Interner Bericht

Complexity of local solution of integral equations

K. Frank, S. Heinrich
University of Kaiserslautern

241/94

Fachbereich Informatik

Universität Kaiserslautern · Postfach 3049 · D-67653 Kaiserslautern

Complexity of local solution of integral equations

K. Frank, S. Heinrich
University of Kaiserslautern

241/94

Universität Kaiserslautern
AG Numerische Algorithmen
Postfach 30 49
67653 Kaiserslautern

März 1994

Herausgeber: AG Numerische Algorithmen
Leiter: Professor Dr. S. Heinrich

Complexity of local solution of integral equations

Karin Frank
Stefan Heinrich

Fachbereich Informatik
Universität Kaiserslautern
Postfach 3049, D–67653 Kaiserslautern
e-mail: frank@informatik.uni-kl.de
heinrich@informatik.uni-kl.de

28th March 1994

Abstract

We study the complexity of local solution of Fredholm integral equations. This means that we want to compute not the full solution, but rather a functional (weighted mean, value in a point) of it. For certain Sobolev classes of multivariate periodic functions we prove matching upper and lower bounds and construct an algorithm of the optimal order, based on Fourier coefficients and a hyperbolic cross approximation.

1 Introduction

Information–based complexity theory studies the intrinsic difficulty of numerical problems — the minimal computational cost (number of operations, etc.) needed to solve a problem approximately, up to a given precision. Many basic numerical problems were investigated from this point of view (see [TWW88]). Among them is the solution of Fredholm integral equations, which was first analyzed in a complexity theoretic setting by Emelyanov and Ilin [EI67]. They considered the case of standard information and continuously differentiable kernels and right–hand sides and determined the order of the complexity both in the case of full and of local solution (see the next section for definitions). This was the first proof of lower bounds for integral equations. The upper estimate was based on a two–grid iteration. Multigrid techniques for integral equations were first developed by Brakhage [BRA60]. Recently, Pereverzev [PER88], [PER89], [PER91] considered the general situation of linear information and determined the complexity of full solution for various Sobolev classes of data. For the case of linear information the problem of local solution was considered for the first time in [HEI93], [HEI94]. For the class C^r of r–times continuously differentiable data an upper bound was derived. What concerns lower bounds, only an equivalence to an (unsolved) problem in n–widths could be shown.

In this paper we consider data belonging to $H^r(G)$ (or in other notation $W_2^r(G)$), the Hilbertian Sobolev space of periodic functions on the d–dimensional torus. We prove matching (with respect to the order) upper and lower bounds and provide an algorithm of optimal order. This algorithm incorporates a two level iteration, which is constructed from a variant of hyperbolic cross (or sparse) approximation based on Fourier coefficients.

2 Notation and formulation of the result

Let $d \in \mathbb{N}$, $G = [0,1]^d$, and let $L_2(G)$ be the space of square–integrable with respect to the Lebesgue measure functions on G. We set

$$e_0(t) \equiv 1$$

and for $n \in \mathbb{N}$

$$\begin{aligned} e_n(t) &= \sqrt{2}\cos 2\pi n t \\ e_{-n}(t) &= \sqrt{2}\sin 2\pi n t. \end{aligned}$$

Given a multiindex $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ we define $e_i \in L_2(G)$ by

$$e_i(t) = e_{i_1}(t_1) \cdot \ldots \cdot e_{i_d}(t_d) \qquad (t = (t_1, \ldots, t_d) \in G).$$

The Fourier coefficients of $f \in L_2(G)$ are given by

$$\hat{f}(i) = (f, e_i) \qquad (i \in \mathbb{Z}^d).$$

Let us further denote

$$|i| = (i_1^2 + \ldots + i_d^2)^{\frac{1}{2}}$$

and define for $r \in \mathbb{R}$ the Sobolev space $H^r(G)$ as

$$H^r(G) = \left\{ f \in L_2(G) : \|f\|_r = \left(\sum_{i \in Z^d} (1 + |i|^2)^r \hat{f}(i)^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

When $r \in \mathbb{N}$, $H^r(G)$ consists of all periodic functions on $[0,1]^d$ with the property that $f \in L_2(G)$ and the generalized derivatives

$$\frac{\partial^p f}{\partial^{p_1} t_1 \ldots \partial^{p_d} t_d} \qquad (0 \le p_1 + \ldots + p_d = p \le r)$$

also belong to $L_2(G)$. We consider only the case of real-valued functions, but the complex case can be treated in the same way. We shall sometimes use the following abbreviations: $L_2=L_2(G)$, $H^r=H^r(G)$, $\mathcal{H}^r=H^r(G^2)$.

