Interner Bericht

Gauss Frame Offsets

G. Farin
Arizona State University
H. Hagen
University of Kaiserslautern
D. Hansford
Arizona State Univestiy

Fachbereich Informatik

Universität Kaiserslautern · Postfach 3049 · D-6750 Kaiserslautern

Gauss Frame Offsets

G. Farin
Arizona State University
H. Hagen
University of Kaiserslautern
D. Hansford
Arizona State University

223/92

Universität Kaiserslautern
AG Computergraphik
Postfach 30 49
6750 Kaiserslautern

Mai 1992

Herausgeber: AG Graphische Datenverarbeitung und Computergeometrie
Leiter: Professor Dr. H. Hagen

Gauss Frame Offsets

G. Farin
Arizona State University
H. Hagen
University of Kaiserslautern
D. Hansford
Arizona State Univestiy

Universität Kaiserslautern
AG Computergraphik
Postfach 30 49
6750 Kaiserslautern

Mai 1992

Herausgeber: **AG Graphische Datenverarbeitung und Computergeometrie**
Leiter: Professor Dr. H. Hagen

Gauss Frame Offsets

G. Farin
Arizona State University

H. Hagen
University of Kaiserslautern

D. Hansford
Arizona State University

May 1992

Abstract

We describe a method to approximate the offset of a tensor product surface by a bicubic Bézier patch. The method takes advantage of the geometric controls of the Bernstein Bézier form, using the Gauss frame and a least squares fit to sampled exact offsets points.

1 Introduction

The offset surfac^ problem is one of the most fundamental ones in the area of Geometry Processing. Since these offsets cannot (except for some very special cases) be computed exactly, approximations are called for, and are, for example employed by Chen and Ravani [3], Farouki [5], Rossignac and Requicha [11], and Sabin [12]. Pham [10] presents a survey of these methods. Interset in having a surface representation of the offset approximation stems from a need to apply algorithms for checking for self-intersections (see Aomura and Uehara [1], Barnhill *et al.* [2], or Farouki and Neff [7, 6]) or algorithms for curvature analysis (see Farouki [5]).

Our method makes specific use of the geometry offered by the Bernstein Bézier form. The progenitor surface (the one whose offset is to be computed) may be of any form – all we assume is that we can evaluate points and first derivatives over a rectangular domain. The method is described in Section 2. Examples and conclusions are given in Section 3.

2 Gauss Frame Offsets

We will describe a method to compute an offset approximation to any surface that is continuously differentiable and defined over a rectangular domain $(u, v) \in [u_0, u_1] \times [v_0, v_1]$. For the sake of concreteness, however, we have worked with progenitors **s** that are Bernstein Bézier patches of various degrees:

$$\mathbf{s}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{s}_{ij} B_i^n(u) B_j^m(v), \qquad (u,v) \in [u_0, u_1] \times [v_0, v_1] \tag{1}$$

where

$$B_i^n(u) = \binom{n}{i}\left(\frac{u-u_0}{u_1-u_0}\right)^i \left(1 - \frac{u-u_0}{u_1-u_0}\right)^{n-i}. \tag{2}$$

The following development is for progenitors of this form. In particular, we take advantage of the fact that the derivatives of **s** are easy to obtain, see Farin [4]. If the progenitor happens not to be in this form, the explicit derivatives in Bernstein Bézier form will have to be replaced by the desired ones.

The offset of **s** at a distance d is

$$\mathbf{s}_d(u,v) = \mathbf{s}(u,v) + d\mathbf{N}(u,v) \tag{3}$$

where $\mathbf{N}(u, v)$ is the outward unit normal to $\mathbf{s}(u, v)$.

We will determine a bicubic approximation **b** to the offset $\mathbf{s}_d$

$$\mathbf{b}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{b}_{ij} B_i^3(u) B_j^3(v) \tag{4}$$

Some notation: we will set $\mathbf{b}_u = \frac{\partial \mathbf{b}(u,v)}{\partial u}$ etc.

The offset approximation, $\mathbf{b}(u, v)$, is constructed to satisfy two conditions.

Condition 1: Tangent planes of **s** and **b** are parallel at the patch corners.

We can trivially achieve that the offset is exact at the corners:

$$\mathbf{b}(u_r, v_t) = \mathbf{s}(u_r, v_t) + d\mathbf{N}(u_r, v_t) \tag{5}$$

for $r, t \in \{0, 1\}$. This condition also demands that

$$\begin{aligned} \mathbf{b}_u(u_r, v_t) &= \lambda_{rt}\mathbf{s}_v(u_r, v_t) + \lambda^\star_{rt}\mathbf{s}_u(u_r, v_t) \qquad (6)\\ \mathbf{b}_v(u_r, v_t) &= \mu_{rt}\mathbf{s}_v(u_r, v_t) + \mu^\star_{rt}\mathbf{s}_u(u_r, v_t) \qquad (7) \end{aligned}$$

for $r, t \in \{0, 1\}$. The scalars $\lambda_{rt}, \lambda^\star_{rt}, \mu_{rt}, \mu^\star_{rt}$ are unknowns to be computed by a least squares fitting routine.

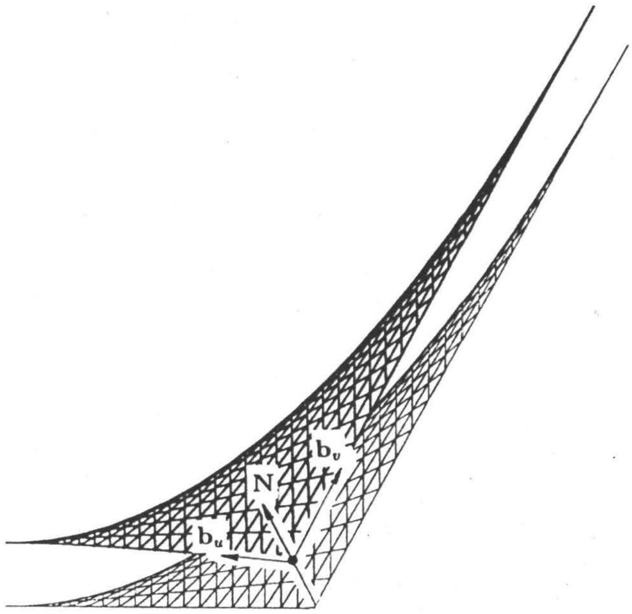

Figure 1: Gauss frames: the relationship between the progenitor and the offset at the corner (u_0, v_0).

Note that the corner derivatives of **s** and **b** do not have to be parallel! However, as is with the exact offset, the offset approximation will have the same normal at the corners as the progenitior.

This condition can be considered a generalization of the work of Hoschek [9] in the case of offsets of curves.

Condition 2: Gauss frame twists.

The twists of $\mathbf{b}(u, v)$ are expressed in terms of the Gauss frame $\mathbf{b}_u, \mathbf{b}_v, \mathbf{N}$, constructed at each corner:

$$\mathbf{b}_{u,v}(u_r, v_t) = \alpha_{rt}\mathbf{N}(u_r, v_t) + \beta_{rt}\mathbf{b}_u(u_r, v_t) + \gamma_{rt}\mathbf{b}_v(u_r, v_t) \tag{8}$$

where $\alpha_{rt}, \beta_{rt}, \gamma_{rt}$ with $r, t \in \{0, 1\}$ are unknowns to be computed by a least squares fitting routine. Figure 1 shows the geometry behind these equations. Again we use the geometric controls of the Bernstein Bézier form by writing the twists of the patch in terms of the known corner vectors.

Steps of the Method.

We now outline the main components of our offset procedure.

1. Evaluate the progenitor in a $(k+1) \times (k+1)$ grid within the boundary of the domain. Compute the exact offset at these points, and call them $\mathbf{EI}_{ij}$. Likewise, evaluate the progenitor in $k+1$ places along each boundary (excluding the endpoints), and compute the exact offset. The exact offset along the $v = v_0$ edge is called $\mathbf{E}_{i0}$. The other boundaries follow similarly.

