Interner Bericht

Composition of Tensor Product Bézier Representations

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Abstract. Trimming of surfaces and volumes, curve and surface modeling via Bézier's idea of destortion, segmentation, reparametrization, geometric continuity are examples of applications of functional composition. This paper shows how to compose polynomial and rational tensor product Bézier representations. The problem of composing Bézier splines and B-spline representations will also be addressed in this paper.

Keywords. Bézier, tensor product, composition, trimming, free-form deformation

I. Introduction

The purpose of this paper is to show how the composition of two Bézier representations can again be represented as a tensor product Bézier representation (of higher degree). Fig. 1 illustrates the idea of composition in the case of a planar Bézier curve $\mathrm{K}(t): \mathbb{R} \to \mathbb{R}^2$ and a tensor product Bézier surface $\mathrm{F}(u,v): \mathbb{R}^2 \to \mathbb{R}^d$, $d = 2, 3$.

Some applications of composition have been pointed out by deRose [Ros 88], who is discussing the composition of Bézier simplex forms. Some simple examples of composition are the evaluation, subdivision and polynomial/rational reparametrization of polynomial/rational Bézier representations. The later one might be of importance in context of GC^r-continuity.

A more interesting application is given by the idea of curve and surface modeling in the sense of free-form deformations (FFDs), first described by Bézier [Béz 78], and in the following subject of [Sed 86], [Gri 89] and [Coq 90]. The FFD idea is to embed an object in a deformable medium and, then manipulate the object by deforming the medium that surrounds it (see also [Far 90], [Hos 91]).

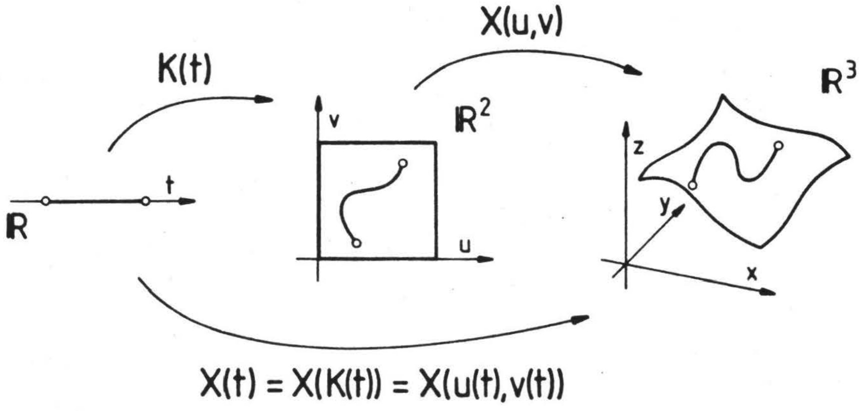

Fig. 1. Composition $\mathbf{F}(\mathbf{K}(t))$ of a planar Bézier curve $\mathbf{K}(t)$ and a tensor product Bézier surface $\mathbf{F}(u,v)$

A second, very important application concerns the subject of trimmed surfaces and trimmed volumes which are of high interest in solid modeling and surface design. Different aspects of trimmed subjects have been discussed in [Cas 87], [Sha 88], [Roc 89], [Hos 90] and [Nis 90].

The results of this paper can be used for example to exactly represent FFDs as well as trim curves and trim surfaces in coordinate space, in the sense that given Theorems and Corollaries can be used to directly and exactly compute the control points for deformed subjects and trim curves and surfaces, respectively. Both implementations are presently under development and are being tested against an approximative method and also against a combination of the exact with an approximative method (see [Las 91a, 91b]).

Section II reviews definitions of tensor product Bézier representations and introduces the notation of this paper. In Sections III and IV explicit representations of the composition of polynomial and of rational tensor product representations are given. Section V is concerned with spline representations, but no in-depth treatment is given there. The composition of B-spline representations is described in detail in [Las 91].

