Ramsey Numbers of K_m versus (n,k)-graphs and the Local Density of Graphs not Containing a K_m

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Abstract

In this paper generalized Ramsey numbers of complete graphs K_m versus the set $\langle ,n,k\rangle$ of (n,k)-graphs are investigated. The value of $r(K_m, \langle n,k\rangle)$ is given in general for (relative to n) values of k small compared to n using a correlation with Turán numbers.

These generalized Ramsey numbers can be used to determine the local densities of graphs not containing a subgraph K_m.

1 Introduction

Let m, l, n and k be positive integers with $0 \leq l \leq \binom{m}{2}$ and $0 \leq k \leq \binom{n}{2}$ and let $\langle n,k\rangle$ denote the set of all (n,k)-graphs, i.e. the set of all graphs with n vertices and k edges. The Ramsey number $r(\langle m,l\rangle, \langle n,k\rangle)$ is defined as the smallest integer p such that in every red-green coloring of the edges of the complete graph K_p a green (m,l)-graph or a red (n,k)-graph occurs, i.e. a green graph with m vertices and l edges or a red graph with n vertices and k edges. Note that $r(\langle m,l\rangle, \langle n,k\rangle)$ is the classical Ramsey number $r(K_m, K_n)$ if $l = \binom{m}{2}$ and $k = \binom{n}{2}$.

Generalized Ramsey numbers $r(\langle m,l\rangle, \langle n,k\rangle)$ have been thoroughly investigated by many authors, an important reason being they lead to some insight in the behaviour of classical Ramsey numbers.

Particularly for small values of m and n several results have been obtained: In [1] the Ramsey numbers $r(\langle 4,l\rangle, \langle 5,k\rangle)$ have been given for all possible values (l,k) except for $(l,k) \in \{(6,9),(6,10)\}$. The two missing numbers can be found in [8] and [22]. The value of $r(\langle 5,l\rangle, \langle 5,k\rangle)$ has been determined in [15] for the special case that $l = k \leq 8$, in [5] for $l = k = 9$ and in [14] for all other possible values of (l,k) with $l \leq 6$ or $l = 7$ and $k \leq 9$. As a more general result, $r(\langle n,s\rangle, \langle n, \binom{n}{2} - s + k\rangle)$ has been given for $k \leq 3$ and $2 \leq s \leq \frac{1}{2}\binom{n}{2} + k$ in [16] and the values of $r(K_n, \langle n,k\rangle)$ have been given in [9] for $1 \leq k \leq n$.

The special case $m = 3$, i.e. the Ramsey numbers $r(K_3, \langle n, k\rangle)$, has drawn much attention. For $n \leq 8$ all values are known [3, 4, 10, 11, 12, 13, 17, 19, 21]. Moreover a more general result has been given in [19] which allows the determination of $r(K_3, \langle n, k\rangle)$ for arbitrary values of n and small values of k (compared to n).

A similar problem has been introduced in [7], where the *local density* of graphs not containing specific subgraphs is investigated:

Let $\alpha \in I\!R$ with $0 < \alpha < 1$ and let $\beta(\alpha, p) \in I\!R$ be the smallest positive number with property $(P1)$ or (more general) $(P2)$:

$(P1)$ If G is a graph with $p \geq 3$ vertices such that every subset of $\lfloor \alpha p \rfloor$ vertices spans more than $\beta(\alpha, p)\, p^2$ edges, then G contains a triangle.

$(P2)$ If G is a graph with $p \geq m$ vertices ($m \geq 3$) such that every subset of $\lfloor \alpha p \rfloor$ vertices spans more than $\beta(\alpha, p)\, p^2$ edges, then G contains a K_m.

To distinguish both cases we will denote $\beta(\alpha, p)$ by $\beta_{K_3}(\alpha, p)$ or $\beta_{K_m}(\alpha, p)$, respectively. Here $\beta_{K_m}(\alpha, p)$ is called the *local density* of a K_m-free graph of order p. Furthermore the local density $\beta_{K_m}(\alpha)$ of a graph with arbitrary order is defined for $m \geq 3$:

$$\beta_{K_m}(\alpha) := \sup_{p \geq m} \beta_{K_m}(\alpha, p). \tag{1}$$

The connexion between the local density of a graph not containing a K_m and the generalized Ramsey numbers $r(K_m, \langle n, k\rangle)$ can be seen as follows:

$$\begin{aligned} r(K_m, \langle \lfloor \alpha p \rfloor, \tbinom{\lfloor \alpha p \rfloor}{2} - \beta_{K_m}(\alpha, p)\, p^2 + 1\rangle) &> p \qquad \text{and} \\ r(K_m, \langle \lfloor \alpha p \rfloor, \tbinom{\lfloor \alpha p \rfloor}{2} - \beta_{K_m}(\alpha, p)\, p^2\rangle) &\leq p. \end{aligned}$$