For a function $k \in L_2(G^2)$ let the integral operator $T_k : L_2 \to L_2$ be defined by

$$T_k g = \int_G k(s,t)g(t)dt \qquad (g \in L_2).$$

We shall consider the integral equation

$$u - T_k u = f \tag{1}$$

with the unknown function $u \in L_2$. We are given a fixed $\chi \in L_2$ and we seek to approximate not the full solution u, but rather the scalar product (u, χ). Thus we want to compute e.g. means, moments etc. of u. We call this the problem of local solution of the integral equation (1). To formulate the complexity problem precisely, we have to introduce the sets of data (i.e. kernels k and right-hand sides f) for which we are going to solve (1).

We shall consider smooth data. Fix $r \in \mathbb{R}$, $r > 0$, $\alpha > 0$, $\beta > 1$, $\gamma > 0$, and put

$$\begin{aligned} K_0 &= \{k \in H^r(G^2) : \|k\|_r \le \alpha, \|(I - T_k)^{-1} : L_2(G) \to L_2(G)\| \le \beta\}, \\ F_0 &= \{f \in H^r(G) : \|f\|_r \le \gamma\}, \\ X_0 &= K_0 \times F_0. \end{aligned}$$

Given $\chi \in L_2(G)$, we define the operator $S_\chi : X_0 \to \mathbb{R}$ by

$$S_\chi(k, f) = (u, \chi) = ((I - T_k)^{-1} f, \chi). \tag{2}$$

So S_χ maps the data (k, f) to the exact solution (u, χ). S_χ is called the local solution operator. In contrast to S_χ, the global solution operator $S : X_0 \to L_2$, mentioned in the introduction, is defined by

$$S(k, f) = (I - T_k)^{-1} f.$$

In this paper we shall consider only S_χ.

Next we recall the framework of information-based complexity theory for the analysis of (1), (2). For details the reader is referred to [TWW88]. We consider approximations to S_χ of the form $\varphi \circ N$, where

$$N(k, f) = (N_1 k, N_2 f),$$

and N_1, N_2 are of the form

$$\begin{aligned} N_1 k &= ((k, g_1), \ldots, (k, g_{n_1})), \\ N_2 f &= ((f, h_1), \ldots, (f, h_{n_2})), \end{aligned}$$

with $g_1, \ldots, g_{n_1} \in \mathrm{H}^r(\mathrm{G}^2)^* = \mathrm{H}^{-r}(\mathrm{G}^2)$, $h_1, \ldots, h_{n_2} \in \mathrm{H}^r(\mathrm{G})^* = \mathrm{H}^{-r}(\mathrm{G})$ being arbitrary continuous linear functionals on the spaces of data, and $n_1 + n_2 = n$. (We denote by E^* the dual of a Banach space E.) The mapping N stands for the information about (k, f) which we use for the computation, as e. g. point values (if $r > \frac{d}{2}$), Fourier coefficients, scalar products with polynomials, piecewise polynomials etc. If N consists only of point evaluations, it is called standard information. If N is general, as above, it is called linear information. Furthermore, $\varphi : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary mapping. It represents the approximation to $S_\chi(k, f)$ which is computed on the basis of the information $N(k, f)$. For a fixed n we let M_n denote the class of all such pairs (N, φ). The error of (N, φ) is defined as

$$e(S_\chi, N, \varphi) = \sup_{(k,f)\in X_0} |S_\chi(k, f) - \varphi(N(k, f))|$$

and the minimal error over M_n as

$$e_n(S_\chi) = \inf_{(N,\varphi)\in M_n} e(S_\chi, N, \varphi).$$

This is the crucial quantity to be analyzed in information-based complexity theory. It is the minimal error which can be reached by methods which use not more than n information functionals. Let us now adopt a model of computation, in which we are allowed to carry out the basic arithmetic operations (at unit cost) and to obtain scalar products (at fixed constant cost $c \geq 1$ — think of a subroutine which supplies the function values or computes Fourier coefficients in some fast way, e. g. symbolically). Then it is obvious that a program for the approximate computation of S_χ of complexity (i. e. cost) n cannot use more than n information functionals, and hence its error is bounded from below by $e_n(S_\chi)$. Hence $e_n(S_\chi)$ serves as a general lower bound. In many concrete cases (as in ours here) one can find implementable algorithms of cost n which reach the order of $e_n(S_\chi)$, thus providing, up to constant factors, also tight upper bounds.

