theorem 2.
Let $\mathcal{J} \in \mathcal{Q}(n+1), |\mathcal{J}| = m \leq n$ and define $\mathcal{Q}(n+1) := \mathcal{Q}(n+1) \setminus \{\mathcal{J}\}$. Let $\mathcal{P} := \emptyset$. For the intersection of level sets $L_{\leq}$ corresponding to $\mathcal{J}$ and the given point x we distinguish three cases.

Case I $d = 0$ (see Figure 1)

Then $L_{\leq} = \{x\}$ and therefore $x \in \mathcal{X}_{s-Par}(f^{q_1}, \ldots, f^{q_m})$ by Theorem 1. By Corollary 1 we conclude that $x \in \mathcal{X}_{s-Par}\left(f^1, \ldots, f^Q\right)$, in particular x is Pareto optimal.

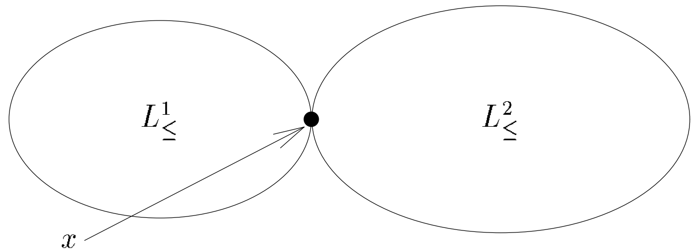

Figure 1: Case I when $n = 2$ and $m = 2$

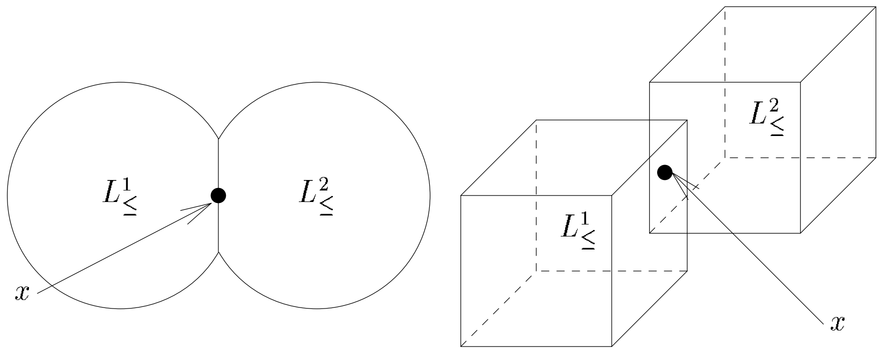

Figure 2: Case II when $n = 2, m = 2$ (left hand side) and $n = 3, m = 2$ (right hand side)

Case II $0 < d < \min\limits_i d_i$ (see Figure 2)

We choose another level curve $L^{q_{m+1}}_{\leq}$ through x and consider three sub-cases, i.e. let $q_{m+1} \in \mathcal{Q} \setminus \mathcal{J}$ and consider $L^{q_{m+1}}_{\leq}$. We let $\mathcal{J} := \mathcal{J} \cup \{q_{m+1}\}$ and delete $\mathcal{J}$ from $\mathcal{Q}(n+1)$, i.e. $\mathcal{Q}(n+1) := \mathcal{Q}(n+1) \setminus \{\mathcal{J}\}$.

a) $\dim(L_\leq \cap L^{q_{m+1}}_{\leq}) = 0$

Then $x \in \mathcal{X}_{s-Par}(f^{q_1}, \ldots, f^{q_{m+1}})$ and by Corollary 1 and since $\mathcal{X}_{s-Par} \subset \mathcal{X}_{Par}$ we conclude that $x \in \mathcal{X}_{Par}\left(f^1, \ldots, f^Q\right)$.

b) $0 < \dim(L_\leq \cap L^{q_{m+1}}_{\leq}) < d$

Then if $m < n$ update $m := m + 1$ and start again with the new $\mathcal{J}$ defined at the beginning of Case II.

If $m = n$ we distinguish two possible situations.

