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Abstract. *The CAD/CAM-based design of free-form surfaces is the beginning of a chain of operations, which ends with the numerically controlled (NC-) production of the designed object. During this process the shape control is an important step to amount efficiency. Several surface interrogation methods already exist to analyze curvature and continuity behaviour of the shape. This paper deals with a new aspect of shape control: the stability of surfaces with respect to infinitesimal bendings. Each infinitesimal bending of a surface determines a so called instability surface, which is used for the stability investigations. The kinematic meaning of this instability surface will be discussed and we present algorithms to calculate it.*

§1. Introduction

The design of free-form surfaces in CAD/CAM technologies is the beginning of a chain of operations, which ends with the numerically controlled (NC-) production of the designed object. The design of free form surfaces is often combined with certain conditions or constraints: Some points as output of a scanning process have to be interpolated or have to be approximated. There are boundary conditions, e.g. given boundary curves or given tangents. Continuity conditions, such as C^1, C^2 or curvature continuities, are often necessary. Furthermore, another class of surface criteria exists, the so-called minimization criteria, to minimize some energy functionals.

During the CAD/CAM design process the quality control of the shape is an important step which amounts the efficiency. Many well known surface interrogation methods like focal surfaces, orthotomics or reflection lines [1] are used to analyze continuity or curvature behaviour and to detect aesthetically unwanted behaviour of the shape as little dents or flat points.

A new aspect of shape control is the stability of surfaces with respect to infinitesimal bendings. There are two possibilities to define the stability of a surface. The first one relates the stability with the strength of the surface. In this case the material the surface is manufactured from plays the important role. The second one takes the stability as a property of the surface itself, as it is done with the bending energy of a surface.

Our stability investigations belong to the second case. Only the shape of the surface decides on the stability. These investigations are based on *infinitesimal bendings*. An infinitesimal bending of a surface is a small continuous deformation of this surface without stretching it. In other words, infinitesimal bendings are deformations which keep the length of any arbitrary surface curve unchanged in first order. Deforming a surface means that it can either be moved as a rigid body or it can change its shape, i.e. it can bend. Surfaces which don't allow bendings are called rigid surfaces. A classical theorem of infinitesimal bendings says that all closed convex analytic surfaces which are connected, are rigid. The first proof was published in 1900 by H. Liebmann [2].

Open surfaces are generally not rigid. They are nevertheless more or less stable, i.e. they can be bended more or less. To record quantitatively how a surface is likely to bend is possible with the *instability surface* of an infinitesimal bending. A stable rigid surface only allows trivial infinitesimal bendings, i.e. it can only be moved as a rigid surface. In this case the instability surface degenerates to a single point.

The following paragraph of this paper gives a short introduction in the theory of infinitesimal bendings. We will see that with each infinitesimal bending a rotation vector field is associated. These rotation vectors taken as radius vectors from the origin describe a surface, the so called *instability surface*. Each radius vector of the instability surface is the rotation axis of the rotation of the corresponding surface element in the moment of the deformation.

This kinematic meaning of the instability surface is the subject of paragraph 3. Because our stability investigations are based on the instability surface, we need to calculate it with an appropriate approximation.

In paragraph 4 we present two numerical approaches to determine the rotation vectors of a surface under infinitesimal bendings. The first one is a finite differences approach. The idea of the finite differences is the discretization of the problem and it consists of a linear approximation of the partial derivatives. The second algorithm is based on a least-square fitting and a B-Spline representation for the instability surface. Both methods have advantages and disadvantages which are then discussed.

This paper closes with a visualization of these results in order to compare the stability of surfaces.

§2. Introduction to the theory of infinitesimal bendings

In this chapter some fundamentals of infinitesimal bendings are presented. A general survey is given in [3].

Parametric surfaces are represented as vector valued functions of class C^3

$$\begin{aligned} X : G &\to \mathbb{R}^3 \\ (u,w) &\mapsto X(u,w) \end{aligned} \tag{2.1}$$

where G is a connected domain of $\mathbb{R}^2$. X is assumed to be regular.

Definition (2.2): *A one-parameter family $\{X_t\}$ of mappings $X_t : G \to \mathbb{R}^3$ with $t \in I := [0,a)$, $a > 0$ and $X_0 = X$, is called* **deformation** *of X.*

Remark:

Instead of a family of mappings we can see the deformation as a mapping of three unknowns

$$\begin{aligned} \widetilde{X} : G \times I &\to \mathbb{R}^3 \\ (u,w,t) &\mapsto X_t(u,w) \end{aligned}$$

which allows to look at continuity and derivatives with respect to t.

We assume now that the surface X is contained in a **continuous** family of surfaces $\{X_t\}_{t\geq 0}$.

Definition (2.3): *Given a deformation*

$$\widetilde{X}(u,w,\varepsilon) := X(u,w) + \varepsilon Z(u,w)\,, \quad \varepsilon \in \mathbb{R} \tag{2.4}$$

where Z is a vector field of class $C^3(G)$. Z is called **deformation vector field.**

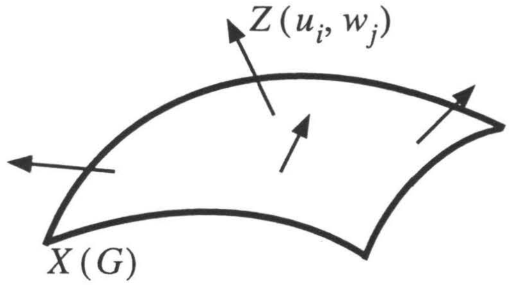


Figure 1: Deformation of the surface X

The basic concept of infinitesimal deformations is to neglect all infinitesimal small quantities of the surface X_ε of order two or higher in ε. At the same time we are only interested in bendings.