2. Compute the exact offset at each corner point of the progenitor. These correspond to the corner Bézier points of the offset approximation. For example,
$$\mathbf{b}_{00} = \mathbf{s}_{00} + d\mathbf{N}(u_0, v_0). \tag{9}$$

3. Compute the offset approximation Bézier points along each boundary. The following details are for the $v = v_0$ edge. We can compute the corner derivatives:
$$\mathbf{b}_u(u_0, v_0) = \frac{3}{u_1 - u_0}(\mathbf{b}_{10} - \mathbf{b}_{00}) \tag{10}$$
$$\mathbf{b}_u(u_1, v_0) = \frac{-3}{u_1 - u_0}(\mathbf{b}_{20} - \mathbf{b}_{30}). \tag{11}$$

   Combining (6) with (10) and (7) with (11), we obtain
$$\frac{3}{u_1 - u_0}(\mathbf{b}_{10} - \mathbf{b}_{00}) = \lambda_{10}\frac{m}{v_1 - v_0}(\mathbf{s}_{01} - \mathbf{s}_{00}) + \lambda^\star_{10}\frac{n}{u_1 - u_0}(\mathbf{s}_{10} - \mathbf{s}_{00}) \tag{12}$$
$$\frac{-3}{u_1 - u_0}(\mathbf{b}_{20} - \mathbf{b}_{30}) = \lambda_{20}\frac{m}{v_1 - v_0}(\mathbf{s}_{n1} - \mathbf{s}_{n0}) - \lambda^\star_{20}\frac{n}{u_1 - u_0}(\mathbf{s}_{(n-1)0} - \mathbf{s}_{n0}) \tag{13}$$

   Now we use (12) and (13) to determine expressions for $\mathbf{b}_{10}$ and $\mathbf{b}_{20}$.

   The offset curve will not pass through the computed points $\mathbf{E}_{i0} = \mathbf{s}(u_i, v_0) + d\mathbf{N}(u_i, v_0)$ exactly. The local error (i.e., by how much each sample point is missed) is written as
$$\delta_i = \mathbf{E}_{i0} - \sum_{j=0}^{3} \mathbf{b}_{j0} B^3_j(u_i). \tag{14}$$

   Then the global error is
$$\delta = \sum_{i=0}^{k} \delta_i^2 = \sum_{i=0}^{k} \Big[\mathbf{D}_{i0} - \lambda_{10}\frac{m}{3}\frac{u_1 - u_0}{v_1 - v_0}(\mathbf{s}_{01} - \mathbf{s}_{00})B^3_1(u_i) \tag{15}$$

$$
\begin{aligned}
&- \lambda_{10}^{\star}\frac{n}{3}(\mathbf{s}_{10}-\mathbf{s}_{00})B_1^3(u_i) \\
&+ \lambda_{20}\frac{m}{3}\frac{u_1-u_0}{v_1-v_0}(\mathbf{s}_{n1}-\mathbf{s}_{n0})B_2^3(u_i) \\
&- \lambda_{20}^{\star}\frac{n}{3}(\mathbf{s}_{(n-1)0}-\mathbf{s}_{n0})B_2^3(u_i)\Big]^2,
\end{aligned}
$$

where

$$\mathbf{D}_{i0} = \mathbf{E}_{i0} - \mathbf{b}_{00}B_0^3(u_i) - \mathbf{b}_{00}B_1^3(u_i) - \mathbf{b}_{30}B_2^3(u_i) - \mathbf{b}_{30}B_3^3(u_i). \quad (16)$$

Minimizing this global error leads to the 4×4 normal equations

$$
\begin{aligned}
\frac{\partial\delta}{\partial\lambda_{10}} = 0, \quad & \frac{\partial\delta}{\partial\lambda_{10}^{\star}} = 0, \\
\frac{\partial\delta}{\partial\lambda_{20}} = 0, \quad & \frac{\partial\delta}{\partial\lambda_{20}^{\star}} = 0,
\end{aligned} \quad (17)
$$

which we solve for $\lambda_{10}, \lambda_{10}^{\star}, \lambda_{20}, \lambda_{20}^{\star}$.

The precise system is given in the Appendix, along with the systems for the other three edges.

4. Compute the twist Bézier points of the offset approximation. Giving more detail for the (u_0, v_0) corner, and using (8) we define

$$\Pi = \frac{3}{(u_1-u_0)}\frac{3}{(v_1-v_0)}. \quad (18)$$

The twist of $\mathbf{b}$ at (u_0, v_0) is given by

$$
\begin{aligned}
\Pi(\mathbf{b}_{00}-\mathbf{b}_{10}-\mathbf{b}_{01}+\mathbf{b}_{11}) &= \alpha_{00}\mathbf{N}_{00} \\
&+ \beta_{00}\frac{3}{u_1-u_0}(\mathbf{s}_{10}-\mathbf{s}_{00}) \\
&+ \gamma_{00}\frac{3}{v_1-v_0}(\mathbf{s}_{01}-\mathbf{s}_{00}).
\end{aligned} \quad (19)
$$

We can easily solve (19) for the twist point $\mathbf{b}_{11}$. A similar procedure is followed for the twist points $\mathbf{b}_{12}$, $\mathbf{b}_{21}$, and $\mathbf{b}_{22}$.

The least squares local error is

$$\delta_{rt} = \mathbf{EI}_{rt} - \sum_{i=0}^{3}\sum_{j=0}^{3}\mathbf{b}_{ij}B_i^3(u_r)B_j^3(v_t), \quad (20)$$

and the global error is given by

$$
\begin{aligned}
\delta = \sum_{r=0}^{k}\sum_{t=0}^{k}\delta_{rt}^2 = \sum_{r=0}^{k}\sum_{t=0}^{k}\Big[\mathbf{DI}_{rt} \quad & - \quad \alpha_{00}\Pi^{-1}\mathbf{N}_{00}B_1^3(u_r)B_1^3(v_t) \\
& - \quad \beta_{00}\frac{v_1 - v_0}{3}(\mathbf{b}_{10} - \mathbf{b}_{00})B_1^3(u_r)B_1^3(v_t) \\
& - \quad \gamma_{00}\frac{u_1 - u_0}{3}(\mathbf{b}_{01} - \mathbf{b}_{00})B_1^3(u_r)B_1^3(v_t) \\
& + \quad \alpha_{01}\Pi^{-1}\mathbf{N}_{03}B_1^3(u_r)B_2^3(v_t) \\
& + \quad \beta_{01}\frac{v_1 - v_0}{3}(\mathbf{b}_{13} - \mathbf{b}_{03})B_1^3(u_r)B_2^3(v_t) \\
& - \quad \gamma_{01}\frac{u_1 - u_0}{3}(\mathbf{b}_{02} - \mathbf{b}_{03})B_1^3(u_r)B_2^3(v_t) \\
& + \quad \alpha_{10}\Pi^{-1}\mathbf{N}_{30}B_2^3(u_r)B_1^3(v_t) \\
& - \quad \beta_{10}\frac{v_1 - v_0}{3}(\mathbf{b}_{20} - \mathbf{b}_{30})B_2^3(u_r)B_1^3(v_t) \\
& + \quad \gamma_{10}\frac{u_1 - u_0}{3}(\mathbf{b}_{31} - \mathbf{b}_{30})B_2^3(u_r)B_1^3(v_t) \\
& - \quad \alpha_{11}\Pi^{-1}\mathbf{N}_{33}B_2^3(u_r)B_2^3(v_t) \\
& + \quad \beta_{11}\frac{v_1 - v_0}{3}(\mathbf{b}_{23} - \mathbf{b}_{33})B_2^3(u_r)B_2^3(v_t) \\
& + \quad \gamma_{11}\frac{u_1 - u_0}{3}(\mathbf{b}_{32} - \mathbf{b}_{33})B_2^3(u_r)B_2^3(v_t)\Big]^2,
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
\mathbf{DI}_{rt} = \mathbf{EI}_{rt} \quad & - \quad \mathbf{b}_{00}B_0^3(u_r)B_0^3(v_t) - \mathbf{b}_{10}B_1^3(u_r)B_0^3(v_t) \\
& - \quad \mathbf{b}_{20}B_2^3(u_r)B_0^3(v_t) - \mathbf{b}_{30}B_3^3(u_r)B_0^3(v_t) \\
& - \quad (-\mathbf{b}_{00} + \mathbf{b}_{10} + \mathbf{b}_{01})B_1^3(u_r)B_1^3(v_t) \\
& - \quad (\mathbf{b}_{20} - \mathbf{b}_{30} + \mathbf{b}_{31})B_2^3(u_r)B_1^3(v_t) \\
& - \quad \mathbf{b}_{01}B_0^3(u_r)B_1^3(v_t) - \mathbf{b}_{31}B_3^3(u_r)B_1^3(v_t) \\
& - \quad \mathbf{b}_{02}B_0^3(u_r)B_2^3(v_t) - \mathbf{b}_{32}B_3^3(u_r)B_2^3(v_t) \\
& - \quad (\mathbf{b}_{02} - \mathbf{b}_{03} + \mathbf{b}_{13})B_1^3(u_r)B_2^3(v_t) \\
& - \quad (\mathbf{b}_{32} + \mathbf{b}_{23} - \mathbf{b}_{33})B_2^3(u_r)B_2^3(v_t) \\
& - \quad \mathbf{b}_{03}B_0^3(u_r)B_3^3(v_t) - \mathbf{b}_{13}B_1^3(u_r)B_3^3(v_t) \\
& - \quad \mathbf{b}_{23}B_2^3(u_r)B_3^3(v_t) - \mathbf{b}_{33}B_3^3(u_r)B_3^3(v_t).
\end{aligned} \tag{22}
$$