- The point x is not Pareto optimal for the objectives in $\mathcal{J}$. Then we start with a new choice of $\mathcal{J}$. (There must exist at least one subset of n criteria for which x is Pareto optimal in order for x to be Pareto optimal for all objectives). Let $\mathcal{Q}(n+1) := \mathcal{Q}(n+1) \setminus \{\mathcal{J}\}$.
- The point x is Pareto optimal for the objectives in $\mathcal{J}$. Then $\mathcal{P} := \mathcal{P} \cup \mathcal{J}$. If $\mathcal{P} = \mathcal{Q}$ then x is Pareto optimal for all Q objectives and we can stop. Otherwise we continue with a new set $\mathcal{J}$ and let $\mathcal{Q}(n+1) := \mathcal{Q}(n+1) \setminus \{\mathcal{J}\}$.

c) $\dim(L_{\leq} \cap L_{\leq}^{q_{m+1}}) = d$

We have three possibilities

- x is not Pareto optimal for $f^{q_1}, \ldots, f^{q_m}$.
  Then x is not Pareto optimal for $f^{q_1}, \ldots, f^{q_{m+1}}$ and we proceed as in Case b) above.
- $x \in \mathcal{X}_{Par}(f^{q_1}, \ldots, f^{q_m})$ and $L_{<}^{q_{m+1}} \cap L_{\leq} \neq \emptyset$.
  Then x is not Pareto optimal for $f^{q_1}, \ldots, f^{q_{m+1}}$ by Lemma 3 and again we proceed as in Case b) above.
- $x \in \mathcal{X}_{Par}(f^{q_1}, \ldots, f^{q_m})$ and $L_{<}^{q_{m+1}} \cap L_{\leq} = \emptyset$.
  Then x is also in $\mathcal{X}_{Par}(f^{q_1}, \ldots, f^{q_{m+1}})$. In this case we choose an index $q'_{m+1} \in \mathcal{Q} \setminus \mathcal{J}$ (i.e. $q'_{m+1} \neq q_{m+1}$) and consider the new index set $\mathcal{J} := (\mathcal{J} \setminus \{q_{m+1}\}) \cup \{q'_{m+1}\}$. Finally we update $\mathcal{Q}(n+1) := \mathcal{Q}(n+1) \setminus \{\mathcal{J}\}$ and let $m := m+1$. We consider the corresponding level curves and level sets and start with this $\mathcal{J}$.

Case III $d = \min_i d_i$ (see Figure 3)

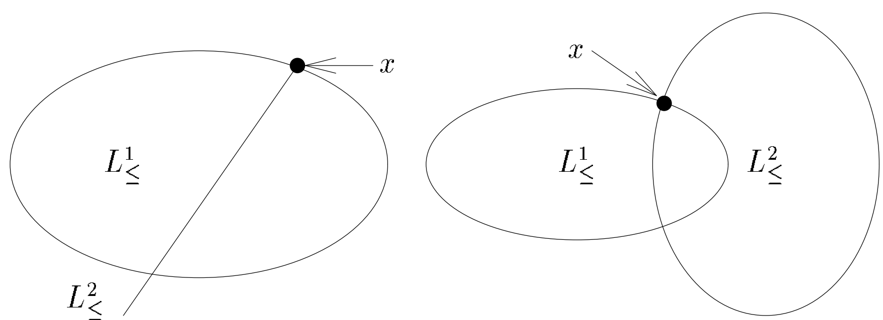

Figure 3: Two possibilities for Case III when $n = 2$ and $m = 2$

We choose another level curve $L_{=}^{q_{m+1}}$ through x and consider three sub-cases.

a) $\dim(L_{\leq} \cap L_{\leq}^{q_{m+1}}) = 0$

Then $x \in \mathcal{X}_{s-Par}(f^{q_1}, \ldots, f^{q_{m+1}})$ and by Lemma 1 we conclude that $x \in \mathcal{X}_{Par}\left(f^1, \ldots, f^Q\right)$.

b) $0 < \dim(L_{\leq} \cap L_{\leq}^{q_{m+1}}) < d$

Since $d = \min_i d_i$ there must be a subset $\{q'_1, \ldots, q'_{m'}\} \subseteq \{q_1, \ldots, q_{m+1}\}$ such that $dim(\bigcap_{i=1}^{m'} L_{\leq}^{q'_i}) < d$. We choose this subset as new index set, $\mathcal{J} := \{q'_1, \ldots, q'_{m'}\}$, update $\mathcal{Q}(n+1) := \mathcal{Q}(n+1) \setminus \{\mathcal{J}\}$ and $m := m'$. We consider the corresponding level sets (and curves) and start with this $\mathcal{J}$.

c) $\dim(L_{\leq} \cap L_{\leq}^{q_{m+1}}) = d$

Then x is not Pareto optimal for $f^{q_1}, \ldots, f^{q_m}$ and x is also not Pareto optimal for $f^{q_1}, \ldots, f^{q_{m+1}}$. We proceed as in Case II b) above.

