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An Optimal Algorithm for the Local Solution of Integral Equations

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Abstract

The local solution problem of multivariate Fredholm integral equations is studied. Recent research proved that for several function classes the complexity of this problem is closely related to the Gelfand numbers of some characterizing operators. The generalization of this approach to the situation of arbitrary Banach spaces is the subject of the present paper. Furthermore, an iterative algorithm is described which – under some additional conditions – realizes the optimal error rate. The way these general theorems work is demonstrated by applying them to integral equations in a Sobolev space of periodic functions with dominating mixed derivative of various order.

1 Introduction

Fredholm integral equations of the second kind appear in many physical applications, e.g. in boundary value problems. Local solution of integral equations means that instead of the solution function on the whole domain only the value of a single linear functional applied to it is to be computed. This may be e.g. the value in a single point, some single basis coefficient of the solution function or a weighted mean.

Usually, in numerical applications we do not have full information about both the kernel function and the right hand side, but rather partial information such as point values or a set of basis coefficients. Information–based complexity theory considers this situation, studying the question how many information and operations are at least required to find an approximation to the solution with an error of at most ε. This quantity, which is called ε–complexity, characterizes the intrinsic difficulty of a numerical problem.

Local solution of Fredholm integral equations was first considered by Heinrich [Hei93], later on by Frank and Heinrich [FH94], [Fra94]. In these papers, for each special function class similar theorems were shown stating the equivalence of the n–th minimal radius of information to Gelfand numbers of some operators (for definitions see sections 2 and 3). Furthermore, for several function classes algorithms were found realizing the optimal error, which base on common principles. Similar algorithms proved to be optimal also for the full solution problem in certain function classes (see [EI67], [Fra95]). It seemed to be quite natural to try to generalize this approach to a wider class of equations. This is the aim of the present paper.

The paper leads out as follows: In section 2, the problem is formulated in the setting of information–based complexity. There also the most important definitions are recalled. In section 3, under some very general assumptions the equivalence of the n–th minimal radius of information and the Gelfand numbers of three operators characterizing the problem is shown. An algorithm is proposed in section 4 whose optimality in the sense of information complexity is shown in section 5 under certain additional conditions. These general results are applied to Fredholm integral equations of the second kind in Sobolev spaces of periodic functions with dominating mixed derivative $\mathcal{H}^{r_1,\dots,r_d}([0,1]^d)$ in section 6, where the exact order of the n–th minimal radius of information is derived.

2 Formulation of the problem

Let V, E and K be Banach spaces. As usual, for the space of all bounded linear operators from V into E we shall write $L(V,E)$, and $L(E) = L(E,E)$. E^* denotes the dual space to E, B_E the unit ball in E. We assume that V is continuously embedded into E, i.e. there exists a linear and continuous embedding operator $J_V : V \to E$. Moreover, there is some linear continuous operator T assigning to each element $k \in K$ an operator $T_k \in L(E)$. We consider such subsets $V_0 \subset V$, $K_0 \subset K$, that the operator $(I - T_k)^{-1} : E \to E$ is bounded for any $k \in K_0$. Here and further I denotes the identity operator. Setting $X_0 = K_0 \times V_0$ we study the class of operator equations

$$u - T_k u = f\,, \quad (k,f) \in X_0\,. \tag{1}$$

The problem has to be formulated within the framework of information–based complexity theory (IBC). Here only the most important definitions are outlined, for more details the reader is referred to [TWW88].

Instead of searching the solution u of equation (1) on the whole domain, we are interested only in the value of one single functional $\chi \in E^*$ applied to the solution $u \in E$. This problem setting is called local solution of the operator equation (1). In special cases, when V, E are function spaces, the functional χ can be e.g. a δ_{t_0}– functional which gives the value of u in a single point t_0, or a weighted mean, or a single coefficient in the representation of u in some basis in E. The operator $S_\chi : X_0 \to \mathbb{R}$ defined as

$$S_\chi(k,f) = \langle (I - T_k)^{-1} f, \chi \rangle \tag{2}$$

is called the local solution operator of equation (1).

Usually, we have no full information about the data $k \in K_0$, $f \in V_0$, but only partial information. We assume, that we are given linear information about $(k,f) \in X_0$, so we define the information operator $\bar{N} = (N, M)$, $\bar{N} : X_0 \to \mathbb{R}^{n_1+n_2}$ as

$$\begin{aligned} Nk &= (\langle k, \mu_i \rangle)_{i=1,..,n_1} \, , \; \mu_i \in K^* \, , \\ Mf &= (\langle f, \nu_i \rangle)_{i=1,..,n_2} \, , \; \nu_i \in V^* \, . \end{aligned} \tag{3}$$

Any operator $\varphi : \mathbb{R}^{n_1+n_2} \to \mathbb{R}$ is called an algorithm assigning an approximate solution $\varphi(\bar{N}(k,f))$ to any information vector $\bar{N}(k,f)$. The error of an algorithm $\varphi : \mathbb{R}^{n_1+n_2} \to \mathbb{R}$ is defined as the worst–case error

$$e\,(S_\chi, \bar{N}, \varphi) \;=\; \sup_{(k,f)\in X_0} |S_\chi(k,f) - \varphi(\bar{N}(k,f))| \, .$$

An essential quantity in IBC is the so–called radius of information. It denotes the minimal error which can be reached by any algorithm using the information $\bar{N}(k,f)$ and is defined as

$$r\,(S_\chi, \bar{N}) \;=\; \inf_{\varphi:\mathbb{R}^n\to\mathbb{R}} e\,(S_\chi, \bar{N}, \varphi) \, ,$$

where $n = n_1 + n_2$. The n–th minimal radius of information

$$r_n(S_\chi) \;=\; \inf_{\bar{N}:X_0\to\mathbb{R}^n} r\,(S_\chi, \bar{N})$$

describes the minimal error, which can be reached by any algorithm, using as information any linear information operator with cardinality n. This quantity is closely related to the ε–complexity of a problem. Furthermore, we shall use the diameter of information

$$d\,(S_\chi, \bar{N}) \;=\; \sup_{\substack{k,h\in K_0 \\ Nk=Nh}} \; \sup_{\substack{f_1,f_2\in V_0 \\ Mf_1=Mf_2}} |S_\chi(k,f_1) - S_\chi(h,f_2)| \, ,$$

as well as the well–known relation

$$\frac{1}{2}\, d\,(S_\chi, \bar{N}) \;\le r\,(S_\chi, \bar{N}) \;\le\; d\,(S_\chi, \bar{N}) \, . \tag{4}$$

At last, we have to introduce a notation often used in the following. Let $a(n), b(n)$ be functions defined on $\mathbb{Z}$. Then we write $a(n) \prec b(n)$ or $a(n) = O(b(n))$, if there is a constant $c > 0$ such that for all $n \in \mathbb{Z}$: $a(n) \le c \cdot b(n)$. Moreover, we write $a(n) \asymp b(n)$ if $a(n) \prec b(n)$ and $b(n) \prec a(n)$.

3 The complexity of the local solution problem

In this section, we shall establish an equivalence between the n–th radius of information and Gelfand numbers of certain operators. This equivalence was shown earlier for some special cases, see [Hei93], [FH94], [Fra94]. It proved to be a useful tool in getting lower bounds for the complexity of the local solution of Fredholm integral equations of the second kind. Now we want to show it under much more general conditions. Our goal is to get an analogon to Theorem 1 by [FHP95], where the problem of full solution was studied.

For this end, we have to make some assumptions about the properties of the subsets K_0 and V_0. We set $V_0 = B_V$ and assume that there exist constants $\rho_1, .., \rho_7 > 0$, $\rho_4, \rho_6 > 1$ such that

(I) $\rho_1 \cdot B_K \subseteq K_0 \subseteq \rho_2 \cdot B_K$;

(II) $\forall k \in K_0 : \|T_k : E \to E\| \leq \rho_3$, $\|(I - T_k)^{-1} : E \to E\| \leq \rho_4$;
$\|T_k : V \to V\| \leq \rho_5$, $\|(I - T_k)^{-1} : V \to V\| \leq \rho_6$;

(III) $\forall k_1, k_2 \in K \; \exists k \in K : T_{k_1} T_{k_2} = T_k$ with $\|k\|_K \leq \rho_7 \cdot \|k_1\|_K \cdot \|k_2\|_K$.

Let Z be the normed linear space defined by the semi–norm

$$\|f\|_Z \; = \; \sup_{k \in B_K} |\langle f, T_k^* \chi \rangle| \, , \; f \in E \, ,$$

i.e. Z is the quotient space $Z = E/\{f : \|f\|_Z = 0\}$. The operator $Q_Z : E \to Z$ denotes the quotient mapping from E onto Z. By assumption (III) for each $k \in K_0$ the operator T_k is correctly defined also in Z, and we have

$$\|T_k : Z \to Z\| \; \leq \; \text{const} \, .$$

We demand the same property for the operator $(I - T_k)^{-1}$, which does not follow from (I)–(III). So we can formulate the last assumption

(IV) $\exists \rho_8, \rho_9 > 0 : \forall k \in K_0 : \|T_k : Z \to Z\| \leq \rho_8$, $\|(I - T_k)^{-1} : Z \to Z\| \leq \rho_9$.