Minimizing this global error, we arrive at a 12×12 system of normal equations

$$
\frac{\partial \delta}{\partial \alpha_{rt}} = 0, \qquad \frac{\partial \delta}{\partial \beta_{rt}} = 0, \qquad \frac{\partial \delta}{\partial \gamma_{rt}} = 0, \tag{23}
$$

for $r, t \in \{0, 1\}$.

This system must be solved for the unknows $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$. The precise system is listed in the Appendix.

3 Results

Since the offset of a rectangular patch is typically not representable as a bicubic, we will only obtain an approximation. Therefore, a method for determining the error is necessary. First note that the parametrization of the offset approximation does not in general correspond to the parametrization of the progenitor. In particular, the difference vector, between the exact offset points and the offset approximation at the same (u, v) pair, will not be parallel to the normal of the progenitor[1]. In order to find the $(\hat{u}, \hat{v})$ of the approximation which yields the point that is closest to a given exact offset point, a method of parameter correction must be implemented. Such methods are described in Hoschek [8] and Sarkar and Menq [13]. The corrected parameter values can be used in two ways. First, the error at each given exact offset point can be (more accurately) evaluated. Second, the corrected parameter values can be used in iterative application of the least squares process to obtain a better offset approximation.

It is typically the case, when dealing with offset surfaces, that primary interest lies in maintaining the maximum error below a given tolerance. In the following examples, the maximum error between known exact offset points and the offset approximation is denoted as "max error". The average error between known exact offset points and offset approximation is denoted as "ave error". The first example in Figure 2 is a bicubic progenitor. The surface is one unit large in each dimension, and the offset distance is 0.1. After one application of the method combined with (Hoschek's) parameter correction:

$$\text{max error} = 0.005 \quad (24)$$

$$\text{ave error} = 0.002. \quad (25)$$

The control polygon of Figure 2 demonstrates the results of our method. After ten iterative applications of our method with parameter correction at each step the results improve:

$$\text{max error} = 0.001 \quad (26)$$

$$\text{ave error} = 0.0005. \quad (27)$$

In Figure 3 (a patch from a hair dryer) is the second example, shown in an interesting view. This data has dimensions of approximately $20 \times 120 \times 20$ units in x,y, and z respectively. An offset distance of -3.1 is used here. The progenitor is a biquintic surface. After one application of our method with

[1] In setting-up the least squares systems, we assigned parameter values to the exact offset points. These were guesses only.

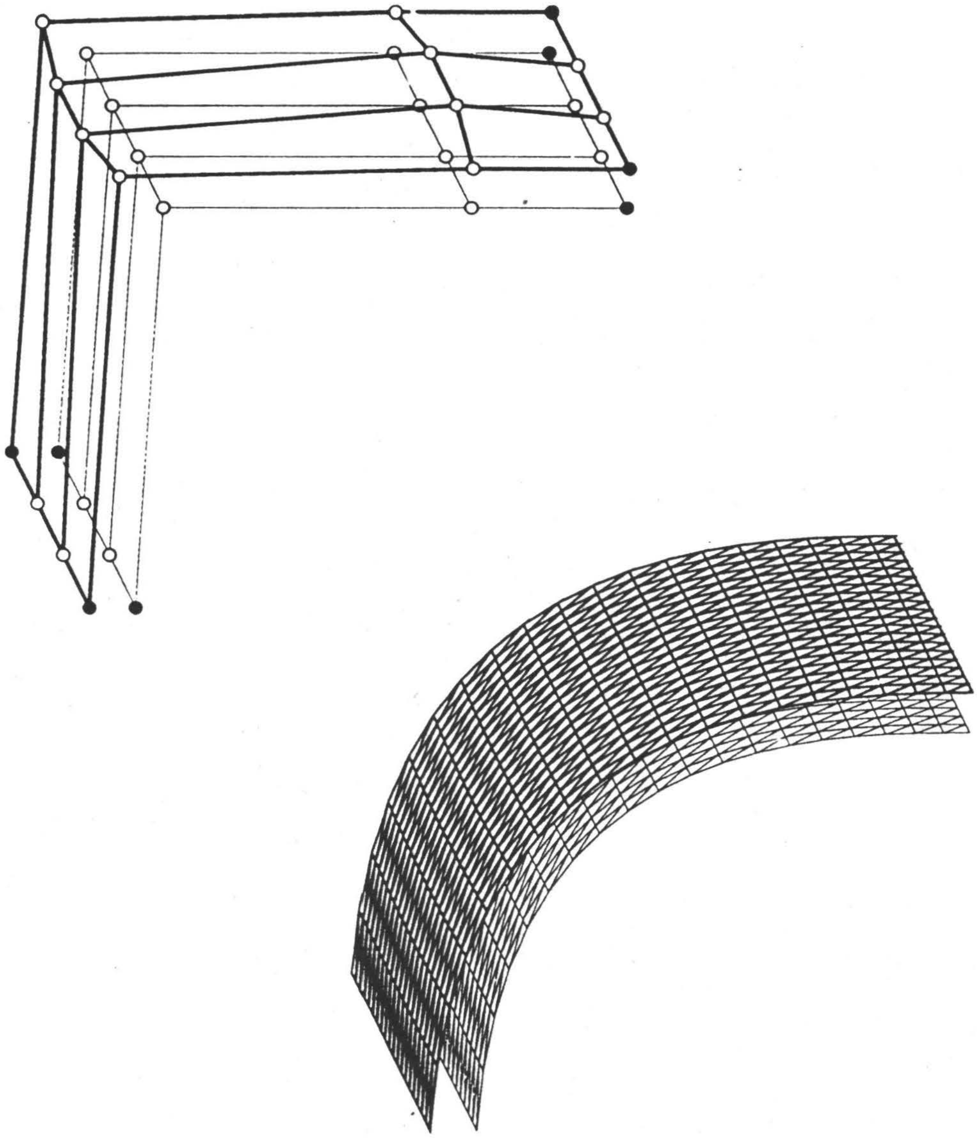

Figure 2: A bicubic progenitor drawn in the finer line width. The Bézier control polygon of the progenitor and offset are shown on the left and the wireframes are on the right.