Let us introduce some further notation which we shall use. Given a Banach space E, we let B_E be its unit ball. Given another Banach space F, we denote the space of all bounded linear operators from E to F by L(E,F). For $T \in$ L(E,F) and $n \in \mathbb{N}$ the n-th Gelfand number of T is defined as

$$c_n(T) = \inf_{\lambda_1,\ldots,\lambda_{n-1}\in E^*} \sup_{\substack{x\in B_E \\ \lambda_1(x)=\ldots=\lambda_{n-1}(x)=0}} \|Tx\|.$$

For details on these numbers we refer to [PIE78], [PIE87]. Given $T \in$ L(E,F), the adjoint operator is denoted by $T^* \in$ L(F*,E*).

Let us briefly review what is known about the minimal error of the full problem S on the classes K_0 and F_0. (Note that $e_n(S)$ can be defined in a way similar to $e_n(S_\chi)$ — we just replace the target space $\mathbb{R}$ of φ by L_2 and the absolute value by the L_2-Norm.) Using the method of Emelyanov and Ilin [EI67], it can be shown that $e_n^{st}(S)$, the minimal error when only standard informations (i. e. function values) are admitted, has the order $n^{-\frac{r}{2d}}$. If we consider arbitrary linear information as we do in this paper, then $e_n(S)$ is (up to logarithmic factors) of order $n^{-\frac{r}{d}}$, so the convergence rate decreases essentially by a factor of $n^{-\frac{r}{2d}}$, once scalar products are available. This follows from [PER88], [PER89]. Surprisingly, this rate decreases further, by another factor of $n^{-\frac{r}{2d}}$, when we restrict the problem to the computation of one functional of the solution. This is the contents of our main result stated below.

In the case of many variables, the computation of functionals of solutions is a traditional situation for the application of Monte Carlo methods. Thus our result about the speed-up is also of interest with respect to the comparative analysis of deterministic and stochastic methods (see [HEI94]).

Now we can state the main result:

Theorem 1 *Let $r > 0$. For each $\chi \in L_2(G)$, $\chi \neq 0$, there exist constants $c_1, c_2 > 0$ such that for all n*

$$c_1 \cdot n^{-\frac{3r}{2d}} \leq e_n(S_\chi) \leq c_2 \cdot n^{-\frac{3r}{2d}}.$$

3 Proof of the lower bound

Let the mapping

$$\Phi : \mathrm{H}^r(\mathrm{G}^2) \rightarrow \mathrm{L}(\mathrm{H}^r(\mathrm{G}), \mathrm{H}^{-r}(\mathrm{G}))$$

be defined by

$$\Phi k = \widetilde{T_k},$$

where $\widetilde{T_k}$ is the integral operator T_k, considered as acting from $\mathrm{H}^r(\mathrm{G})$ to $\mathrm{H}^{-r}(\mathrm{G})$. For the case of C^r instead of H^r it was shown in [HEI93] that there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$

$$c_1 \cdot c_{3n+2}(\Phi) \leq e_{3n+1}(S_\chi) \leq c_2 \cdot c_{n+1}(\Phi). \tag{3}$$

As mentioned there, the proof carries over to other situations. Our present one is such. However, we are not going to repeat the proof step by step. Instead, we supply a lemma, which contains all those details about the H^r case, which make the proof of [HEI93] working for this case as well.

Let us first introduce some notation: If $a(x)$ and $b(x)$ are functions defined on some set X, we write

$$a(x) \prec b(x)$$

if there is a constant $c > 0$ such that for all $x \in X$

$$a(x) \leq c \cdot b(x).$$

We write $a(x) \asymp b(x)$, if $a(x) \prec b(x)$ and $b(x) \prec a(x)$. Moreover, for simplicity we often use the same symbol for possibly different constants.

Lemma 1

(i) There exist constants $c_1, c_2 > 0$ *such that for all* $k \in K_0$

$$c_1 B_{H^r} \subseteq (I - T_k)^{-1} B_{H^r} \subseteq c_2 B_{H^r}.$$

(ii) There exist $c_1, c_2 > 0$ *such that*

$$c_1 B_{H^r} \subseteq \{T_k^* \chi : k \in B_{\mathcal{H}^r}\} \subseteq c_2 B_{H^r}.$$

(iii) There exists $c_1 > 0$ *such that*

$$\{(I - T_k)^{-1} T_k : k \in K_0\} \subseteq \{T_h : h \in c_1 B_{\mathcal{H}^r}\},$$

and for each $\delta > 0$ *there is a* $c_2 > 0$ *with*

$$\{T_h : h \in c_2 B_{\mathcal{H}^r}\} \subseteq \{(I - T_k)^{-1} T_k : k \in \delta B_{\mathcal{H}^r}\}.$$

Proof:
All these statements are easily verified. We omit (i), since it is discussed in connection with Lemmas 3 and 4 of the next section. The right-hand inclusion of (ii) also follows from Lemma 3. To see the left-hand relation, just consider elements of $\mathcal{H}^r$ of the form

$$k(s,t) = e_{i_0}(s) f(t)$$

where i_0 is a fixed index chosen in such a way that $(\chi, e_{i_0}) \neq 0$, and $f \in \mathrm{H}^r$. Now we shall give details of (iii), since this requires an approach different from that in [HEI93].