Note that assumption (III) implies that $\rho_8 = \rho_2 \cdot \rho_7$. The following operators turn out to be important:

$$\begin{aligned} \Psi &: K \to L(V, Z) \\ \Psi k &= Q_Z \, T_k \, J_V \, ; \\ \Phi &: V \to Z \\ \Phi f &= Q_Z \, J_V \, f \, ; \\ \Theta &: K \to V^* \\ \Theta k &= (T_k^* \chi) \, J_V \, , \text{ i.e. for } v \in V : \langle v, \Theta k \rangle = \langle J_V v, T_k^* \chi \rangle \, . \end{aligned}$$

As a last preparation we have to introduce the so–called Gelfand numbers of a linear operator. Given an operator $S \in L(E, V)$ and a linear information operator $N : E \to \mathbb{R}^m$ with arbitrary m, we define the Gelfand radius

$$c\,(S, N) \; = \; \sup_{\substack{x \in B_E \\ Nx = 0}} \|Sx\|_V \, .$$

Then the n–th Gelfand number of the linear operator $S : E \to V$ is

$$c_n(S) \;=\; \inf_{N:E\to \mathbb{R}^{n-1}} c\,(S, N)\,.$$

Now we are ready to formulate the following general result. Before we start the proof of Theorem 1 two lemmas have to be provided. The first lemma follows immediately from assumption (II), whereas the proof of the second one is given in more detail.

Theorem 1 *There are constants $d_1, d_2 > 0$ such that for any information operator $\bar{N} = (N, M)$:*

$$\begin{aligned} r\,(S_\chi, \bar{N}) \;&\geq\; d_1 \cdot [c\,(\Psi, N) + c\,(\Phi, M) + c\,(\Theta, N) + c\,(\chi J_V, M)] \\ r\,(S_\chi, \bar{N}) \;&\leq\; d_2 \cdot [c\,(\Psi, N) + c\,(\Phi, M) + c\,(\Theta, N) + c\,(\chi J_V, M)]\;. \end{aligned}$$

Lemma 1 *For any $k \in K_0$: $\frac{1}{1+\rho_5} \cdot B_V \;\subseteq\; (I - T_k)^{-1} B_V \;\subseteq\; \rho_6 \cdot B_V$.*

Lemma 2 *There is a constant $C_2 > 0$ such that*
$$\forall g \in C_2 \cdot B_K \,:\, \exists k \in \rho_1 \cdot B_K \,:\, T_g \;=\; T_k (I - T_k)^{-1}.$$

Proof: Let $0 < C_2 \leq \frac{\rho_1}{1+\rho_1\rho_7}$. Define S as

$$S \;:=\; T_g (I + T_g)^{-1} \;=\; \sum_{i=1}^{\infty} (-1)^{i+1} T_g^i\,.$$

Then by assumption (III) there are such functions $k_i \in K$, that $T_g^i = T_{k_i}$, and they satisfy

$$\|k_i\|_K \;\leq\; \rho_7^{i-1} \|g\|_K^i \;\leq\; \frac{(C_2 \cdot \rho_7)^i}{\rho_7}\,.$$

We define the function k as

$$k \;=\; \sum_{i=1}^{\infty} (-1)^i k_i\,.$$

The absolute convergence of this series in the norm of K follows from $C_2 \cdot \rho_7 < 1$. Since K is a Banach space, the limit k belongs to K. Since $T : K \to L(E)$ is linear and continuous we get $T_k = S$, and from

$$\|k\|_K \;\leq\; \sum_{i=1}^{\infty} \|k_i\|_K \;\leq\; \frac{C_2}{1 - C_2 \cdot \rho_7} \;\leq\; \rho_1\,,$$

we conclude that $k \in \rho_1 \cdot B_K$. ◁

Proof (Theorem 1):

We shall use relation (4), in which we write the diameter of information as a sum of four terms as follows:

$$\begin{aligned} d(S_\chi, \bar{N}) &= \sup_{\substack{k,h\in K_0\\ Nk=Nh}} \sup_{\substack{f_1,f_2\in B_V\\ Mf_1=Mf_2}} |S_\chi(k,f_1) - S_\chi(h,f_2)| \\ &= \mathrm{SUP}\left|\sum_{i=1}^{4} A_i(x)\right| . \end{aligned} \tag{5}$$

All over this proof SUP serves as an abbreviation for $\sup_{\substack{k,h\in K_0\\ Nk=Nh}} \sup_{\substack{f_1,f_2\in B_V\\ Mf_1=Mf_2}}$. The terms $A_i(x)$, $x = (k, f_1, h, f_2) \in X_0 \times X_0$, have the following form:

$$\begin{aligned} A_1(x) &= \langle (T_k - T_h)(I - T_h)^{-1} f_1, \chi\rangle \\ A_2(x) &= \langle T_k(I-T_k)^{-1}(T_k - T_h)(I - T_h)^{-1} f_1, \chi\rangle \\ A_3(x) &= \langle f_1 - f_2, \chi\rangle \\ A_4(x) &= \langle T_h(I-T_h)^{-1}(f_1 - f_2), \chi\rangle \end{aligned}$$

To show the upper bound of the theorem, we will estimate each of these terms from above:

$$d(S_\chi, \bar{N}) \leq \sum_{i=1}^{4} \mathrm{SUP}|A_i(x)|$$

Using assumption (I) as well as Lemma 1, we have

$$\begin{aligned} \mathrm{SUP}|A_1(x)| &= \mathrm{SUP}|\langle (T_k - T_h)(I-T_h)^{-1} f_1, \chi\rangle| \\ &\overset{\text{L.1}}{\leq} 2\rho_6 \sup_{\substack{k\in\rho_2\cdot B_K\\ Nk=0}} \sup_{f\in B_V} |\langle T_k f, \chi\rangle| \\ &\leq 2\rho_6\rho_2 \cdot c(\Theta, N) . \end{aligned} \tag{6}$$

By means of assumptions (I) and (IV) we can derive

$$\begin{aligned} \mathrm{SUP}|A_2(x)| &= \mathrm{SUP}|\langle T_k(I-T_k)^{-1}(T_k - T_h)(I-T_h)^{-1} f_1, \chi\rangle| \\ &\overset{\text{L.1}}{\leq} \rho_6 \sup_{\substack{k,h\in K_0\\ Nk=Nh}} \sup_{f\in B_V} |\langle T_k(I-T_k)^{-1}(T_k - T_h) f, \chi\rangle| \\ &\leq \rho_6 \sup_{k\in K_0} \sup_{\substack{h\in 2\rho_2 B_K\\ Nh=0}} \sup_{f\in B_V} |\langle (I-T_k)^{-1} T_h f, T_k^* \chi\rangle| \\ &\leq 2\rho_2^2\rho_6 \sup_{k\in K_0} \sup_{\substack{h\in B_K\\ Nh=0}} \sup_{f\in B_V} \|(I-T_k)^{-1} T_h f\|_Z \\ &\overset{\text{(IV)}}{\leq} 2\rho_2^2\rho_6\rho_9 \sup_{\substack{h\in B_K\\ Nh=0}} \sup_{f\in B_V} \|T_h f\|_Z \\ &= 2\rho_2^2\rho_6\rho_9 \cdot c(\Psi, N) \end{aligned} \tag{7}$$

It is easy to see that

$$\begin{aligned}
\mathrm{SUP}|A_3(x)| &= \mathrm{SUP}|\langle f_1 - f_2, \chi\rangle| \\
&\le 2 \cdot \sup_{\substack{f\in B_V \\ Mf=0}} |\langle f, \chi\rangle| \\
&= 2 \cdot c\,(\chi J_V, M) \qquad (8)
\end{aligned}$$

The last estimate we get again using assumptions (I) and (IV):

$$\begin{aligned}
\mathrm{SUP}|A_4(x)| &= \mathrm{SUP}|\langle T_h(I-T_h)^{-1}(f_1 - f_2), \chi\rangle| \\
&\le 2 \sup_{\substack{f\in B_V \\ Mf=0}} \sup_{h\in K_0} |\langle (I-T_h)^{-1} f, T_h^* \chi\rangle| \\
&\le 2\rho_2 \sup_{\substack{f\in B_V \\ Mf=0}} \sup_{h\in K_0} \|(I-T_h)^{-1} f\|_Z \\
&\overset{(IV)}{\le} 2\rho_2\rho_9 \sup_{\substack{f\in B_V \\ Mf=0}} \|f\|_Z \\
&= 2\rho_2\rho_9 \cdot c\,(\Phi, M) \qquad (9)
\end{aligned}$$

The relations (6)–(9) prove the upper bound of the theorem. To verify the lower bound, we estimate:

$$\begin{aligned}
d\,(S_\chi, \bar{N}) &= \mathrm{SUP}\left|\sum_{i=1}^{4} A_i(x)\right| \\
&\overset{f_1:=f_2}{\ge} \sup_{\substack{k,h\in K_0 \\ Nk=Nh}} \sup_{f_1\in B_V} |A_1(x) + A_2(x)| \\
&\overset{k:=0}{\ge} \sup_{\substack{h\in K_0 \\ Nh=0}} \sup_{f\in B_V} |\langle T_h(I-T_h)^{-1} f, \chi\rangle| \\
&\overset{L.1}{\ge} \sup_{\substack{h\in K_0 \\ Nh=0}} \sup_{f\in \frac{1}{1+\rho_5}B_V} |\langle T_h f, \chi\rangle| \\
&\ge \frac{1}{1+\rho_5} \sup_{\substack{h\in \rho_1 B_K \\ Nh=0}} \sup_{f\in B_V} |\langle T_h f, \chi\rangle| \\
&= \frac{\rho_1}{1+\rho_5} \cdot c\,(\Theta, N) \\
&\overset{(6)}{\ge} \bar{c} \cdot \mathrm{SUP}|A_1(x)| \qquad (10)
\end{aligned}$$

There $\bar{c}$ is some positive constant, independent of N. On the other hand,

$$\begin{aligned}
d\,(S_\chi, \bar{N}) &\overset{f_1:=f_2}{\ge} \sup_{\substack{k,h\in K_0 \\ Nk=Nh}} \sup_{f_1\in B_V} |A_1(x) + A_2(x)| \\
&\ge \sup_{\substack{k,h\in K_0 \\ Nk=Nh}} \sup_{f_1\in B_V} |A_2(x)| - \sup_{\substack{k,h\in K_0 \\ Nk=Nh}} \sup_{f_1\in B_V} |A_1(x)| \qquad (11)
\end{aligned}$$