Definition (2.5): *The deformation (2.4) is called* **infinitesimal bending of first order** *of the surface X, if the length of any arbitrary smooth curve on the surface keeps unchanged in first order in ε, i.e. $L(c_\varepsilon) = L(c) + o(\varepsilon)$, where c is an arbitrary smooth surface curve, c_ε the deformed curve and L the length of the curves.*

To give some properties of infinitesimal bendings, we need the following definition of the first variation of a function f:

Definition (2.6): *Let $\{X_t\}_{t\in I}$ $(I \subset \mathbb{R})$ be a deformation. Let $\{f_t\}_{t\in I}$ be a family of functions f_t defined on G such that $\left(\begin{smallmatrix} f\ :\ G\times I \to \mathbb{R} \\ (u,w,t)\mapsto f_t(u,w)\end{smallmatrix}\right)$ is continuous and has*

continuous partial derivatives.
The function δf defined on G by

$$\delta f := \frac{\partial f}{\partial t}|_{t=0}$$

is called the **1. variation of** f.

This definition enables us to characterize the infinitesimal bendings with the following theorem :

Theorem (2.7): *Let Z be a deformation vector field of the surface X. Let $\tilde{g}_{ij}$ be the coefficients of the first fundamental form of the deformation $\widetilde{X} = X + \varepsilon Z$ and denote the first partial derivatives of X and Z by $X_u := \partial X/\partial u$ and $Z_u := \partial Z/\partial u$. The following statements are equivalent:*

i) Z *defines an infinitesimal bending of first order of* X

ii) $\delta \tilde{g}_{ij} = 0 \quad i, j = 1, 2$ *(2.8)*

iii) $< X_u, Z_u >= 0$

$< X_w, Z_w >= 0$ *(2.9)*

$< X_u, Z_w > + < X_w, Z_u >= 0$

where $<\ ,\ >$ denotes the dot product.

Proof: see Efimov [3]

Remark:

- Equation (2.9) is also called the differential equation of the infinitesimal bendings.
- Equation (2.8) means that the length of any surface curve doesn't change in first order in ε during the deformation, because the first fundamental measures distances on the surface [4].

§3. Rotation vector field - Stable rigid surfaces

An essential theorem for the stability investigations is presented in the beginning of this paragraph. Furthermore the definition and kinetic meaning of the rotation vector field is given.

Theorem (3.1): - Existence Theorem - *If the deformation vector field Z verifies the three equations (2.9), then there exists an unique vector field Y with the following properties:*

$$[Y, X_u] = Z_u \quad \text{and} \quad [Y, X_w] = Z_w \,, \tag{3.2}$$

where [,] denotes the vector product.

Proof: see Efimov [3]

Definition (3.3): *The vector field $Y(u, w)$ in theorem (3.1) is called* **rotation vector field.**

Kinematic meaning of Y:
The kinetic meaning of the rotation vector field is given by its name. The next proposition precises this meaning.

Proposition (3.4): *$\varepsilon Y(u, w)$ is the rotation vector field of the infinitesimal bending of the surface $X(u, w)$ into the surface $X_\varepsilon(u, w)$.*
The direction of the rotation vector is the axis of the rotation of the surface element in the point (u, w) during the bending.
The length and the orientation of the rotation vector determine the angle of rotation, except for quantities of higher order.

Proof:
Equation (3.2) in differentials has the form $[Y, dX] = dZ$, and a surface element of the deformation surface $X_\varepsilon(u, w)$ is given by

$$dX_\varepsilon = dX + \varepsilon \, [Y, dX] \, . \tag{3.5}$$

Let now

- $\boldsymbol{e}$ be an unit vector ($\boldsymbol{e} = \frac{\boldsymbol{Y}}{\|\boldsymbol{Y}\|}$), and
- α be the rotation angle in positive direction, and
- $\boldsymbol{v}$ be an arbitrary vector with its starting point on the rotation axis.

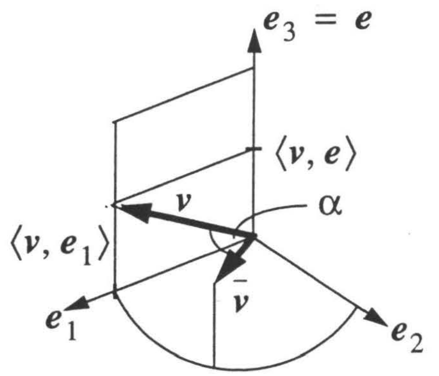


Figure 2a

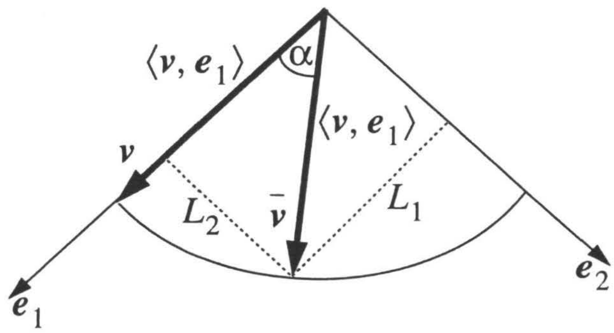


Figure 2b: Projection of $\overline{\boldsymbol{v}}$ in the $(\boldsymbol{e}_1, \boldsymbol{e}_2)$-plane.

We can state that $\boldsymbol{v}$ after rotation ($\boldsymbol{e}$ is the rotation axis) turns into $\overline{\boldsymbol{v}}$, with

$$\overline{\boldsymbol{v}} = \cos\alpha \boldsymbol{v} + \langle \boldsymbol{v}, \boldsymbol{e}\rangle (1 - \cos\alpha)\boldsymbol{e} + \sin\alpha\,[\boldsymbol{e}, \boldsymbol{v}] \;. \tag{3.6}$$

Proof of (3.6):
We choose an orthonormal basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ such that the vector $\boldsymbol{v}$ lies in the $(\boldsymbol{e}_1, \boldsymbol{e}_3)$-plane, i.e.