Thus if the values of the generalized Ramsey numbers $r(K_m, \langle n, k\rangle)$ are known for all values of n and k, $\beta_{K_m}(\alpha, p)$ can be evaluated for arbitrary values of α and p using the following transformation (2):

$$\beta_{K_m}(\alpha, p) = \min \left\{ \beta \ : \ r(K_m, \langle n, k\rangle) \leq p \text{ where } n = \lfloor \alpha p \rfloor \text{ and } k = \binom{\lfloor \alpha p \rfloor}{2} - \beta p^2 \right\}. \tag{2}$$

On the other hand, if $r(K_m, \langle n, k\rangle) \leq cn$ holds for a constant c, $r(K_m, \langle n, k\rangle)$ can be obtained if $\beta_{K_m}(\alpha, p)$ is known for all values of α and p:

$$r(K_m, \langle n, k\rangle) = \min \left\{ p \ : \ \lfloor \alpha p \rfloor = n \text{ and } \binom{n}{2} - \beta_{K_m}(\alpha, p) p^2 = k \right\}. \tag{3}$$

Some notations will be used in the following. A red-green coloring of K_p is called a $(K_m, \langle n, k\rangle)$-coloring, if it contains neither a green K_m nor a red (n, k)-graph. We use $V(G)$ to denote the set of vertices of G and define $N_g(v)$ and $N_r(v)$, for all $v \in V(G)$, to be the sets of green and red neighbours of v, respectively. The number of green (or red) edges incident to v is denoted by $g(v)$ (or $r(v)$). Δ_g (or Δ_r) is the maximal degree with respect to the green (or red) subgraph of G and $g(G)$ (or $r(G)$) is the number of green (or red) edges in G.

In the following we will determine the Ramsey numbers $r(K_m, \langle n,k\rangle)$ for $m \geq 3$ and (compared to $\binom{n}{2}$) small values of k. Sections 3 and 4 are devoted to generalizations of this result whereas consequences for the local density of K_m-free graphs are derived in Section 5.

2 Ramsey numbers $r(K_m, \langle n,k\rangle)$ with $n \geq m \geq 3$

There is a close relationship between Ramsey numbers $r(K_m, \langle n,k\rangle)$ for small k and special Turán graphs. After defining the Turán problem we will draw the connexion to the given Ramsey problem and deduce previously unknown values for $r(K_m, \langle n,k\rangle)$.

The Turán problem asks for the maximal number $t_G(n)$ of edges in a graph of order n not containing a subgraph isomorphic to G. A Turán graph for G and n is defined to be a graph of order n and size $t_g(n)$ that does not contain a subgraph G. Turán's Theorem [24] solves the problem in case of $G = K_m$. If $n \geq m \geq 3$ then

$$t_{K_m}(n) = \frac{m-2}{2(m-1)}(n^2 - \beta^2) + \binom{\beta}{2}, \tag{4}$$

where β is defined by

$$n = \alpha(m-1) + \beta, \quad \alpha \in I\!N_0, \quad 0 \leq \beta \leq m-2. \tag{5}$$

Furthermore, the complete $(m-1)$-partite graph $K_{n_1,\ldots,n_{m-1}} = T_{K_m}(n)$ where $n_1 = \cdots = n_\beta = \alpha + 1$ and $n_{\beta+1} = \cdots = n_{m-1} = \alpha$ is the only Turán graph for K_m and n.

A very simple connexion between $t_{K_m}(n)$ and $r(K_m, \langle n,k\rangle)$ can be derived for $k \leq \binom{n}{2} - t_{K_m}(n)$:

Theorem 1 *Let $n \geq m \geq 3$ be positive integers. Then*

$$r(K_m, \langle n,k\rangle) = n \qquad \textit{for} \qquad 0 \leq k \leq \binom{n}{2} - t_{K_m}(n).$$

This correlation will be extended in the following to higher values of k. For this purpose some further definitions are useful.

For $n \geq m \geq 3$ let $\{s_0, s_1, s_2, \ldots\}$, $s_0 < s_1 < s_2 < \cdots$, be the set of non-negative integers such that $n + s_i \pmod{m-1} \neq m-2$. Using the representation of n in (5) we obtain

$$s_i = i + \left\lfloor \frac{i+\beta}{m-2} \right\rfloor \qquad i \geq 0. \tag{6}$$

There are uniquely defined integers γ_i and δ_i such that

$$n + s_i =: \gamma_i(m-1) + \delta_i \qquad \gamma_i \in I\!N_0, \ \ 0 \le \delta_i \le m-3 \tag{7}$$

and for $s_i \le \gamma_i$ (which is true for $s_i \le \frac{n}{m}$) we define

$$k_i := \delta_i \binom{\gamma_i + 1}{2} + (m - \delta_i - 2)\binom{\gamma_i}{2} + \binom{\gamma_i - s_i}{2}. \tag{8}$$