After a finite number of applications of this procedure we have checked at most all subsets of m, $2 \leq m \leq n+1$, level curves through x and the corresponding intersections of the level sets. The conclusion is that either x is a strict Pareto optimal solution (Cases I, IIa), IIIa)), x is a Pareto optimal solution (Cases IIc) and IIb)) or there was no set of $n+1$ objective functions such that $\bigcap_{i=1}^{n+1} L_<^{q_i} = \emptyset$ implying that x is not Pareto optimal ($\mathcal{Q}(n+1)$ is empty without the conclusion that x is Pareto optimal). Note that if in Case II c) the last sub-case occurs for every choice of $L_{\leq}^{q_{m+1}}$ then $x \in \mathcal{X}_{Par}\left(f^1, \ldots, f^Q\right)$.
The number of intersections of level sets checked is bounded by

$$|\mathcal{Q}(n+1)| = \sum_{i=1}^{n+1} \binom{Q}{i} = O(Q^{n+1}).$$

□

We would like to emphasize that intersecting level curves and level sets is essentially testing Pareto optimality for $m \leq n+1$ criteria. Therefore, in designing applications of the procedure given in the proof of Theorem 2, any algorithm to solve quasi-convex (MOP) with no more than $n+1$ objectives may be used.

Lemma 2 and Theorem 2 show how to decide (weak) Pareto optimality, by checking subsets of $n+1$ level sets. To conclude this section we briefly address the case of strict Pareto optimality. Here the geometrical characterization provided by Theorem 1 seems to be easier again. But the following example shows that up to $2n$ level sets have to be intersected, in order to prove strict Pareto optimality.

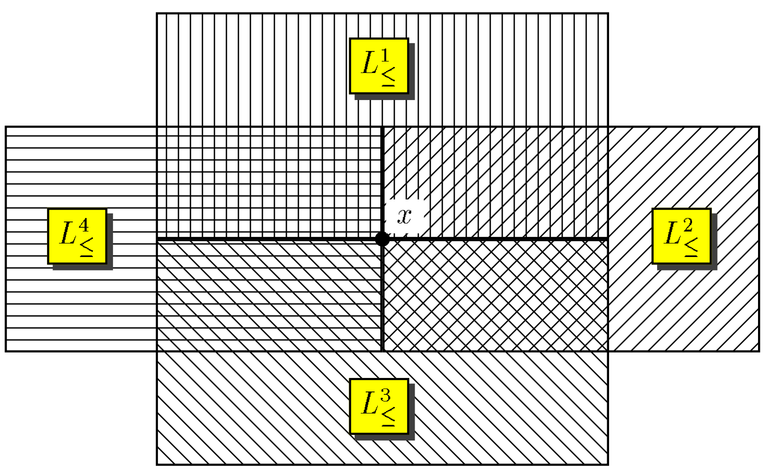

Figure 4: Intersection of $2n$ level sets

In Figure 4, x belongs to both $\mathcal{X}_{Par}(f^1, f^3)$ and $\mathcal{X}_{Par}(f^2, f^4)$ as well as $\mathcal{X}_{s-Par}(f^1, \ldots, f^4)$ but is not Pareto optimal for any set of three objectives. The example also illustrates the importance of index set $\mathcal{P}$ in Theorem 2. After checking $\mathcal{J} = \{1, 3\}$ and $\mathcal{J} = \{2, 4\}$, the procedure will stop with the conclusion that x is a Pareto solution, not detecting strict Pareto optimality of x.

4 An Illustrative Example

We consider an Example with four objectives and $\mathcal{C} = [-5, 5] \times [-5, 5] \subseteq \mathrm{IR}^2$. The objectives are

$$
\begin{aligned}
f^1(x, y) &= (x - 2.5)^2 + (y - 0.5)^2 - 6.5 \\
f^2(x, y) &= 2x^2 + \frac{1}{2}(y - 3)^2 - 4.5 \\
f^3(x, y) &= y - 5 \\
f^4(x, y) &= 2x + 3
\end{aligned}
$$