Given $k \in \mathrm{L}_2(\mathrm{G}^2)$, we define for $j \in \mathbb{Z}^d$ the function $k_j \in \mathrm{L}_2(\mathrm{G})$ as the function with Fourier coefficients

$$\hat{k}_j(i) = \hat{k}(i,j).$$

In other words,

$$k_j(s) = (k(s,\cdot), e_j) = \int_G k(s,t) e_j(t) dt.$$

Now observe that for $k \in \mathcal{H}^r$,

$$\begin{aligned}
\|k\|_r^2 &= \sum_{i,j \in Z^d} (1 + |i|^2 + |j|^2)^r \, \hat{k}(i,j)^2 \\
&\asymp \sum_{i,j \in Z^d} ((1 + |i|^2)^r + (1 + |j|^2)^r) \, \hat{k}(i,j)^2 \\
&= \sum_{j \in Z^d} \left(\sum_{i \in Z^d} (1 + |i|^2)^r \hat{k}(i,j)^2 + (1 + |j|^2)^r \sum_{i \in Z^d} \hat{k}(i,j)^2 \right) \\
&= \sum_{j \in Z^d} (\|k_j\|_r^2 + (1 + |j|^2)^r \|k_j\|_0^2) \qquad (4)
\end{aligned}$$

Given $k \in K_0$, we let $h \in L_2(G^2)$ be the function defined by

$$(I - T_k)^{-1} T_k = T_h.$$

If h_j is defined as k_j above, it follows that

$$h_j = T_h e_j = (I - T_k)^{-1} T_k e_j = (I - T_k)^{-1} k_j.$$

Hence

$$\begin{aligned} \|h\|_r^2 &\prec \sum_{j \in Z^d} (\|h_j\|_r^2 + (1 + |j|^2)\|h_j\|_0^2) \\ &\prec \sum_{j \in Z^d} (\|k_j\|_r^2 + (1 + |j|^2)\|k_j\|_0^2) \\ &\prec \|k\|_r^2. \end{aligned}$$

This implies the first inclusion of (iii). To see the second one, we choose c_2 small enough, take $h \in c_2 B_{\mathcal{H}^r}$ and define $k \in L_2(G^2)$ by

$$T_k = T_h(I + T_h)^{-1}.$$

Then

$$T_h = (I - T_k)^{-1} T_k,$$

and the same argument as above gives $k \in \delta B_{\mathcal{H}^r}$. This proves the lemma. □

Now we shall prove the lower bound. The basic idea is the following: Using (3), we shall estimate the Gelfand numbers of Φ. We do this by transforming the problem into the estimation of Gelfand numbers of another operator, for which they are already known.
For $n \in \mathbb{N}$ we define