II. Bézier Representations

A **Bézier curve** of degree l in u is defined by

$$\mathbf{X}(u) = \sum_{i=0}^{l} \mathbf{b}_i \, B_i^l(u), \qquad u \in [0,1],$$

where $\mathbf{b}_i \in \mathbb{R}^d$, $d \in \mathbb{N}$, and

$$B_i^l(u) = \binom{l}{i} u^i (1-u)^{l-i}$$

are the (ordinary) **Bernstein polynomials** of degree l in u. The coefficients $\mathbf{b}_i$ are called **Bézier points**. They form in their natural ordering, given by their subscripts, the vertices of the **Bézier polygon**.

All properties of Bézier curves are a direct consequence of properties of the Bernstein polynomials. We list the ones which are of importance for the following calculations:

Recursion formula:

$$B_I^N(t) = (1-t)\, B_I^{N-1}(t) + t\, B_{I-1}^{N-1}(t)$$

Partition of unity:

$$\sum_{I=0}^{N} B_I^N(t) = 1$$

Product formula:

$$\prod_{k=1}^{\alpha} B_{I_k}^{N_k}(t) = \frac{\prod_{k=1}^{\alpha} \binom{N_k}{I_k}}{\binom{|\mathbf{N}|}{|\mathbf{I}|}}\, B_{|\mathbf{I}|}^{|\mathbf{N}|}(t)$$

where $\mathbf{I} = (I_1, \ldots, I_\alpha)$, $|\mathbf{I}| = I_1 + \ldots + I_\alpha$, and $\mathbf{N} = (N_1, \ldots, N_\alpha)$, $|\mathbf{N}| = N_1 + \ldots + N_\alpha$.

The Bézier description of a curve is a very powerful tool because the expansion in terms of Bernstein polynomials yields, firstly, a numerically very stable behavior of all the curve algorithms. And, secondly, a geometric relationship between a curve and its defining Bézier points.

A tensor product Bézier surface - briefly **TPB-surface** - of degree (l, m) is defined by

$$\mathbf{X}(u,v) = \sum_{i=0}^{l} \sum_{j=0}^{m} \mathbf{b}_{i,j}\, B_i^l(u)\, B_j^m(v)\,, \qquad u, v \in [0,1]\,,$$

and **a tensor product Bézier volume** - briefly **TPB-volume** - of degree (l, m, n) is defined by

$$\mathbf{X}(u,v,w) = \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} \mathbf{b}_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w)\,, \qquad u, v, w \in [0,1]\,.$$

By reason of the tensor product definition the properties of Bézier surfaces and volumes are similar to the ones for curves and can easily be deduced from properties of the underlying Bézier curve scheme.
Also, as a consequence of the tensor product definition algorithms in u, in v and in w commute, and the result is independent of the order.

A rational Bézier curve of degree l in u is defined by

$$\mathbf{X}(u) = \frac{\sum_{i=0}^{l} \beta_i \mathbf{b}_i\, B_i^l(u)}{\sum_{i=0}^{l} \beta_i\, B_i^l(u)}\,, \qquad u \in [0,1]\,,$$

with weights $\beta_i \in \mathbb{R}$, and **rational TPB-surfaces** and **TPB-volumes** analogously.

If we demand positive weights, we have all the properties and algorithms for rational Bézier curves, surfaces and volumes which we have for non-rational representations.

For an extensive coverage of properties of Bernstein polynomials and Bézier representations see e.g. [Far 90], [Hos 91].

III. Composition of polynomial TPB-representations

III.1 Composition of Bézier curves and TPB-surfaces

Theorem 1 is fundamental for the composition $\mathbf{F}(t) = \mathbf{F}(\mathbf{K}(t)) = \mathbf{F}(u(t), v(t))$ of Bézier curves $\mathbf{K}(t)$ and TPB-surfaces $\mathbf{F}(u,v)$:

Theorem 1. Bézier curves and TPB-surfaces
Let $\mathbf{K}(t) : \mathbb{R} \to \mathbb{R}^2$ be a planar polynomial Bézier curve of degree N,

$$\mathbf{K}(t) \;=\; \sum_{I=0}^{N} \mathbf{k}_I \, B_I^N(t)\,, \qquad t \in [0,1]\,,$$

where $\mathbf{K}(t) = (u(t), v(t))$, and Bézier points $\mathbf{k}_I = (u_I, v_I)$. And let $\mathbf{F}(u,v) : \mathbb{R}^2 \to \mathbb{R}^d$, $d = 2,\, 3,$ be a polynomial TPB-surface of degree (l, m),

$$\mathbf{F}(u,v) \;=\; \sum_{i=0}^{l}\sum_{j=0}^{m} \mathbf{b}_{i,j} \, B_i^l(u)\, B_j^m(v)\,, \qquad u, v \in [0,1]\,,$$

where $\mathbf{F}(u,v) = (x(u,v), y(u,v), z(u,v))$, and with Bézier points $\mathbf{b}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})$. If $d = 2$ (2D-Solid): $z_{i,j} = 0$, for all i, j.

For each $r = \alpha + \beta$ where $\alpha \in \{0, \ldots, l\}$, $\beta \in \{0, \ldots, m\}$, i.e. $r \in \{0, \ldots, l+m\}$, we have

$$\mathbf{F}(t) \;=\; \mathbf{F}(\mathbf{K}(t)) \;=\; \sum_{R=0}^{rN} \mathbf{B}_R \, B_R^{rN}(t)\,, \tag{1}$$

where

$$\mathbf{B}_R \;=\; \sum_{|\mathbf{I}|=R} C_R^{\alpha,\beta}(N, \mathbf{I})\, \mathbf{F}^{\alpha,\beta} \tag{2}$$

with

$$\mathbf{F}^{\alpha,\beta} \;=\; \sum_{i=0}^{l-\alpha}\sum_{j=0}^{m-\beta} \mathbf{b}_{i,j}^{i+\alpha, j+\beta} \, B_i^{l-\alpha}(u(t))\, B_j^{m-\beta}(v(t))\,, \tag{3}$$

where $\mathbf{b}_{i,j}^{i+\alpha,j+\beta} = \mathbf{b}_{i,j}^{i+\alpha,j+\beta}(u^{\alpha}_{\mathbf{I}^u}, v^{\beta}_{\mathbf{I}^v})$, and with constants

$$C_R^{\alpha,\beta}(N, \mathbf{I}) \;=\; \frac{\displaystyle\prod_{Q^u=1}^{\alpha} \binom{N}{I^u_{Q^u}} \prod_{Q^v=1}^{\beta} \binom{N}{I^v_{Q^v}}}{\displaystyle\binom{rN}{R}}\,.$$

$\sum_{|\mathbf{I}|=R}$ has the meaning of summation over all $\mathbf{I} = (\mathbf{I}^u, \mathbf{I}^v)$ where $\mathbf{I}^u = (I_1^u, \ldots, I_\alpha^u)$, $\mathbf{I}^v = (I_1^v, \ldots, I_\beta^v)$ and where $0 \le I_1^u, \ldots, I_\alpha^u \le N$ and $0 \le I_1^v, \ldots, I_\beta^v \le N$ and $|\mathbf{I}| = |\mathbf{I}^u| + |\mathbf{I}^v| = I_1^u + \ldots + I_\alpha^u + I_1^v + \ldots + I_\beta^v = R$.

The $\mathbf{b}_{i,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^\alpha, v_{\mathbf{I}^v}^\beta)$ are defined recursively by de Casteljau's construction, i.e. for the u parameter direction by

$$\mathbf{b}_{i,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^\alpha, v_{\mathbf{I}^v}^\beta) = (1 - u_{I_\alpha^u})\, \mathbf{b}_{i,j}^{i+\alpha-1,j+\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^\beta) + u_{I_\alpha^u}\, \mathbf{b}_{i+1,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^\beta),$$

and for the v parameter direction by

$$\mathbf{b}_{i,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^\alpha, v_{\mathbf{I}^v}^\beta) = (1 - v_{I_\beta^v})\, \mathbf{b}_{i,j}^{i+\alpha,j+\beta-1}(u_{\mathbf{I}^u}^\alpha, v_{\mathbf{I}^v}^{\beta-1}) + v_{I_\beta^v}\, \mathbf{b}_{i,j+1}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^\alpha, v_{\mathbf{I}^v}^{\beta-1}),$$

where $\mathbf{b}_{i,j}^{i,j} = \mathbf{b}_{i,j}$.