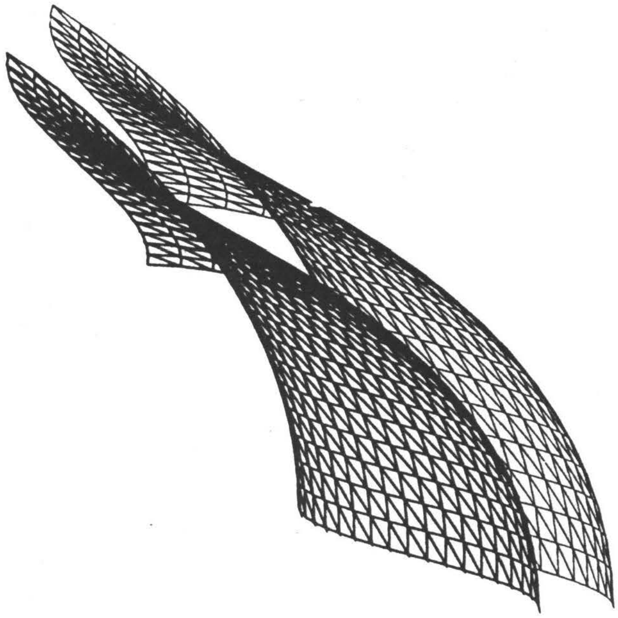

Figure 3: A biquintic progenitor drawn in the finer line width. The wireframe of the progenitor and offset are shown.

parameter correction:

$$\text{max error} = 0.09 \quad (28)$$
$$\text{ave error} = 0.03. \quad (29)$$

Clearly it is possible to find examples where a bicubic approximation is inadequate (as the second example). A subdivision scheme must be incorporated in order to reduce the progenitor complexity.

An unresolved problem is that of choosing the sampling size k. We have obtained good results for $k = 10$. As done in Chen and Ravani [3], it is advantageous to select more points in areas of large variation of curvature, although it is more costly to detect these regions. We experimented with different values of k, examining the effect on the error. For relatively simple progenitors, increasing k had little or no effect. Therefore, if a subdivision scheme is used, k can remain small.

Because the offset approximation is only bicubic, it is not possible to construct tangent continuous approximations (during the subdivision process). (This was done by Farouki [5].) We argue that if the approximations satisfy stringent enough tolerances, then suitable continuity will be obtained.

We finally note that all linear systems that occur in our method are invertible due to the fact that they stem from least squares problems. Our method is therefore stable in this sense.

4 Appendix

The symmetric system for the $v = v_0$ edge. The system is of the form $\mathbf{C}\lambda = \mathbf{A}$ where

$$\begin{aligned} \lambda &= (\lambda_{10}, \lambda_{10}^\star, \lambda_{20}, \lambda_{20}^\star)^T \\ \mathbf{A} &= (A_{10}, A_{10}^\star, A_{20}, A_{20}^\star)^T. \end{aligned}$$

We will use the abbreviation

$$\Delta = (u_1 - u_0)/(v_1 - v_0).$$

The elements of the matrix are given by:

$$\begin{aligned}
C_{00}^1 &= 2(\frac{m}{3}\Delta)^2 \sum_{i=0}^{k} (\mathbf{s}_{01} - \mathbf{s}_{00})^2 (B_1^3(u_i))^2 \\
C_{10}^1 &= 2\frac{n}{3}\frac{m}{3}\Delta \sum_{i=0}^{k} (\mathbf{s}_{01} - \mathbf{s}_{00})(\mathbf{s}_{10} - \mathbf{s}_{00})(B_1^3(u_i))^2 \\
C_{20}^1 &= -2(\frac{m}{3}\Delta)^2 \sum_{i=0}^{k} (\mathbf{s}_{01} - \mathbf{s}_{00})(\mathbf{s}_{n1} - \mathbf{s}_{n0}) B_1^3(u_i) B_2^3(u_i) \\
C_{30}^1 &= 2\frac{n}{3}\frac{m}{3}\Delta \sum_{i=0}^{k} (\mathbf{s}_{01} - \mathbf{s}_{00})(\mathbf{s}_{(n-1)0} - \mathbf{s}_{n0}) B_1^3(u_i) B_2^3(u_i) \\
C_{11}^1 &= 2(\frac{n}{3})^2 \sum_{i=0}^{k} (\mathbf{s}_{10} - \mathbf{s}_{00})^2 (B_1^3(u_i))^2 \\
C_{21}^1 &= -2\frac{n}{3}\frac{m}{3}\Delta \sum_{i=0}^{k} (\mathbf{s}_{10} - \mathbf{s}_{00})(\mathbf{s}_{n1} - \mathbf{s}_{n0}) B_1^3(u_i) B_2^3(u_i) \\
C_{31}^1 &= 2(\frac{n}{3})^2 \sum_{i=0}^{k} (\mathbf{s}_{10} - \mathbf{s}_{00})(\mathbf{s}_{(n-1)0} - \mathbf{s}_{n0}) B_1^3(u_i) B_2^3(u_i) \\
C_{22}^1 &= 2(\frac{m}{3}\Delta)^2 \sum_{i=0}^{k} ((\mathbf{s}_{n1} - \mathbf{s}_{n0}))^2 (B_2^3(u_i))^2 \\
C_{32}^1 &= -2\frac{n}{3}\frac{m}{3}\Delta \sum_{i=0}^{k} (\mathbf{s}_{n1} - \mathbf{s}_{n0})(\mathbf{s}_{(n-1)0} - \mathbf{s}_{n0})(B_2^3(u_i))^2 \\
C_{33}^1 &= 2(\frac{n}{3})^2 \sum_{i=0}^{k} (\mathbf{s}_{(n-1)0} - \mathbf{s}_{n0})^2 (B_2^3(u_i))^2
\end{aligned}$$

The right-hand side is given by:

$$
\begin{aligned}
A_{10} &= 2\frac{m}{3}\Delta\sum_{i=0}^{k}(\mathbf{s}_{01}-\mathbf{s}_{00})\mathbf{D}_{i0}B_1^3(u_i) \\
A_{10}^{\star} &= 2\frac{n}{3}\sum_{i=0}^{k}(\mathbf{s}_{10}-\mathbf{s}_{00})\mathbf{D}_{i0}B_1^3(u_i) \\
A_{20} &= -2\frac{m}{3}\Delta\sum_{i=0}^{k}(\mathbf{s}_{n1}-\mathbf{s}_{n0})\mathbf{D}_{i0}B_2^3(u_i) \\
A_{20}^{\star} &= 2\frac{n}{3}\sum_{i=0}^{k}(\mathbf{s}_{(n-1)0}-\mathbf{s}_{n0})\mathbf{D}_{i0}B_2^3(u_i),
\end{aligned}
$$

where

$$
\mathbf{D}_{i0} = \mathbf{E}_{i0} - \mathbf{b}_{00}B_0^3(u_i) - \mathbf{b}_{00}B_1^3(u_i) - \mathbf{b}_{30}B_2^3(u_i) - \mathbf{b}_{30}B_3^3(u_i). \qquad (30)
$$

The system for the $v = v_1$ edge:

The matrix:

$$
\begin{aligned}
C_{00}^3 &= 2(\frac{m}{3}\Delta)^2\sum_{i=0}^{k}(\mathbf{s}_{0(m-1)}-\mathbf{s}_{0m})^2(B_1^3(u_i))^2 \\
C_{10}^3 &= -2\frac{n}{3}\frac{m}{3}\Delta\sum_{i=0}^{k}(\mathbf{s}_{0(m-1)}-\mathbf{s}_{0m})(\mathbf{s}_{1m}-\mathbf{s}_{0m})(B_1^3(u_i))^2 \\
C_{20}^3 &= -2(\frac{m}{3}\Delta)^2\sum_{i=0}^{k}(\mathbf{s}_{0(m-1)}-\mathbf{s}_{0m})(\mathbf{s}_{n(m-1)}-\mathbf{s}_{nm})B_1^3(u_i)B_2^3(u_i) \\
C_{30}^3 &= -2\frac{n}{3}\frac{m}{3}\Delta\sum_{i=0}^{k}(\mathbf{s}_{0(m-1)}-\mathbf{s}_{0m})(\mathbf{s}_{(n-1)m}-\mathbf{s}_{nm})B_1^3(u_i)B_2^3(u_i) \\
C_{11}^3 &= 2(\frac{n}{3})^2\sum_{i=0}^{k}(\mathbf{s}_{1m}-\mathbf{s}_{0m})^2(B_1^3(u_i))^2 \\
C_{21}^3 &= 2\frac{n}{3}\frac{m}{3}\Delta\sum_{i=0}^{k}(\mathbf{s}_{1m}-\mathbf{s}_{0m})(\mathbf{s}_{n(m-1)}-\mathbf{s}_{nm})B_1^3(u_i)B_2^3(u_i) \\
C_{31}^3 &= 2(\frac{n}{3})^2\sum_{i=0}^{k}(\mathbf{s}_{1m}-\mathbf{s}_{0m})(\mathbf{s}_{(n-1)m}-\mathbf{s}_{nm})B_1^3(u_i)B_2^3(u_i) \\
C_{22}^3 &= 2(\frac{m}{3}\Delta)^2\sum_{i=0}^{k}(\mathbf{s}_{n(m-1)}-\mathbf{s}_{nm})^2(B_2^3(u_i))^2 \\
C_{32}^3 &= 2\frac{n}{3}\frac{m}{3}\Delta\sum_{i=0}^{k}(\mathbf{s}_{n(m-1)}-\mathbf{s}_{nm})(\mathbf{s}_{(n-1)m}-\mathbf{s}_{nm})(B_2^3(u_i))^2
\end{aligned}
$$

$$C_{33}^3 = 2(\frac{n}{3})^2 \sum_{i=0}^{k} (\mathbf{s}_{(n-1)m} - \mathbf{s}_{nm})^2 (B_2^3(u_i))^2$$

The right-hand side:

$$\begin{aligned}
A_{13} &= -2\frac{m}{3}\Delta \sum_{i=0}^{k} (\mathbf{s}_{0(m-1)} - \mathbf{s}_{0m}) \mathbf{D}_{i3} B_1^3(u_i) \\
A_{13}^\star &= 2\frac{n}{3} \sum_{i=0}^{k} (\mathbf{s}_{1m} - \mathbf{s}_{0m}) \mathbf{D}_{i3} B_1^3(u_i) \\
A_{23} &= 2\frac{m}{3}\Delta \sum_{i=0}^{k} (\mathbf{s}_{n(m-1)} - \mathbf{s}_{nm}) \mathbf{D}_{i3} B_2^3(u_i) \\
A_{23}^\star &= 2\frac{n}{3} \sum_{i=0}^{k} (\mathbf{s}_{(n-1)m} - \mathbf{s}_{nm}) \mathbf{D}_{i3} B_2^3(u_i),
\end{aligned}$$

where

$$\mathbf{D}_{i3} = \mathbf{E}_{i3} - \mathbf{b}_{03} B_0^3(u_i) - \mathbf{b}_{03} B_1^3(u_i) - \mathbf{b}_{33} B_2^3(u_i) - \mathbf{b}_{33} B_3^3(u_i) \qquad (31)$$

The system for the $u = u_0$ edge:

The matrix:

$$\begin{aligned}
C_{00}^4 &= 2(\frac{m}{3})^2 \sum_{i=0}^{k} (\mathbf{s}_{01} - \mathbf{s}_{00})^2 (B_1^3(v_i))^2 \\
C_{10}^4 &= 2\frac{n}{3}\frac{m}{3}\Delta^{-1} \sum_{i=0}^{k} (\mathbf{s}_{01} - \mathbf{s}_{00})(\mathbf{s}_{10} - \mathbf{s}_{00})(B_1^3(v_i))^2 \\
C_{20}^4 &= 2(\frac{m}{3})^2 \sum_{i=0}^{k} (\mathbf{s}_{01} - \mathbf{s}_{00})(\mathbf{s}_{0(m-1)} - \mathbf{s}_{0m}) B_1^3(v_i) B_2^3(v_i) \\
C_{30}^4 &= -2\frac{n}{3}\frac{m}{3}\Delta^{-1} \sum_{i=0}^{k} (\mathbf{s}_{01} - \mathbf{s}_{00})(\mathbf{s}_{1m} - \mathbf{s}_{0m}) B_1^3(v_i) B_2^3(v_i) \\
C_{11}^4 &= 2(\frac{n}{3}\Delta^{-1})^2 \sum_{i=0}^{k} (\mathbf{s}_{10} - \mathbf{s}_{00})^2 (B_1^3(v_i))^2 \\
C_{21}^4 &= 2\frac{n}{3}\frac{m}{3}\Delta^{-1} \sum_{i=0}^{k} (\mathbf{s}_{10} - \mathbf{s}_{00})(\mathbf{s}_{0(m-1)} - \mathbf{s}_{0m}) B_1^3(v_i) B_2^3(v_i) \\
C_{31}^4 &= -2(\frac{n}{3}\Delta^{-1})^2 \sum_{i=0}^{k} (\mathbf{s}_{10} - \mathbf{s}_{00})(\mathbf{s}_{1m} - \mathbf{s}_{0m}) B_1^3(v_i) B_2^3(v_i)
\end{aligned}$$

$$C^4_{22} = 2(\frac{m}{3})^2\sum_{i=0}^{k}(\mathbf{s}_{0(m-1)} - \mathbf{s}_{0m})^2(B^3_2(v_i))^2$$

$$C^4_{32} = -2\frac{n}{3}\frac{m}{3}\Delta^{-1}\sum_{i=0}^{k}(\mathbf{s}_{0(m-1)} - \mathbf{s}_{0m})(\mathbf{s}_{1m} - \mathbf{s}_{0m})(B^3_2(v_i))^2$$

$$C^4_{33} = 2(\frac{n}{3}\Delta^{-1})^2\sum_{i=0}^{k}(\mathbf{s}_{1m} - \mathbf{s}_{0m})^2(B^3_2(v_i))^2$$

The right-hand side:

$$A_{01} = 2\frac{m}{3}\sum_{i=0}^{k}(\mathbf{s}_{01} - \mathbf{s}_{00})\mathbf{D}_{0i}B^3_1(v_i)$$

$$A^*_{01} = 2\frac{n}{3}\Delta^{-1}\sum_{i=0}^{k}(\mathbf{s}_{10} - \mathbf{s}_{00})\mathbf{D}_{0i}B^3_1(v_i)$$

$$A_{02} = 2\frac{m}{3}\sum_{i=0}^{k}(\mathbf{s}_{0(m-1)} - \mathbf{s}_{0m})\mathbf{D}_{0i}B^3_2(v_i)$$

$$A^*_{02} = -2\frac{n}{3}\Delta^{-1}\sum_{i=0}^{k}(\mathbf{s}_{1m} - \mathbf{s}_{0m})\mathbf{D}_{0i}B^3_2(v_i),$$

where

$$\mathbf{D}_{0i} = \mathbf{E}_{0i} - \mathbf{b}_{00}B^3_0(v_i) - \mathbf{b}_{00}B^3_1(v_i) - \mathbf{b}_{03}B^3_2(v_i) - \mathbf{b}_{03}B^3_3(v_i). \tag{32}$$

The system for the $u = u_1$ edge:

The matrix:

$$C^2_{00} = 2(\frac{m}{3})^2\sum_{i=0}^{k}(\mathbf{s}_{n1} - \mathbf{s}_{n0})^2(B^3_1(v_i))^2$$

$$C^2_{10} = -2\frac{n}{3}\frac{m}{3}\Delta^{-1}\sum_{i=0}^{k}(\mathbf{s}_{n1} - \mathbf{s}_{n0})(\mathbf{s}_{(n-1)0} - \mathbf{s}_{n0})(B^3_1(v_i))^2$$

$$C^2_{20} = 2(\frac{m}{3})^2\sum_{i=0}^{k}(\mathbf{s}_{n1} - \mathbf{s}_{n0})(\mathbf{s}_{n(m-1)} - \mathbf{s}_{nm})B^3_1(v_i)B^3_2(v_i)$$

$$C^2_{30} = 2\frac{n}{3}\frac{m}{3}\Delta^{-1}\sum_{i=0}^{k}(\mathbf{s}_{n1} - \mathbf{s}_{n0})(\mathbf{s}_{(n-1)m} - \mathbf{s}_{nm})B^3_1(v_i)B^3_2(v_i)$$

$$C^2_{11} = 2(\frac{n}{3}\Delta^{-1})^2\sum_{i=0}^{k}(\mathbf{s}_{(n-1)0} - \mathbf{s}_{n0})^2(B^3_1(v_i))^2$$

$$
\begin{aligned}
C^2_{21} &= -2\frac{n}{3}\frac{m}{3}\Delta^{-1}\sum_{i=0}^{k}(\mathbf{s}_{(n-1)0}-\mathbf{s}_{n0})(\mathbf{s}_{n(m-1)}-\mathbf{s}_{nm})B^3_1(v_i)B^3_2(v_i) \\
C^2_{31} &= -2(\frac{n}{3}\Delta^{-1})^2\sum_{i=0}^{k}(\mathbf{s}_{(n-1)0}-\mathbf{s}_{n0})(\mathbf{s}_{(n-1)m}-\mathbf{s}_{nm})B^3_1(v_i)B^3_2(v_i) \\
C^2_{22} &= 2(\frac{m}{3})^2\sum_{i=0}^{k}(\mathbf{s}_{n(m-1)}-\mathbf{s}_{nm})^2(B^3_2(v_i))^2 \\
C^2_{32} &= 2\frac{n}{3}\frac{m}{3}\Delta^{-1}\sum_{i=0}^{k}(\mathbf{s}_{n(m-1)}-\mathbf{s}_{nm})(\mathbf{s}_{(n-1)m}-\mathbf{s}_{nm})(B^3_2(v_i))^2 \\
C^2_{33} &= 2(\frac{n}{3}\Delta^{-1})^2\sum_{i=0}^{k}(\mathbf{s}_{(n-1)m}-\mathbf{s}_{nm})^2(B^3_2(v_i))^2
\end{aligned}
$$

The right-hand side:

$$
\begin{aligned}
A_{31} &= 2\frac{m}{3}\sum_{i=0}^{k}(\mathbf{s}_{n1}-\mathbf{s}_{n0})\mathbf{D}_{3i}B^3_1(v_i) \\
A^\star_{31} &= -2\frac{n}{3}\Delta^{-1}\sum_{i=0}^{k}(\mathbf{s}_{(n-1)0}-\mathbf{s}_{n0})\mathbf{D}_{3i}B^3_1(v_i) \\
A_{32} &= 2\frac{m}{3}\sum_{i=0}^{k}(\mathbf{s}_{n(m-1)}-\mathbf{s}_{nm})\mathbf{D}_{3i}B^3_2(v_i) \\
A^\star_{32} &= 2\frac{n}{3}\Delta^{-1}\sum_{i=0}^{k}(\mathbf{s}_{(n-1)m}-\mathbf{s}_{nm})\mathbf{D}_{3i}B^3_2(v_i),
\end{aligned}
$$

where

$$
\mathbf{D}_{3i} = \mathbf{E}_{3i} - \mathbf{b}_{30}B^3_0(v_i) - \mathbf{b}_{30}B^3_1(v_i) - \mathbf{b}_{33}B^3_2(v_i) - \mathbf{b}_{33}B^3_3(v_i) \qquad (33)
$$

The symmetric system for the twists:
It is of the form $\mathbf{C}\lambda = \mathbf{A}$ where

$$
\begin{aligned}
\lambda &= (\alpha_{00},\beta_{00},\gamma_{00},\alpha_{01},\beta_{01},\gamma_{01},\alpha_{10},\beta_{10},\gamma_{10},\alpha_{11},\beta_{11},\gamma_{11})^T \\
\mathbf{A} &= (A_{00},A^\star_{00},A^\dagger_{00},A_{01},A^\star_{01},A^\dagger_{01},A_{10},A^\star_{10},A^\dagger_{10},A_{11},A^\star_{11},A^\dagger_{11})^T.
\end{aligned}
$$

Setting

$$
\Pi = \frac{3}{(u_1-u_0)}\frac{3}{(v_1-v_0)}, \qquad (34)
$$

the 12×12 system matrix is given by:

$$
\begin{aligned}
C_{00} &= 2(\Pi^{-1})^2 \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{N}_{00})^2(B_1^3(u_r)B_1^3(v_t))^2 \\
C_{10} &= 2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}(\mathbf{b}_{10}-\mathbf{b}_{00})(B_1^3(u_r)B_1^3(v_t))^2 \\
C_{20} &= 2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}(\mathbf{b}_{01}-\mathbf{b}_{00})(B_1^3(u_r)B_1^3(v_t))^2 \\
C_{30} &= -2(\Pi^{-1})^2\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}\mathbf{N}_{03}(B_1^3(u_r))^2B_1^3(v_t)B_2^3(v_t) \\
C_{40} &= -2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}(\mathbf{b}_{13}-\mathbf{b}_{03})(B_1^3(u_r))^2B_1^3(v_t)B_2^3(v_t) \\
C_{50} &= 2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}(\mathbf{b}_{02}-\mathbf{b}_{03})(B_1^3(u_r))^2B_1^3(v_t)B_2^3(v_t) \\
C_{60} &= -2(\Pi^{-1})^2\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}\mathbf{N}_{30}B_1^3(u_r)B_2^3(u_r)(B_1^3(v_t))^2 \\
C_{70} &= 2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}(\mathbf{b}_{20}-\mathbf{b}_{30})B_1^3(u_r)B_2^3(u_r)(B_1^3(v_t))^2 \\
C_{80} &= -2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}(\mathbf{b}_{31}-\mathbf{b}_{30})B_1^3(u_r)B_2^3(u_r)(B_1^3(v_t))^2 \\
C_{90} &= 2(\Pi^{-1})^2\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}\mathbf{N}_{33}B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t) \\
C_{10,0} &= -2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}(\mathbf{b}_{23}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t) \\
C_{11,0} &= -2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{00}(\mathbf{b}_{32}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t) \\
C_{11} &= 2(\frac{v_1-v_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10}-\mathbf{b}_{00})^2(B_1^3(u_r)B_1^3(v_t))^2 \\
C_{21} &= 2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10}-\mathbf{b}_{00})(\mathbf{b}_{01}-\mathbf{b}_{00})(B_1^3(u_r)B_1^3(v_t))^2 \\
C_{31} &= -2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10}-\mathbf{b}_{00})\mathbf{N}_{03}(B_1^3(u_r))^2B_1^3(v_t)B_2^3(v_t)
\end{aligned}
$$