Note that $k_0 = \binom{n}{2} - t_{K_m}(n)$ is the highest value of k such that $r(K_m, \langle n, k\rangle)$ is determined by Theorem 1. Moreover, for $i \ge 1$

$$k_i = \begin{cases} k_{i-1} + s_i; & i + \beta \not| \ m-2 \\ k_{i-1} + s_i - 1; & i + \beta \ | \ m-2 \end{cases} \tag{9}$$

which implies that

$$k_i = k_0 + \sum_{j=1}^{i} s_j - \left\lfloor \frac{i+\beta}{m-2} \right\rfloor \ge \binom{n}{2} - t_{K_m}(n) + \frac{i(i+1)}{2}. \tag{10}$$

It is easily checked that k_i can also be written as

$$k_i = \frac{1}{2}\left[-(m-1)\left\lfloor \frac{n+s_i}{m-1} \right\rfloor^2 + (2n - m + 1)\left\lfloor \frac{n+s_i}{m-1} \right\rfloor + s_i^2 + s_i \right]. \tag{11}$$

The following lower bound for $r(K_m, \langle n, k\rangle)$ is easily obtained using the Turán graphs T_{K_m}.

Theorem 2 *Let $m \ge n \ge 3$, $i \in I\!N_0$ and let s_i, k_i be defined by (6) and (8). Then*

$$r(K_m, \langle n, k\rangle) > n + s_i \qquad \text{if} \quad k > k_i \quad \text{and} \quad s_i \le \frac{n}{m}.$$

Proof: No green K_m occurs in the red-green coloring of K_{n+s_i} in which the green subgraph is isomorphic to the Turán graph $T_{K_m}(n+s_i)$. On the other hand it is easy to verify that the maximal number of red edges in a subgraph with n vertices is exactly k_i.

□

This lower bound is sharp for relative to $\binom{n}{2}$ small values of k.

Theorem 3 *Let $m \ge n \ge 3$, $i \in I\!N_0$ and let s_i, k_i be defined by (6) and (8). Then*

$$r(K_m, \langle n, k\rangle) = n + s_i + 1 \quad \text{if} \ \ k_i < k \le k_{i+1} \ \ \text{and} \ \ s_i \le s_{\max} := \sqrt{\frac{n+2}{m-1} - \frac{2m-3}{m-2}}.$$

Proof: Considering the lower bound for $r(K_m, \langle n,k\rangle)$ given in Theorem 2 it suffices to show that $r(K_m, \langle n,k\rangle) \leq n+s_i+1$ is true for $k \leq k_{i+1}$.
Assume that there exists a $(K_m, \langle n, k_{i+1}\rangle)$-coloring C of K_{n+s_i+1}. Using (7) we obtain that $n+s_i+1 = \gamma(m-1)+\delta$, where $\gamma=\gamma_i$ and $\delta = \delta_i + 1$, $1 \leq \delta \leq m-2$. It follows that $k_{i+1} = \delta\binom{\gamma+1}{2} + (m-\delta-2)\binom{\gamma}{2} + \binom{\gamma-s_i-1}{2}$ for each value of i.
Let $v \in V(K_{n+s_i+1})$ be a vertex with $g(v) = \Delta_g$. Furthermore let $H_1 := [N_g(v)]$ and $H_2 := [N_r(v)+v]$. Since there is no green subgraph $K_m \subset K_{n+s_i+1}$ in C, there is no green subgraph $K_{m-1} \subseteq H_1$.
We distinguish two cases:

1. $\Delta_g \geq n+s_i+1-\gamma$, i.e. $|V(H_2)| \leq \gamma$.

   If $\Delta_g \geq n$, any subgraph G with $|V(G)| = n$ of H_1 contains at most $t_{K_{m-1}}(n)$ green edges. Thus $r(G) \geq \binom{n}{2} - t_{K_{m-1}}(n) \geq k_{i+1}$.

   In the case that $\Delta_g < n$ any subgraph G containing H_1 and $n-\Delta_g$ arbitrary vertices from H_2 has at most $t_{K_{m-1}}(\Delta_g) + (n-\Delta_g)\Delta_g$ green edges where $n-\Delta_g \leq \gamma - s_i - 1$. Thus $g(G)$ is maximal (and $r(G)$ minimal) if Δ_g is minimal, i.e. if $\Delta_g = n+s_i+1-\gamma$. Therefore

   $$\begin{aligned} g(G) &\leq t_{K_{m-1}}(n+s_i+1-\gamma) + (n+s_i+1-\gamma)(s_i+1-\gamma) \\ &= \binom{n}{2} - \delta\binom{\gamma+1}{2} + (m-\delta-2)\binom{\gamma}{2} + \binom{\gamma-s_i-1}{2} \end{aligned}$$

   and

   $$r(G) \geq \delta\binom{\gamma+1}{2} + (m-\delta-2)\binom{\gamma}{2} + \binom{\gamma-s_i-1}{2} = k_{i+1}.$$

   Thus G contains a red (n, k_{i+1})-graph contradicting the assumption.