$$\boldsymbol{v} = \langle \boldsymbol{v}, \boldsymbol{e}_1\rangle\, \boldsymbol{e}_1 + \langle \boldsymbol{v}, \boldsymbol{e}\rangle\, \boldsymbol{e}\,, \tag{$*$}$$

where $\boldsymbol{e}_3 = \boldsymbol{e}$.
In Figure 2b we see that $\cos\alpha = \frac{L_1}{\langle \boldsymbol{v}, \boldsymbol{e}_1\rangle}$ and $\sin\alpha = \frac{L_2}{\langle \boldsymbol{v}, \boldsymbol{e}_1\rangle}$ hold. It follows

$$\overline{\boldsymbol{v}} = \langle \boldsymbol{v}, \boldsymbol{e}\rangle\, \boldsymbol{e} + \cos\alpha\,\langle \boldsymbol{v}, \boldsymbol{e}_1\rangle\, \boldsymbol{e}_1 + \sin\alpha\,\langle \boldsymbol{v}, \boldsymbol{e}_1\rangle\, \boldsymbol{e}_2\,. \tag{$**$}$$

If we put $(*)$ in the second term in $(**)$ we get

$$\overline{\boldsymbol{v}} = \cos\alpha \boldsymbol{v} + (1 - \cos\alpha)\,\langle \boldsymbol{v}, \boldsymbol{e}\rangle\, \boldsymbol{e} + \sin\alpha\,\langle \boldsymbol{v}, \boldsymbol{e}_1\rangle\, \boldsymbol{e}_2\,.$$

So that $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ form a right-handed coordinate system we put $\boldsymbol{e}_2 := [\boldsymbol{e}, \boldsymbol{e}_1]$.
With

$$\begin{aligned}\langle \boldsymbol{v}, \boldsymbol{e}_1\rangle\, \boldsymbol{e}_2 &= \langle \boldsymbol{v}, \boldsymbol{e}_1\rangle\, [\boldsymbol{e}, \boldsymbol{e}_1] = [\boldsymbol{e}, \langle \boldsymbol{v}, \boldsymbol{e}_1\rangle\, \boldsymbol{e}_1] \\ &= [\boldsymbol{e}, \boldsymbol{v} - \langle \boldsymbol{v}, \boldsymbol{e}\rangle\, \boldsymbol{e}] = [\boldsymbol{e}, \boldsymbol{v}]\;.\end{aligned}$$

follows now equation (3.6) □

If α is very small we get in first order $\cos\alpha = 1$ and $\sin\alpha = \alpha$, therefore we can write

$$\overline{\boldsymbol{v}} = \boldsymbol{v} + \alpha\,[\boldsymbol{e}, \boldsymbol{v}]\,. \tag{3.7}$$

The infinitesimal vector $\alpha\boldsymbol{e}$ is called the **rotation vector**. And in comparism with (3.5) we see that Y is the rotation vector of the surface element of X in the point (u, w). The rotation angle is given by $\alpha = \|Y\|\varepsilon$.

□

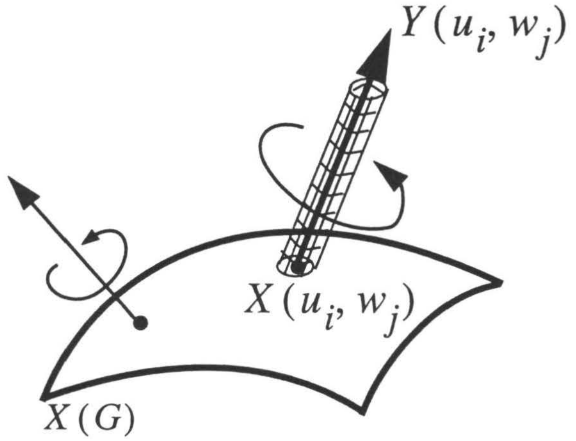


Figure 3: Rotation vector field

In terms of the rotation vectors we can now define the **stability of surfaces**. It is obvious that the differential equation (2.9) of infinitesimal bendings $< dX, dZ >= 0$ has always the solution

$$Z = [C, X] + D\,, \tag{3.8}$$

where C and D are constant vectors. These deformations don't cause inner deformations of the surface, because (3.8) defines a rigid (infinitesimal) motion of the surface. If an infinitesimal bending is of the form (3.8), then the corresponding rotation vector field is constant.

Definition (3.9): *If $\widetilde{X} = X + \varepsilon Z$ is an infinitesimal bending with $Z = [C, X] + D$, where C, D are constant vectors, i.e. $Y = C = const$, then $\widetilde{X}$ is called* **trivial infinitesimal bending** *or* **infinitesimal motion** *of the surface X.*

All the bundles of line elements have the same momentary rotation if the rotation vector field is constant. This context leads to the following

Definition (3.10): *A surface which allows only trivial infinitesimal bendings is called* **infinitesimal rigid** *under infinitesimal bendings.*

Another way to treat infinitesimal bendings is to apply the fundamental equations of this theory. We start with the relations $Z_u = [Y, X_u]$ and $Z_w = [Y, X_w]$ known from the existence theorem (3.1). To get a **jerk free deformation**, we need

$$Z_{uw} = Z_{wu}\,, \tag{3.11}$$

and it follows

$$[Y_u, X_w] = [Y_w, X_u]\,. \tag{3.12}$$

This relation means that Y_u, Y_w, X_u and X_w (the partial derivatives of X and Y) are coplanar. Therefore some real functions α, β, γ and δ exist such that

$$\exists\, \alpha, \beta, \gamma, \delta : G \to I\!R \ :\ Y_u = \alpha X_u + \beta X_w\ ,\ Y_w = \gamma X_u + \delta X_w\,.$$

From (3.12) follows $\alpha = -\delta$, i.e.

$$\begin{aligned} Y_u &= \alpha X_u + \beta X_w \\ Y_w &= \gamma X_u - \alpha X_w\,. \end{aligned} \tag{3.13}$$

A further condition to impose is

$$Y_{uw} = Y_{wu} \tag{3.14}$$

which is equivalent to the equation:

$$(\alpha_w - \gamma_u)X_u + (\beta_w + \alpha_u)X_w - \gamma X_{uu} + 2\alpha X_{wu} + \beta X_{ww} = 0\,. \tag{3.15}$$

By expressing the derivatives of the vectors X_u, X_v in the basis $\{X_u,\, X_w,\, N\}$ (N is the unit normal vector of the surface), we obtain

$$\begin{aligned} X_{uu} &= \Gamma^1_{11}X_u + \Gamma^2_{11}X_v + h_{11}N \\ X_{uw} &= \Gamma^1_{12}X_u + \Gamma^2_{12}X_v + h_{12}N \\ X_{ww} &= \Gamma^1_{22}X_u + \Gamma^2_{22}X_v + h_{22}N\,, \end{aligned} \tag{3.16}$$

where the coefficients Γ_{ij}^k, $i, j, k = 1, 2$ are called the Christoffel symbols of X and where h_{ij} $(i, j = 1, 2)$ are the coefficients of the second fundamental form of X. (The reader who isn't familiar with differential geometry can find the fundamentals in [4], [5]).