By summation of (10) + $\bar{c}\cdot$ (11) it follows that

$$(1+\bar{c})\cdot \sup_{\substack{k,h\in K_0\\ Nk=Nh}} \sup_{f_1\in B_V} |A_1(x)+A_2(x)| \;\geq\; \bar{c}\cdot \sup_{\substack{k,h\in K_0\\ Nk=Nh}} \sup_{f_1\in B_V} |A_2(x)| \tag{12}$$

From (10) and (12) we get

$$\begin{aligned}
d\,(S_\chi,\bar{N}) \;&\geq\; \frac{\bar{c}}{1+\bar{c}}\cdot \sup_{\substack{k,h\in K_0\\ Nk=Nh}} \sup_{f\in B_V} |\langle T_k(I-T_k)^{-1}(T_k-T_h)(I-T_h)^{-1}f,\chi\rangle| \\
&\overset{\text{L.1}}{\geq}\; \frac{\bar{c}}{1+\bar{c}} \sup_{\substack{k,h\in K_0\\ Nk=Nh}} \sup_{f\in \frac{1}{1+\rho_5}B_V} |\langle T_k(I-T_k)^{-1}(T_k-T_h)f,\chi\rangle| \\
&\geq\; \frac{\bar{c}}{1+\bar{c}} \sup_{\substack{h\in\frac{1}{2}\rho_1\cdot B_K\\ Nh=0}} \sup_{k\in\frac{1}{2}\rho_1 B_K} \sup_{f\in\frac{1}{1+\rho_5}B_V} |\langle T_hf,(I^*-T_k^*)^{-1}T_k^*\chi\rangle| \\
&\overset{\text{L.2}}{\geq}\; \frac{2\bar{c}\rho_1}{(1+\bar{c})} \sup_{\substack{h\in B_K\\ Nh=0}} \sup_{g\in C_2B_K} \sup_{f\in\frac{1}{1+\rho_5}B_V} |\langle T_hf,T_g^*\chi\rangle| \\
&\geq\; \frac{2\bar{c}C_2\rho_1}{(1+\bar{c})(1+\rho_5)} \sup_{\substack{h\in B_K\\ Nh=0}} \sup_{g\in B_K} \sup_{f\in B_V} |\langle T_hf,T_g^*\chi\rangle| \\
&\geq\; \frac{2\bar{c}C_2\rho_1}{(1+\bar{c})(1+\rho_5)}\cdot c\,(\Psi,N) \qquad\qquad (13)
\end{aligned}$$

The remaining two needed inequalities we derive using a similar argument as above:

$$\begin{aligned}
d\,(S_\chi,\bar{N}) \;&\overset{k:=h}{\geq}\; \sup_{k\in K_0} \sup_{\substack{f_1,f_2\in B_V\\ Mf_1=Mf_2}} |A_3(x)+A_4(x)| \\
&\overset{k:=0}{\geq}\; 2\sup_{\substack{f\in B_V\\ Mf=0}} |\langle f,\chi\rangle| \\
&=\; 2\cdot c\,(\chi J_V,M) \qquad\qquad (14) \\
&\overset{(8)}{\geq}\; \mathrm{SUP}|A_3(x)|
\end{aligned}$$

Hence

$$\begin{aligned}
d\,(S_\chi,\bar{N}) \;&\geq\; \frac{1}{2}\,\mathrm{SUP}|A_4(x)| \\
&\overset{f_2:=0}{\geq}\; \frac{1}{2}\sup_{k\in\rho_1 B_K} \sup_{\substack{f\in B_V\\ Mf=0}} |\langle T_k(I-T_k)^{-1}f,\chi\rangle| \\
&\overset{\text{L.2}}{\geq}\; \frac{1}{2}\sup_{g\in C_2B_K} \sup_{\substack{f\in B_V\\ Mf=0}} |\langle f,T_g^*\chi\rangle| \\
&\geq\; \frac{C_2}{2}\sup_{g\in B_K} \sup_{\substack{f\in B_V\\ Mf=0}} |\langle f,T_g^*\chi\rangle|
\end{aligned}$$

$$\geq \quad \frac{C_2}{2} \cdot c(\Phi, M) \tag{15}$$

Relations (10)–(15) together with

$$\forall a, b, c, d \in \mathbb{R}: \ \max\{a, b, c, d\} \geq \frac{1}{4}(a + b + c + d)$$

show the lower bound of the theorem. ◁

Corollary 1 *There are constants $d_1, d_2 > 0$ such that for all $n \in \mathbb{N}$:*

$$d_1(c_{3n+2}(\Theta) + c_{3n+2}(\Phi) + c_{3n+2}(\Psi)) \leq r_{3n+1}(S_\chi) \leq d_2(c_{n+1}(\Theta) + c_{n+1}(\Phi) + c_{n+1}(\Psi)).$$

Proof: One possible information operator is $\bar{N} = (N, M)$, where

$$\begin{aligned} Mf &= (\langle f, \nu_1 \rangle, ..., \langle f, \nu_n \rangle, \langle f, \chi \rangle), \\ M'f &= (\langle f, \nu_1 \rangle, ..., \langle f, \nu_n \rangle), \end{aligned}$$

and the functionals $\nu_1, ..., \nu_n \in V^*$ are chosen in such a way that

$$c(M', \Phi) \leq 2 \cdot c_{n+1}(\Phi). \tag{16}$$

Then

$$c_{n+2}(\chi J_V) = \inf_{M: V \to \mathbf{R}^{n+1}} \sup_{\substack{f \in B_V \\ Mf = 0}} |\langle f, \chi \rangle| = 0. \tag{17}$$

The information functionals $\mu_1, ..., \mu_n, \mu_{n+1}, ..., \mu_{2n} \in K^*$ are chosen in such a way, that

$$\begin{aligned} N'k &= (\langle k, \mu_1 \rangle, ..., \langle k, \mu_n \rangle), \\ N''k &= (\langle k, \mu_{n+1} \rangle, ..., \langle k, \mu_{2n} \rangle), \\ N &= (N', N''); \\ c(N', \Psi) &\leq 2 \cdot c_{n+1}(\Psi), && (18) \\ c(N'', \Theta) &\leq 2 \cdot c_{n+1}(\Theta). && (19) \end{aligned}$$

Relations (16)–(19) show the upper bound. The lower bound follows immediately from the definition of Gelfand numbers. ◁

4 An iterative algorithm

In earlier papers, for some special cases optimal algorithms were provided, based on a two-level iteration, where the finer level represented an optimal approximation to the kernel function of a Fredholm integral equation. In this section we will generalize this approach and propose an iterative algorithm working in the general setting formulated in section 3.

Let E, K, $V \subseteq E$, $K_0 \subseteq K$, $V_0 = B_V$ be as in the previous section, with assumptions (I)–(IV). We suppose the sequence of elements $(e_l)_{l=0}^{\infty}$, $e_l \in E$ to be a Schauder basis in E (for definitions see [LT77]). Then there exists a biorthogonal sequence in E^* consisting of elements $(e_l^*)_{l=0}^{\infty}$, $e_l^* \in E^*$, satisfying $\langle e_i, e_j^* \rangle = \delta_{ij}$ for all $i, j \in \mathbb{N}$. The biorthogonal sequence is uniquely defined, and each element $x \in E$ can be represented as

$$x = \sum_{i=0}^{\infty} \langle x, e_i^* \rangle e_i \,. \tag{20}$$

Define for $n \in \mathbb{N}$ sets $C_n, D_n \subseteq \mathbb{N}^2$ with $C_n \subset D_n$ and $\mathrm{card}(D_n) = O(n^{\alpha})$ for some $\alpha > 0$. Let $(k, f) \in X_0$ be arbitrary, but fixed, and define approximations of the operator T_k by:

$$\begin{aligned} \langle T_g e_j, e_i^* \rangle &= \begin{cases} \langle T_k e_j, e_i^* \rangle & (i,j) \in C_n \\ 0 & \text{otherwise;} \end{cases} \\ \langle T_h e_j, e_i^* \rangle &= \begin{cases} \langle T_k e_j, e_i^* \rangle & (i,j) \in D_n \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{21}$$

Note that the index sets C_n, D_n do not depend on the concrete data (k, f), but only on the function spaces we are working in. They describe the approximation of the operator T_k used in the iterative algorithm below. The choice of these index sets is the crucial point in the definition of the algorithm, and it reflects the typical behaviour of the values $\langle T_k e_j, e_i^* \rangle$ for an element k of the subset $K_0 \subseteq K$. Usually, the cardinalities of both C_n and D_n are monotonely increasing functions of the parameter n, and for the sake of simplicity we will use n instead of $\mathrm{card}(D_n)$ to characterize the accuracy of approximation in (22).

We suppose that there exist functions $g, h \in K$ satisfying (21). Moreover, we suppose, that their norms are bounded by some constants $\alpha_1, \alpha_2 > 0$ for any $n \in \mathbb{N}$ and for any $(k, f) \in X_0$: $\|g\|_K \le \alpha_1$, $\|h\|_K \le \alpha_1$, $\|f_0\|_V \le \alpha_2$. Furthermore, we define a projection f_0 of the right-hand side f as

$$\begin{aligned} A_n &= \{ i \in \mathbb{N} \mid \exists j \in \mathbb{N} : (i,j) \in C_n \} \,, \\ B_n &= \{ i \in \mathbb{N} \mid \exists j \in \mathbb{N} : (i,j) \in D_n \} \,, \\ f_0 &= \sum_{i \in B_n} \langle f, e_i^* \rangle e_i \,. \end{aligned}$$

The projection f_0 is assumed to belong to V.

