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Complexity of local solution of multivariate integral equations

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Abstract

In this paper the complexity of the local solution of Fredholm integral equations is studied. For certain Sobolev classes of multivariate periodic functions with dominating mixed derivative we prove matching lower and upper bounds. The lower bound is shown using relations to s -numbers. The upper bound is proved in a constructive way providing an implementable algorithm of optimal order based on Fourier coefficients and a hyperbolic cross approximation.

1 Introduction

One of the standard problems considered in information-based complexity theory is the solution of Fredholm integral equations of the second kind. These equations often appear in physical applications, e.g. boundary value problems can be formulated in this form. To get a general idea of the existing results, we start with a short overview.

Within the framework of information-based complexity several cases are distinguished. The first distinction is made with respect to the required result. One can either be interested in full solution, i.e. in computing an approximation to the solution function on the whole domain, or in local solution, i.e. in computing the value of some functional applied to the solution function. This functional can be e.g. the value of the solution function at a single point or a weighted mean. The second distinction is made between different types of knowledge about the input data: Either only values of the kernel and the right-hand side at some points are known (this is called standard information), or the values of some linear functionals both of the kernel and the right-hand side are given (this is called linear information). Note that the permission of linear information includes a wider class of algorithms.

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The first work on complexity of Fredholm problems where lower bounds were shown, was the paper of Emelyanov and Ilin [EI67], in which the class of r -times continuously differentiable data with standard information was considered, both for full and local solution. The upper bound was shown by a two-grid iteration. For the more general case of full solution with linear information some results were obtained by Pereverzev [Per88], [Per89], [Per91]. Werschulz (1985) discussed the problem of full solution of integral equations with fixed kernel and varying right-hand side, both with standard and linear information of the right-hand side.

The problem of local solution with linear information was first studied by Heinrich [Hei93], [Hei94]. For the class of r -times continuously differentiable data an upper bound was derived. Concerning the lower bound, only an equivalence to an open problem in s -numbers could be shown. However, replacing the class C^r by the Hilbertian Sobolev class W_2^r , this approach could be extended by Frank and Heinrich [FH94], resulting in the proof of matching upper and lower bounds.

In the present paper, the Sobolev class of periodic functions with dominating mixed derivative is discussed. This class of functions was recently studied by Pereverzev, who got some results on the complexity of the full solution for the case of linear information. The aim of our paper is to obtain upper and lower bounds of the same order for the complexity of local solution in the general situation of linear information.

To show the lower bound of the Theorem, we use an s -number technique, which is based on the fact that the radius of information of the problem is bounded from both sides by the so-called Gelfand numbers of some operator (see Section 3 for definitions). There exist various types of s -numbers, e.g. Gelfand numbers or Kolmogorov numbers, whose relation to linear problems is well-known [TWW88]. Recently, s -number methods were applied by Heinrich [Hei93] to the problem of complexity of integral equations.

Probably this method could also be used to proof the upper bound of the Theorem. However, we prefer the constructive and more intelligible way to estimate the radius of information from above by the error of a concrete algorithm of optimal order. The algorithm is based on a two-grid iteration, where the kernel is represented by a specific hyperbolic cross approximation. Approximations of that type were introduced by Babenko [Bab60].

2 Formulation of the problem and the main result

2.1 The problem

Let us first introduce some notations. Let $G = [0, 1]^d$ with $d \in \mathbb{N}$, and $L_2(G)$ be the space of square summable with respect to the Lebesgue measure functions on G . We consider the orthonormal trigonometric basis in $L_2([0, 1])$

$$e_0(\tau) \equiv 1$$

$$\begin{aligned} e_n(\tau) &= \sqrt{2} \cos 2\pi n\tau \\ e_{-n}(\tau) &= \sqrt{2} \sin 2\pi n\tau \end{aligned}$$

for $n \in \mathbb{N}$. Then for a given multiindex $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$ the basis function $e_i \in L_2(G)$ is defined by

$$e_i(t) = e_{i_1}(t_1) \cdot \dots \cdot e_{i_d}(t_d) \quad (t = (t_1, \dots, t_d) \in G).$$

The Fourier coefficients of $f \in L_2(G)$ are given by

$$\hat{f}(i) = (f, e_i) \quad (i \in \mathbb{Z}^d).$$

Similarly, an orthonormal basis $\{e_{ij}\}_{i,j \in \mathbb{Z}^d}$ in $L_2(G^2)$ is defined by

$$e_{ij}(s, t) = e_i(s) \cdot e_j(t) \quad (s, t \in G)$$

Then, the Fourier coefficients of $k \in L_2(G^2)$ are of the following form

$$\hat{k}(i, j) = (k, e_{ij}) \quad (i, j \in \mathbb{Z}^d).$$

Now we shall define the class of data to be discussed. Therefore, given a multiindex $i = (i_1, i_2, \dots, i_d) \in \mathbb{Z}^d$ we set $|i| = \max(1, |i_1|) \cdot \max(1, |i_2|) \cdot \dots \cdot \max(1, |i_d|)$, where $|i_k|$ denotes the ordinary absolute value of $i_k \in \mathbb{Z}$. Let $r \geq 0$. Then the function spaces $\mathcal{H}^r(G)$ and $\mathcal{H}^{r,r}(G^2)$ are defined as

