Interner Bericht

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Abstract. The simulation of random fields has many applications in computer graphics such as e.g. ocean wave or turbulent wind field modeling. We present a new and strikingly simple synthesis algorithm for random fields on rank-1 lattices that requires only one Fourier transform independent of the dimension of the support of the random field. The underlying mathematical principle of discrete Fourier transforms on rank-1 lattices breaks the curse of dimension of the standard tensor product Fourier transform, i.e. the number of function values does not exponentially depend on the dimension, but can be chosen linearly.

1 Introduction

For producing visual effects it often suffices to efficiently simulate a random field with the properties of a physical phenomenon instead of simulating the phenomenon itself at high cost. The most prominent example is the simulation of the ocean surface [FR86, Tes00] by random fields as used in the movies Titanic, Waterworld, or The Devil's Advocate. The same principle has been applied for modeling turbulent wind fields and various other phenomena [SF91, SF93, Sta95, Sta97].

The above examples use the fast Fourier transform to synthesize realizations of random fields and require one fast Fourier transform for each dimension of the parameter domain. For example a turbulent wind field simulation in space-time requires 4 transforms. Thus the efficiency of the simulation of random fields exponentially decreases with the dimension of the parameter domain. This bad effect is called the *curse of dimension*.

We introduce a new and very simple synthesis technique that is independent of the dimension of the parameter domain since it requires only one fast Fourier transform for the realization of a random field thus breaking the curse of dimension.

In the sequel we briefly recapture the definition of periodic random fields, introduce the new algorithm of realizing periodic random fields on rank-1 lattices, and illustrate the new technique by the example of ocean wave simulation.

2 Fourier Synthesis of Random Fields

A random field (also called random function)

$$\begin{aligned} f : \Omega &\to C(s,d) \\ \omega &\mapsto \boldsymbol{f}_\omega \end{aligned}$$

maps the space of elementary events Ω into the space of continuous functions $C(s,d)$. Thus the realization $\boldsymbol{f}_\omega$ of a random field f is a continuous function mapping elements of the parameter domain $\mathbb{R}^s$ into $\mathbb{R}^d$. A random field can be regarded as the generalization of a random variable, i.e. a parameter dependent random variable.

We focus on realizations of random fields $\boldsymbol{f}_\omega(\boldsymbol{x})$ of period $\mathbf{1}$, i.e.

$$\boldsymbol{f}_\omega(\boldsymbol{x}) = \boldsymbol{f}_\omega(\boldsymbol{x} + \boldsymbol{z})$$

for $\boldsymbol{x} \in \mathbb{R}^s$ and $\boldsymbol{z} \in \mathbb{Z}^s$ that, given a realization of the Fourier coefficients $\hat{\boldsymbol{f}}_\omega(\boldsymbol{k})$ for the wave vectors $\boldsymbol{k} \in \mathbb{Z}^s$, can be expressed in terms of the Fourier series

$$\boldsymbol{f}_\omega(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^s} \hat{\boldsymbol{f}}_\omega(\boldsymbol{k}) e^{2\pi i \boldsymbol{k}^T \cdot \boldsymbol{x}}.$$

Typical applications [Sta95] of random field simulation are the modeling of stationary or time-dependent height fields that lack a foldover structure, i.e. $h_\omega : \mathbb{R}^2 \to \mathbb{R}$ or $h_\omega : \mathbb{R}^3 \to \mathbb{R}$ as used for terrain or ocean surface modeling, respectively. Evolving cloud or wind fields efficiently can be simulated using scalar density $\rho_\omega : \mathbb{R}^4 \to \mathbb{R}$ or vector random fields $\boldsymbol{v}_\omega : \mathbb{R}^4 \to \mathbb{R}^3$ over space-time, too.

These phenomena typically are described by spectral models, and are realized by applying a discrete Fourier transform to the Fourier coefficients that are obtained by filtering white noise using the given spectrum of the phenomenon [Sta95, Tes00]. Using the standard tensor product approach the fast Fourier transform is applied to each dimension of the parameter domain separately in order to compute the sum of harmonics. This approach already is fast, since it requires only $\mathcal{O}(M^s \log M)$ operations, where M is the number of function values in one dimension. However the order of operations exponentially depends on the dimension s of the parameter domain.