It is reasonable to demand that for increasing cardinality the operators defined by (21) converge to T_k in the norm of $L(E, E)$, hence we suppose that there are constants $\alpha_3, \alpha_4 > 0$ such that for any $k \in K_0$

$$\begin{aligned} \|T_k - T_g : E \to E\| &= O(n^{-\alpha_3}) \,; \\ \|T_k - T_h : E \to E\| &= O(n^{-\alpha_4}) \,, \end{aligned} \tag{22}$$

where the constants in the $O(\cdot)$–expressions are independent of $(k, f) \in X_0$. Furthermore, the operators T_g, T_h shall preserve the properties of T_k at least so far that there are constants $\alpha_5, \alpha_6 > 0$, $n_0 \in \mathbb{N}$ such that for any $k \in K_0$

$$\begin{aligned} &\|T_h : Z \to Z\| \le \alpha_5 \quad ; \\ &\|T_g : Z \to Z\| \le \alpha_5 \quad ; \quad \text{for all } n \ge n_0 : \|(I - T_g)^{-1} : Z \to Z\| \le \alpha_6 \,. \end{aligned} \tag{23}$$

From these conditions the following property can be derived.

Lemma 3 *There are constants $C_3 > 0$, $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$*

$$\|(I - T_g)^{-1} : E \to E\| \le C_3$$

Proof: From relations (22) we get that for some $N_0 \in \mathbb{N}$ for all $n \ge N_0$

$$\|T_k - T_g : E \to E\| \le c \cdot n^{-\alpha_3} \le \frac{1}{2\rho_4} \,.$$

Moreover, for $k \in K_0$ we know by assumption (II)

$$\|(I - T_k)^{-1} : E \to E\| \le \rho_4 \,.$$

Since

$$(I - T_g)^{-1} = (I + (I - T_k)^{-1}(T_k - T_g))^{-1}(I - T_k)^{-1} \,,$$

we can derive

$$\|(I - T_g)^{-1} : E \to E\| \le 2\rho_4 \,. \qquad \triangleleft$$

Now we can correctly formulate the algorithm $\phi : \mathbb{R}^{card(\bar{N})} \to \mathbb{R}$, where card$(\bar{N})$ denotes the cardinality of the information operator $\bar{N}$ specified below. First an iterative solution v_L approximating the solution function $u = (I - T_k)^{-1} f$ is computed by repeating

$$(I - T_g) v_l = f_0 + (T_h - T_g) v_{l-1} \,, \tag{24}$$

for $l = 1, \ldots, L$, $v_0 = 0$. After that an approximation to the exact local solution $S_\chi(k, f)$ is calculated by

$$\phi(Nk, Mf) = \langle f, \chi \rangle + \langle v_L, T_k^* \chi \rangle \,. \tag{25}$$

By Lemma 3 equation (24) is uniquely solvable, hence both v_L and the approximate solution $\phi(Nk, Mf)$ are uniquely defined for any $(k, f) \in X_0$ under condition (22). Using the representation (20) and the notations $\hat{f}(i) = \langle f, e_i^* \rangle$, $\hat{k}(i, j) = \langle T_k e_j, e_i^* \rangle$, we can rewrite the algorithm ϕ

in a less abstract form as follows. The equations

$$i \in \mathbb{Z}^d \setminus B_n \ : \ \hat{v}_l(i) = 0\,, \tag{26}$$

$$i \in B_n \setminus A_n \ : \ \hat{v}_l(i) = \hat{f}(i) + \sum_{j:(i,j)\in D_n} \hat{k}(i,j)\hat{v}_{l-1}(j)\,, \tag{27}$$

$$i \in A_n \ : \ \hat{v}_l(i) - \sum_{j:(i,j)\in C_n} \hat{k}(i,j)\hat{v}_l(j) = \hat{f}(i) + \sum_{j:(i,j)\in D_n\setminus C_n} \hat{k}(i,j)\hat{v}_{l-1}(j)\,, \tag{28}$$

describe the iteration (24), and the final computation (25) transforms into

$$\phi(Nk, Mf) = \langle f, \chi\rangle + \sum_{j\in B_n} \langle T_k e_j, \chi\rangle \hat{v}_L(j)\,. \tag{29}$$

Equation (24) is equivalent to equations (26) – (28), since $\hat{v}_l(i) = 0$ for all $i \notin B_n$, as a consequence of $\hat{f}_0(i) = 0$ and $\langle T_h e_j, e_i^*\rangle = \langle T_g e_j, e_i^*\rangle = 0$ for all $i, j \in \mathbb{N}$ with $i \notin B_n$. For the same reason, equation (25) is equivalent to equation (29). The information operator $\bar{N} = (N, M)$ used by this algorithm is

$$\begin{aligned} N &= (N_1, N_2, N_3)\,, & Mf &= (M_1 f, \langle f, \chi\rangle)\,,\\ N_1 k &= \left(\hat{k}(i,j)\right)_{(i,j)\in C_n}\,, & M_1 f &= \left(\hat{f}(i)\right)_{i\in B_n}\,,\\ N_2 k &= \left(\hat{k}(i,j)\right)_{(i,j)\in D_n\setminus C_n}\,, & &\\ N_3 k &= (\langle T_k e_i, \chi\rangle)_{i\in B_n}\,. & & \end{aligned}$$

Note that the cardinality of information as well as the number of arithmetic operations required by ϕ depend only on the definition of the index sets C_n and D_n.

5 Optimality of the algorithm ϕ

In this section we will show that in the setting formulated above the algorithm ϕ is always optimal in the sense, that its error is bounded by the Gelfand numbers of the same operators as in Theorem 1. For this end, we formulate the following theorem.

Theorem 2 *Let $\bar{N} = (N, M)$ be an information operator as defined in section 4, with $n \geq \max\{N_0, n_0\}$. Assume there exist constants $L \in \mathbb{N}$, $L > 1$, $c > 0$ such that for this information operator $\bar{N} : X_0 \to \mathbb{R}^{card(\bar{N})}$:*

$$[c\,(T : K \to L(E), N_1)]^L \leq c \cdot c\,(\Psi, (N_1, N_2))\,.$$

Then there is a constant $C > 0$ such that for $\bar{N}$:

$$e\,(S_\chi, \bar{N}, \phi) \ \leq \ C \cdot (c\,(\Psi, (N_1, N_2)) + c\,(\Phi, M_1))\,.$$

Proof (Theorem 2): The error of the algorithm ϕ can be represented in the following form:

$$\begin{aligned}
e(S_\chi, \bar{N}, \phi) &= \sup_{(k,f)\in X_0} |S_\chi(k,f) - \phi(Nk, Mf)| \\
&= \sup_{(k,f)\in X_0} |\langle (I-T_k)^{-1} f, \chi\rangle - \langle f, \chi\rangle - \langle v_L, T_k^* \chi\rangle| \\
&= \sup_{(k,f)\in X_0} |\langle (I-T_k)^{-1} f - v_L, T_k^* \chi\rangle| \\
&= \sup_{(k,f)\in X_0} \left|\sum_{i=1}^{3} A_i(k,f)\right|
\end{aligned}$$

where the terms $A_i(k,f)$ are

$$\begin{aligned}
A_1(k,f) &= \langle (I-T_k)^{-1}(f-f_0), T_k^*\chi\rangle\,; \\
A_2(k,f) &= \left\langle \left((I-T_g)^{-1}(T_h-T_g)\right)^L (I-T_k)^{-1} f_0, T_k^*\chi \right\rangle\,; \\
A_3(k,f) &= \left\langle \sum_{l=0}^{L-1} \left((I-T_g)^{-1}(T_h-T_g)\right)^l (I-T_g)^{-1}(T_k-T_h)(I-T_k)^{-1} f_0, T_k^*\chi \right\rangle .
\end{aligned}$$

Here we used the basic property $(I-T_k)^{-1} = I + T_k(I-T_k)^{-1}$ and a similar transformation as in [FH94]. Note that the functions g, h and f_0 are as defined in section 4 and depend only on k, f and the choice of the information operator $\bar{N}$. Then

$$e(S_\chi, \bar{N}, \phi) = \sup_{(k,f)\in X_0} \left|\sum_{i=1}^{3} A_i(k,f)\right| \leq \sum_{i=1}^{3} \sup_{(k,f)\in X_0} |A_i(k,f)|,$$

and all three terms will be estimated one by one.

$$\begin{aligned}
\sup_{(k,f)\in X_0} |A_1(k,f)| &= \sup_{(k,f)\in X_0} |\langle (I-T_k)^{-1}(f-f_0), T_k^*\chi\rangle| \\
&\leq \rho_2 \sup_{(k,f)\in X_0} \|(I-T_k)^{-1}(f-f_0)\|_Z \\
&\overset{(IV)}{\leq} \rho_2\rho_9 \sup_{\substack{\bar{f}\in(1+\alpha_2)B_V \\ M_1\bar{f}=0}} \|\bar{f}\|_Z \\
&\leq (1+\alpha_2)\rho_2\rho_9 \cdot c(\Phi, M_1) \qquad (30)
\end{aligned}$$