$$A_n = \{i \in \mathbb{Z}^d : n \leq |i| < 2n\}.$$

Let

$$m = m(n) = |A_n|$$

be the number of elements of A_n. Clearly

$$m(n) \asymp n^d \tag{5}$$

(meaning that the constants involved do not depend on n). Furthermore, let

$$p : \{1, \ldots, m\} \to A_n$$

be a one-to-one mapping, and define the operator $U_n : l_2^m \to H^r(G)$ by

$$U_n b_l = e_{p(l)} \qquad (l = 1, \ldots, m).$$

Here (b_l) is the unit vector basis of l_2^m. It follows from the definition of H^r and of A_n that
$$\|U_n\| \asymp n^r.$$
Let $V_n : \mathrm{H}^{-r}(\mathrm{G}) \to l_2^m$ be such that
$$V_n f = (f, e_{p(l)})_{l=1}^m \qquad (f \in \mathrm{H}^{-r}(\mathrm{G})).$$
We have
$$\|V_n\| \asymp n^r.$$
We use these operators to define
$$Y_n : \mathrm{L}(\mathrm{H}^r(\mathrm{G}), \mathrm{H}^{-r}(\mathrm{G})) \to \mathrm{L}(l_2^m)$$
by
$$Y_n(T) = V_n T U_n.$$
From the above we get
$$\|Y_n\| \prec n^{2r}. \tag{6}$$
Finally, we let $W_n : l_2^{m^2} \to \mathrm{H}^r(\mathrm{G}^2)$ be given by
$$W_n b_{kl} = e_{p(k)p(l)} \qquad (k, l = 1, \ldots, m).$$
As in the case of U_n we deduce
$$\|W_n\| \asymp n^r. \tag{7}$$
We compose these operators in the following way:
$$l_2^{m^2} \xrightarrow{W_n} \mathrm{H}^r(\mathrm{G}^2) \xrightarrow{\Phi} \mathrm{L}(\mathrm{H}^r(\mathrm{G}), \mathrm{H}^{-r}(\mathrm{G})) \xrightarrow{Y_n} \mathrm{L}(l_2^m).$$
The composition $Y_n \Phi W_n$ is just the operator (denoted by J_m) which sends a matrix $A = (a_{kl})_{k,l=1}^m$ to the operator on l_2^m generated by A. Consequently, by basic properties of the Gelfand numbers,
$$c_l(J_m) \le \|Y_n\| c_l(\Phi) \|W_n\|,$$
and thus
$$c_l(\Phi) \ge \|W_n\|^{-1} \|Y_n\|^{-1} c_l(J_m) \tag{8}$$
for all $l \in \mathbb{N}$. To derive a lower bound on $c_l(J_m)$, we let Q_m be the inverse operator of J_m, but considered as acting from $\mathrm{L}(l_2^m)$ into l_∞^m. Obviously, $\|Q_m\| \le 1$, which implies
$$\begin{aligned} c_l(J_m) &\ge c_l(Q_m J_m) = c_l(\mathrm{Id} : l_2^{m^2} \to l_\infty^{m^2}) \\ &= \left(\frac{m^2 - l + 1}{m^2} \right)^{\frac{1}{2}} \end{aligned} \tag{9}$$

for $1 \le l \le m^2+1$. The last equality is contained in [PIE78], 11.11.8, 11.7.4 and 11.5.2. Summarizing relations (6)–(9), we get for $l = [\frac{m^2}{2}]$

$$c_{[\frac{m^2}{2}]}(\Phi) \succ n^{-3r}.$$

By (5), $m^2 \asymp n^{2d}$, and standard reasoning using the monotonicity of the Gelfand numbers (i.e. $c_{l+1}(T) \le c_l(T)\ \forall l \in \mathbb{N}$) yields

$$c_l(\Phi) \succ l^{-\frac{3r}{2d}}$$

for $k \in \mathbb{N}$.

4 Proof of the upper bound

We could use relation (3) and estimate the Gelfand numbers of Φ from above. Instead, we choose a more constructive way: We provide a concrete, implementable algorithm and estimate the number of information functionals, the error and the complexity. The algorithm is based on Fourier coefficients and it turns out to be optimal not only with respect to the order of the error, but also with respect to arithmetic complexity: The number of arithmetic operations is of the same order as the number of Fourier coefficients required (compare the remarks after the definition of $e_n(S_\chi)$ in section 2).

The basic idea of the algorithm is taken from [HEI94]. That method, however, was based on spline bases and the analysis was technically very complicated. The analysis of the Hilbertian case is different, and, as we think, more transparent, so that the role of the main ingredient of the algorithm — a suitable "compression" of the kernel, a sparse approximation — becomes more apparent. In addition, our algorithm matches the lower bound up to constant factors, so that no logarithmic gap is left. First we describe the algorithm in a functional–analytic form.

Let $k \in K_0$ and $f \in F_0$ be given. Fix $n \in \mathbb{N}$ and put

$$\begin{aligned} A_n &= \{i \in \mathbb{Z}^d : |i| \le n^{\frac{2}{3}}\} \\ B_n &= \{i \in \mathbb{Z}^d : |i| \le n^{\frac{3}{2}}\} \\ C_n &= \{(i,j) \in \mathbb{Z}^{2d} : \max(|i|,|j|) \le n^{\frac{2}{3}}\} \\ D_n &= \{(i,j) \in \mathbb{Z}^{2d} : \max(1,|i|)\cdot\max(1,|j|)\cdot\max(|i|,|j|) \le n^3\}. \end{aligned}$$

Define $f_0 \in \mathrm{H}^r(\mathrm{G})$ by

$$\hat{f}_0(i) = \begin{cases} \hat{f}(i) & \text{if } i \in B_n \\ 0 & \text{otherwise,} \end{cases}$$

$g \in \mathrm{H}^r(\mathrm{G}^2)$ by

$$\hat{g}(i,j) = \begin{cases} \hat{k}(i,j) & \text{if } (i,j) \in C_n \\ 0 & \text{otherwise,} \end{cases}$$

and $h \in \mathrm{H}^r(\mathrm{G}^2)$ by

$$\hat{h}(i,j) = \begin{cases} \hat{k}(i,j) & \text{if } (i,j) \in D_n \\ 0 & \text{otherwise.} \end{cases}$$