3 The Fast Fourier Transform on Rank-1 Lattices

Using rank-1 lattices [SJ94] with a suited choice of wave vectors it is possible to perform a fast Fourier transform on an s-dimensional parameter domain using only one standard one-dimensional fast Fourier transform. Therefore we introduce the notion of rank-1 lattices, describe how the usage of lattice points modifies the Fourier series, and show how to choose a suited set of wave vectors to make the scheme work.

3.1 Rank-1 Lattices

We choose a point set $P_N := \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\} \subset \mathbb{R}^s$ of N points. As mentioned before, we are considering functions of period $\mathbf{1}$ and such the sum $L := P_N + \mathbb{Z}^s$ covers the parameter domain $\mathbb{R}^s$ in a $\mathbf{1}$-periodic way. If the discrete subset $L \in \mathbb{R}^s$ is closed under addition and subtraction, it is called a *lattice* [SJ94]. For a suited choice of the *generator vector* $\boldsymbol{g} \in \mathbb{N}^s$ rank-1 lattices can be generated by

$$\boldsymbol{x}_n := \frac{n}{N}\boldsymbol{g}. \tag{1}$$

This concept is illustrated in figure 1, where we plotted the points $\boldsymbol{x}_n \pmod{\mathbf{1}} \in P_N \pmod{\mathbf{1}} := L \cap [0,1)^s$ of a regular grid aside to a rank-1 lattice for $s = 2$ dimensions. While the regular grid is a rank-2 lattice, since at least two generator vectors are required to describe it, the rank-1 lattice in contrast requires only one generator vector.

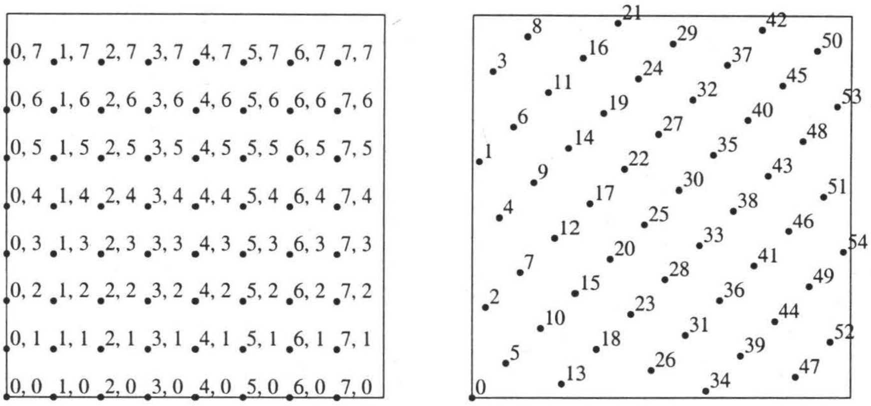

Fig. 1. On the left we display a rectangular grid of $N = 64$ points on $[0,1)^2$ as it arises from tensor product techniques. Because at least two basis vectors are required to describe the grid (as indicated by the indices), it is a rank-2 lattice. On the right a Fibonacci lattice at $N = F_{10} = 55$ is displayed. Since only one basis vector is required to describe the lattice, it is a rank-1 lattice.

3.2 Fourier Synthesis

The central idea of the scheme is the special choice of the wave vectors. We select a set $K_N := \{\boldsymbol{k}_0, \dots, \boldsymbol{k}_{N-1}\} \subset \mathbb{Z}^s$ of wave vectors such that

$$\boldsymbol{k}_m \in Z_m := \{\boldsymbol{k} \in \mathbb{Z}^s \mid \boldsymbol{k}^T \cdot \boldsymbol{g} \equiv m \pmod{N}\} \tag{2}$$

where $\boldsymbol{g}$ is the generator vector of the set P_N as defined in (1) and in consequence

$$\boldsymbol{k}_m^T \cdot \boldsymbol{x}_n = \boldsymbol{k}_m^T \cdot \frac{n}{N}\boldsymbol{g} = (m + l_m N)\frac{n}{N}$$

for $\boldsymbol{x}_n \in P_N$ and some $l_m \in \mathbb{Z}$. Given the Fourier coefficients $\hat{\boldsymbol{f}}_m := \hat{\boldsymbol{f}}(\boldsymbol{k}_m)$, synthesizing a function $\boldsymbol{f}$ on the lattice L by

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x}_n) &= \sum_{m=0}^{N-1} \hat{\boldsymbol{f}}(\boldsymbol{k}_m) e^{2\pi i \boldsymbol{k}_m^T \cdot \boldsymbol{x}_n} = \sum_{m=0}^{N-1} \hat{\boldsymbol{f}}(\boldsymbol{k}_m) e^{2\pi i (m\frac{n}{N} + l_m n)} \\ &= \sum_{m=0}^{N-1} \hat{\boldsymbol{f}}_m e^{2\pi i m \frac{n}{N}} \end{aligned}$$

in fact turns out to be a one dimensional finite Fourier series independent of the dimension s of the domain of the lattice, that of course can be computed by only one fast Fourier transform.