The second term we estimate using Lemma 1 and Lemma 3:

$$\begin{aligned}
\sup_{(k,f)\in X_0} |A_2(k,f)| &= \sup_{(k,f)\in X_0} \left|\left\langle \left((I-T_g)^{-1}(T_h-T_g)\right)^L (I-T_k)^{-1} f_0, T_k^*\chi \right\rangle\right| \\
&\overset{L.1}{\leq} \alpha_2\rho_6 \sup_{(k,f)\in X_0} \left|\left\langle \left((I-T_g)^{-1}(T_h-T_g)\right)^L f, T_k^*\chi \right\rangle\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_2\rho_2\rho_6 \sup_{(k,f)\in X_0} \| \left((I-T_g)^{-1}(T_h-T_g)\right)^L f\|_Z \\
&\leq \alpha_2\rho_2\rho_6\|Q_Z\|\|J_V\| \cdot \sup_{k\in K_0} \|(I-T_g)^{-1}(T_h-T_g)\|_{E\to E}^L \\
&\leq \alpha_2\rho_2\rho_6\|Q_Z\|\|J_V\| \cdot \sup_{k\in K_0} \|(I-T_g)^{-1}\|_{E\to E}^L \cdot \|T_h-T_g\|_{E\to E}^L \\
&\overset{\text{L.3}}{\leq} \alpha_2\rho_2\rho_6 C_3^L\|Q_Z\|\|J_V\| \cdot \sup_{\substack{g,h\in\alpha_1 B_K \\ N_1 g = N_1 h}} \|T_h-T_g\|_{E\to E}^L \\
&\leq (2\alpha_1 C_3)^L \alpha_2\rho_2\rho_6\|Q_Z\|\|J_V\| \cdot [c\,(T:K\to L(E), N_1)]^L \\
&\leq (2\alpha_1 C_3)^L \alpha_2\rho_2\rho_6\|Q_Z\|\|J_V\| \cdot c \cdot c\,(\Psi,(N_1,N_2))\,, \qquad (31)
\end{aligned}
$$

as follows from the condition of the theorem. It remains to estimate $A_3(k,f)$ by means of Lemma 1 and conditions (23):

$$
\begin{aligned}
&\sup_{(k,f)\in X_0} |A_3(k,f)| = \\
&= \sup_{(k,f)\in X_0} \left|\left\langle \sum_{l=0}^{L-1} \left((I-T_g)^{-1}(T_h-T_g)\right)^l (I-T_g)^{-1}(T_k-T_h)(I-T_k)^{-1} f_0, T_k^*\chi\right\rangle\right| \\
&\leq \sum_{l=0}^{L-1} \sup_{(k,f)\in X_0} \left|\left\langle \left((I-T_g)^{-1}(T_h-T_g)\right)^l (I-T_g)^{-1}(T_k-T_h)(I-T_k)^{-1} f_0, T_k^*\chi\right\rangle\right| \\
&\overset{\text{L.1}}{\leq} \rho_2\rho_6\alpha_2 \sum_{l=0}^{L-1} \sup_{(k,f)\in X_0} \|(I-T_g)^{-1}(T_h-T_g)\|_{Z\to Z}^l \|(I-T_g)^{-1}\|_{Z\to Z} \|(T_k-T_h)f\|_Z \\
&\overset{(23)}{\leq} \rho_2\rho_6\alpha_2 \sum_{l=0}^{L-1} (2\alpha_5\alpha_6)^l \alpha_6(\alpha_1+\rho_2) \sup_{\substack{\bar k\in B_K \\ N_1\bar k = N_2\bar k = 0}} \|T_{\bar k}\|_{V\to Z} \\
&\leq \rho_2\rho_6\alpha_2 \sum_{l=0}^{L-1} (2\alpha_5\alpha_6)^l \alpha_6(\alpha_1+\rho_2) \cdot c\,(\Psi,(N_1,N_2))\,. \qquad (32)
\end{aligned}
$$

Relations (30)–(32) show the theorem. ◁

Corollary 2 *If there is an information operator $\bar N : X_0 \to \mathbb{R}^{card(\bar N)}$ with $card(\bar N) = m$ satisfying the conditions of Theorem 2 as well as the conditions*

$$
\begin{aligned}
c\,(\Phi, M_1) &\leq c \cdot c_{m+1}(\Phi) \\
c\,(\Psi,(N_1,N_2)) &\leq c \cdot c_{m+1}(\Psi)
\end{aligned}
$$

for some constant $c > 0$, then the algorithm $\phi : \mathbb{R}^m \to \mathbb{R}$ using this information operator $\bar N = (N, M)$ realizes the optimal error up to a constant, i.e. it is of optimal order in the sense of information complexity.

Proof: The statement follows immediately from Theorem 2 and Corollary 1. ◁

6 The local solution problem in the space $\mathcal{H}^{r_1,\dots,r_d}$

As was already mentioned, in [Hei93], [Fra94] and [FH94] integral equations with data from several function classes were studied, showing analogous theorems to Theorem 1 and partly analyzing algorithms working on the same principle as the optimal algorithm ϕ. Therefore, these papers could serve as examples of the application of Theorems 1 and 2. However, to illustrate the application of the theory developed above, we consider another class of data.

Let $G = [0,1]^d$, $d \in \mathbb{N}$, and $L_2([0,1])$ be the space of periodic, square summable functions over $[0,1]$ with the orthonormal trigonometric basis:

$$\begin{aligned} e_0(\tau) &\equiv 1 \\ e_n(\tau) &= \sqrt{2}\cos(2\pi n\tau) \\ e_{-n}(\tau) &= \sqrt{2}\sin(2\pi n\tau)\,, \end{aligned}$$

where $n \in \mathbb{N}$, $\tau \in [0,1]$. Given a multiindex $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$, the i–th basis function in $L_2(G)$ is defined as

$$e_i(t) = e_{i_1}(t_1) \cdot \ldots \cdot e_{i_d}(t_d)\,,$$

where $t = (t_1, \dots, t_d) \in G$. Similarly, a basis in $L_2(G^2)$ is described by

$$e_{ij}(s,t) = e_i(s) \cdot e_j(t)\ ,\ i,j \in \mathbb{Z}^d\,,\ s,t \in G\,.$$

The Fourier coefficients of the functions $f \in L_2(G)$, $k \in L_2(G^2)$ for $i,j \in \mathbb{Z}^d$ are denoted by $\hat{f}(i) = \langle f, e_i\rangle$, $\hat{k}(i,j) = \langle k, e_{ij}\rangle$.

Let $r = (r_1, ..., r_d)$ be a vector of nonnegative real numbers $r_k \geq 0$, $k = 1, .., d$. For an arbitrary multiindex $i \in \mathbb{Z}^d$ we set $\rho_r(i) = \max(1, |i_1|^{r_1}) \cdot \ldots \cdot \max(1, |i_d|^{r_d})$. Then we define the following function spaces

$$\begin{aligned} \mathcal{H}^{r_1,\dots,r_d}(G) &= \left\{ f \in L_2(G)\ :\ \|f\|_r^2 = \sum_{i \in \mathbb{Z}^d} \rho_r(i)^2 \hat{f}(i)^2 < \infty \right\}; \\ \mathcal{H}^{r_1,\dots,r_d,r_1,\dots,r_d}(G^2) &= \left\{ k \in L_2(G^2)\ :\ \|k\|_r^2 = \sum_{i,j \in \mathbb{Z}^d} \rho_r(i)^2 \rho_r(j)^2 \hat{k}(i,j)^2 < \infty \right\}. \end{aligned}$$

For simplicity, we will often use the notations $\mathcal{H}^r = \mathcal{H}^{r_1,\dots,r_d}(G)$, $\mathcal{H}^{r,r} = \mathcal{H}^{r_1,\dots,r_d,r_1,\dots,r_d}(G^2)$, $L_2 = L_2(G)$. Note that if the vector $r = (r_1, ..., r_d)$ consists of natural numbers $r_k \in \mathbb{N}$, $k = 1, ..., d$, the space $\mathcal{H}^r$ constitutes the Sobolev space of periodic on G functions which together with its mixed generalized derivative $\frac{\partial^{(r_1+\dots+r_d)} f}{\partial x_1^{r_1} \dots \partial x_d^{r_d}}$ belong to $L_2(G)$. These spaces are called Sobolev spaces with dominating mixed derivative. Without loss of generality we assume, that the dimensions are ordered in such a way, that $r_1 = \ldots = r_\nu = \varrho$ and $r_{\nu+1}, \dots, r_d > \varrho$, where $\varrho = \min\{r_1, \dots, r_d\}$. This assumption is made for technical reasons only. We concentrate on the case $d > \nu$, since the special case $d = \nu$ was considered already in [Fra94], where an exact rate of $r_n(S_\chi) \asymp n^{-2\varrho}(\log n)^{2\varrho(2d-1)}$ was derived. Here we want to generalize this result.

By $\mathcal{H}^{-r} = (\mathcal{H}^r)^*$ we denote the dual space of $\mathcal{H}^r$. L_2 imbeds into $\mathcal{H}^{-r}$ in a canonical way, and the $\mathcal{H}^{-r}$–norm of a function $f \in L_2$ is given by

$$\|f\|_{-r}^2 = \sum_{i \in \mathbf{Z}^d} \rho_r(i)^{-2} \hat{f}(i)^2 . \tag{33}$$

Note, that L_2 is a dense subspace of $\mathcal{H}^{-r}$. We set $E = L_2$, $V = \mathcal{H}^r$, $K = \mathcal{H}^{r,r}$, and define the basic subsets

$$V_0 = \{f \in \mathcal{H}^r : \|f\|_r \leq 1\}, \tag{34}$$
$$K_0 = \{k \in \mathcal{H}^{r,r} : \|k\|_r \leq \beta_1 , \|(I - T_k)^{-1} : L_2 \to L_2\| \leq \beta_2\} . \tag{35}$$

with $\beta_1 > 0$, $\beta_2 > 1$. The linear operator $T : K \to L(E)$ shall assign to a kernel function $k \in \mathcal{H}^{r,r}$ the integral operator $Tk = T_k : L_2 \to L_2$ with

$$T_k f = \int_G k(s,t)\, f(t)\, dt .$$

That means we study Fredholm integral equations of the second kind

$$u(s) - \int_G k(s,t)\, u(t)\, dt = f(s) . \tag{36}$$

with data $(k, f) \in K_0 \times V_0$. Instead of the quotient space Z we will consider the space $\mathcal{H}_0^{-r}$:

$$\mathcal{H}_0^{-r} = \{f \in \mathcal{H}^{-r} : f \in L_2\},$$

that means, the space of all L_2–functions with the norm (33) induced by $\mathcal{H}^{-r}$. The correctness of this procedure will be shown in Lemma 6.