2. $\Delta_g \leq n+s_i-\gamma$, i.e. $|V(H_2)| > \gamma$.

   Construct a subgraph $G \subset K_{n+s_i+1}$ by successive deletion of s_i+1 vertices of maximal degree with respect to the green subgraph. As the number of green edges in the removed subgraph cannot exceed $t_{K_m}(s_i+1)$, the number of green edges in G cannot exceed the following upper bound:

   $$\begin{aligned} g(G) &\leq \frac{1}{2}n\Delta_g - \left(\frac{1}{2}(s_i+1)\Delta_g - t_{K_m}(s_i+1)\right) \\ &\leq \frac{1}{2}(n-s_i-1)(n+s_i-\gamma) + \frac{m-2}{2(m-1)}(s_i+1)^2. \end{aligned}$$

   G contains a red (n, k_{i+1})-graph contradicting the assumption if

   $$g(G) \leq \binom{n}{2} - k_{i+1} = \binom{n}{2} - \left[\delta\binom{\gamma+1}{2} + (m-\delta-2)\binom{\gamma}{2} + \binom{\gamma-s_i-1}{2}\right].$$

   Transformation of the above equations show that this is true if

   $$s_i^2 + \frac{4m-6}{m-2}s_i - \frac{m-1}{m-2}(m-1-\delta)\gamma + \frac{3m-4}{m-2} \leq 0.$$

Because of $\frac{m-1}{m-2} > 1$ and $m - 1 - \delta \geq 1$, this is satisfied if

$$s_i^2 + \frac{4m-6}{m-2}s_i - \frac{m-1}{m-2}\gamma + \frac{3m-4}{m-2} \leq 0.$$

With $\gamma = \lfloor \frac{n+s_i+1}{m-1} \rfloor$ it is easy to see that G contains a red (n, k_{i+1})-graph if $s_i \leq \sqrt{\frac{n+2}{m-1}} - \frac{2m-3}{m-2}$.

□

The following Lemma gives a bound for the values of k for which the Ramsey number $r(K_m, \langle n, k\rangle)$ can be determined using Theorem 3.

Lemma 1 *Theorem 3 can be applied for all positive integers m, n, k with $n \geq m \geq 3$ and $\binom{n}{2} - t_{K_m}(n) < k \leq k_{\max}$ where*

$$k_{\max} \geq \binom{n}{2} - t_{K_m}(n) \ + \frac{(m-2)^2}{2(m-1)^3}(n+2) - \frac{5(m-2)}{2(m-1)^{\frac{3}{2}}}\sqrt{n+2} + 3.$$

Proof: From the upper bound for s_i an upper bound for i can be derived. Using (10) a straightforward calculation leads to the given lower bound for $k_{\max}$.

□

Using the description of $r(K_m, \langle n, k\rangle)$ in Theorem 3 it is not possible to give $r(K_m, \langle n, k\rangle)$ explicitly. Anyway, given a value of k with $0 \leq k \leq k_{\max}$ the value of $r(K_m, \langle n, k\rangle)$ can easily be evaluated using (9) and (11) and Theorem 3.

3 Ramsey numbers $r(\langle m, l\rangle, \langle n, k\rangle)$

The results obtained in Section 2 strongly depend on the Turán graph $T_{K_m}(n)$. Theorems 2 and 3 use the special structure of this graph. They can thus be transferred to more general cases where the Turán graphs have a similar structure.
One generalization of the problem of Turán is that of finding the maximal number of edges $t_{\mathcal{G}}(n)$ in a graph with n vertices not containing a subgraph $G \in \mathcal{G}$ where $\mathcal{G}$ is a given set of graphs. A Turán graph for $\mathcal{G}$ and n is a graph of order n and size $t_{\mathcal{G}}(n)$ that does not contain a subgraph $G \in \mathcal{G}$. (Note that in this general formulation a Turán graph is not necessarily unique.) For the case that $\mathcal{G} = \langle m, \binom{m}{2} - \lambda\rangle$ with $n \geq m+1$ and $0 \leq \lambda \leq \frac{m-3}{2}$ it has been shown in [6] that the graph $T_{K_{m-\lambda}}(n)$ is the only Turán graph for $\mathcal{G}$ and n. Thus analogously to Theorems 1 to 3 the following statements can be proven:

Theorem 4 *Let $n, m \in \mathbb{N}$ and $\lambda \in \mathbb{N}_0$ with $n \geq m+1 \geq 4$ and $\lambda \leq \frac{m-3}{2}$. Then*

$$r(\langle m, \binom{m}{2} - \lambda\rangle, \langle n, k\rangle) = n \quad \text{if} \quad 0 \leq k \leq \binom{n}{2} - t_{K_{m-\lambda}}(n).$$

Theorem 5 *Let $n, m \in I\!N$ and $\lambda, i \in I\!N_0$ with $n \geq m+1 \geq 4$ and $\lambda \leq \frac{m-3}{2}$. Furthermore let s_i and k_i be defined as in (6) and (8), where m is replaced by $m-\lambda$ in (5)-(8). Then*

$$r(\langle m, \binom{m}{2} - \lambda\rangle, \langle n, k\rangle) > n + s_i \qquad \text{if} \quad k > k_i \quad \text{and} \quad s_i \leq \frac{n}{m-\lambda}.$$

For small values of k this bound is sharp.

Theorem 6 *Let $n, m \in I\!N$ and $\lambda \in I\!N_0$ with $n \geq m+1 \geq 4$, $\lambda \leq \frac{m-4}{2}$ and let $s_i, k_i \in I\!N_0$ as in Theorem 5. Then*

$$r(\langle m, \tbinom{m}{2} - \lambda\rangle, \langle n, k\rangle) = n + s_i + 1 \qquad \text{for} \qquad k_i < k \leq k_{i+1}$$

$$\text{and} \qquad s_i \leq s_{\max} := \sqrt{\tfrac{n+2}{m-\lambda-1}} - \tfrac{2(m-\lambda)-3}{m-\lambda-2}.$$

Proof: Theorem 6 can be proven analogously to Theorem 3. As the Turán number $t_{\langle m-1, \binom{m-1}{2} - \lambda\rangle}(\bullet)$ is needed, the value of λ must be bounded by $\lambda \leq \frac{m-4}{2}$.

□

4 Ramsey numbers $r(K_{i_1}, \ldots, K_{i_l}, \langle n, k\rangle)$

Let $t_{\mathcal{G}_1,\ldots,\mathcal{G}_l}(n)$ denote the maximal number of edges in a graph G with n vertices such that there exists an l-coloring of the edges of G not containing a monochromatic subgraph $G_i \in \mathcal{G}_i$ in color i for all $i = 1, \ldots, l$. This number can also be regarded as generalized Turán number and the graphs in question of size $t_{\mathcal{G}_1,\ldots,\mathcal{G}_l}(n)$ are also called Turán graphs. In the case that $\mathcal{G}_j = \{K_{i_j}\}$ ($j = 1, \ldots, l$) the only Turán graph for $\mathcal{G}_1, \ldots, \mathcal{G}_l$ and n is the "classical" Turán graph $T_{K_r}(n)$ where $r = r(K_{i_1}, \ldots, K_{i_l})$ is the Ramsey number of $K_{i_1}, \ldots, K_{i_l}$ (see [23]). Thus Theorems 1 to 3 can be transferred to Ramsey numbers $r(K_{i_1}, \ldots, K_{i_l}, \langle n, k\rangle)$ by replacing the value of m by the value of r.

Theorem 7 *Let $l, i_1, \ldots, i_l, r, n \in I\!N$ with $l \geq 2$, $r := r(K_{i_1}, \ldots, K_{i_l}) \geq 3$ and $n \geq r$. Then*

$$r(K_{i_1}, \ldots, K_{i_l}, \langle n, k\rangle) = n \quad \text{if} \quad 0 \leq k \leq \binom{n}{2} - t_{K_r}(n).$$

Theorem 8 *Let $l, i_1, \ldots, i_l, r, n \in I\!N$ and $i \in I\!N_0$ with $l \geq 2$, $r := r(K_{i_1}, \ldots, K_{i_l}) \geq 3$ and $n \geq r$. Furthermore let s_i and k_i be defined as in (6) and (8), where m is replaced by r in (5)-(8). Then*

$$r(K_{i_1}, \ldots, K_{i_l}, \langle n, k\rangle) > n + s_i \quad \text{if} \quad k > k_i \quad \text{and} \quad s_i \leq \frac{n}{r}.$$

Theorem 9 *Let $l, i_1, \ldots, i_l, r, n \in I\!N$ with $l \geq 2$, $r := r(K_{i_1}, \ldots, K_{i_l}) \geq 3$, $n \geq r$ and*

$$\sum_{j=1}^{l} \Big(r(K_{i_1}, \ldots, K_{i_{j-1}}, K_{i_j - 1}, K_{i_{j+1}}, \ldots, K_{i_l}) - 1 \Big) \leq r - 2. \tag{12}$$

Then

$$\begin{array}{lll} r(K_{i_1}, \ldots, K_{i_l}, \langle n, k \rangle) = n + s_i + 1 & \textit{for} & k_i < k \leq k_{i+1} \\ & \textit{and} & s_i \leq s_{\max} := \sqrt{\frac{n+2}{r-1}} - \frac{2r-3}{r-2}. \end{array}$$