According to (1), $\mathbf{F}(\mathbf{K}(t))$ is polynomial and can be represented as Bézier curve of degree rN. Bézier points of this representation are given as, (2), convex combinations of auxiliary points $\mathbf{F}^{\alpha,\beta}$ which are calculated, (3), for parameter values $(u_{\mathbf{I}^u}^\alpha, v_{\mathbf{I}^v}^\beta)$ via the *blossoming principle* (see e.g. [Ram 87]). This means: The argument $(u_{\mathbf{I}^u}^\alpha, v_{\mathbf{I}^v}^\beta)$ has the meaning that $\mathbf{b}_{i,j}^{i+\alpha,j+\beta}$ has to be calculated by performing α de Casteljau constructions in u direction for the u parameter values given by the indices $\mathbf{I}^u = (I_1^u, \ldots, I_\alpha^u)$, i.e. for the parameter values $u_{I_1^u}, \ldots, u_{I_\alpha^u}$ and β de Casteljau constructions in v direction for the v parameter values given by the indices $\mathbf{I}^v = (I_1^v, \ldots, I_\beta^v)$, i.e. for the parameter values $v_{I_1^v}, \ldots, v_{I_\beta^v}$. Calculations for different parameter values commute, and the order of performed calculations does not affect the final result.

To further illustrate the notation, we give the example of $l = m = 3$, $N = 3$ and $\alpha = 3$, $\beta = 2$. In this case $\mathbf{I}^u$ and $\mathbf{I}^v$ can be given by

$$\mathbf{I}^u = (0,0,2),\ \ \mathbf{I}^v = (0,1) \ \rightarrow\ \mathbf{b}_{i,j}^{i+3,j+2}(u_{\mathbf{I}^u}^3, v_{\mathbf{I}^v}^2) = \mathbf{b}_{i,j}^{i+3,j+2}(u_0, u_0, u_2, v_0, v_1)$$
$$\mathbf{I}^u = (0,1,1),\ \ \mathbf{I}^v = (0,3) \ \rightarrow\ \mathbf{b}_{i,j}^{i+3,j+2}(u_{\mathbf{I}^u}^3, v_{\mathbf{I}^v}^2) = \mathbf{b}_{i,j}^{i+3,j+2}(u_0, u_1, u_1, v_0, v_3)$$

for example, etc.

Next we prove the statement of Theorem 1.

Proof of Theorem 1. By induction on $r = \alpha + \beta$.

Base case. $r = 0$, i.e. $\alpha = \beta = 0$. This is trivially true, because for $r = 0$, Theorem 1 yields $\mathbf{F}(u, v)$.

Inductive hypothesis. We assume,

$$\mathbf{F}(t) = \mathbf{F}(\mathbf{K}(t)) = \sum_{R=0}^{rN} \mathbf{B}_R\, B_R^{rN}(t)$$

is valid. On the one side for $r = (\alpha - 1) + \beta$, i.e.

$$\mathbf{B}_R = \sum_{|\mathbf{I}|=R} C_R^{\alpha-1,\beta}(N, \mathbf{I})\, \mathbf{F}^{\alpha-1,\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^\beta)$$

and

$$\mathbf{F}^{\alpha-1,\beta} = \sum_{i=0}^{l-\alpha+1}\sum_{j=0}^{m-\beta} \mathbf{b}_{i,j}^{i+\alpha-1,j+\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta})\, B_i^{l-\alpha+1}(u(t))\, B_j^{m-\beta}(v(t))\,,$$

where $\mathbf{I}^u = (I_1^u, \ldots, I_{\alpha-1}^u)$ and $\mathbf{I}^v = (I_1^v, \ldots, I_\beta^v)$; on the other side for $r = \alpha + (\beta - 1)$, i.e.