Replacing equations (3.16) in (3.15) and decomposition of the resulting equation with respect to the basis $\{X_u, X_w, N\}$, give the *fundamental equations*:

$$\begin{aligned} &h_{11}\gamma - 2h_{12}\alpha - h_{22}\beta = 0 \\ &\Gamma_{11}^1\gamma - 2\Gamma_{12}^1\alpha - \Gamma_{22}^1\beta = \alpha_w - \gamma_u \\ &\Gamma_{11}^2\gamma - 2\Gamma_{12}^2\alpha - \Gamma_{22}^2\beta = \alpha_u + \beta_w \,. \end{aligned} \tag{3.17}$$

If α, β, γ are solutions of the system (3.17), one gets the rotation vector field Y by integration and by a second integration the deformation vector field Z. The surface X allows only the trivial infinitesimal bendings if and only if α, β, $\gamma \equiv 0$ is the unique solution of (3.17).

§4. The instability surface

It is also possible to think of $Y(u, w)$ as a parametric surface. In this case $Y(u, w)$ is called the **instability surface** of the infinitesimal bending. A main property of this instability surface is the fact, that a stable surface has an instability surface reduced to a single point.

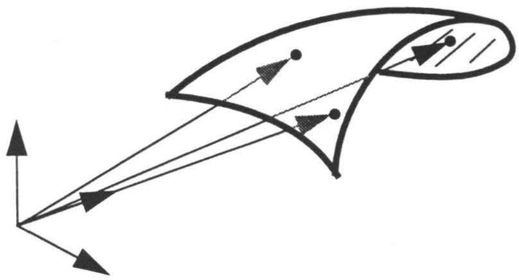


Figure 4: Instability surface

$Y(u, w)$ is not only of central importance for the calculation of the infinitesimal bendings but also a perfect visualization tool. But before visualizing some results about stability we have to calculate the rotation vector fields of a surface. There are two possible accesses to the rotation vector field. The first one is the fundamental system (3.17). The solutions α, β and γ have to be calculated in order to get Y_u and Y_w. After this step we can integrate to get $Y(u, w)$. The second possibility is based on the existence theorem (3.1) in combination with the jerk free deformation condition (3.11). The advantage of the second way is that one can get $Y(u, w)$ more directly without calculating first some functions as necessary in the first method. We choose the second method and the problem now states as follows:

(4.1): *Given an open connected surface* X *with* $X : G \to \mathbb{R}^3$, $X \in C^3(G)$ *and* $G = [a_1, a_2] \times [b_1, b_2] \subset \mathbb{R}^2$.
Wanted the rotation vector fields $Y : G \to \mathbb{R}^3$, $Y \in C^2(G)$ *correspon-ding to all possible infinitesimal bendings* $Z \in C^3(G)$ *of* X, *i.e. all* Y *verifying the equation*

$$[Y_u, X_w] = [Y_w, X_u] .$$

Obviously, $Y = const$ is always a solution for all surfaces X. To solve the system of equations, we present in the following two methods which we can use.

4.1 A discrete solution

The idea of this method is based on the difference methods. The application area of these methods is the numerical solution of differential equations, mainly partial differential equations of second order with boundary conditions [6], [7] or ordinary differential equations [8]. The concept of difference methods is based on a discretization of the problem and an approximation of the derivatives by differences. For the discretization one takes a regular mesh. The given and wanted functions are replaced by their values at the discrete mesh points. Furthermore their derivatives are approximated by numerical differentiation at the grid points.

The application of difference methods to our problem (4.1) leads to a linear system of equations with a non symmetric band matrix. The solutions are discrete values of $Y(u, w)$ for certain parameter values (u_i, w_j). The parameter domain is a two dimensional domain $G = [a_1, a_2] \times [b_1, b_2]$. To simplify we suppose G to be rectangular, but a domain G with an arbitrary border is also possible. In this case one has to treat the grid points near the border in a special manner.

Derivation of the system of equations:

In a first step we need for discretization the following quantities:

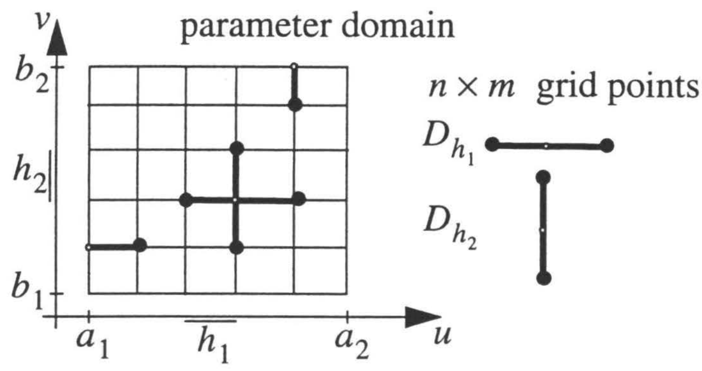


Figure 5: Discretization with centered differences

$n_1, n_2 \in \mathbb{N}$: number of grid points in u- and w-direction
$h_1 := \frac{a_2 - a_1}{n_1}$: grid wideness in u-direction
$h_2 := \frac{b_2 - b_1}{n_2}$: grid wideness in w-direction
$G_{h_1 h_2} := \{(u_i, w_j) = (a_1 + ih_1, b_1 + jh_2) / \; i, j \in \mathbb{N}_0, \; 0 \leq i < n_1, \; 0 \leq j < n_2\}$
set of grid points
$X_{ij} := X(u_i, w_j)$: function values at the grid points $\quad i = 0, \ldots, n_1 - 1$
$Y_{ij} := Y(u_i, w_j)$: analogous $\quad j = 0, \ldots, n_2 - 1$

The second step consists of approximating the partial derivatives by differences. We can choose between three kinds of differences, the forward, the backward and the centered differences (see [9]). They are motivated by the Taylor expansion of a bivariate function.