$$\begin{aligned} \mathcal{H}^r(G) &= \{f \in L_2(G) : \|f\|_r^2 = \sum_{i \in \mathbb{Z}^d} |i|^{2r} \hat{f}(i)^2 < \infty\}, \\ \mathcal{H}^{r,r}(G^2) &= \{k \in L_2(G^2) : \|k\|_{r,r}^2 = \sum_{i,j \in \mathbb{Z}^d} |i|^{2r} |j|^{2r} \hat{k}(i,j)^2 < \infty\}. \end{aligned}$$

For simplicity, we will often use the following notation: $\mathcal{H}^r = \mathcal{H}^r(G)$, $\mathcal{H}^{r,r} = \mathcal{H}^{r,r}(G^2)$, $L_2 = L_2(G)$. Note that for $r \in \mathbb{N}$ the space $\mathcal{H}^r(G)$ constitutes the Sobolev space of periodic functions f on $[0, 1]^d$, for which both f and the generalized mixed derivative $\frac{\partial^{dr} f}{\partial t_1^r \dots \partial t_d^r}$ belong to L_2 . These spaces are called Sobolev spaces with dominating mixed derivative.

By $\mathcal{H}^{-r} = (\mathcal{H}^r)^*$ we denote the dual space of \mathcal{H}^r . L_2 imbeds into \mathcal{H}^{-r} in a canonical way, and the \mathcal{H}^{-r} -norm of a function $f \in L_2$ is given by

$$\|f\|_{-r}^2 = \sum_{i \in \mathbb{Z}^d} |i|^{-2r} \hat{f}(i)^2.$$

Note that L_2 is a dense subspace of \mathcal{H}^{-r} .

Finally, we define subsets $F_0 \subset \mathcal{H}^r(G)$, $K_0 \subset \mathcal{H}^{r,r}(G^2)$ of the form

$$\begin{aligned} F_0 &= \{f \in \mathcal{H}^r(G) : \|f\|_r \leq \gamma\}, \\ K_0 &= \{k \in \mathcal{H}^{r,r}(G^2) : \|k\|_{r,r} \leq \alpha, \|(I - T_k)^{-1} : L_2 \rightarrow L_2\| \leq \beta\}, \end{aligned}$$

where $\alpha, \gamma > 0$ and $\beta > 1$.

Now we are ready to state the problem to be studied. We consider integral equations of the form

$$u - T_k u = f, \quad (1)$$

where $f \in F_0$, $k \in K_0$, and T_k denotes the integral operator

$$\begin{aligned} T_k &: L_2(G) \rightarrow L_2(G) \\ T_k u &= \int_G k(s, t) u(t) dt. \end{aligned}$$

The problem is to be formulated within the framework of information-based complexity theory. Here only the most important definitions are outlined, referring to [TWW88] for further notations.

Since we are interested not in the full solution of (1), but rather in the value of one linear functional χ of it, we have to consider the so-called local solution operator

$$\begin{aligned} S_\chi &: K_0 \times F_0 \rightarrow \mathbb{R} \\ S_\chi(k, f) &= ((I - T_k)^{-1} f, \chi), \end{aligned}$$

where $\chi \in L_2$ is a given non-zero linear functional. For example, this may be a Fourier coefficient or a weighted mean. We permit linear information on the data, i.e. the information operator is defined by $N : K_0 \times F_0 \rightarrow \mathbb{R}^n$, $N = (N_1, N_2)$ with

$$\begin{aligned} N_1 k &= ((k, g_1), \dots, (k, g_{n_1})) , \quad g_k \in \mathcal{H}^{r,r}(G^2)^* \quad (k = 1, \dots, n_1) \\ N_2 f &= ((f, h_1), \dots, (f, h_{n_2})) , \quad h_l \in \mathcal{H}^{-r}(G) \quad (l = 1, \dots, n_2) \end{aligned}$$

where $n_1 + n_2 = n$. Here, $\mathcal{H}^{r,r}(G^2)^*$ denotes the dual space of $\mathcal{H}^{r,r}(G^2)$.

An approximation to the exact solution $S_\chi(k, f)$ is to be computed. An arbitrary mapping $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, which combines the information $N(k, f)$ and computes an approximation $\varphi(N(k, f))$ to S_χ , is called an algorithm. Then the error of an approximation $\varphi(N(k, f))$ is defined by

$$e(S_\chi, N, \varphi) = \sup_{f \in F_0, k \in K_0} |S_\chi(k, f) - \varphi(N(k, f))| .$$

Let us agree about the model of computation. We assume, that standard arithmetical operations, including comparisons, can be performed with unit cost, while linear functionals on the input data can be computed with constant cost $c(d)$. Imagine a subroutine which supplies the computation of one linear functional on the data.