$$
\begin{aligned}
C_{41} &= -2(\frac{v_1 - v_0}{3})^2 \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10} - \mathbf{b}_{00})(\mathbf{b}_{13} - \mathbf{b}_{03})(B_1^3(u_r))^2 B_1^3(v_t)B_2^3(v_t) \\
C_{51} &= 2\Pi^{-1} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10} - \mathbf{b}_{00})(\mathbf{b}_{02} - \mathbf{b}_{03})(B_1^3(u_r))^2 B_1^3(v_t)B_2^3(v_t) \\
C_{61} &= -2\Pi^{-1} \frac{v_1 - v_0}{3} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10} - \mathbf{b}_{00})\mathbf{N}_{30} B_1^3(u_r)B_2^3(u_r)(B_1^3(v_t))^2 \\
C_{71} &= 2(\frac{v_1 - v_0}{3})^2 \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10} - \mathbf{b}_{00})(\mathbf{b}_{20} - \mathbf{b}_{30}) B_1^3(u_r)B_2^3(u_r)(B_1^3(v_t))^2 \\
C_{81} &= -2\Pi^{-1} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10} - \mathbf{b}_{00})(\mathbf{b}_{31} - \mathbf{b}_{30}) B_1^3(u_r)B_2^3(u_r)(B_1^3(v_t))^2 \\
C_{91} &= 2\Pi^{-1} \frac{v_1 - v_0}{3} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10} - \mathbf{b}_{00})\mathbf{N}_{33} B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t) \\
C_{10,1} &= -2(\frac{v_1 - v_0}{3})^2 \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10} - \mathbf{b}_{00})(\mathbf{b}_{23} - \mathbf{b}_{33}) B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t) \\
C_{11,1} &= -2\Pi^{-1} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{10} - \mathbf{b}_{00})(\mathbf{b}_{32} - \mathbf{b}_{33}) B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t) \\
C_{22} &= 2(\frac{u_1 - u_0}{3})^2 \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01} - \mathbf{b}_{00})^2 (B_1^3(u_r)B_1^3(v_t))^2 \\
C_{32} &= -2\Pi^{-1} \frac{u_1 - u_0}{3} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01} - \mathbf{b}_{00})\mathbf{N}_{03}(B_1^3(u_r))^2 B_1^3(v_t)B_2^3(v_t) \\
C_{42} &= -2\Pi^{-1} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01} - \mathbf{b}_{00})(\mathbf{b}_{13} - \mathbf{b}_{03})(B_1^3(u_r))^2 B_1^3(v_t)B_2^3(v_t) \\
C_{52} &= 2(\frac{u_1 - u_0}{3})^2 \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01} - \mathbf{b}_{00})(\mathbf{b}_{02} - \mathbf{b}_{03})(B_1^3(u_r))^2 B_1^3(v_t)B_2^3(v_t) \\
C_{62} &= -2\Pi^{-1} \frac{u_1 - u_0}{3} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01} - \mathbf{b}_{00})\mathbf{N}_{30} B_1^3(u_r)B_2^3(u_r)(B_1^3(v_t))^2 \\
C_{72} &= 2\Pi^{-1} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01} - \mathbf{b}_{00})(\mathbf{b}_{20} - \mathbf{b}_{30}) B_1^3(u_r)B_2^3(u_r)(B_1^3(v_t))^2 \\
C_{82} &= -2(\frac{u_1 - u_0}{3})^2 \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01} - \mathbf{b}_{00})(\mathbf{b}_{31} - \mathbf{b}_{30}) B_1^3(u_r)B_2^3(u_r)(B_1^3(v_t))^2 \\
C_{92} &= 2\Pi^{-1} \frac{u_1 - u_0}{3} \sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01} - \mathbf{b}_{00})\mathbf{N}_{33} B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)
\end{aligned}
$$

$$C_{10,2} = -2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01}-\mathbf{b}_{00})(\mathbf{b}_{23}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{11,2} = -2(\frac{u_1-u_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{01}-\mathbf{b}_{00})(\mathbf{b}_{32}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{33} = 2(\Pi^{-1})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{N}_{03})^2(B_1^3(u_r)B_2^3(v_t))^2$$

$$C_{43} = 2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{03}(\mathbf{b}_{13}-\mathbf{b}_{03})(B_1^3(u_r)B_2^3(v_t))^2$$

$$C_{53} = -2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{03}(\mathbf{b}_{02}-\mathbf{b}_{03})(B_1^3(u_r)B_2^3(v_t))^2$$

$$C_{63} = 2(\Pi^{-1})^2\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{03}\mathbf{N}_{30}B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{73} = -2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{03}(\mathbf{b}_{20}-\mathbf{b}_{30})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{83} = 2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{03}(\mathbf{b}_{31}-\mathbf{b}_{30})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{93} = -2(\Pi^{-1})^2\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{03}\mathbf{N}_{33}B_1^3(u_r)B_2^3(u_r)(B_2^3(v_t))^2$$

$$C_{10,3} = 2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{03}(\mathbf{b}_{23}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)(B_2^3(v_t))^2$$

$$C_{11,3} = 2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{03}(\mathbf{b}_{32}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)(B_2^3(v_t))^2$$

$$C_{44} = 2(\frac{v_1-v_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{13}-\mathbf{b}_{03})^2(B_1^3(u_r)B_2^3(v_t))^2$$

$$C_{54} = -2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{13}-\mathbf{b}_{03})(\mathbf{b}_{02}-\mathbf{b}_{03})(B_1^3(u_r)B_2^3(v_t))^2$$

$$C_{64} = 2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{13}-\mathbf{b}_{03})\mathbf{N}_{30}B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{74} = -2(\frac{v_1-v_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{13}-\mathbf{b}_{03})(\mathbf{b}_{20}-\mathbf{b}_{30})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{84} = 2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{13}-\mathbf{b}_{03})(\mathbf{b}_{31}-\mathbf{b}_{30})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{94} = -2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{13}-\mathbf{b}_{03})\mathbf{N}_{33}B_1^3(u_r)B_2^3(u_r)(B_2^3(v_t))^2$$

$$C_{10,4} = 2(\frac{v_1-v_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{13}-\mathbf{b}_{03})(\mathbf{b}_{23}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)(B_2^3(v_t))^2$$

$$C_{11,4} = 2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{13}-\mathbf{b}_{03})(\mathbf{b}_{32}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)(B_2^3(v_t))^2$$

$$C_{55} = 2(\frac{u_1-u_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{02}-\mathbf{b}_{03})^2(B_1^3(u_r)B_2^3(v_t))^2$$

$$C_{65} = -2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{02}-\mathbf{b}_{03})\mathbf{N}_{30}B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{75} = 2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{02}-\mathbf{b}_{03})(\mathbf{b}_{20}-\mathbf{b}_{30})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{85} = -2(\frac{u_1-u_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{02}-\mathbf{b}_{03})(\mathbf{b}_{31}-\mathbf{b}_{30})B_1^3(u_r)B_2^3(u_r)B_1^3(v_t)B_2^3(v_t)$$

$$C_{95} = 2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{02}-\mathbf{b}_{03})\mathbf{N}_{33}B_1^3(u_r)B_2^3(u_r)(B_2^3(v_t))^2$$

$$C_{10,5} = -2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{02}-\mathbf{b}_{03})(\mathbf{b}_{23}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)(B_2^3(v_t))^2$$

$$C_{11,5} = -2(\frac{u_1-u_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{02}-\mathbf{b}_{03})(\mathbf{b}_{32}-\mathbf{b}_{33})B_1^3(u_r)B_2^3(u_r)(B_2^3(v_t))^2$$

$$C_{66} = 2(\Pi^{-1})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{N}_{30})^2(B_2^3(u_r)B_1^3(v_t))^2$$

$$C_{76} = -2\Pi^{-1}\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{30}(\mathbf{b}_{20}-\mathbf{b}_{30})(B_2^3(u_r)B_1^3(v_t))^2$$

$$C_{86} = 2\Pi^{-1}\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{30}(\mathbf{b}_{31}-\mathbf{b}_{30})(B_2^3(u_r)B_1^3(v_t))^2$$

$$C_{96} = -2(\Pi^{-1})^2\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{30}\mathbf{N}_{33}(B_2^3(u_r))^2B_1^3(v_t)B_2^3(v_t)$$