The centered difference is the mean of the two others

$$D_{h_1} X := \frac{X(u+h_1, w) - X(u-h_1, w)}{2h_1}, \quad D_{h_2} X := \frac{X(u, w+h_2) - X(u, w-h_2)}{2h_2} \tag{4.2}$$

Remark:

The forward and backward differences have a truncation error of order h while the centered differences D_h have a truncation error of order h^2.

Not only the smaller local discretization error recommend the centered differences but it also allows a symmetric treating of the border points of $G_{h_1 h_2}$.

Therefore we get for the inner points

$$\begin{aligned} Y_u &\approx D_{h_1} Y_{ij} = \frac{Y_{i+1,j} - Y_{i-1,j}}{2h_1} \,, \\ Y_w &\approx D_{h_2} Y_{ij} = \frac{Y_{i,j+1} - Y_{i,j-1}}{2h_2} \,, \end{aligned} \tag{4.3}$$

and for the border points

$$\begin{aligned} D_{h_1} Y_{0j} &= \frac{Y_{1,j} - Y_{0,j}}{h_1} \,, \qquad & D_{h_1} Y_{n_1-1,j} &= \frac{Y_{n_1-1,j} - Y_{n_1-2,j}}{h_1} \,, \\ D_{h_2} Y_{i0} &= \frac{Y_{i,1} - Y_{i,0}}{h_2} \,, \qquad & D_{h_2} Y_{i,n_2-1} &= \frac{Y_{i,n_2-1} - Y_{i,n_2-2}}{h_2} \,. \end{aligned} \tag{4.4}$$

The system (4.1) to solve (for the inner grid points) is given by

$$\left(\frac{[Y_{i+1,j}, X_{w_{i,j}}]}{2h_1} - \frac{[Y_{i-1,j}, X_{w_{i,j}}]}{2h_1} - \frac{[Y_{i,j+1}, X_{u_{i,j}}]}{2h_2} + \frac{[Y_{i,j-1}, X_{u_{i,j}}]}{2h_2} \right) = 0 \,, \tag{4.5}$$

where you have to replace the corresponding summands by (4.4) at the border points. The system (4.5) consists now of $3n_1n_2$ homogenous linear equations in $3n_1n_2$ unknowns $Y_{ij} = (y_{ij}^0, y_{ij}^1, y_{ij}^2)$. The coefficient matrix is a band matrix as usual with difference methods. It has a symmetric structure but is not a symmetric matrix, because the vector products of (4.5) destroy all symmetry of such equations. The solutions we get are discrete values Y_{ij} of the rotation vectors at the points (u_i, w_j).

4.2 A B-spline based least-square method

With this method we get an integral solution for $Y(u, w)$, i.e. a C^2-continuous function on G. We take Y as a bicubic B-spline surface with $p \cdot q$ control points and an uniform knot vector $T = \{\bar{u}_0, \dots, \bar{u}_{p+4}, \bar{w}_0, \dots, \bar{w}_{q+4}\}$

$$Y(u, w) := \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} d_{ij} N_i^3(u) N_j^3(w) \,. \tag{4.6}$$

The partial derivatives of Y are given by

$$
\begin{aligned}
Y_u(u,w) &= 3\sum_{i=1}^{p-1}\sum_{j=0}^{q-1}\frac{d_{ij}-d_{i-1,j}}{\bar{u}_{i+3}-\bar{u}_i}N_i^2(u)N_j^3(w)\,,\\
Y_w(u,w) &= 3\sum_{i=0}^{p-1}\sum_{j=1}^{q-1}\frac{d_{ij}-d_{i,j-1}}{\bar{w}_{i+3}-\bar{w}_i}N_i^3(u)N_j^2(w)\,.
\end{aligned}
\tag{4.7}
$$

The system (4.1) is given now as

$$
Q_{uw}(\ldots,d_{ij}^r,\ldots) := [X_u(u,w),Y_w(u,w)] - [X_w(u,w),Y_u(u,w)] = 0\,, \tag{4.8}
$$

where the unknowns $d_{ij} = (d_{ij}^1, d_{ij}^2, d_{ij}^3)$ are the control points of the B-spline representation of Y. To solve (4.8) we use a least-square fitting, which is also an approximative solution of the problem. Instead of (4.8) we solve now the problem

$$
\sum_{\alpha=1}^{f_1}\sum_{\beta=1}^{f_2}\Big\|\big[X_u(u_\alpha,w_\beta),Y_w(u_\alpha,w_\beta)\big] - \big[X_w(u_\alpha,w_\beta),Y_u(u_\alpha,w_\beta)\big]\Big\|^2 \longrightarrow \min\,. \tag{4.9}
$$

$f_1 \cdot f_2$ denote the number of points, where the function (4.9) should be minimized.