2.2 The main result

Our main theorem provides estimates for the radius of information of the given problem. This quantity describes the minimal error, which can be obtained by any algorithm φ using at most n information functionals:

$$e_n(S_\chi) = \inf_{N(k,f) \in R^n} \inf_{\varphi: R^n \rightarrow R} e(S_\chi, N, \varphi)$$

This is the crucial quantity to be analyzed in information-based complexity. Since any algorithm of cost n can use at most n information functionals due to the model of computation, $e_n(S_\chi)$ serves as a general lower bound for the error of any algorithm of cost n .

Theorem 1 *Let $r > 0$. For each $\chi \in L_2(G)$, $\chi \neq 0$, there exist constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$*

$$c_1 \cdot n^{-2r} \log^{2r(2d-1)} n \leq e_n(S_\chi) \leq c_2 \cdot n^{-2r} \log^{2r(2d-1)} n.$$

3 Proof of the lower bound

To prove the lower bound of the Theorem, we are going to use a method of Heinrich, which was originally developed for the class of r times continuously differentiable data (see [Hei93]) and extended to other situations in [FH94]. For this end, let us define some mapping Φ by

$$\begin{aligned} \Phi & : \mathcal{H}^{r,r}(G^2) \rightarrow L(\mathcal{H}^r(G), \mathcal{H}^{-r}(G)) \\ \Phi k & = T_k : \mathcal{H}^r(G) \rightarrow \mathcal{H}^{-r}(G) \end{aligned}$$

and introduce the so-called Gelfand numbers of an operator. Given two Banach spaces E and F , let B_E denote the unit ball of E and $L(E, F)$ the space of all bounded linear operators from E to F . Then for an operator $T \in L(E, F)$ and $n \in \mathbb{N}$ the n -th Gelfand number of T is defined by

$$c_n(T) = \inf_{\lambda_1, \dots, \lambda_{n-1} \in E^*} \sup_{\substack{x \in B_E \\ \lambda_1(x) = \dots = \lambda_{n-1}(x) = 0}} \|Tx\|.$$

For details on these numbers we refer to [Pie78], [Pie87].

The relation of Gelfand numbers to the radius of information for linear problems with arbitrary linear information is well-known [TWW88]. Since in our case the solution operator S_χ is nonlinear, this result is not applicable and we need the theorem below, which states an equivalence of the radius of information of our problem and the Gelfand numbers of the operator Φ . The proof of this theorem will not be given in detail. Instead, we shall show several lemmas, which make the proof of [Hei93] work in this case as well.

Theorem 2 *There are constants $a_1, a_2 > 0$ such that for all $n \in \mathbb{N}$:*

$$a_1 \cdot c_{3n+2}(\Phi) \leq e_{3n+1}(S_\chi) \leq a_2 \cdot c_{n+1}(\Phi).$$

First, an agreement about the notation of constants is to be made. If $a(x)$ and $b(x)$ are functions defined on some set X , the notation

$$a(x) \prec b(x)$$

means that there is a constant $c > 0$ such that $a(x) \leq c \cdot b(x)$ for all $x \in X$. One writes

$$a(x) \asymp b(x)$$

if $a(x) \prec b(x)$ and $b(x) \prec a(x)$. For simplicity, we will often use the same symbol for possibly different constants.

Lemma 1 *There are constants $c_1, c_2 > 0$ such that for all $k \in K_0$:*

$$c_1 \cdot B_{\mathcal{H}^r} \subseteq (I - T_k)^{-1} B_{\mathcal{H}^r} \subseteq c_2 \cdot B_{\mathcal{H}^r}.$$

Proof:

By assumption

$$\|(I - T_k)^{-1} : L_2 \rightarrow L_2\| \leq \beta.$$

Furthermore,

$$\|T_k : L_2 \rightarrow \mathcal{H}^r\| \leq c, \tag{2}$$

which can be proved using the Fourier coefficients. For this purpose, let $(\xi_{ij})_{i,j \in \mathbb{Z}^d}$ be a sequence of numbers with $\xi_{ij} = |i|^r |j|^r \hat{k}(i, j)$. Then $\|k\|_r^2 = \sum_{i,j \in \mathbb{Z}^d} \xi_{ij}^2 \leq \alpha^2$ and

$$\begin{aligned} \|T_k f\|_r^2 &= \sum_{i \in \mathbb{Z}^d} |i|^{2r} \left(\sum_{j \in \mathbb{Z}^d} \hat{f}(j) \hat{k}(i, j) \right)^2 \\ &= \sum_{i \in \mathbb{Z}^d} |i|^{2r} \left(\sum_{j \in \mathbb{Z}^d} \hat{f}(j) |i|^{-r} |j|^{-r} \xi_{ij} \right)^2 \\ &\leq \sum_{i \in \mathbb{Z}^d} \left(\sum_{j \in \mathbb{Z}^d} \hat{f}(j) \xi_{ij} \right)^2 \\ &\leq \|f\|_0^2 \cdot \alpha^2. \end{aligned}$$