Lemma 4 *For the subsets $V_0 \subseteq \mathcal{H}^r$, $K_0 \subseteq \mathcal{H}^{r,r}$ defined in (34), (35) there are constants $\rho_1, ..., \rho_9 > 0$ such that assumptions (I) – (IV) are satisfied.*

Proof:

(I) Immediately from the definition (35) we conclude $\rho_2 = \beta_1$. For $0 < \rho_1 \leq \min\{\beta_1, 1 - \frac{1}{\beta_2}\}$ we can show that each $k \in \rho_1 \cdot B_K$ is in K_0, since for each $k \in \rho_1 \cdot B_K$ it holds $\|k\|_r \leq \beta_1$ and

$$\begin{aligned} \|T_k f\|_{L_2}^2 &= \sum_{i \in \mathbf{Z}^d} \left(\sum_{j \in \mathbf{Z}^d} \hat{k}(i,j) \hat{f}(j) \right)^2 \\ &\leq \sum_{i,j \in \mathbf{Z}^d} \hat{k}(i,j)^2 \cdot \sum_{i \in \mathbf{Z}^d} \hat{f}(i)^2 \qquad (37) \\ &= \|k\|_{L_2}^2 \cdot \|f\|_{L_2}^2 \\ &\leq \|k\|_r^2 \cdot \|f\|_{L_2}^2 , \end{aligned}$$

so $\|T_k : L_2 \to L_2\| \le 1 - \frac{1}{\beta_2} < 1$ and using the Neumann series we conclude

$$\|(I - T_k)^{-1} : L_2 \to L_2\| \ \le \ \sum_{l=0}^{\infty} \|T_k : L_2 \to L_2\|^l \ \le \ \beta_2 \,.$$

(II) Relation (37) shows also $\|T_k : L_2 \to L_2\| \le \|k\|_{L_2} \le \|k\|_r \le \beta_1$, so $\rho_3 = \beta_1$. By definition (35) we have $\rho_4 = \beta_2$. Moreover, for $k \in \beta_1 B_{\mathcal{H}^{r,r}}$ we can find real numbers $(\xi_{ij})_{i,j\in\mathbf{Z}^d}$ such that $\hat{k}(i,j) = \rho_r(i)^{-1}\rho_r(j)^{-1}\xi_{ij}$ with $\sum_{i,j\in\mathbf{Z}^d} \xi_{ij}^2 \le \beta_1^2$. Then

$$\begin{aligned}
\|T_k f\|_r^2 &= \sum_{i\in\mathbf{Z}^d} \rho_r(i)^2 \left(\sum_{j\in\mathbf{Z}^d} \hat{k}(i,j)\hat{f}(j) \right)^2 \\
&= \sum_{i\in\mathbf{Z}^d} \rho_r(i)^2 \left(\sum_{j\in\mathbf{Z}^d} \rho_r(i)^{-1}\rho_r(j)^{-1}\xi_{ij}\hat{f}(j) \right)^2 \\
&= \sum_{i\in\mathbf{Z}^d} \left(\sum_{j\in\mathbf{Z}^d} \rho_r(j)^{-1}\xi_{ij}\hat{f}(j) \right)^2 \\
&\le \sum_{i,j\in\mathbf{Z}^d} \xi_{ij}^2 \cdot \sum_{i\in\mathbf{Z}^d} \hat{f}(i)^2 \\
&\le \beta_1^2 \cdot \|f\|_r^2 \,.
\end{aligned}$$

These calculations show also

$$\|T_k \ : \ L_2 \to \mathcal{H}^r\| \ \le \ \beta_1 \,, \tag{38}$$

and from $(I - T_k)^{-1} = I + T_k\,(I - T_k)^{-1}$ it follows that for $\rho_6 \ge 1 + \beta_1\beta_2$

$$\|(I - T_k)^{-1} \ : \mathcal{H}^r \to \mathcal{H}^r\| \ \le \ \rho_6 \,.$$

(III) The existence of $k \in L_2(G^2)$ follows from

$$\begin{aligned}
T_k f &= T_{k_1} T_{k_2} f \\
&= \int_G k_1(s,\tau) \int_G k_2(\tau,t) f(t)\, dt\, d\tau \\
&= \int_G \left(\int_G k_1(s,\tau) k_2(\tau,t)\, d\tau \right) f(t)\, dt \\
&= \int_G k(s,t) f(t)\, dt \,.
\end{aligned}$$

To find the constant ρ_7, we introduce $k_l^{(j)}(s) = T_{k_l} e_j$ for $l = 1,2$. Then

$$\|k_l\|_r^2 \ = \ \sum_{i,j\in\mathbf{Z}^d} \rho_r(i)^2 \rho_r(j)^2 \hat{k}_l(i,j)^2$$

$$
\begin{aligned}
&= \sum_{j\in\mathbf{Z}^d} \rho_r(j)^2 \sum_{i\in\mathbf{Z}^d} \rho_r(i)^2 \hat{k_l}(i,j)^2 \\
&= \sum_{j\in\mathbf{Z}^d} \rho_r(j)^2 \|k_l^{(j)}\|_r^2 ,
\end{aligned}
$$

for both $l = 1,2$. Since $T_k = T_{k_1} T_{k_2}$, we have $k^{(j)}(s) = T_k e_j = T_{k_1}(T_{k_2} e_j) = T_{k_1} k_2^{(j)}(s)$ and so

$$
\begin{aligned}
\|k\|_r^2 &= \sum_{j\in\mathbf{Z}^d} \|k^{(j)}\|_r^2 \\
&\leq \|T_{k_1} : \mathcal{H}^r \to \mathcal{H}^r\| \cdot \sum_{j\in\mathbf{Z}^d} \rho_r(j)^2 \|k_2^{(j)}\|_r^2 \\
&\leq \|k_1\|_r^2 \cdot \|k_2\|_r^2 .
\end{aligned}
$$

(IV) From relation (38) we get by duality $\|T_k : \mathcal{H}_0^{-r} \to L_2\| \leq \beta_1$, and the equation $(I - T_k)^{-1} = I + (I - T_k)^{-1} T_k$ gives us

$$
\|(I - T_k)^{-1} : \mathcal{H}_0^{-r} \to \mathcal{H}_0^{-r}\| \leq 1 + \beta_2 \beta_1 . \qquad \triangleleft
$$

Now we define the operators

$$
\begin{aligned}
\Psi &: \mathcal{H}^{r,r} \to L(\mathcal{H}^r, \mathcal{H}_0^{-r}) \\
\Psi k &= Q_{\mathcal{H}_0^{-r}} T_k J_{\mathcal{H}^r} ;
\end{aligned}
$$

$$
\begin{aligned}
\Phi &: \mathcal{H}^r \to \mathcal{H}_0^{-r} \\
\Phi f &= Q_{\mathcal{H}_0^{-r}} J_{\mathcal{H}^r} f ;
\end{aligned}
$$

$$
\begin{aligned}
\Theta &: \mathcal{H}^{r,r} \to \mathcal{H}_0^{-r} \\
\Theta k &= (T_k^* \chi) J_{\mathcal{H}^r} , \text{ i.e. for } v \in \mathcal{H}^r : \langle v, \Theta k\rangle = \langle J_{\mathcal{H}^r} v, T_k^* \chi\rangle .
\end{aligned}
$$

Note that there instead of the quotient space Z the space $\mathcal{H}_0^{-r} = \mathcal{H}^{-r} \cap L_2$ considered with the norm (33) is used. Nevertheless, Theorems 1 and 2 remain valid for these operators as well, because the norms $\|\cdot\|_Z$ and $\|\cdot\|_{-r}$ are equivalent. This fact is shown in the following two lemmas.

Lemma 5 *There are constants $c_1, c_2 > 0$ such that*

$$
c_1 \cdot B_{\mathcal{H}^r} \subseteq \{T_k^* \chi : k \in B_{\mathcal{H}^{r,r}}\} \subseteq c_2 \cdot B_{\mathcal{H}^r} .
$$

Proof: Since $\chi \in L_2$, the right–hand side follows from

$$
\|T_k^* : L_2 \to \mathcal{H}^r\| \leq \beta_1 ,
$$

which can be derived from inequality (38) using a symmetry argument.

The left–hand side can be shown considering kernels $k \in \mathcal{H}^{r,r}$ of the form

$$k(s,t) \;=\; e_{i_0}(s) \cdot f(t)\,,$$

where $f \in \mathcal{H}^r$ and $i_0 \in \mathbb{Z}^d$ is a fixed index with $\langle e_{i_0}, \chi\rangle \neq 0$. Since $\chi \neq 0$, such an i_0 exists, and

$$\begin{aligned} T_k^* \chi &= \int_G k(t,s)\, \chi(t)\, dt \\ &= \int_G f(s)\, e_{i_0}(t)\, \chi(t)\, dt \\ &= f(s) \cdot \langle e_{i_0}, \chi\rangle\,. \end{aligned}$$

◁

Lemma 6 *There are constants $c_1, c_2 > 0$ such that for all $f \in L_2$*

$$c_1 \cdot \|f\|_{-r} \;\leq\; \|f\|_Z \;\leq\; c_2 \cdot \|f\|_{-r}\,.$$

Proof: The right–hand side can be shown using the right–hand side of Lemma 5:

$$\begin{aligned} \|f\|_Z &= \sup_{k \in B_{\mathcal{H}^{r,r}}} |\langle f, T_k^* \chi\rangle| \\ &\leq \|f\|_{-r} \cdot \|T_k^* \chi\|_r \\ &\overset{\text{L.5}}{\leq} c_2 \cdot \|f\|_{-r}\,. \end{aligned}$$

To show the left–hand side, the left inequality of Lemma 5 is needed:

$$\begin{aligned} \|f\|_Z &= \sup_{k \in B_{\mathcal{H}^{r,r}}} |\langle f, T_k^* \chi\rangle| \\ &\overset{\text{L.5}}{\geq} \sup_{\mu \in c_1 B_{\mathcal{H}^r}} |\langle f, \mu\rangle| \\ &= c_1 \cdot \|f\|_{-r}\,. \end{aligned}$$

◁

Now we are ready to formulate the following theorem, which states the exact rate of the n–th minimal radius of information for equations of type (36).