In a clearer form we can write the function (4.9) with (4.8)

$$
F(\ldots,d_{ij}^r,\ldots) := \sum_{\alpha,\beta} < Q_{u_\alpha w_\beta}(\ldots,d_{ij}^r,\ldots), Q_{u_\alpha w_\beta}(\ldots,d_{ij}^r,\ldots) > \longrightarrow \min\,. \tag{4.10}
$$

F is a convex, quadratic form in the unknowns d_{ij}^r ($i = 0,\ldots,p-1$; $j = 0,\ldots,q-1$; $r = 0,1,2$) and (4.10) is therefore equivalent to the system of $3pq$ equations

$$
\begin{aligned}
\frac{\partial F}{\partial d_{kl}^s} &= \frac{\partial}{\partial d_{kl}^s}\Big(\sum_{\alpha,\beta} < Q_{u_\alpha w_\beta}, Q_{u_\alpha w_\beta} >\Big)\\
&= 2\sum_{\alpha,\beta} < \frac{\partial Q_{u_\alpha w_\beta}}{\partial d_{kl}^s}, Q_{u_\alpha w_\beta} >= 0
\end{aligned}
\tag{4.11}
$$

with $k = 0,\ldots,p-1$; $l = 0,\ldots,q-1$; $s = 0,1,2$. (4.11) are linear equations, because $Q_{u_\alpha w_\beta}$ is linear in the unknowns.

The next step is now to calculate the coefficient matrix. The problem is to extract the unknowns d_{ij}^r from the scalar product of Q and $\frac{\partial Q}{\partial d_{kl}^s}$. Beginning with Q:

$$Q_{u_\alpha w_\beta}(\ldots, d^r_{ij}, \ldots) = 3\sum_{i=0}^{p-1}\sum_{j=1}^{q-1} \frac{N_i^3(u_\alpha)N_j^2(w_\beta)}{\bar{w}_{j+3}-\bar{w}_j}\Big([X_u, d_{ij}] - [X_u, d_{i,j-1}]\Big)$$

$$-3\sum_{i=1}^{p-1}\sum_{j=0}^{q-1} \frac{N_i^2(u_\alpha)N_j^3(w_\beta)}{\bar{u}_{i+3}-\bar{u}_i}\Big([X_w, d_{ij}] - [X_w, d_{i-1,j}]\Big). \tag{4.12}$$

After index transformation we get

$$\begin{aligned} Q_{u_\alpha w_\beta}(\ldots, d^r_{ij}, \ldots) = {} & 3\sum_{i=0}^{p-1}\sum_{j=1}^{q-1} \frac{N_i^3(u_\alpha)N_j^2(w_\beta)}{\bar{w}_{j+3}-\bar{w}_j}[X_u, d_{ij}] \\ & -3\sum_{i=0}^{p-1}\sum_{j=0}^{q-2} \frac{N_i^3(u_\alpha)N_{j+1}^2(w_\beta)}{\bar{w}_{j+4}-\bar{w}_{j+1}}[X_u, d_{ij}] \\ & -3\sum_{i=1}^{p-1}\sum_{j=0}^{q-1} \frac{N_i^2(u_\alpha)N_j^3(w_\beta)}{\bar{u}_{i+3}-\bar{u}_i}[X_w, d_{ij}] \\ & +3\sum_{i=0}^{p-2}\sum_{j=0}^{q-1} \frac{N_{i+1}^2(u_\alpha)N_j^3(w_\beta)}{\bar{u}_{i+4}-\bar{u}_{i+1}}[X_w, d_{ij}]\,. \end{aligned} \tag{4.13}$$

If we fix the indices i, j, the corresponding summand of $Q_{u_\alpha w_\beta}$ is given by

$$i, j \text{ fixed} : \quad \underbrace{3\left[\frac{N_i^3(u_\alpha)N_j^2(w_\beta)}{\bar{w}_{j+3}-\bar{w}_j} - \frac{N_i^3(u_\alpha)N_{j+1}^2(w_\beta)}{\bar{w}_{j+4}-\bar{w}_{j+1}}\right]}_{\phi_{ij}}[X_u, d_{ij}]$$

$$\underbrace{-3\left[\frac{N_i^2(u_\alpha)N_j^3(w_\beta)}{\bar{u}_{i+3}-\bar{u}_i} - \frac{N_{i+1}^2(u_\alpha)N_j^3(w_\beta)}{\bar{u}_{i+4}-\bar{u}_{i+1}}\right]}_{\psi_{ij}}[X_w, d_{ij}]\,,$$

i.e.

$$3\phi_{ij}[X_u, d_{ij}] - 3\psi_{ij}[X_w, d_{ij}] \tag{4.14}$$

with

$$\phi_{ij} := \frac{N_i^3(u_\alpha)N_j^2(w_\beta)}{\bar{w}_{j+3}-\bar{w}_j} - \frac{N_i^3(u_\alpha)N_{j+1}^2(w_\beta)}{\bar{w}_{j+4}-\bar{w}_{j+1}} \qquad \text{for} \qquad \begin{array}{l} i = 0, \ldots, p-1 \\ j = 1, \ldots, q-2 \end{array}$$

$$\phi_{i0} := \qquad - \frac{N_i^3(u_\alpha)N_1^2(w_\beta)}{\bar{w}_4 - \bar{w}_1} \tag{4.15}$$

$$\phi_{i,q-1} := \frac{N_i^3(u_\alpha)N_{q-1}^2(w_\beta)}{\bar{w}_{q-1+3} - \bar{w}_{q-1}}$$

and

$$\begin{aligned}
\psi_{ij} &:= \frac{N_i^2(u_\alpha)N_j^3(w_\beta)}{\bar{u}_{i+3} - \bar{u}_i} - \frac{N_{i+1}^2(u_\alpha)N_j^3(w_\beta)}{\bar{u}_{i+4} - \bar{u}_{i+1}} \qquad \text{for} \quad \begin{matrix} i = 1, \dots, p-2 \\ j = 0, \dots, q-1 \end{matrix} \\
\psi_{0j} &:= \qquad - \frac{N_1^2(u_\alpha)N_j^3(w_\beta)}{\bar{u}_4 - \bar{u}_1} \\
\psi_{p-1,j} &:= \frac{N_{p-1}^2(u_\alpha)N_j^3(w_\beta)}{\bar{u}_{p-1+3} - \bar{u}_{p-1}} .
\end{aligned} \tag{4.16}$$

It is not possible to extract d_{ij}^r of this equation without writing the vector product in its components. In the following calculations it is shown how to avoid this.