Together with the relation

$$(I - T_k)^{-1} = I + T_k(I - T_k)^{-1}$$

these inequalities imply the boundedness of the operator $(I - T_k)^{-1} : \mathcal{H}^r \rightarrow \mathcal{H}^r$ and in that way the right-hand side of the lemma. The left-hand side follows from the continuity of the linear operator $(I - T_k) : \mathcal{H}^r \rightarrow \mathcal{H}^r$, which is a consequence of (2). \square

Lemma 2 *There are constants $c_1, c_2 > 0$ such that*

$$c_1 \cdot B_{\mathcal{H}^r} \subseteq \{T_k^* \chi : k \in B_{\mathcal{H}^{r,r}}\} \subseteq c_2 \cdot B_{\mathcal{H}^r},$$

where T_k^* denotes the adjoint operator of T_k .

Proof:

Since $\chi \in L_2$, the right-hand side follows from

$$\|T_k^* : L_2 \rightarrow \mathcal{H}^r\| \leq c,$$

which can be derived from inequality (2) using a symmetry argument.

The left-hand side can be shown considering kernels $k \in \mathcal{H}^{r,r}$ of the form

$$k(s, t) = e_{i_0}(s)f(t),$$

where $f \in \mathcal{H}^r$ and $i_0 \in \mathbb{Z}^d$ is a fixed index such that $(e_{i_0}, \chi) \neq 0$. □

Lemma 3

$$(i) \exists c_1 > 0 : \{(I - T_k)^{-1}T_k : k \in K_0\} \subseteq \{T_h : h \in c_1 \cdot B_{\mathcal{H}^{r,r}}\}$$

$$(ii) \forall \delta > 0 \exists c_2 > 0 : \{T_h : h \in c_2 \cdot B_{\mathcal{H}^{r,r}}\} \subseteq \{(I - T_k)^{-1}T_k : k \in \delta \cdot B_{\mathcal{H}^{r,r}}\}$$

Proof:

To prove the first statement, the function $k_j \in L_2(G)$ ($j \in \mathbb{Z}^d$) is defined by the relation

$$\hat{k}_j(i) = \hat{k}(i, j) \quad (i \in \mathbb{Z}^d).$$

Then

$$k_j(s) = T_k e_j = \int_G k(s, t) e_j(t) dt. \quad (3)$$

Notice that

$$\begin{aligned} \|k\|_{r,r}^2 &= \sum_{i,j \in \mathbb{Z}^d} |i|^{2r} |j|^{2r} \hat{k}(i, j)^2 \\ &= \sum_{j \in \mathbb{Z}^d} |j|^{2r} \sum_{i \in \mathbb{Z}^d} |i|^{2r} \hat{k}(i, j)^2 \\ &= \sum_{j \in \mathbb{Z}^d} |j|^{2r} \|k_j\|_r^2. \end{aligned}$$

Given $k \in K_0$, let $h \in L_2(G^2)$ be the function defined by

$$(I - T_k)^{-1}T_k = T_h.$$

Combining this with (3) yields

$$h_j = T_h e_j = (I - T_k)^{-1}T_k e_j = (I - T_k)^{-1}k_j$$

and so

$$\begin{aligned}\|h\|_{r,r}^2 &= \sum_{j \in \mathbb{Z}^d} |j|^{2r} \|h_j\|_r^2 \\ &\prec \sum_{j \in \mathbb{Z}^d} |j|^{2r} \|k_j\|_r^2 \\ &= \|k\|_{r,r}^2.\end{aligned}$$

This implies the first statement.

To prove the second one, we take $h \in c_2 B_{\mathcal{H}^r}$ and define $k \in L_2(G^2)$ by

$$T_k = T_h(I + T_h)^{-1}.$$

Then

$$T_h = (I - T_k)^{-1} T_k$$

and for a well-chosen c_2 the same argument as above gives $k \in \delta B_{\mathcal{H}^r}$. \square

Now, we shall estimate the Gelfand numbers of Φ to show the lower bound. For this end, operators

$$\begin{aligned}W &: l_2(\mathbb{Z}^{2d}) \rightarrow \mathcal{H}^{r,r}(G^2), \\ V &: L(\mathcal{H}^r(G), \mathcal{H}^{-r}(G)) \rightarrow l_\infty(\mathbb{Z}^{2d}),\end{aligned}$$

are constructed and composed with Φ to a diagonal operator

$$\begin{aligned}D &: l_2(\mathbb{Z}^{2d}) \rightarrow l_\infty(\mathbb{Z}^{2d}), \\ D &= V\Phi W.\end{aligned}\tag{4}$$

Then the Gelfand numbers of Φ are estimated by the Gelfand numbers of D , which are much easier to determine.