First of all we show that none of the objectives is nonessential. In the following table we list the 4 objectives f^i in the first column. The second and third columns show four points x^i which are Pareto solutions for the (MOP)

$$\min_{x \in \mathcal{C}} (f^1, f^2, f^3, f^4)$$

and their respective objective values. The fourth and fifth column show points x_d^i which dominate x^i in the (MOP)

$$\min_{x \in \mathcal{C}} \hat{f}_i,$$

where $\hat{f}_i$ denotes the criterion mapping f with f^i dropped.

f^i	x^i	$f(x^i)$	x_d^i	$\hat{f}_i(x_d^i)$
f^1	(2.5, 0.5)	(-6.5, 11.25, -4.5, 8)	(0, 0)	(• 0, -5, 3)
f^2	(0, 3)	(6, -4.5, -2, 3)	(0, 1)	(0, • -4, 3)
f^3	(1, -5)	(26, 29.5, -10, 5)	(0, -2)	(6, 8, •, 3)
f^4	(-5, 1)	(50, 47.5, -4, -7)	(-4, 0)	(36, 32, -5, •)

Note that x^1 and x^2 are Pareto optimal because they are the (unique) minimizers of f^1 and f^2, respectively. Points x^3 and x^4 are minimizers of f^3 and f^4 over $\mathcal{C}$, respectively. Furthermore they are Pareto optimal for

$$\min\{\hat{f}_3(x, y) : (x, y) \in \mathcal{C}, y = -5\},$$

the Pareto set of which is $[-5, 2.5] \times \{-5\}$ and for

$$\min\{\hat{f}_4(x, y) : (x, y) \in \mathcal{C}, x = -5\}$$

with the Pareto set $\{-5\} \times [-5, 3]$, respectively. Both facts imply that x^3 and x^4 are Pareto optimal for the original (MOP). Now we apply the procedure described in the proof of Theorem 2 for two feasible points. First, we consider $x = (0, 0)$. The corresponding level curves are shown in Figure 5.

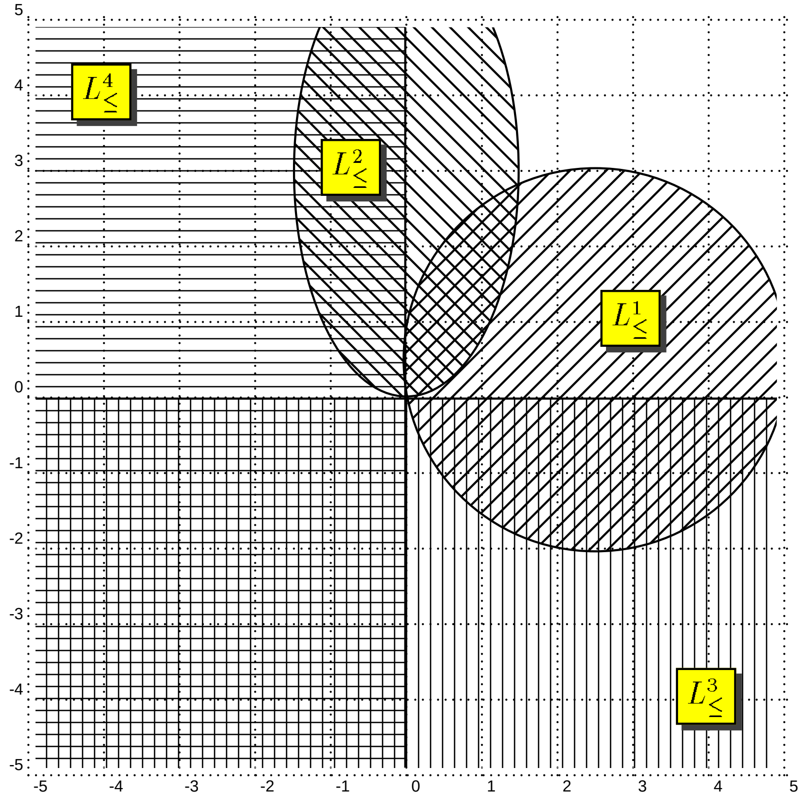

Figure 5: Level sets for $x = (0, 0)$

From $\bigcap_{q=1}^{4} L^q_\leq(x) = \{x\}$ with Theorem 1 we see that x is a Pareto solution. Let us apply the procedure of the proof of Theorem 2.
We choose $\mathcal{J} = \{1, 4\}$. As

$$\dim \bigcap_{q \in \mathcal{J}} L^q_\leq = 2$$

we are in Case III (the three points $(0, 0), (0, 1)$ and $(-0.5, 2 - \sqrt{6.5})$ belong to both level sets). We choose $q_{m+1} = 2$. Now $\dim(L_\leq \cap L^2_\leq) = 2$ and x is not Pareto optimal for (f^1, f^2, f^4). Since $m = n = 2$ we select a new $\mathcal{J}$. We choose $\mathcal{J} = \{1, 3\}$ and again

$$dim \bigcap_{q \in \mathcal{J}} L^q_\leq = 2.$$

Now in Case III let us choose $q_{m+1} = 2$. Then $\dim(L_\leq \cap L^2_\leq) = 0$ implying that x is strictly Pareto optimal.