$$\begin{aligned}
\frac{\partial Q_{u_\alpha, w_\beta}}{\partial d_{kl}^s}(\dots, d_{ij}^r, \dots) = {} & 3\, \frac{N_k^3(u_\alpha)N_l^2(w_\beta)}{\bar{w}_{l+3} - \bar{w}_l} \frac{\partial}{\partial d_{kl}^s} [X_u, d_{kl}] \\
& - 3\, \frac{N_k^3(u_\alpha)N_{l+1}^2(w_\beta)}{\bar{w}_{l+4} - \bar{w}_{l+1}} \frac{\partial}{\partial d_{kl}^s} [X_u, d_{kl}] \\
& - 3\, \frac{N_k^2(u_\alpha)N_l^3(w_\beta)}{\bar{u}_{k+3} - \bar{u}_k} \frac{\partial}{\partial d_{kl}^s} [X_w, d_{kl}] \\
& + 3\, \frac{N_{k+1}^2(u_\alpha)N_l^3(w_\beta)}{\bar{u}_{k+4} - \bar{u}_{k+1}} \frac{\partial}{\partial d_{kl}^s} [X_w, d_{kl}]
\end{aligned} \tag{4.17}$$

$$k = 0, \dots, p-1$$
$$l = 0, \dots, p-1$$
$$s = 0, 1, 2$$

are constant values.

We introduce now the following notations

$$C_0 := \frac{\partial}{\partial d^0_{kl}}\,[X_u, d_{kl}] = \begin{bmatrix} 0 \\ x^3_u \\ -x^2_u \end{bmatrix}, \qquad \overline{C}_0 := \frac{\partial}{\partial d^0_{kl}}\,[X_w, d_{kl}] = \begin{bmatrix} 0 \\ x^3_w \\ -x^2_w \end{bmatrix},$$

$$C_1 := \frac{\partial}{\partial d^1_{kl}}\,[X_u, d_{kl}] = \begin{bmatrix} -x^3_u \\ 0 \\ x^1_u \end{bmatrix}, \qquad \overline{C}_1 := \frac{\partial}{\partial d^1_{kl}}\,[X_w, d_{kl}] = \begin{bmatrix} -x^3_w \\ 0 \\ x^1_w \end{bmatrix}, \tag{4.18}$$

$$C_2 := \frac{\partial}{\partial d^2_{kl}}\,[X_u, d_{kl}] = \begin{bmatrix} x^2_u \\ -x^1_u \\ 0 \end{bmatrix}, \qquad \overline{C}_2 := \frac{\partial}{\partial d^2_{kl}}\,[X_w, d_{kl}] = \begin{bmatrix} x^2_w \\ -x^1_w \\ 0 \end{bmatrix}$$

and one gets from (4.17)

$$\frac{\partial Q_{u_\alpha, w_\beta}}{\partial d^s_{kl}}(\ldots, d^r_{ij}, \ldots) = 3\left[\frac{N^3_k(u_\alpha)N^2_l(w_\beta)}{\bar{w}_{l+3} - \bar{w}_l} - \frac{N^3_k(u_\alpha)N^2_{l+1}(w_\beta)}{\bar{w}_{l+4} - \bar{w}_{l+1}}\right] \cdot C_s$$

$$- 3\left[\frac{N^2_k(u_\alpha)N^3_l(w_\beta)}{\bar{u}_{k+3} - \bar{u}_k} - \frac{N^2_{k+1}(u_\alpha)N^3_l(w_\beta)}{\bar{u}_{k+4} - \bar{u}_{k+1}}\right] \cdot \overline{C}_s$$

$$\iff \qquad \frac{\partial Q_{u_\alpha, w_\beta}}{\partial d^s_{kl}}(\ldots, d^r_{ij}, \ldots) = 3\phi_{kl} C_s - 3\psi_{kl}\overline{C}_s\,. \tag{4.19}$$

ϕ, ψ are given in (4.15) and (4.16). The system (4.11) can now be written as

$$\sum_{\alpha,\beta} < \frac{\partial Q_{u_\alpha, w_\beta}}{\partial d^s_{kl}}, Q_{u_\alpha, w_\beta} >= 0 \qquad \text{for} \qquad \begin{matrix} k = 0, \ldots, p-1 \\ l = 0, \ldots, p-1 \\ s = 0, 1, 2 \end{matrix}$$

$$\iff \qquad \sum_{\alpha,\beta} 9 \left\langle\, \phi_{kl} C_s - \psi_{kl}\overline{C}_s\,,\ \sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\Big(\phi_{ij}[X_u, d_{ij}] - \psi_{ij}[X_w, d_{ij}]\Big)\,\right\rangle = 0$$

$$\iff \qquad \sum_{\alpha,\beta}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\Big(\phi_{kl}\phi_{ij} < C_s, [X_u, d_{ij}] > - \,\phi_{kl}\psi_{ij} < C_s, [X_w, d_{ij}] >$$

$$- \,\psi_{kl}\phi_{ij} < \overline{C}_s, [X_u, d_{ij}] > + \,\psi_{kl}\psi_{ij} < \overline{C}_s, [X_w, d_{ij}] > \Big) = 0$$

Now it's easy to see that we don't have to calculate the vector products of (4.14), because we can get a scalar product with d_{ij}:

$$\begin{aligned}\sum_{\alpha,\beta}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\Big(&\phi_{kl}\phi_{ij} < [C_s, X_u], d_{ij} > \\ &- \phi_{kl}\psi_{ij} < [C_s, X_w], d_{ij} > \\ &- \psi_{kl}\phi_{ij} < [\overline{C}_s, X_u], d_{ij} > \\ &+ \psi_{kl}\psi_{ij} < [\overline{C}_s, X_w], d_{ij} >\Big) = 0\,,\end{aligned} \tag{4.20}$$

where $\phi_{kl} = \phi_{kl}(u_\alpha, w_\beta)$; ψ_{kl}, C_s, $\overline{C}_s$, X_u, X_w analogous; denote the function values at the parameter values (u_α, w_β). (4.20) is the homogenous linear system to solve

$$A \cdot d = 0 \tag{4.21}$$

with $A = \left(A_{row,\, column}\right)_{\substack{row=0,\dots,3pq-1\\ column=0,\dots,3pq-1}}$

and $d = [d_{00}^0, d_{00}^1, d_{00}^2, d_{01}^0, \dots, d_{ij}^r, \dots, d_{3pq-1,3pq-1}^2]^T$.