Let $\{b_{ij}\}_{i,j \in \mathbb{Z}^d}$ be the unit vector basis of $l_2(\mathbb{Z}^{2d})$ and define

$$\begin{aligned}Wb_{ij} &= |i|^{-r} |j|^{-r} e_i(s) \cdot e_j(t), \\ V(T) &= (\eta_{ij})_{i,j \in \mathbb{Z}^d},\end{aligned}$$

where

$$\eta_{ij} = |i|^{-r} |j|^{-r} (Te_j, e_i).$$

Hence, the operator W is an isometry, so $\|W\| = 1$, and the operator V is an injection with $\|V\| \leq 1$. The operator D , defined by (4), has the following form:

$$\begin{aligned}Db_{ij} &= \xi_{ij} \cdot b_{ij}, \\ \xi_{ij} &= |i|^{-2r} |j|^{-2r}.\end{aligned}$$

Now we define a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \dots$ by

$$\begin{aligned}\lambda_n &= \inf \{ \varepsilon : |\{(i, j) : \xi_{ij} \geq \varepsilon\}| < n \} \\ &= \max_{\substack{A \subset \mathbb{Z}^{2d} \\ |A|=n}} \min \{ \xi_{ij} : (i, j) \in A \}.\end{aligned}$$

For our purpose, we need the set A_n , which is defined by

$$\begin{aligned}A_n &= \{(i, j) : \xi_{ij} \geq n^{-4r}\} \\ &= \{(i, j) : |j|^{-2r} |i|^{-2r} \geq n^{-4r}\} \\ &= \{(i, j) : |i| \cdot |j| \leq n^2\}.\end{aligned}$$

From the definition of $\{\lambda_n\}_{n \in \mathbb{N}}$ follows immediately:

$$\lambda_{|A_n|} \geq n^{-4r} > \lambda_{|A_n|+1}.$$

Moreover, as can be seen easily, the cardinality of the set A_n is

$$|A_n| \asymp n^2 \log^{2d-1} n.$$

Hence,

$$\lambda_{[n^2 \log^{2d-1} n]} \asymp n^{-4r}$$

which can be transformed in a standard way into

$$\lambda_N \asymp N^{-2r} (\log N)^{2r(2d-1)}.$$

From the Theorem (11.11.7) in [Pie78] follows

$$\begin{aligned}c_n(D : l_2(\mathbb{Z}^{2d}) \rightarrow l_\infty(\mathbb{Z}^{2d})) &\asymp \lambda_n \\ &\asymp n^{-2r} (\log n)^{2r(2d-1)}.\end{aligned}\tag{5}$$

Furthermore, by basic properties of Gelfand numbers,

$$c_l(D) \leq \|V\| c_l(\Phi) \|W\|$$

and thus

$$c_l(\Phi) \geq \frac{c_l(D)}{\|V\| \cdot \|W\|}$$

for all $l \in \mathbb{N}$. Finally, using (5) we get

$$c_n(\Phi) \succ n^{-2r} (\log n)^{2r(2d-1)}$$

for all $n \in \mathbb{N}$. This proves the lower bound of the Theorem.

4 Proof of the upper bound

The upper bound is proved by providing a concrete algorithm and estimating the number of required information functionals, the error, and the complexity of the method. Our algorithm constitutes a modification of the algorithm used in [FH94]. The structure of the set of Fourier coefficients taking part in the approximation of the kernel is essentially changed according to the different function spaces considered. Hence the basic index sets are to be redefined and new norm estimates have to be derived.

Let $k \in K_0$, $f \in F_0$ be given. Fix $n \in \mathbb{N}$ and put

$$\begin{aligned} A_n &= \{i \in \mathbb{Z}^d : |i| \leq n^{\frac{1}{3}}\} \\ B_n &= \{i \in \mathbb{Z}^d : |i| \leq n^2\} \\ C_n &= \{(i, j) \in \mathbb{Z}^{2d} : \max(|i|, |j|) \leq n^{\frac{1}{3}}\} \\ D_n &= \{(i, j) \in \mathbb{Z}^{2d} : |i| \cdot |j| \leq n^2\} \end{aligned}$$

Remember that $|i|$ is defined by $|i| = \max(1, |i_1|) \cdot \max(1, |i_2|) \cdot \dots \cdot \max(1, |i_d|)$ in contrast to [FH94]. So the cardinalities of these sets are

$$\begin{aligned} |A_n| &\asymp n^{\frac{1}{3}} \log^{d-1} n \\ |B_n| &\asymp n^2 \log^{d-1} n \\ |C_n| &\asymp n^{\frac{2}{3}} \log^{2(d-1)} n \\ |D_n| &\asymp n^2 \log^{2d-1} n \end{aligned}$$