Proof: The colors $1, \ldots, l$ are identified with the color "green" in the proof of Theorem 3. The vertex v in the first part of the proof is therefore incident to Δ_g edges of the colors $1, \ldots, l$, i.e. to Δ_g "green" edges. In the induced subgraph $N_j(v)$ not more than $t_{K_{i_1}, \ldots, K_{i_{j-1}}, K_{i_j - 1}, K_{i_{j+1}}, \ldots, K_{i_l}}(|V(N_j(v))|)$ edges are "green" ($j = 1, \ldots, l$). With (12) we have

$$\sum_{j=1}^{l} t_{K_{i_1}, \ldots, K_{i_{j-1}}, K_{i_j - 1}, K_{i_{j+1}}, \ldots, K_{i_l}}(|V(N_j(v))|) \leq t_{K_{r-1}}(\Delta_g)$$

and the proof of Theorem 9 can be completed analogously to the proof of Theorem 3.

□

Unfortunately the Ramsey numbers in inequality (12) are not known in general. Nevertheless, inequality (12) is true e.g. for (K_3, K_3), (K_3, K_3, K_3) and (K_4, K_4). The corresponding Ramsey numbers are $r(K_3, K_2) = 3$, $r(K_3, K_3) = r(K_3, K_3, K_2) = 6$, $r(K_3, K_3, K_3) = 17$, $r(K_4, K_3) = 7$ and $r(K_4, K_4) = 18$.

Corollary 1

$$\begin{array}{rcllll} r(K_3, K_3, \langle n, k \rangle) & = & n + s_i + 1 & \textit{for} \;\; k_i < k \leq k_{i+1} & \textit{and} & s_i \leq \frac{1}{\sqrt{5}}\sqrt{n+2} - \frac{7}{3}, \\ r(K_3, K_3, K_3, \langle n, k \rangle) & = & n + s_i + 1 & \textit{for} \;\; k_i < k \leq k_{i+1} & \textit{and} & s_i \leq \frac{1}{4}\sqrt{n+2} - \frac{31}{15}, \\ r(K_4, K_4, \langle n, k \rangle) & = & n + s_i + 1 & \textit{for} \;\; k_i < k \leq k_{i+1} & \textit{and} & s_i \leq \frac{1}{\sqrt{17}}\sqrt{n+2} - \frac{33}{16}. \end{array}$$

5 The local density of K_m-free graphs

In this section the connexion between Ramsey numbers $r(K_m, \langle n, k \rangle)$ and the local density $\beta_{K_m}(\alpha, p)$ of graphs not containing a subgraph K_m given in Section 1, equations (2) and (3) will be used to derive some new results concerning the local density of K_m-free graphs ($m \geq 3$).

In [7] the following conjecture about the values of $\beta_{K_3}(\alpha)$ has been raised:

Conjecture 1 ([7])

$$\beta_{K_3}(\alpha) = \begin{cases} (2\alpha-1)/4 & \textit{if} \quad 17/30 \le \alpha \le 1, \\ (5\alpha-2)/25 & \textit{if} \quad 53/120 \le \alpha \le 17/30. \end{cases}$$

For sufficiently large values of p and $0.648 \le \alpha \le 1$ Conjecture 1 has been proven in [7] and for $3/5 \le \alpha \le 1$ it has been proven in [20]. Conjecture 1 is not true if $0.442 \simeq 53/120 \le \alpha < 474/1000$ as has been shown in [2].

For the more general case that G does not contain a subgraph K_m, $m \ge 3$, the following upper bounds for $\beta_{K_m}(\alpha, p)$ have been given in [7]:

Theorem 10 ([7]) *Let G be a graph with p vertices not containing a subgraph K_m ($m \ge 3$) and let $0 < \alpha < 1$. Furthermore let δ be a positive real number such that $\delta(m-2) < 1$. Then the following upper bound for $\beta_{K_m}(\alpha)$ can be given for p sufficiently large:*

$$\begin{aligned} \beta_{K_m}(\alpha,p) &\le \max\{[(m-3)/(2m-4)]\alpha^2, \alpha^3/2\}, \\ \beta_{K_m}(\alpha,p) &\le (1/2)\alpha^{2+\delta}. \end{aligned}$$

Using the results for Ramsey numbers $r(K_m, \langle n,k\rangle)$ obtained in Section 2, the values of $\beta_{K_m}(\alpha, p)$ can now be determined exactly for some values of α and p. For this purpose equation (2) is applied:

Theorem 11 *Let $\alpha \in I\!R$ with $0 < \alpha \le 1$ and let $\beta_{K_m}(\alpha,p) \in I\!R$ be the smallest positive number with property ($P2$). Furthermore let αp be a positive integer such that $\alpha p \ge m$. Then*

$$\begin{aligned} \beta_{K_m}(\alpha,p) &= \alpha\left(1 - \frac{1}{p}\left\lfloor\frac{p+\sigma}{m-1}\right\rfloor\right) - \left(\frac{1}{2} - \frac{m-1}{2p^2}\left\lfloor\frac{p+\sigma}{m-1}\right\rfloor^2\right) \\ &\quad - \frac{1}{p}\left(\frac{1}{2} + (1-\alpha)\sigma - \frac{m-1}{2p}\left\lfloor\frac{p+\sigma}{m-1}\right\rfloor\right) - \frac{1}{2p^2}\left(\sigma^2+\sigma\right) \\ &\le \frac{m-2}{m-1}\left(\alpha - \frac{1}{2}\right) \end{aligned}$$

where $\sigma := \begin{cases} 1 & \textit{if} \quad m-1 \mid p+1 \\ 0 & \textit{else} \end{cases}$

and $\quad \alpha \ge \alpha_{\min} := 1 - \dfrac{\sqrt{(p+1)(m-1)+\frac{1}{4}} - m + \frac{1}{2}}{p(m-1)}.$

Proof: Let s_i, k_i be defined as in (6) and (8). The following statement is an easy consequence of (2) and Theorem 3:

$$\beta_{K_m}(\alpha,p) = \frac{1}{p^2}\left[\binom{\lfloor \alpha p\rfloor}{2} - k_{i+1}\right] \qquad \text{with} \quad i = \max\{i \ge 0 \;:\; p \ge n + s_i + 1\}. \tag{13}$$

We will distinguish two cases:

1. $m-1 \not| p$: From (6) we know that there exists s_i such that $p = n + s_i + 1$. Thus we have $s_i = (1-\alpha)p - 1$ and

$$s_{i+1} = \begin{cases} s_i + 1 = (1-\alpha)p & \text{if} \quad m-1 \not| p+1 \\ s_i + 2 = (1-\alpha)p + 1 & \text{else} \end{cases}$$

Using (11) and (13) and defining $s := s_{i+1}$, we get

$$\beta_{K_m}(\alpha, p) = \frac{1}{2p^2}\left(\alpha^2 p^2 - \alpha p + (m-1)\left\lfloor \frac{\alpha p + s}{m-1} \right\rfloor^2 - (2\alpha p - m + 1)\left\lfloor \frac{\alpha p + s}{m-1} \right\rfloor - s^2 - s\right)$$

and an easy calculation using $\sigma := s - (1-\alpha)p$ gives

$$\begin{aligned} \beta_{K_m}(\alpha, p) &= \alpha\left(1 - \frac{1}{p}\left\lfloor \frac{p+\sigma}{m-1} \right\rfloor\right) - \left(\frac{1}{2} - \frac{m-1}{2p^2}\left\lfloor \frac{p+\sigma}{m-1} \right\rfloor^2\right) \\ &\quad - \frac{1}{p}\left(\frac{1}{2} + (1-\alpha)\sigma - \frac{m-1}{2p}\left\lfloor \frac{p+\sigma}{m-1} \right\rfloor\right) - \frac{1}{2p^2}\left(\sigma^2 + \sigma\right) \\ &\leq \alpha\left(1 - \frac{1}{m-1}\right) - \frac{1}{2}\left(1 - \frac{1}{m-1}\right) \\ &= \frac{m-2}{m-1}\left(\alpha - \frac{1}{2}\right). \end{aligned}$$

2. $m-1 \mid p$: In this case there exists s_i with $p - 1 = n + s_i + 1$. Furthermore there is no $\tilde{s}_i$ such that $p = n + \tilde{s}_i + 1$. It follows that $s_i = (1-\alpha)p - 2$ and using $m - 1 \mid p$

$$s_{i+1} = s_i + 2 = (1-\alpha)p.$$

Using (11) and (13), an easy calculation analogous to case 1 completes the proof.

□

In the special case of $m = 3$ and $1 \geq \alpha \geq \alpha_{\min} = 1 - \frac{\sqrt{2(p+1)+0.25}-1.5}{2p}$ the result stated in Conjecture 1 and proven in [7] is also obtained by Theorem 11:

$$\beta_{K_3}(\alpha, p) = \begin{cases} \frac{\alpha}{2} - \frac{1}{4}, & \text{if } p \text{ even} \\ \frac{\alpha}{2} - \frac{1}{4} - \frac{1-\alpha}{2p} - \frac{1}{4p^2} & \text{if } p \text{ odd.} \end{cases}$$

A similar result with a better lower bound for α can also be found in [18].