In this case the coefficient matrix is symmetric and with (4.20) we are able to calculate each element independently from each other.

§5. Visualizing Stability

As already mentioned above it is obvious that $\boldsymbol{Y} = constant$ is always a solution of problem (4.1). The matrix is therefore singular and usually has more than one linearly independent solution. If there is at least one non constant rotation vector field, then the surface is not stable because the corresponding infinitesimal bending is not trivial. But if all solutions are constant rotation vector fields, then the surface is stable. In this case the instability surface degenerates into a single point.

Parametric surfaces have different bending behaviours: there are surfaces which are more likely to bend and others are more likely to resist "pressure". With the help of the rotation vector fields, we want to visualize the bending property. Indeed the notion of stability is closely related to the rotation vector field of infinitesimal bendings: A stable surface in the sense of infinitesimal bendings has a constant rotation vector field. Moreover, the more the rotation vectors vary in their directions the less the surface is stable. This behaviour provides a visual test of stability as illustrated in the following examples.

We want to compare two surfaces. In Figure 6a we see a bicubic Bézier surface with its instability surface in Figure 6b. The rotation vectors are attached on their corresponding parameter values in the parameter domain seen from a special point of view. The different directions of the rotation vectors are easily seen and indicate that they vary a lot (see Figure 6c).

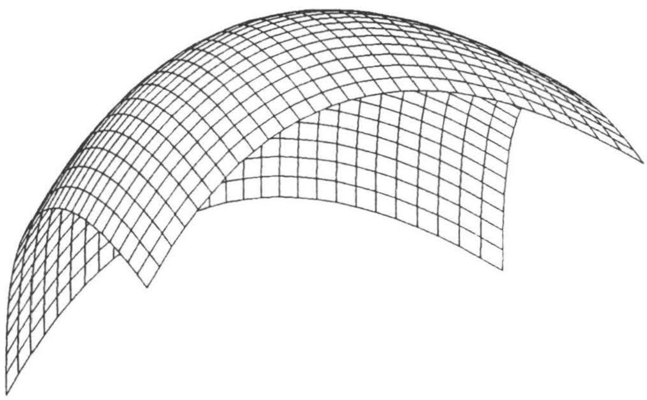


Figure 6a: Test surface

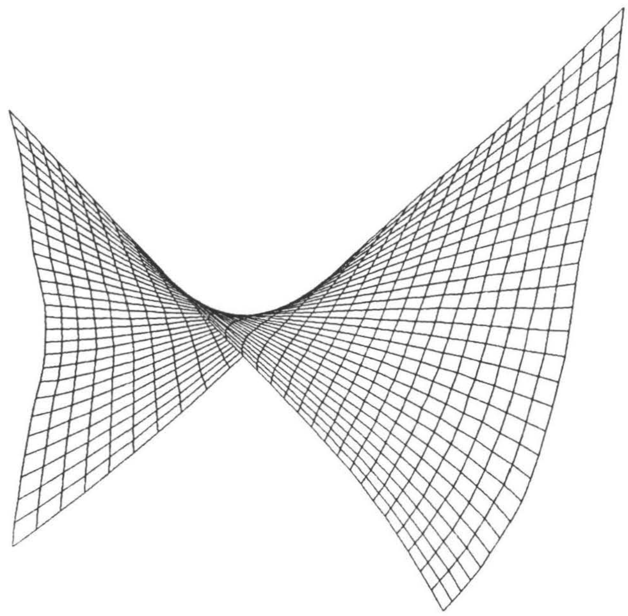
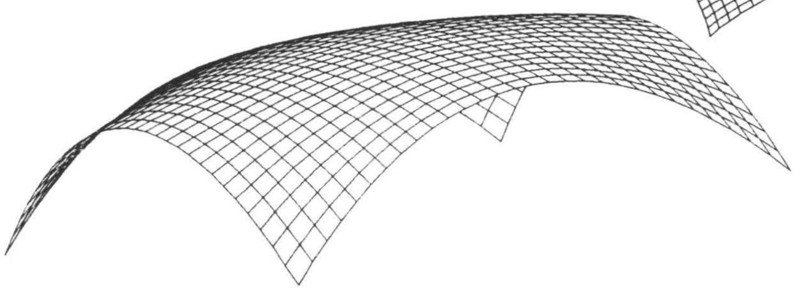


Figure 6b: Instability surface

Figure 6c: Rotation vectors

The second example is also a bicubic Bézier surface (see Figure 7a). Its instability surface looks similar. We show here instability surfaces which are 'well looking'. But it also happens that this surface can look total crazy, so that it is not possible to visualize it. More important are the rotation vectors in this case. In Figure 7c we can see that the rotation vectors seen from the same view point don't vary so much in their directions.



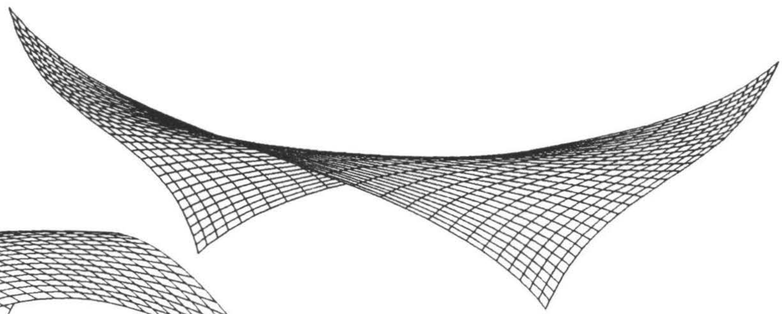


Figure 7a: Test surface

Figure 7b: Instability surface

Figure 7c: Rotation vectors

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