$$\mathbf{B}_R = \sum_{|\mathbf{I}|=R} C_R^{\alpha,\beta-1}(N, \mathbf{I})\, \mathbf{F}^{\alpha,\beta-1}(u_{\mathbf{I}^u}^{\alpha}, v_{\mathbf{I}^v}^{\beta-1})$$

and

$$\mathbf{F}^{\alpha,\beta-1} = \sum_{i=0}^{l-\alpha}\sum_{j=0}^{m-\beta+1} \mathbf{b}_{i,j}^{i+\alpha,j+\beta-1}(u_{\mathbf{I}^u}^{\alpha}, v_{\mathbf{I}^v}^{\beta-1})\, B_i^{l-\alpha}(u(t))\, B_j^{m-\beta+1}(v(t))\,,$$

where $\mathbf{I}^u = (I_1^u, \ldots, I_\alpha^u)$ and $\mathbf{I}^v = (I_1^v, \ldots, I_{\beta-1}^v)$.

Inductive proof. Both cases are done similarly. We only prove the first case: Applying the recursive definition of the Bernstein polynomials to $B_i^{l-\alpha+1}(u(t))$ gives

$$B_i^{l-\alpha+1}(u(t)) = (1 - u(t))\, B_i^{l-\alpha}(u(t)) + u(t)\, B_{i-1}^{l-\alpha}(u(t))\,.$$

With that and an index transformation, the sum over i is

$$\sum_{i=0}^{l-\alpha} \left[(1 - u(t))\, \mathbf{b}_{i,j}^{i+\alpha-1,j+\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta}) + u(t)\, \mathbf{b}_{i+1,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta})\right] B_i^{l-\alpha}(u(t))\,.$$

Using the Bernstein representation of $u(t)$ and the partition of unity property of the Bernstein polynomials, the term in squared brackets can be written as

$$\sum_{I_\alpha^u=0}^{N} \left\{(1 - u_{I_\alpha^u})\, \mathbf{b}_{i,j}^{i+\alpha-1,j+\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta}) + u_{I_\alpha^u}\, \mathbf{b}_{i+1,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta})\right\} B_{I_\alpha^u}^{N}(t)\,.$$

Substituting

$$\mathbf{b}_{i,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^{\alpha}, v_{\mathbf{I}^v}^{\beta}) = (1 - u_{I_\alpha^u})\, \mathbf{b}_{i,j}^{i+\alpha-1,j+\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta}) + u_{I_\alpha^u}\, \mathbf{b}_{i+1,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta})\,,$$

where $\mathbf{I}^u = (I_1^u, \ldots, I_{\alpha-1}^u, I_\alpha^u)$, on the left side, while $\mathbf{I}^u = (I_1^u, \ldots, I_{\alpha-1}^u)$, on the right side, results in

$$\sum_{i=0}^{l-\alpha} \left[\sum_{I_\alpha^u=0}^{N} \mathbf{b}_{i,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^{\alpha}, v_{\mathbf{I}^v}^{\beta})\, B_{I_\alpha^u}^{N}(t)\right] B_i^{l-\alpha}(u(t))\,.$$

Therefore, the expression for $\mathbf{B}_R$ becomes

$$\mathbf{B}_R = \sum_{|\mathbf{I}|=R} C_R^{\alpha-1,\beta}(N, \mathbf{I}) \sum_{I_\alpha^u=0}^{N} \mathbf{F}^{\alpha,\beta}(u_{\mathbf{I}^u}^{\alpha}, v_{\mathbf{I}^v}^{\beta})\, B_{I_\alpha^u}^{N}(t)$$

with

$$\mathbf{F}^{\alpha,\beta} = \sum_{i=0}^{l-\alpha}\sum_{j=0}^{m-\beta} \mathbf{b}_{i,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^{\alpha}, v_{\mathbf{I}^v}^{\beta})\, B_i^{l-\alpha}(u(t))\, B_j^{m-\beta}(v(t))\,.$$