The algorithm first computes an approximation v to u in an iterative way: We put $v_0 = 0$ and determine v_l $(l = 1, \ldots, l_0)$ from

$$(I - T_g)v_l = f_0 + (T_h - T_g)v_{l-1}. \tag{10}$$

We set $v = v_{l_0}$ and compute the final approximation $\psi_n(k,f)$ to $S_\chi(k,f) = (u,\chi)$ by

$$\psi_n(k,f) = (f,\chi) + (v, T_k^*\chi). \tag{11}$$

For the smoothness classes we consider, $l_0 = 5$ iterations suffice. Next we convert this method from the operator form into an arithmetic procedure and estimate the complexity. Taking Fourier coefficients, (10) turns into

$$\hat{v}_l(i) - \sum_{j \in A_n} \hat{k}(i,j)\hat{v}_l(j) = \hat{f}(i) + \sum_{j:(i,j) \in D_n \setminus C_n} \hat{k}(i,j)\hat{v}_{l-1}(j) \tag{12}$$

for $i \in A_n$, and

$$\hat{v}_l(i) = \hat{f}(i) + \sum_{j:(i,j) \in D_n} \hat{k}(i,j)\hat{v}_{l-1}(j) \tag{13}$$

for $i \in B_n \setminus A_n$.

Since $\hat{f}_0(i) = 0$ for $|i| > n^{\frac{3}{2}}$ and $\hat{h}(i,j) = \hat{g}(i,j) = 0$ if $\max(|i|,|j|) > n^{\frac{3}{2}}$, equation (10) gives

$$\hat{v}_l(i) = 0 \qquad (|i| > n^{\frac{3}{2}}).$$

Consequently, (10) is equivalent to (12) and (13). The relations (12) constitute a system of linear equations, from which one determines $\hat{v}_l(i)$ $(i \in A_n)$, e. g. by Gaussian elimination. The unique solvability of (12) follows from that of (10) which, in turn, is a consequence of Lemma 4(ii) below. The remaining coefficients $\hat{v}_l(i)$ $(i \in B_n \setminus A_n)$ are obtained explicitly from (13) by computing the right-hand side. Finally, from (11)

$$\psi_n(k,f) = (f,\chi) + \sum_{j \in B_n} \hat{k}(\chi,j)\hat{v}(j) \tag{14}$$

where we denoted

$$\hat{k}(\chi,j) = (k, \chi \otimes e_j) = \int\int_{G^2} k(s,t)\chi(s)e_j(t)dsdt.$$

The information operator $N = (N_1, N_2)$ of this algorithm is the following:

$$\begin{aligned} N_1 k &= \left(\left(\hat{k}(i,j)\right)_{(i,j) \in D_n}, \left(\hat{k}(\chi,j)\right)_{j \in B_n} \right) \\ N_2 f &= \left(\left(\hat{f}(i)\right)_{i \in B_n}, (f,\chi) \right). \end{aligned}$$

Note that we assume the knowledge of (f, χ) and $k(\chi, j)$ $(j \in B_n)$. This seems to be a natural assumption — to know the value of χ on the data k and f, once we are to compute the value of χ on the solution u (think, e.g., of χ being some e_i itself). The mapping φ is simply defined by

$$\varphi(N_1 k, N_2 f) = \psi_n(k, f).$$

Clearly, the cardinalities of the sets involved are

$$\begin{aligned} |A_n| &\asymp n^{\frac{2d}{3}} \\ |B_n| &\asymp n^{\frac{3d}{2}} \\ |C_n| &\asymp n^{\frac{4d}{3}} \end{aligned}$$

and it is an easy exercise in series summation to verify that

$$|D_n| \asymp n^{2d}.$$

So we have a total of $O(n^{2d})$ information functionals, the solution of (12) requires $O(n^{2d})$ arithmetic operations. The computation of the remaining Fourier coefficients from (13) can be accomplished in $O(|D_n|) = O(n^{2d})$ operations. Finally, (14) requires $O(n^{\frac{3d}{2}})$ operations. So the number of information functionals and of arithmetic operations are of the same order n^{2d}, and the algorithm will be of optimal order, if its error satisfies

$$e(S_\chi, N, \varphi) \asymp n^{-3r}. \tag{15}$$

This is what we shall prove in the rest of the section. Put

$$\begin{aligned} Y &= (I - T_g)^{-1}(T_h - T_g) \\ Z &= (I - T_g)^{-1}(T_k - T_h) \end{aligned}$$