Theorem 11 suggests the following conjecture for the local density of K_m-free graphs:

Conjecture 2 *Let $m \geq 3$ be a positive integer and $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$. Then*

$$\beta_{K_m}(\alpha) = \frac{m-2}{m-1}\left(\alpha - \frac{1}{2}\right)$$

for some values of α close to 1.

Analogous to the results about Ramsey numbers $r(K_m, \langle n, k\rangle)$ given in Section 2, the more general concepts for Ramsey numbers $r(\langle m, l\rangle, \langle n, k\rangle)$ described in Section 3 can be transferred to the concept of local densities. For this purpose let $\mathcal{G}$ be a given set of graphs and let $\beta_{\mathcal{G}}(\alpha, p) \in \mathbb{R}$ be defined as the smallest positive number with the following property:

$(P3)$ If G is a graph with $p \geq 3$ vertices such that every subset of $\lfloor \alpha p \rfloor$ vertices spans more than $\beta_{\mathcal{G}}(\alpha, p)\, p^2$ edges, then G contains a subgraph $G_i \in \mathcal{G}$.

In the case that $\mathcal{G} = \langle m, l\rangle$, i.e. $\mathcal{G}$ equals the set of all graphs with m vertices and l edges, the value of $\beta_{\mathcal{G}}(\alpha, p)$ can be interpreted as the local density of a graph G whose local density with respect to subgraphs of order m does not exceed $l-1$.

Theorem 12 *Let $\mathcal{G}$ be the set of all graphs with $m \geq 3$ vertices and $\binom{m}{2} - \lambda$ edges, where $\lambda \in \mathbb{N}_0$ and $\lambda \leq \frac{m-4}{2}$, i.e. $\mathcal{G} := \langle m, \binom{m}{2} - \lambda\rangle$.*
Let $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$ and let $\beta_{\mathcal{G}}(\alpha, p) \in \mathbb{R}$ be the smallest positive number with property $(P3)$. Furthermore let αp be a positive integer such that $\alpha p \geq m$. Then

$$\begin{aligned}
\beta_{\mathcal{G}}(\alpha, p) &= \alpha\left(1 - \frac{1}{p}\left\lfloor\frac{p+\sigma}{m-\lambda-1}\right\rfloor\right) - \left(\frac{1}{2} - \frac{m-\lambda-1}{2p^2}\left\lfloor\frac{p+\sigma}{m-\lambda-1}\right\rfloor^2\right) \\
&\quad - \frac{1}{p}\left(\frac{1}{2} + (1-\alpha)\sigma - \frac{m-\lambda-1}{2p}\left\lfloor\frac{p+\sigma}{m-\lambda-1}\right\rfloor\right) - \frac{1}{2p^2}\left(\sigma^2 + \sigma\right) \\
&\leq \frac{m-\lambda-2}{m-\lambda-1}\left(\alpha - \frac{1}{2}\right)
\end{aligned}$$

$$\textit{where } \sigma := \begin{cases} 1 & \textit{if } \; m-\lambda-1 \mid p+1 \\ 0 & \textit{else} \end{cases}$$

$$\textit{and} \quad \alpha \geq \alpha_{\min} := 1 - \frac{\sqrt{(p+1)(m-\lambda-1)+\frac{1}{4}} - m - \lambda + \frac{1}{2}}{p(m-\lambda-1)}.$$

Proof: Theorem 12 can be proven analogously to Theorem 11 by using Theorem 6 and replacing m by $m - \lambda$.

$\square$

Thus Conjecture 2 can be generalized:

Conjecture 3 *Let $\mathcal{G} := \langle m, \binom{m}{2} - \lambda\rangle$ be the set of graphs with $m \geq 3$ vertices and $\binom{m}{2} - \lambda$ edges, where $\lambda \in \mathbb{N}_0$ and $\lambda \leq \frac{m-4}{2}$. Furthermore, let $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$. Then*

$$\beta_{\mathcal{G}}(\alpha) = \frac{m-\lambda-2}{m-\lambda-1}\left(\alpha - \frac{1}{2}\right)$$

for some values of α close to 1.

6 Conclusions

An interesting behaviour of the values of $r(K_m, \langle n, k\rangle)$ depending on k has been proven in this paper. Starting with $k = \binom{n}{2} - t_{K_m}(n)$ the values of $r(K_m, \langle n, k\rangle)$ grow rapidly whereas with increasing values of k (and therefore s_i) the slope of $r(K_m, \langle n, k\rangle)$ decreases again.

At least one important question remains open: There must exist a point of inflection, i.e. a value of k for which the slope of $r(K_m, \langle n, k\rangle)$ starts to increase again. The known lower bounds for $r(K_m, K_n)$ show that at least one point like this must exist.
Furthermore the results obtained for $r(K_m, \langle n, k\rangle)$ have been transferred to the concept of the local density $\beta_{K_m}(\alpha, p)$ of K_m-free graphs, for which some previously unknown values have been determined.

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