Regrouping the terms of the sums $\sum_{|\mathbf{I}|=R}$ and $\sum_{I_\alpha^u=0}^{N}$ into one sum and considering the product formula for Bernstein polynomials for the product of $B_R^{rN}(t)$ and $B_{I_\alpha^u}^N(t)$ completes the proof. □

Setting $\alpha = l$ and $\beta = m$ results in:

Corollary 1. Bézier curves and TPB-surfaces
Let $\mathbf{F}(u,v)$ and $\mathbf{K}(t)$ be given as in Theorem 1. For $\mathbf{F}(t) = \mathbf{F}(\mathbf{K}(t)) = \mathbf{F}(u(t), v(t))$ we have

$$\mathbf{F}(t) = \sum_{R=0}^{rN} \mathbf{B}_R \, B_R^{rN}(t),$$

where $r = l + m$ and with Bézier points

$$\mathbf{B}_R = \sum_{|\mathbf{I}|=R} C_R^{l,m}(N, \mathbf{I}) \, \mathbf{b}_{0,0}^{l,m}(u_{\mathbf{I}^u}^l, v_{\mathbf{I}^v}^m).$$

Remark 1. Bézier curves and surfaces

- **Triangle Bézier surface of degree n**
  Parameter lines and lines of parameter space in general position map to Bézier curves of degree n.
- **TPB-surface of degree (l,m)**
  Parameter lines map to Bézier curves of degree l or m, respectively.
  Lines of parameter space in general position map to Bézier curves of degree $l + m$.

Example 1. Straight Line
Let $\mathbf{F}(u,v)$ be a biquadratic tensor product surface, i.e. $l = m = 2 \rightarrow r = 4$. And let $\mathbf{K}(t)$ be a straight line (in general position), i.e. $N = 1 \rightarrow rN = 4$. Bézier points $\mathbf{B}_R$ of the quartic Bézier curve $\mathbf{F}(t)$ are convex combinations of points $\mathbf{b}_{0,0}^{2,2}(u_{\mathbf{I}^u}^2, v_{\mathbf{I}^v}^2)$:

$$\begin{aligned}
\mathbf{B}_0 &= \mathbf{b}_{0,0}^{2,2}(u_0, u_0, v_0, v_0) \\
\mathbf{B}_1 &= \tfrac{1}{4}\Big[\mathbf{b}_{0,0}^{2,2}(u_0, u_0, v_0, v_1) + \mathbf{b}_{0,0}^{2,2}(u_0, u_0, v_1, v_0) \\
&\qquad + \mathbf{b}_{0,0}^{2,2}(u_0, u_1, v_0, v_0) + \mathbf{b}_{0,0}^{2,2}(u_1, u_0, v_0, v_0)\Big] \\
\mathbf{B}_2 &= \tfrac{1}{6}\Big[\mathbf{b}_{0,0}^{2,2}(u_0, u_0, v_1, v_1) + \mathbf{b}_{0,0}^{2,2}(u_0, u_1, v_0, v_1) + \mathbf{b}_{0,0}^{2,2}(u_0, u_1, v_1, v_0) \\
&\qquad + \mathbf{b}_{0,0}^{2,2}(u_1, u_0, v_0, v_1) + \mathbf{b}_{0,0}^{2,2}(u_1, u_0, v_1, v_0) + \mathbf{b}_{0,0}^{2,2}(u_1, u_1, v_0, v_0)\Big] \\
\mathbf{B}_3 &= \tfrac{1}{4}\Big[\mathbf{b}_{0,0}^{2,2}(u_0, u_1, v_1, v_1) + \mathbf{b}_{0,0}^{2,2}(u_1, u_0, v_1, v_1) \\
&\qquad + \mathbf{b}_{0,0}^{2,2}(u_1, u_1, v_0, v_1) + \mathbf{b}_{0,0}^{2,2}(u_1, u_1, v_1, v_0)\Big] \\
\mathbf{B}_4 &= \mathbf{b}_{0,0}^{2,2}(u_1, u_1, v_1, v_1)
\end{aligned}$$

The auxiliary points $\mathbf{b}_{0,0}^{2,2}(u_{\mathbf{I}^u}^2, v_{\mathbf{I}^v}^2)$ are a result of the merging of several de Casteljau algorithms, i.e. they are given by the polar form (blossom) values of $\mathbf{F}(u,v)$ for parameter values $u \in \{u_0, u_1\}$ and $v \in \{v_0, v_1\}$.