Let us shortly recall the idea of the algorithm. The projections g, h and f_0 of k and f , respectively, are defined by

$$\begin{aligned} f_0 \in \mathcal{H}^r(G) &: \hat{f}_0(i) = \begin{cases} \hat{f}(i) & \text{if } i \in B_n \\ 0 & \text{otherwise,} \end{cases} \\ g \in \mathcal{H}^{r,r}(G^2) &: \hat{g}(i, j) = \begin{cases} \hat{k}(i, j) & \text{if } (i, j) \in C_n \\ 0 & \text{otherwise,} \end{cases} \\ h \in \mathcal{H}^{r,r}(G^2) &: \hat{h}(i, j) = \begin{cases} \hat{k}(i, j) & \text{if } (i, j) \in D_n \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (6)$$

First, the algorithm computes an approximation v to $(I - T_k)^{-1} f$ by a two-grid iteration, setting $v_0 = 0$ and determining v_l ($l = 1, \dots, l_0$) from

$$(I - T_g) v_l = f_0 + (T_h - T_g) v_{l-1}. \quad (7)$$

Then, taking $v = v_{l_0}$, the final approximation is calculated by

$$\psi_n(k, f) = (f, \chi) + (v, T_k^* \chi). \quad (8)$$

In this case, $l_0 = 12$ iterations are sufficient. The unique solvability of (7) follows from Lemma 6(ii) below. In terms of Fourier coefficients, the algorithm looks like the following ($l = 1, \dots, l_0$):

$$\hat{v}_l(i) - \sum_{j \in A_n} \hat{k}(i, j) \hat{v}_l(j) = \hat{f}(i) + \sum_{j: (i, j) \in D_n \setminus C_n} \hat{k}(i, j) \hat{v}_{l-1}(j) \quad (9)$$

for $i \in A_n$, and

$$\hat{v}_l(i) = \hat{f}(i) + \sum_{j: (i, j) \in D_n} \hat{k}(i, j) \hat{v}_{l-1}(j) \quad (10)$$

for $i \in B_n \setminus A_n$. Finally, (8) turns into

$$\psi_n(k, f) = (f, \chi) + \sum_{j \in B_n} \hat{k}(\chi, j) \hat{v}(j) \quad (11)$$

where

$$\hat{k}(\chi, j) = (k, \chi \otimes e_j) = \int \int_{G^2} k(s, t) \chi(s) e_j(t) ds dt.$$

Since for $|i| > n^2$ both $\hat{f}_0(i) = 0$ and $\hat{g}(i, j) = \hat{h}(i, j) = 0$ ($j \in \mathbb{Z}^d$), from (7) follows $\hat{v}_l(i) = 0$ for $i \notin B_n$. Hence, the system (9) and (10) is equivalent to equation (7). Note that the system of linear equations (9) is to be solved only for a comparatively small set of unknowns $\hat{v}_l(i)$ ($i \in A_n$), whereas the main part of Fourier coefficients $\hat{v}_l(i)$ ($i \in B_n \setminus A_n$) can be computed directly from (10). So the number of operations needed for the solution of the system (9) should not exceed the number of operations required for the computation of the remaining Fourier coefficients in (10).

Now, we shall estimate the number of information functionals and of operations needed in the computational process (7) and (8). The information about the functions k, f , required in (9), (10), and (11), can be collected in the information operator $N = (N_1, N_2)$:

$$\begin{aligned} N_1 k &= \left(\left(\hat{k}(i, j) \right)_{(i, j) \in D_n}, \left(\hat{k}(\chi, j) \right)_{j \in B_n} \right) \\ N_2 f &= \left(\left(\hat{f}(i) \right)_{i \in B_n}, (f, \chi) \right). \end{aligned}$$

Consequently, a total of $|D_n| = O(n^2 \log^{2d-1} n)$ information functionals is needed. The solution of the system of linear equations (9), e.g. by Gaussian method, requires $O(|A_n|^3) = O(n \log^{3(d-1)} n)$ operations; the computation of the remaining Fourier coefficients in (10) can be performed in $O(|D_n|) = O(n^2 \log^{2d-1} n)$ operations. The final approximation (11) requires $O(|B_n|) = O(n^2 \log^{d-1} n)$ operations. So we have a total of $O(n^2 \log^{2d-1} n)$ operations and $O(n^2 \log^{2d-1} n)$ information functionals necessary for the computation of (7) and (8).