Theorem 3 *Let $r = (r_1, \ldots, r_d) \in \mathbb{R}_+^d$, with $r_1 = \ldots = r_\nu = \varrho$, $r_{\nu+1}, \ldots, r_d > \varrho$, $1 \leq \nu < d$. Then for any $\chi \in L_2(G)$, $\chi \neq 0$, the n–th minimal radius of information has the order*

$$r_n(S_\chi) \;\asymp\; n^{-2\varrho} (\log n)^{2\varrho(2\nu-1)}\,.$$

Moreover, there is an algorithm $\phi : \mathbb{R}^n \to \mathbb{R}$ which realizes this order-optimal error bound using $O(n)$ arithmetic operations.

Before we start the proof of Theorem 3 we provide the following lemma.

Lemma 7 *Let $\Gamma_n^{(d)}$ be the set of multiindices*

$$\Gamma_n^{(d)} = \{(i,j) \in \mathbb{Z}^{2d} : \rho_r(i) \cdot \rho_r(j) \leq n^{2\varrho}\}.$$

Then for any $d \in \mathbb{N}$, $d \geq \nu$, the cardinality of the set $\Gamma_n^{(d)}$ is

$$|\Gamma_n^{(d)}| \asymp n^2 \cdot \log^{2\nu-1} n.$$

Proof: The cardinality of the set

$$\Gamma_n^{(\nu)} = \{(i,j) \in \mathbb{Z}^{2\nu} : \rho_{(\varrho,\ldots,\varrho)}(i) \cdot \rho_{(\varrho,\ldots,\varrho)}(j) \leq n^{2\varrho}\}$$

is $|\Gamma_n^{(\nu)}| \asymp n^2(\log n)^{2\nu-1}$, as was shown in [Fra94]. For $d = \nu$ this is already the statement of the lemma. If $d > \nu$, we take this fact as start for the induction over m with $m = \nu, \ldots, d$.

Assume that for some m, where $\nu \leq m < d$, it is shown that $|\Gamma_n^{(m)}| \asymp n^2(\log n)^{2\nu-1}$. Then we can compute $|\Gamma_n^{(m+1)}|$:

$$\begin{aligned}
|\Gamma_n^{(m+1)}| &\asymp \sum_{i=1}^{n^{\frac{2\varrho}{r_{m+1}}}} \sum_{j=1}^{\frac{n^{\frac{2\varrho}{r_{m+1}}}}{i}} \Gamma^{(m)}_{\left[n\cdot(ij)^{-\frac{r_{m+1}}{2\varrho}}\right]} \\
&\asymp \int_1^{n^{\frac{2\varrho}{r_{m+1}}}} \int_1^{n\cdot i^{-\frac{r_{m+1}}{2\varrho}}} n^{\frac{2\varrho}{r_{m+1}}} \cdot i^{-1} \cdot v^{1-\frac{2\varrho}{r_{m+1}}} \cdot (\log v)^{2\nu-1}\, dv\, di \\
&\asymp \int_1^n n^{\frac{2\rho}{r_{\nu+1}}} \cdot u^{1-\frac{2\rho}{r_{\nu+1}}} \cdot (\log u)^{2\nu-1}\, du \\
&\asymp n^2(\log n)^{2\nu-1},
\end{aligned}$$

where we used the substitutions $v = n \cdot (ij)^{-\frac{r_{m+1}}{2\varrho}}$, $u = n \cdot i^{-\frac{r_{\nu+1}}{2\varrho}}$, and the fact, that for $m > 1$, $a \neq -1$, $b \in \mathbb{N}$:

$$\int_1^m x^a \cdot (\log x)^b\, dx = \sum_{k=0}^{b} \frac{(-1)^{b-k}\, b!}{(1+a)^{b-k+1}\, k!}\, x^{1+a} (\log x)^k \Big|_1^m \asymp m^{1+a} (\log m)^b. \quad \triangleleft$$

Proof: (Theorem 3)

The lower bound of Theorem 3 we show using Corollary 1 and estimating the Gelfand number $c_{3n+2}(\Psi)$. For this end, we define operators

$$\begin{aligned}
W &: l_2(\mathbb{Z}^{2d}) \to \mathcal{H}^{r,r}(G^2) \\
W b_{ij} &= \rho_r(i)^{-1} \cdot \rho_r(j)^{-1} \cdot e_i(s) \cdot e_j(t)\,; \\
U &: L(\mathcal{H}^r, \mathcal{H}_0^{-r}) \to l_\infty(\mathbb{Z}^{2d}) \\
U(A) &= (\zeta_{ij})_{i,j\in\mathbb{Z}^d}\,,
\end{aligned}$$

where $\{b_{ij}\}_{i,j\in\mathbb{Z}^d}$ is the unit vector basis in $l_2(\mathbb{Z}^{2d})$, and the sequence $(\zeta_{ij})_{i,j\in\mathbb{Z}^d}$ is defined for arbitrary $A \in L(\mathcal{H}^r, \mathcal{H}_0^{-r})$ by

$$\zeta_{ij} = \rho_r(i)^{-1} \cdot \rho_r(j)^{-1} \cdot \langle Ae_j, e_i \rangle .$$

The operator W is an isometry, so $\|W\| = 1$, and the operator U is an injection with $\|U\| \leq 1$. Composing these operators with Ψ we obtain the diagonal operator

$$\begin{aligned} D &: l_2(\mathbb{Z}^{2d}) \to l_\infty(\mathbb{Z}^{2d}) \\ D &= U \Psi W \\ D\, b_{ij} &= \eta_{ij}\, b_{ij} , \end{aligned}$$

where $\eta_{ij} = \rho_r(i)^{-2}\rho_r(j)^{-2}$. In order to make use of Theorem (11.11.7) in [Pie78] about Gelfand numbers of diagonal operators we must rearrange the sequence $\{\eta_{ij}\}$ in nonincreasing order. For this purpose, the sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq \ldots$ is defined as:

$$\begin{aligned} \lambda_n &= \inf\{\varepsilon : |\{(i,j) : \eta_{ij} \geq \varepsilon\}| < n\} \\ &= \max_{\substack{\Gamma_n \subset \mathbb{Z}^{2d} \\ |\Gamma_n| = n}} \min\{\eta_{ij} \ : \ (i,j) \in \Gamma_n\} . \end{aligned}$$

Setting $\Gamma_n = \Gamma_n^{(d)}$ and using Lemma 7, we get $\lambda_{|\Gamma_n|} \geq n^{-4\varrho} > \lambda_{|\Gamma_n|+1}$, hence

$$\lambda_{[n^2(\log n)^{2\nu-1}]} \asymp n^{-4\varrho} ,$$

which can be transformed in a standard way into

$$\lambda_N \asymp N^{-2\varrho}(\log N)^{2\varrho(2\nu-1)} \tag{39}$$

From Theorem (11.11.7) in [Pie78] follows

$$c_n(D : l_2(\mathbb{Z}^{2d}) \to l_\infty(\mathbb{Z}^{2d})) \asymp \lambda_n .$$

Moreover, by basic properties of Gelfand numbers we can conclude for any $n \in \mathbb{N}$

$$\begin{aligned} c_{3n+2}(\Psi) &\geq \frac{c_{3n+2}(D)}{\|U\| \cdot \|W\|} \\ &\succ \lambda_{3n+2} \\ &\succ n^{-2\varrho}(\log n)^{2\varrho(2\nu-1)} . \end{aligned}$$

This proves the lower bound of the theorem.

The upper bound we shall prove in a constructive way by providing an order–optimal algorithm ϕ of the same type as (24), (25), applying Theorem 2 to it and estimating the Gelfand numbers

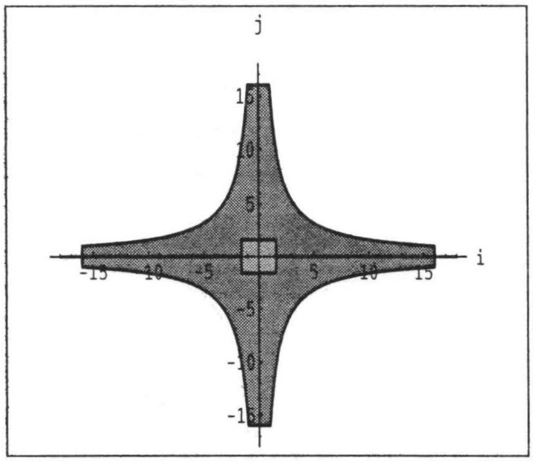

Figure 1: Shape of the hyperbolic cross $\Gamma_n^{(d)}$ and the index set C_n (middle square) for $d = 1$, $n = 16$.