Because de Casteljau constructions commute, and the polar form of $\mathbf{F}(u,v)$ is symmetric in the argument, we have

$$\mathbf{b}_{0,0}^{2,2}(u_0,u_1,v_0,v_0) \;=\; \mathbf{b}_{0,0}^{2,2}(u_0,v_0,u_1,v_0) \;=\; \mathbf{b}_{0,0}^{2,2}(u_1,v_0,u_0,v_0)\,,$$

and so on, and therefore,

$$\begin{aligned}
\mathbf{B}_0 &= \mathbf{b}_{0,0}^{2,2}(u_0,u_0,v_0,v_0)\\
\mathbf{B}_1 &= \tfrac{1}{2}\left[\mathbf{b}_{0,0}^{2,2}(u_0,u_0,v_0,v_1) + \mathbf{b}_{0,0}^{2,2}(u_0,u_1,v_0,v_0)\right]\\
\mathbf{B}_2 &= \tfrac{1}{6}\left[\mathbf{b}_{0,0}^{2,2}(u_0,u_0,v_1,v_1) + 4\,\mathbf{b}_{0,0}^{2,2}(u_0,u_1,v_0,v_1) + \mathbf{b}_{0,0}^{2,2}(u_1,u_1,v_0,v_0)\right]\\
\mathbf{B}_3 &= \tfrac{1}{2}\left[\mathbf{b}_{0,0}^{2,2}(u_0,u_1,v_1,v_1) + \mathbf{b}_{0,0}^{2,2}(u_1,u_1,v_0,v_1)\right]\\
\mathbf{B}_4 &= \mathbf{b}_{0,0}^{2,2}(u_1,u_1,v_1,v_1)
\end{aligned}$$

Fig. 2a illustrates $\mathbf{K}(t) = (u(t), v(t))$, having Bézier points $\mathbf{k}_I = (u_I, v_I)$, embedded in the domain of $\mathbf{F}(u,v)$. Auxiliary points defined by the intersection of parameter lines and given by (u_0,v_1) and (u_1,v_0) are marked, too.
Fig. 2b gives the Bézier net of the biquadratic tensor product Bézier surface.
Fig. 2c shows the result of one de Casteljau step in u-direction and one de Casteljau step in v-direction, for all possible (i.e. four) pairs of parameter values. Thus, Fig. 2c shows the image of the parameter space situation under the affine map of $\mathbf{F}(u,v)$ for each of the quadrilaterals of the Bézier net.