The standard fast Fourier transform tensor product approach requires $\mathcal{O}(M^s \log M)$ operations, where s is the dimension of the parameter domain and M is the number of samples in one dimension. The Fourier synthesis on rank-1 lattices does not depend on the dimension s and consequently breaks the curse of dimension, i.e. the exponential dependence on s. This already has been exploited for the pseudospectral method for solving partial differential equations[1]: Considering the approximation error, for the

[1]Our method is based on this ongoing research work done by Li Dong and Fred Hickernell. See the Acknowledgements.

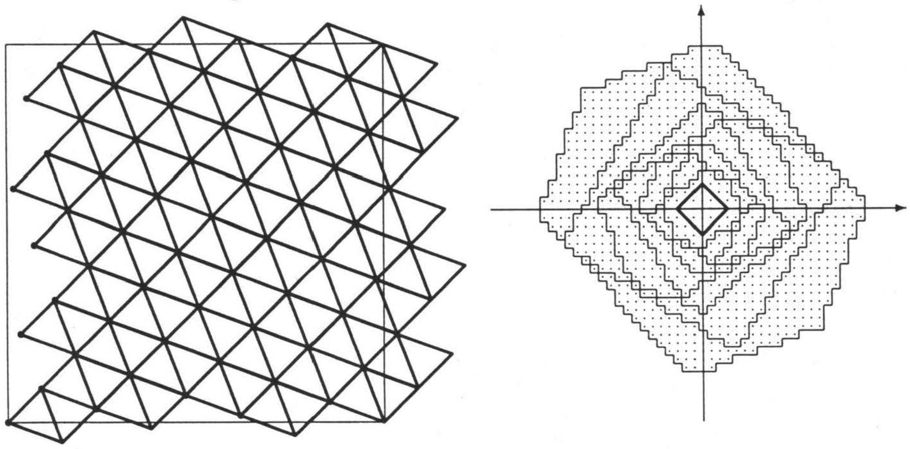

Fig. 2. On the left the periodic Delauney triangulation of the unit square induced by the Fibonacci lattice at $N = F_{10} = 55$ points is displayed. On the right the sets $K_N \subset \mathbb{Z}^2$ of wave vectors associated to the Fibonacci lattices at $N = F_{10}, \dots, F_{17}$ points are outlined. In the middle $\|\boldsymbol{k}\|_1 = 3$ is highlighted as it it used for enumerating the wave vectors.

same number of points the rank-1 lattice method outperforms the standard tensor product approach using a rank-s lattice. This can be explained by considering the discrepancy of the underlying lattices. Whereas regular grids are not of low discrepancy, rank-1 lattices are of low discrepancy [SJ94]. Thus for the same number of points the points of a rank-1 lattice are much better equidistributed than regular grids and in consequence the exponential basis functions centered on the rank-1 lattice points expose an improved coverage of the parameter domain as compared to regular grids.

3.3 Determining the Wave Vectors

By the above condition (2) the wave vectors in K_N are not unique, yet. In order to synthesize a band limited signal, we decide to include the smallest frequencies first. This means that the wave vectors $\boldsymbol{k}_m$ have to be chosen such that their Euclidean norm is minimal, i.e.

$$\|\boldsymbol{k}_m\|_2 = \min_{\boldsymbol{k} \in Z_m} \|\boldsymbol{k}\|_2.$$

Obviously the constant term of the Fourier series is identified by $\boldsymbol{k}_0 = \boldsymbol{0}$. In an implementation the wave vectors are determined by enumerating the candidates $\boldsymbol{k} \in \mathbb{Z}^s$ in a fixed order along the diagonal lines of increasing $\|\boldsymbol{k}\|_1 = 1, 2, 3, \dots$ (see figure 2 for illustration), computing $m = \boldsymbol{k}^T \cdot \boldsymbol{g} \pmod N$, and keeping book of the current minimal norm of $\boldsymbol{k}_m$.