The algorithm φ is simply defined by

$$\varphi(N_1 k, N_2 f) = \psi_n(k, f)$$

and it will be of optimal order, if its error satisfies

$$e(S_\chi, N, \varphi) \prec n^{-4r} \quad (\text{card}(N) = [n^2 \log^{2d-1} n]).$$

Before we start the proof of this error bound, we rewrite the algorithm in a more convenient form. Let

$$\begin{aligned} Y &= (I - T_g)^{-1}(T_h - T_g) \\ Z &= (I - T_g)^{-1}(T_k - T_h) \\ w &= (I - T_k)^{-1}f_0. \end{aligned} \tag{12}$$

Then

$$u - w = (I - T_k)^{-1}(f - f_0). \tag{13}$$

From (12) and (7) we derive

$$\begin{aligned} (I - T_g)w &= f_0 + (T_k - T_g)w, \\ (I - T_g)(w - v_l) &= (T_h - T_g)(w - v_{l-1}) + (T_k - T_h)w. \end{aligned}$$

This implies

$$w - v_l = Y(w - v_{l-1}) + Z w \quad (l = 1, \dots, l_0)$$

and so, for $l = l_0 = 12$ and $v = v_{l_0}$:

$$w - v = Y^{12} w + \sum_{l=0}^{11} Y^l Z w. \tag{14}$$

Now three lemmas will be formulated which hold for all $k \in K_0$, the projections $g, h \in K_0$ introduced in (6) and, if not mentioned otherwise, for any $n \in \mathbb{N}$. The constants involved are independent of k and n . These lemmas shall help us to analyze the behaviour of the operators Y and Z .

Lemma 4

- (i) $\|T_k - T_h : \mathcal{H}^r \rightarrow \mathcal{H}^{-r}\| \prec n^{-4r}$
- (ii) $\|T_k - T_g : L_2 \rightarrow L_2\| \prec n^{-\frac{r}{3}}$
- (iii) $\|T_h - T_g : L_2 \rightarrow L_2\| \prec n^{-\frac{r}{3}}$.

Proof:

In detail we shall give only the proof of the first statement. The other two can be shown in a similar way.

Let $f \in \mathcal{H}^r(G)$, $\|f\|_r \leq 1$, and define ξ_{ij}, η_j such that

$$\begin{aligned}\hat{k}(i, j) &= |i|^{-r} |j|^{-r} \xi_{ij}, \\ \hat{f}(j) &= |j|^{-r} \eta_j.\end{aligned}$$

Since $k \in K_0$, $f \in B_{\mathcal{H}^r}$, it follows that

$$\sum_{i, j \in \mathbb{Z}^d} \xi_{ij}^2 \leq \alpha^2, \quad \sum_{j \in \mathbb{Z}^d} \eta_j^2 \leq 1.$$

Now it turns out that

$$\begin{aligned}\|(T_k - T_h)f\|_{-r}^2 &= \sum_{i \in \mathbb{Z}^d} |i|^{-2r} \left(\sum_{j \in \mathbb{Z}^d} (\hat{k}(i, j) - \hat{h}(i, j)) \hat{f}(j) \right)^2 \\ &= \sum_{i \in \mathbb{Z}^d} |i|^{-2r} \left(\sum_{j: (i, j) \notin D_n} \hat{k}(i, j) \hat{f}(j) \right)^2 \\ &= \sum_{i \in \mathbb{Z}^d} |i|^{-2r} \left(\sum_{j: (i, j) \notin D_n} |i|^{-r} |j|^{-r} |j|^{-r} \xi_{ij} \eta_j \right)^2 \\ &= \sum_{i \in \mathbb{Z}^d} |i|^{-4r} \left(\sum_{j: (i, j) \notin D_n} |j|^{-2r} \xi_{ij} \eta_j \right)^2 \\ &\leq \max_{(i, j) \notin D_n} |i|^{-4r} |j|^{-4r} \cdot \sum_{i \in \mathbb{Z}^d} \left(\sum_{j \in \mathbb{Z}^d} \xi_{ij} \eta_j \right)^2 \\ &\leq n^{-8r} \alpha^2.\end{aligned}$$

This proves the first statement. \square

Lemma 5 For each $T = T_k, T_h, T_g$ the following estimates hold:

$$(i) \|T : L_2 \rightarrow \mathcal{H}^r\| \leq c$$

$$(ii) \|T : \mathcal{H}^{-r} \rightarrow L_2\| \leq c.$$

Proof:

For $T = T_k$ the first statement is already shown in the proof of Lemma 1. The proof for $T = T_h, T_g$ is similar.

(ii) follows from (i) by duality: If $\|T : L_2 \rightarrow \mathcal{H}^r\| \leq c$, then $\|T^* : \mathcal{H}^{-r} \rightarrow L_2\| \leq c$. \square

Lemma 6 *There are constants $c_1, c_2, c_3, c_4 > 0$ and $n_0 \in \mathbb{N}$ such that*

$$(i) \quad \begin{aligned} \|(I - T_k)^{-1} : \mathcal{H}^r \rightarrow \mathcal{H}^r\| &\leq c_1, \\ \|(I - T_k)^{-1} : \mathcal{H}^{-r} \rightarrow \mathcal{H}^{-r}\| &\leq c_2. \end{aligned}$$

$$(ii) \quad \text{For } n \geq n_0, \|(I - T_g)^{-1} : L_2 \rightarrow L_2\| \leq c_3.$$

$$(iii) \quad \text{For } n \geq n_0, \|(I - T_g)^{-1} : \mathcal{H}^{-r} \rightarrow \mathcal{H}^{-r}\| \leq c_4.$$

Proof:

The first statement of (i) follows from the relation

$$(I - T_k)^{-1} = I + T_k(I - T_k)^{-1}$$

and from Lemma 5(i). The second one is implied by duality.