and

$$w = (I - T_k)^{-1} f_0. \tag{16}$$

Then

$$u - w = (I - T_k)^{-1}(f - f_0). \tag{17}$$

Immediately from (16) we derive

$$(I - T_g)w = f_0 + (T_k - T_g)w,$$

and, by subtracting (10), we obtain

$$(I - T_g)(w - v_l) = (T_h - T_g)(w - v_{l-1}) + (T_k - T_h)w.$$

This implies

$$w - v_l = Y(w - v_{l-1}) + Zw \qquad (l = 1, \ldots, l_0)$$

and so, for $l = l_0 = 5$,

$$w - v = Y^5 w + \sum_{l=0}^{4} Y^l Z w. \tag{18}$$

Next we analyze the operators Y and Z. In the following lemmas we obtain estimates which hold for all $k \in K_0$ and $n \in \mathbb{N}$. Recall that g and h as defined above depend on k and the choice of n. So the inequalities below are always meant to hold with constants independent of k, h, g and n.

Lemma 2

(i) $\|T_k - T_h : H^r \to H^{-r}\| \prec n^{-3r}$

(ii) $\|T_k - T_g : L_2 \to L_2\| \prec n^{-\frac{2r}{3}}$

(iii) $\|T_h - T_g : L_2 \to L_2\| \prec n^{-\frac{2r}{3}}$.

Proof:

We shall only verify (i), the proof of the remaining statements is similar — just easier. Let $z \in \mathrm{H}^r$, $\|z\|_r \le 1$ and define ξ_{ij} and η_j $(i, j \in \mathbb{Z}^d)$ by

$$\begin{aligned} \hat{k}(i,j) &= (1 + |i|^2 + |j|^2)^{-\frac{r}{2}} \xi_{ij} \\ \hat{z}(j) &= (1 + |j|^2)^{-\frac{r}{2}} \eta_j. \end{aligned}$$

Since $k \in K_0$, it follows that

$$\sum_{i,j \in \mathbf{Z}^d} \xi_{ij}^2 \le \alpha^2. \tag{19}$$

By the choice of z,

$$\sum_{j \in \mathbf{Z}^d} \eta_j^2 \le 1. \tag{20}$$

Now we have

$$\|(T_k - T_h)z\|_{-r}^2 = \sum_{i \in \mathbf{Z}^d} (1 + |i|^2)^{-r} \left(\sum_{j \in \mathbf{Z}^d} (\hat{k}(i,j) - \hat{h}(i,j)) \hat{z}(j) \right)^2$$

$$= \sum_{i \in \mathbf{Z}^d} (1 + |i|^2)^{-r} \left(\sum_{j:(i,j) \notin D_n} \hat{k}(i,j) \hat{z}(j) \right)^2$$

$$= \sum_{i \in \mathbf{Z}^d} (1 + |i|^2)^{-r} \left(\sum_{j:(i,j) \notin D_n} (1 + |i|^2 + |j|^2)^{-\frac{r}{2}} (1 + |j|^2)^{-\frac{r}{2}} \xi_{ij} \eta_j \right)^2$$

$$\le \sum_{i \in \mathbf{Z}^d} (1 + |i|^2)^{-r} \max_{j:(i,j) \notin D_n} \left((1 + |i|^2 + |j|^2)^{-r} (1 + |j|^2)^{-r} \right) \left(\sum_{j \in \mathbf{Z}^d} |\xi_{ij} \eta_j| \right)^2$$

$$\leq \max_{(i,j)\notin D_n} ((1+|i|^2)^{-r}(1+|i|^2+|j|^2)^{-r}(1+|j|^2)^{-r}) \sum_{i\in Z^d}\left(\sum_{j\in Z^d}|\xi_{ij}\eta_j|\right)^2 .$$

The first factor is of the order n^{-6r}, while the second one is $\leq \alpha^2$. This proves the lemma. □

Lemma 3 *For* $T = T_k, T_h, T_g$:

(i) $\|T : L_2 \to H^r\| \leq c$

(ii) $\|T : H^{-r} \to L_2\| \leq c.$

Proof:
(i) can be easily checked by looking at the Fourier coefficients, as in the proof of Lemma 2.
(ii) follows from (i) by duality: If $\|T : \mathrm{L}_2 \to \mathrm{H}^r\| \leq c$, then $\|T^* : \mathrm{H}^{-r} \to \mathrm{L}_2\| \leq c$. □

Lemma 4

(i) $\|(I-T_k)^{-1} : H^r \to H^r\| \leq c,$
$\|(I-T_k)^{-1} : H^{-r} \to H^{-r}\| \leq c.$