Also note that if $\mathcal{J} = \{2,3\}$ is chosen initially, strict Pareto optimality is immediate, i.e. the result does not provide any information on which objectives really determine Pareto optimality of x.

For a second point, let us try $x = (5,5)$. The corresponding level sets are depicted in Figure 6. Note that $L^3_{\leq} = L^4_{\leq} = \mathcal{C}$ in this case.

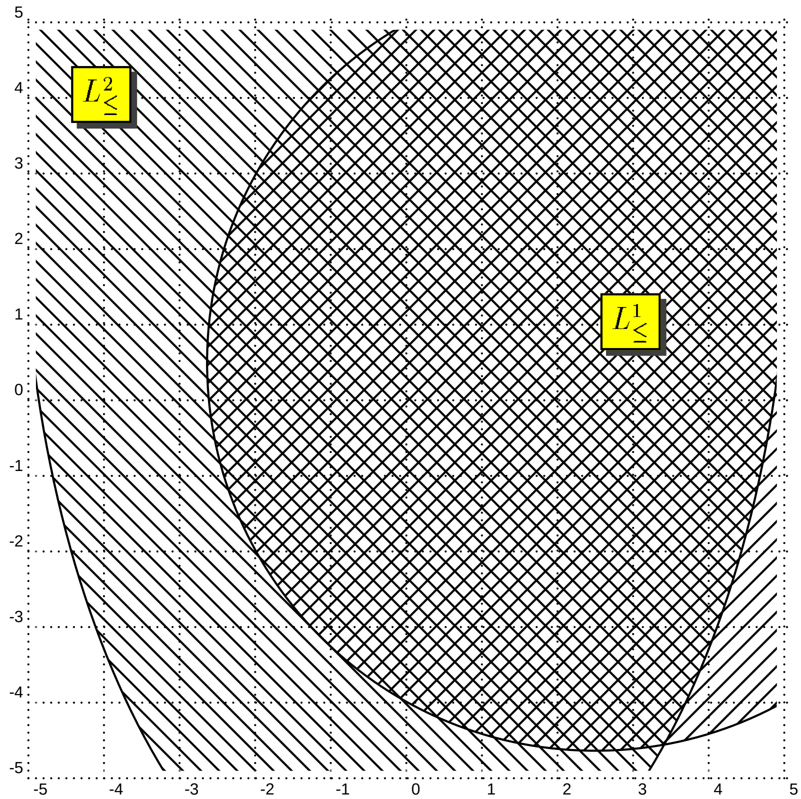

Figure 6: Level sets for $x = (5,5)$

Evidently, whatever the choice of $\mathcal{J}$, $\bigcap_{q\in\mathcal{J}} L^q_{\leq}$ will have dimension two, and the procedure will (correctly) stop without the conclusion that x is Pareto optimal. The geometrical characterization of Theorem 1 is not satisfied for x, confirming this conclusion.

5 Conclusions

In this paper we have discussed methods to decide Pareto optimality for a point in the decision space of (MOP) using only subsets of the set of criteria. For weak Pareto solutions a straightforward corollary to Helly's theorem shows that $n+1$ criteria suffice. This result has been generalized to the case of Pareto solutions. As an important consequence, solving a Q-criteria quasi-convex (MOP) in $\mathbb{R}^n$ can be reduced to solving a polynomial number of $n+1$-criteria problems in $\mathbb{R}^n$.

We remark that in the case $n = 1$, Theorem 2 is not helpful. It is well known that $\mathcal{X}_{Par}$ and $\mathcal{X}_{w-Par}$ are connected, see [17]. Then these sets are intervals and their determination is equivalent to the solution of Q single criterion convex minimization problems in $\mathbb{R}$.
An application of the methods developed has already been started in location theory, see [13], [10], and [15]. Further research will be concerned with applications in more general settings.

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