To show the second part of the Lemma, we conclude from Lemma 4(ii), that

$$\forall n \geq n_0 : \|T_k - T_g : L_2 \rightarrow L_2\| \leq c \cdot n^{-\frac{1}{3}r} \leq \frac{1}{2\beta}.$$

Moreover, for $k \in K_0$ we have

$$\|(I - T_k)^{-1} : L_2 \rightarrow L_2\| \leq \beta.$$

Since

$$(I - T_g)^{-1} = (I + (I - T_k)^{-1}(T_k - T_g))^{-1}(I - T_k)^{-1},$$

we get

$$\|(I - T_g)^{-1} : L_2 \rightarrow L_2\| \leq 2\beta.$$

Using the lemmas 5(ii) and 6(ii) and the relation

$$(I - T_g)^{-1} = I + (I - T_g)^{-1}T_g,$$

we derive the third statement. □

Corollary 1 *For $n \geq n_0$:*

$$\begin{aligned} \|Y : L_2 \rightarrow L_2\| &\prec n^{-\frac{r}{3}} \\ \|Y : \mathcal{H}^{-r} \rightarrow \mathcal{H}^{-r}\| &\leq c \\ \|Z : \mathcal{H}^r \rightarrow \mathcal{H}^{-r}\| &\prec n^{-4r}. \end{aligned}$$

Now we are ready to accomplish the proof of the upper bound. It follows from the definition of f_0 , that

$$\|f - f_0\|_{-r} \leq c \cdot n^{-4r}.$$

Lemma 6(i) gives

$$\begin{aligned} \|u - w\|_{-r} &= \|(I - T_k)^{-1}(f - f_0)\|_{-r} \\ &\leq c \cdot \|f - f_0\|_{-r} \\ &\leq c \cdot n^{-4r}. \end{aligned} \tag{15}$$

Moreover,

$$\|w\|_r = \|(I - T_k)^{-1}f_0\|_r \leq c \cdot \|f_0\|_r \leq c.$$

From Corollary 1 and equation (14) we deduce

$$\begin{aligned} \|w - v\|_{-r} &\leq \|Y^{12}w\|_0 + \left\| \sum_{l=0}^{11} Y^l : \mathcal{H}^{-r} \rightarrow \mathcal{H}^{-r} \right\| \cdot \|Z : \mathcal{H}^r \rightarrow \mathcal{H}^{-r}\| \cdot \|w\|_r \\ &\leq c \cdot n^{-\frac{12}{3}r} + c \cdot n^{-4r} \\ &\leq c \cdot n^{-4r}. \end{aligned}$$

Together with (15) this gives

$$\|u - v\|_{-r} \leq c \cdot n^{-4r}.$$

Finally, we get

$$\begin{aligned} |S_\chi(k, f) - \psi_n(k, f)| &= |((I - T_k)^{-1}f, \chi) - (f, \chi) - (v, T_k^* \chi)| \\ &= |(f, \chi) + (T_k(I - T_k)^{-1}f, \chi) - (f, \chi) - (v, T_k^* \chi)| \\ &= |((I - T_k)^{-1}f, T_k^* \chi) - (v, T_k^* \chi)| \\ &= |(u - v, T_k^* \chi)| \\ &\leq \|u - v\|_{-r} \|T_k^* \chi\|_r \\ &\leq c \cdot \|u - v\|_{-r} \\ &\leq c \cdot n^{-4r}. \end{aligned}$$

This completes the proof of the Theorem.

5 Summary

In the present paper, the complexity of local solution of Fredholm integral equations for a Sobolev class of multivariate periodic functions with dominating mixed derivative is discussed. Matching upper and lower bounds of order $O(n^{-2r} \log^{2r(2d-1)} n)$ are derived. Consequently, the stated problem is tractable in d , i.e. the complexity does not increase exponentially with the dimension [Woz93].

To prove the lower bound, an s -number technique was used, which was applied earlier to two other special classes of functions. So this method seems to be a powerful means for the estimation of the radius of information. Actually, we are trying to find more general conditions, which can guarantee the applicability of this technique to a wider class of problems.

The upper bound was shown constructing an implementable algorithm of optimal order, based on a hyperbolic cross approximation of the kernel. Usually, high-dimensional problems are the domain of Monte-Carlo algorithms. It would be interesting to compare our deterministic algorithm with stochastic ones. Numerical experiments will be reported in a forthcoming paper.

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