$$C_{10,6} = 2\Pi^{-1}\frac{v_1 - v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{30}(\mathbf{b}_{23} - \mathbf{b}_{33})(B_2^3(u_r))^2 B_1^3(v_t)B_2^3(v_t)$$

$$C_{11,6} = 2\Pi^{-1}\frac{u_1 - u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{30}(\mathbf{b}_{32} - \mathbf{b}_{33})(B_2^3(u_r))^2 B_1^3(v_t)B_2^3(v_t)$$

$$C_{77} = 2(\frac{v_1 - v_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{20} - \mathbf{b}_{30})^2(B_2^3(u_r)B_1^3(v_t))^2$$

$$C_{87} = -2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{20} - \mathbf{b}_{30})(\mathbf{b}_{31} - \mathbf{b}_{30})(B_2^3(u_r)B_1^3(v_t))^2$$

$$C_{97} = 2\Pi^{-1}\frac{v_1 - v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{20} - \mathbf{b}_{30})\mathbf{N}_{33}(B_2^3(u_r))^2 B_1^3(v_t)B_2^3(v_t)$$

$$C_{10,7} = -2(\frac{v_1 - v_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{20} - \mathbf{b}_{30})(\mathbf{b}_{23} - \mathbf{b}_{33})(B_2^3(u_r))^2 B_1^3(v_t)B_2^3(v_t)$$

$$C_{11,7} = -2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{20} - \mathbf{b}_{30})(\mathbf{b}_{32} - \mathbf{b}_{33})(B_2^3(u_r))^2 B_1^3(v_t)B_2^3(v_t)$$

$$C_{88} = 2(\frac{u_1 - u_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{31} - \mathbf{b}_{30})^2(B_2^3(u_r)B_1^3(v_t))^2$$

$$C_{98} = -2\Pi^{-1}\frac{u_1 - u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{31} - \mathbf{b}_{30})\mathbf{N}_{33}(B_2^3(u_r))^2 B_1^3(v_t)B_2^3(v_t)$$

$$C_{10,8} = 2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{31} - \mathbf{b}_{30})(\mathbf{b}_{23} - \mathbf{b}_{33})(B_2^3(u_r))^2 B_1^3(v_t)B_2^3(v_t)$$

$$C_{11,8} = 2(\frac{u_1 - u_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{31} - \mathbf{b}_{30})(\mathbf{b}_{32} - \mathbf{b}_{33})(B_2^3(u_r))^2 B_1^3(v_t)B_2^3(v_t)$$

$$C_{99} = 2(\Pi^{-1})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{N}_{33})(B_2^3(u_r)B_2^3(v_t))^2$$

$$C_{10,9} = -2\Pi^{-1}\frac{v_1 - v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{33}(\mathbf{b}_{23} - \mathbf{b}_{33})(B_2^3(u_r)B_2^3(v_t))^2$$

$$C_{11,9} = -2\Pi^{-1}\frac{u_1 - u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{N}_{33}(\mathbf{b}_{32} - \mathbf{b}_{33})(B_2^3(u_r)B_2^3(v_t))^2$$

$$C_{10,10} = 2(\frac{v_1 - v_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{23} - \mathbf{b}_{33})^2(B_2^3(u_r)B_2^3(v_t))^2$$

$$C_{11,10} = 2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{23}-\mathbf{b}_{33})(\mathbf{b}_{32}-\mathbf{b}_{33})(B_2^3(u_r)B_2^3(v_t))^2$$

$$C_{11,11} = 2(\frac{u_1-u_0}{3})^2\sum_{r=0}^{k}\sum_{t=0}^{k}(\mathbf{b}_{32}-\mathbf{b}_{33})^2(B_2^3(u_r)B_2^3(v_t))^2$$

The right-hand side is given by:

$$A_{00} = 2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}\mathbf{N}_{00}B_1^3(u_r)B_1^3(v_t)$$

$$A_{00}^{\star} = 2\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}(\mathbf{b}_{10}-\mathbf{b}_{00})B_1^3(u_r)B_1^3(v_t)$$

$$A_{00}^{\dagger} = 2\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}(\mathbf{b}_{01}-\mathbf{b}_{00})B_1^3(u_r)B_1^3(v_t)$$

$$A_{01} = -2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}\mathbf{N}_{03}B_1^3(u_r)B_2^3(v_t)$$

$$A_{01}^{\star} = -2\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}(\mathbf{b}_{13}-\mathbf{b}_{03})B_1^3(u_r)B_2^3(v_t)$$

$$A_{01}^{\dagger} = 2\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}(\mathbf{b}_{02}-\mathbf{b}_{03})B_1^3(u_r)B_2^3(v_t)$$

$$A_{10} = -2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}\mathbf{N}_{30}B_2^3(u_r)B_1^3(v_t)$$

$$A_{10}^{\star} = 2\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}(\mathbf{b}_{20}-\mathbf{b}_{30})B_2^3(u_r)B_1^3(v_t)$$

$$A_{10}^{\dagger} = -2\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}(\mathbf{b}_{31}-\mathbf{b}_{30})B_2^3(u_r)B_1^3(v_t)$$

$$A_{11} = 2\Pi^{-1}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}\mathbf{N}_{33}B_2^3(u_r)B_2^3(v_t)$$

$$A_{11}^{\star} = -2\frac{v_1-v_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}(\mathbf{b}_{23}-\mathbf{b}_{33})B_2^3(u_r)B_2^3(v_t)$$

$$A_{11}^{\dagger} = -2\frac{u_1-u_0}{3}\sum_{r=0}^{k}\sum_{t=0}^{k}\mathbf{DI}_{rt}(\mathbf{b}_{32}-\mathbf{b}_{33})B_2^3(u_r)B_2^3(v_t)$$

Acknowledgements

Farin and Hansford were supported in part by the Department of Energy through grant DE-FG0287ER25041, and the National Science Foundation through grant DMC-8807747, both awarded to Arizona State University. Additionally, this work was supported in part by a Fulbright Junior Research grant awarded to Hansford, a DFG grant IID3-Ha1491/5-1 awarded to Hagen, and a DFG grant awarded to Farin.

References

[1] S. Aomura and T. Uehara. Self-intersection of an offset surface. *Computer Aided Design*, 22(7):417–422, 1990.

[2] R. Barnhill, T. Frost, and S. Kersey. Self-intersections and offset surfaces. In R. E. Barnhill, editor, *Geometry Processing for Design and Manufacturing*, pages 35–44. SIAM, Philadelphia, 1992.

[3] Y. Chen and B. Ravani. Offset surface generation and contouring in computer-aided design. *J Mechanisms, Transmissions, and Automation in Design*, 3:133–142, 1987.

[4] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design*. Academic Press, 1988. third edition 1992.

[5] R. Farouki. The approximation of non-degenerate offset surfaces. *Computer Aided Geometric Design*, 3(1):15–43, 1986.

[6] R. Farouki and C. Neff. Algebraic properties of plane offset curves. *Computer Aided Geometric Design*, 7(1-4):101–128, 1990.

[7] R. Farouki and C. Neff. Analytic properties of plane offset curves. *Computer Aided Geometric Design*, 7(1-4):83–100, 1990.

[8] J. Hoschek. Offset curves in the plane. *Computer Aided Design*, 17(2):77–82, 1985.

[9] J. Hoschek. Spline approximation of offset curves. *Computer Aided Geometric Design*, 5(1):33–40, 1988.

[10] B. Pham. Offset curves and surfaces: a brief survey. *Computer-Aided Design*, 24(4):223–230, 1992.

[11] J. Rossignac and A. Requicha. Offsetting operations in solid modelling. *Computer Aided Geometric Design*, 3(2):129–148, 1986.

[12] M. Sabin. Offset parametric surfaces. Technical Report VTO/MS/149, British Aircraft Corporation, 1968.

[13] B. Sarkar and C. Menq. Parameter optimization in approximating curves and surfaces to measurement data. *Computer Aided Geometric Design*, 8(4):267–290, 1991.