4 Random Field Synthesis on Rank-1 Lattices: Ocean Waves

The general method derived in the previous section now is applied to the simulation of ocean waves [Tes00]. This simple example is perfectly suited for illustrating the principles of realizing random fields on rank-1 lattices.

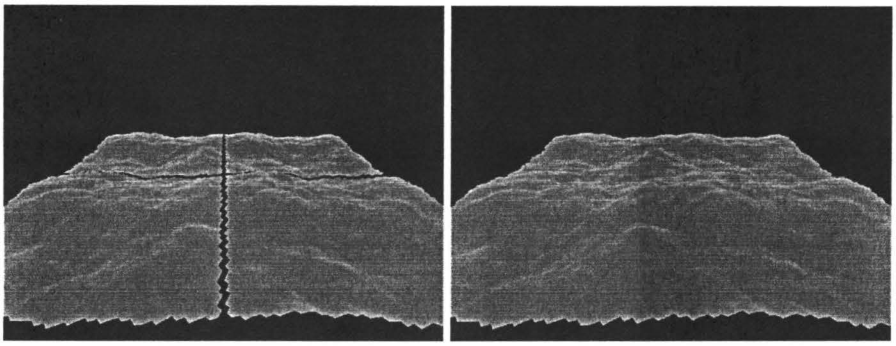

Fig. 3. Periodic tiling property of four identical patches. The random height field is realized using a one-dimensional fast Fourier transform on a rank-1 lattice.

As rank-1 lattice we choose the two dimensional Fibonacci lattice[2], which is defined by the generator vector

$$g = (1, F_{k-1}),$$

at $N = F_k$ points, where the F_k are the Fibonacci numbers defined by $F_1 = F_2 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k > 2, k \in \mathbb{N}$. For $N = F_{10} = 55$ the points $P_{55} \pmod{\mathbf{1}}$ are depicted in the right image in figure 1.

The time-dependent random height field $h_\omega : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ of the ocean surface from [Tes00] formulated for a rank-1 lattice is

$$h_\omega(\boldsymbol{x}_n, t) = \sum_{m=0}^{N-1} \hat{h}_\omega(\boldsymbol{k}_m, t) e^{2\pi i m \frac{n}{N}},$$

where the surface normals in the lattice points $\boldsymbol{x}_n$ at time t are found by normalizing the gradient

$$\nabla h_\omega(\boldsymbol{x}_n, t) = \sum_{m=0}^{N-1} \boldsymbol{i} 2\pi \boldsymbol{k}_m \hat{h}_\omega(\boldsymbol{k}_m, t) e^{2\pi i m \frac{n}{N}}.$$

For brevity of space we do not reprint the explicit formulas for the realization of the random Fourier coefficients $\hat{h}_\omega(\boldsymbol{k}_m, t)$, however these are perfectly described and explained in [Tes00]. Note that although $s = \dim \boldsymbol{x}_n = \dim \boldsymbol{k}_m$, the sum of harmonics is computed using a one-dimensional fast Fourier transform.

In order to render the realization of the random height field a periodic Delauney triangulation (see figure 2) of $P_N \pmod{\mathbf{1}}$ is used, which is found by an $\mathcal{O}(N)$ time scan of the points P_N. By construction h and ∇h are periodic and continuous. Thus the patches periodically can be tiled as illustrated in figure 3. Consequently it is sufficient to keep one master patch in memory and to treat the periodic padding by virtual instancing.

[2]Other lattice constructions can be found in [SJ94].

5 Conclusion and Impact

We presented a new general scheme for the realization of periodic random fields on rank-1 lattices. Exploiting the special structure of rank-1 lattices the algorithm uses only one fast Fourier transform as opposed to the standard tensor product approach using as many Fourier transforms as there are dimensions in the parameter domain of the realization of the random field.

Besides the realization of random fields, the underlying mathematical principle of Fourier transforms on rank-1 lattices affects all algorithms that use a multidimensional Fourier transform, because if a rank-1 lattice can be used in fact only one Fourier transform setup is required. So the Fourier transform on Fibonacci lattices directly transfers to the soft shadow technique presented in [SS98] and Fourier transforms on rank-1 lattices also yield new results in volume rendering using the Fourier slice projection theorem [Mal93, LG95].

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