(ii) For $n \geq n_0$, $\|(I-T_g)^{-1} : L_2 \to L_2\| \leq c.$

(iii) For $n \geq n_0$, $\|(I-T_g)^{-1} : H^{-r} \to H^{-r}\| \leq c.$

Proof:
The first statement of (i) follows from the relation

$$(I-T)^{-1} = I + T(I-T)^{-1} \tag{21}$$

and from Lemma 3(i). The second one follows by duality. The statement (ii) follows from Lemma 2(ii). Finally, (iii) is derived from (ii) and Lemma 3(ii) using again (21). □

Corollary 1 *For* $n \geq n_0$,

$$\begin{aligned} \|Y : L_2 \to L_2\| &\prec n^{-\frac{2r}{3}} \\ \|Y : H^{-r} \to H^{-r}\| &\leq c \\ \|Z : H^r \to H^{-r}\| &\prec n^{-3r}. \end{aligned}$$

Now we can accomplish the proof of the upper bound. It follows readily from the definition of f_0 that

$$\|f - f_0\|_{-r} \prec n^{-3r}.$$

Moreover, Lemma 4(i) gives

$$\begin{aligned} \|u - w\|_{-r} &= \|(I - T_k)^{-1}(f - f_0)\|_{-r} \\ &\leq c \cdot \|f - f_0\|_{-r} \\ &\leq c \cdot n^{-3r}. \end{aligned} \tag{22}$$

Moreover,

$$\|w\|_r = \|(I - T_k)^{-1} f_0\|_r \leq c \cdot \|f_0\|_r \leq c.$$

From (18) and Corollary 1 we deduce

$$\begin{aligned} \|w - v\|_{-r} &\leq \|Y^5 w\|_0 + \|\sum_{l=0}^{4} Y^l : \mathrm{H}^{-r} \to \mathrm{H}^{-r}\| \cdot \|Z : \mathrm{H}^r \to \mathrm{H}^{-r}\| \cdot \|w\|_r \\ &\leq c \cdot n^{-\frac{10r}{3}} + c \cdot n^{-3r} \\ &\leq c \cdot n^{-3r}. \end{aligned}$$

This together with (22) gives

$$\|u - v\|_{-r} \leq c \cdot n^{-3r}.$$

Finally, we get

$$\begin{aligned} |S_\chi(k, f) - \psi_n(k, f)| &= \left|((I - T_k)^{-1} f, \chi) - (f, \chi) - (v, T_k^* \chi)\right| \\ &= \left|(f, \chi) + (T_k(I - T_k)^{-1} f, \chi) - (f, \chi) - (v, T_k^* \chi)\right| \\ &= \left|((I - T_k)^{-1} f, T_k^* \chi) - (v, T_k^* \chi)\right| \\ &= |(u - v, T_k^* \chi)| \\ &\leq \|u - v\|_{-r} \|T_k^* \chi\|_r \\ &\leq c \cdot \|u - v\|_{-r} \\ &\leq c \cdot n^{-3r}. \end{aligned}$$

This completes the proof of the Theorem.

References

[BRA60] H. Brakhage: Über die numerische Behandlung von Integralgleichungen nach der Quadraturformelmethode. Numer. Math. **2**(1960), 183–196.

[EI67] K. V. Emelyanov, A. M. Ilin: On the number of arithmetic operations, necessary for the approximate solution of Fredholm integral equations of the second kind (in Russian). Zh. Vychisl. Mat. i Mat. Fiz. **7**(1967), 905–910.

[HEI93] S. Heinrich: Complexity of integral equations and relations to s–numbers. Journal of Complexity **9**(1993), 141–153.

[HEI94] S. Heinrich: Random approximation in numerical analysis. Proceedings of the Conference "Functional Analysis" Essen 1991 (ed. by K. D. Bierstedt, A. Pietsch, W. M. Ruess, D. Vogt), Marcel Dekker, 1994.

[PER88] S. V. Pereverzev: On the complexity of the problem of finding solutions of Fredholm equations of the second kind with differentiable kernels I (in Russian). Ukrain. Mat. Zh. **40**(1988), 84–91.

[PER89] S. V. Pereverzev: On the complexity of the problem of finding solutions of Fredholm equations of the second kind with differentiable kernels II (in Russian). Ukrain. Mat. Zh. **41**(1989), 189–193.

[PER91] S. V. Pereverzev: Hyperbolic cross and complexity of the approximate solution of Fredholm equations of the second kind with differentiable kernels (in Russian). Sibir. Mat. Zh. **32**(1991), 107–115.

[PIE78] A. Pietsch: Operator Ideals. Deutscher Verlag der Wissenschaften, Berlin 1978.

[PIE87] A. Pietsch: Eigenvalues and s-Numbers. Cambridge University Press, Cambridge 1987.

[TWW88] J. F. Traub, G. W. Wasilkowski, H. Woźniakowski: Information–Based Complexity. Academic Press, New York 1988.