$c(\Psi, (N_1, N_2))$ and $c(\Phi, M_1)$ for the concrete information operator $\bar{N} : (K_0 \times V_0) \to \mathbb{R}^m$ used by this algorithm. Let index sets be given by

$$\begin{aligned} A_n &= \{i \in \mathbb{Z}^d : \rho_r(i) \le n^{\frac{\varrho}{3}}\} \\ B_n &= \{i \in \mathbb{Z}^d : \rho_r(i) \le n^{2\varrho}\} \\ C_n &= \{(i,j) \in \mathbb{Z}^{2d} : \max\{\rho_r(i), \rho_r(j)\} \le n^{\frac{\varrho}{3}}\} \\ D_n &= \Gamma_n^{(d)} . \end{aligned}$$

Then by Lemma 7 (or similar calculations) the cardinalities of these sets are

$$\begin{aligned} |A_n| &\asymp n^{\frac{1}{3}} (\log n)^{\nu-1} \\ |B_n| &\asymp n^2 (\log n)^{\nu-1} \\ |C_n| &\asymp n^{\frac{2}{3}} (\log n)^{2(\nu-1)} \\ |D_n| &\asymp n^2 (\log n)^{2\nu-1} . \end{aligned} \tag{40}$$

Defining projections

$$g \in \mathcal{H}^{r,r} \; : \; \hat{g}(i,j) = \begin{cases} \hat{k}(i,j) & (i,j) \in C_n \\ 0 & \text{otherwise;} \end{cases}$$

$$h \in \mathcal{H}^{r,r} \; : \; \hat{h}(i,j) = \begin{cases} \hat{k}(i,j) & (i,j) \in D_n \\ 0 & \text{otherwise;} \end{cases}$$

$$f_0 \in \mathcal{H}^{r} \; : \; \hat{f}_0(i) = \begin{cases} \hat{f}(i) & i \in B_n \\ 0 & \text{otherwise.} \end{cases}$$

it can be seen easily that they satisfy conditions (21). Furthermore, the norms of g, h in $\mathcal{H}^{r,r}$ and f_0 in $\mathcal{H}^r$, respectively, are

$$\begin{aligned}\|g\|_r^2 &= \sum_{(i,j)\in C_n} \rho_r(i)^2\rho_r(j)^2\hat{k}(i,j)^2 \le \|k\|_r^2 \le \beta_1^2; \ \|h\|_r^2 \le \beta_1^2; \\ \|f_0\|_r^2 &= \sum_{i\in B_n} \rho_r(i)^2\hat{f}(i)^2 \le \|f\|_r^2 \le 1.\end{aligned}$$

So $\alpha_1 = \beta_1$, $\alpha_2 = 1$. The convergence (22) can be shown using the same representation of $\hat{k}(i,j)$ as in Lemma 4:

$$\begin{aligned}\|(T_k - T_h)f\|_{L_2}^2 &= \sum_{i\in \mathbf{Z}^d}\left(\sum_{j:(i,j)\notin D_n} \hat{k}(i,j)\hat{f}(j)\right)^2 \\ &\le \max_{(i,j)\notin D_n} \rho_r(i)^{-2}\rho_r(j)^{-2}\cdot \sum_{i,j\in\mathbf{Z}^d}\xi_{ij}^2 \cdot \sum_{j\in\mathbf{Z}^d}\hat{f}(j)^2 \\ &\le n^{-4\varrho}\cdot\beta_1^2; \\ \|(T_k - T_g)f\|_{L_2}^2 &= \sum_{i\in \mathbf{Z}^d}\left(\sum_{j:(i,j)\notin C_n} \hat{k}(i,j)\hat{f}(j)\right)^2 \\ &\le \max_{(i,j)\notin C_n} \rho_r(i)^{-2}\rho_r(j)^{-2}\cdot \sum_{i,j\in\mathbf{Z}^d}\xi_{ij}^2 \cdot \sum_{j\in\mathbf{Z}^d}\hat{f}(j)^2 \\ &\le n^{-\frac{2\varrho}{3}}\cdot\beta_1^2.\end{aligned}$$

Conditions (23) can be verified in a similar way: For $f \in B_{\mathcal{H}_0^{-r}}$ we have

$$\begin{aligned}\|T_g f\|_{L_2}^2 &= \sum_{i\in \mathbf{Z}^d}\left(\sum_{j:(i,j)\in C_n} \rho_r(i)^{-1}\rho_r(j)^{-1}\xi_{ij}\hat{f}(j)\right)^2 \\ &\le \sum_{i,j\in\mathbf{Z}^d}\xi_{ij}^2\cdot\sum_{j\in\mathbf{Z}^d}\rho_r(j)^{-2}\hat{f}(j)^2 \\ &\le \beta_1^2. \qquad (41)\end{aligned}$$

Hence, $\|T_g : \mathcal{H}_0^{-r} \to \mathcal{H}_0^{-r}\| \le \beta_1$. In the same way we can check $\|T_h : \mathcal{H}_0^{-r} \to \mathcal{H}_0^{-r}\| \le \beta_1$. Using inequality (41) and the relation $(I - T_g)^{-1} = I + (I - T_g)^{-1}T_g$ we conclude

$$\|(I - T_g)^{-1} : \mathcal{H}_0^{-r} \to \mathcal{H}_0^{-r}\| \le 1 + \|(I - T_g)^{-1} : L_2 \to L_2\|\cdot\beta_1.$$

As follows from Lemma 3, for all $n \ge N_0$ this norm is bounded by some constant independent of n and $(k, f) \in X_0$.

By Lemma 3 the algorithm ϕ is correctly defined by equations (24), (25) with the information operator $\bar{N} : (K_0 \times V_0) \to \mathbb{R}^m$, where $\bar{N} = (N, M)$, $N = (N_1, N_2, N_3)$, $Mf = (M_1 f, \langle f, \chi\rangle)$ and

$$N_1 k = \left(\hat{k}(i,j)\right)_{(i,j)\in C_n}, \quad N_2 k = \left(\hat{k}(i,j)\right)_{(i,j)\in D_n\setminus C_n}, \quad N_3 k = (\langle T_k e_i, \chi\rangle)_{i\in B_n},$$

$$M_1 f = \left(\hat{f}(i)\right)_{i\in B_n}.$$

From relations (40) we derive the cardinality of information $m = \mathrm{card}(\bar{N}) \asymp n^2(\log n)^{2\nu-1}$. To apply Theorem 2 it remains to check whether there is such a constant $L \in \mathbb{N}$ that

$$[c(T : \mathcal{H}^{r,r} \to L(L_2), N_1)]^L \prec c(\Psi, (N_1, N_2)).$$

From the first part of the proof we know that

$$n^{-4\varrho} \prec c_m(\Psi) \prec c(\Psi, (N_1, N_2)).$$

By simple calculations we get

$$\begin{aligned} c(T : \mathcal{H}^{r,r} \to L(L_2), N_1) &= \sup_{\substack{\|h\|_r\le 1\\ N_1 h=0}} \sup_{\|f\|_{L_2}\le 1} \|T_h f\|_{L_2} \\ &= \sup_{\|h\|_r\le 1} \sup_{\|f\|_{L_2}\le 1} \left(\sum_{i\in\mathbf{Z}^d}\left(\sum_{j:(i,j)\notin C_n} \hat{h}(i,j)\hat{f}(j)\right)^2\right)^{\frac{1}{2}} \\ &\le \sup_{\|h\|_r\le 1} \sup_{\|f\|_{L_2}\le 1} \left(\max_{(i,j)\notin C_n} \rho_r(i)^{-2}\rho_r(j)^{-2}\right)^{\frac{1}{2}} \\ &\le n^{-\frac{\varrho}{3}}, \end{aligned}$$

and for $L \ge 12$ the condition is satisfied. Hence, by Theorem 2 we have

$$e(S_\chi, \bar{N}, \phi) \prec (c(\Psi, (N_1, N_2)) + c(\Phi, M_1)).$$

We conclude by estimating these Gelfand radii:

$$\begin{aligned} c(\Psi, (N_1, N_2)) &= \sup_{\substack{\|k\|_r\le 1\\ N_1 k=N_2 k=0}} \sup_{\|f\|_r\le 1} \|T_k f\|_{-r} \\ &\le \sup_{\|k\|_r\le 1} \sup_{\|f\|_r\le 1} \left(\sum_{i\in\mathbf{Z}^d} \rho_r(i)^{-2}\left(\sum_{j:(i,j)\notin D_n} \hat{k}(i,j)\hat{f}(j)\right)^2\right)^{\frac{1}{2}} \\ &\le \left(\max_{(i,j)\notin D_n} \rho_r(i)^{-4}\rho_r(j)^{-4}\right)^{\frac{1}{2}} \\ &\le n^{-4\varrho}, \end{aligned}$$

$$
\begin{aligned}
c(\Phi, M_1) &= \sup_{\|f\|_r \le 1,\, M_1 f = 0} \|f\|_{-r} \\
&\le \sup_{\|f\|_r \le 1} \left(\sum_{i \notin B_n} \rho_r(i)^{-2} \hat{f}(i)^2 \right)^{\frac{1}{2}} \\
&\le \left(\max_{i \notin B_n} \rho_r(i)^{-4} \right)^{\frac{1}{2}} \\
&\le n^{-4\varrho}.
\end{aligned}
$$

Together with the cardinality of information and the same standard transformation as in equation (39) this proves the upper bound of the theorem. ◁

Remark: The hyperbolic cross $\Gamma_n^{(d)}$ was originally used by Babenko [Bab60] in connection with the problem of function approximation in the space $\mathcal{H}^{r_1,\ldots,r_d}$.

7 Conclusions

As was demonstrated in section 6, the Gelfand number method based on Theorem 1 is a useful tool to derive lower bounds of the complexity of the local solution problem. However, in the non–Hilbertian case it can prove to be a hard task to estimate the Gelfand numbers of the operator Ψ. Another inconvenience of Corollary 1 is the gap between the $(3n+2)$–th Gelfand numbers in the lower bound and the $(n+1)$–th Gelfand numbers in the upper bound. This does not cause any difficulties as long as the Gelfand numbers behave like $O(n^\alpha(\log n)^\beta)$ for some $\alpha < 0$, $\beta \in \mathbb{R}$, but if they decrease exponentially fast, the upper and lower bound do not match any more. This is the case in the space of analytic functions which was recently considered by Pereverzev and Azizov [PA95], who proved the matching upper bound by analyzing the error of an order–optimal algorithm.

The assumptions (I)–(IV) are formulated sufficiently general, so Theorem 1 seems to be applicable not only to Hilbertian Sobolev spaces, but to a wide class of function spaces frequently used in numerical analysis, as e.g. the C^r–spaces or the L_p–spaces. The conditions under which the algorithm ϕ described in section 4 is optimal are not so general. However, it seems to be possible to weaken them in some degree. This will be a subject of further work, as well as numerical experiments comparing the algorithm ϕ with deterministic and stochastic standard algorithms.

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