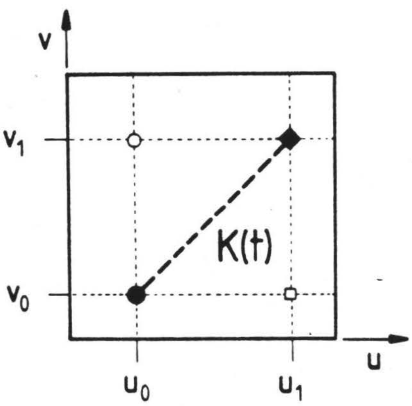

Fig.2a. $\mathbf{K}(t)$ and domain of $\mathbf{F}(u,v)$

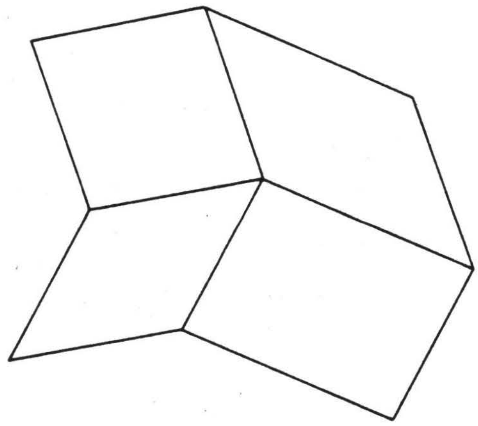
Fig.2b. Bézier net of $\mathbf{F}(u,v)$

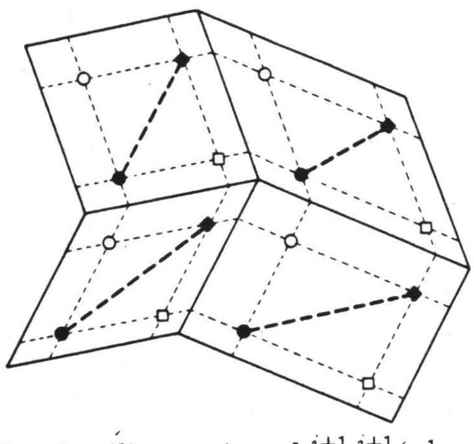
Fig. 2c. Auxiliary points $\mathbf{b}_{i,j}^{i+1,j+1}(u_{\mathbf{I}^u}^1, v_{\mathbf{I}^v}^1)$

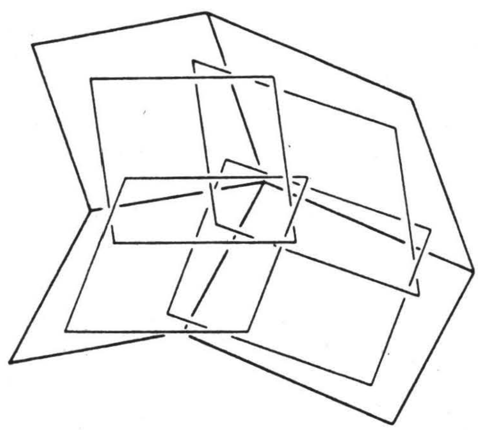
Fig. 2d. New bilinear Bézier nets

Fig. 2d constructs four new bilinear Bézier nets using the auxiliary points $\mathbf{b}_{i,j}^{i+1,j+1}(u^1_{\mathbf{I}^u}, v^1_{\mathbf{I}^v})$ of Fig. 2c.
Fig. 2e repeats the procedure for each of the four bilinear Bézier nets, resulting in 16 auxiliary points $\mathbf{b}_{0,0}^{2,2}(u^2_{\mathbf{I}^u}, v^2_{\mathbf{I}^v})$. Only nine of them are distinct (see the remark above). This is due to the connection with the polar form of $\mathbf{F}(u,v)$.
Fig. 2f depicts the construction of Bézier points $\mathbf{B}_R$ as convex combinations of auxiliary points $\mathbf{b}_{0,0}^{2,2}(u^2_{\mathbf{I}^u}, v^2_{\mathbf{I}^v})$.

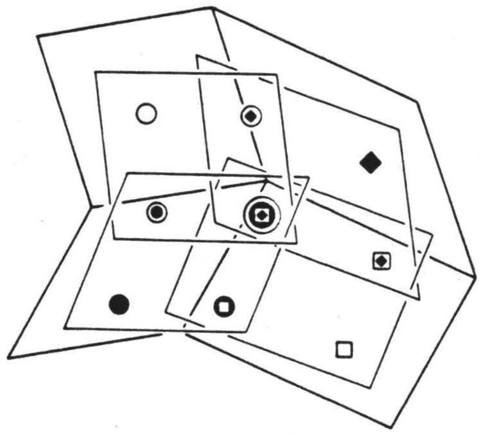

Fig. 2e. Auxiliary points $\mathbf{b}_{0,0}^{2,2}(u^2_{\mathbf{I}^u}, v^2_{\mathbf{I}^v})$

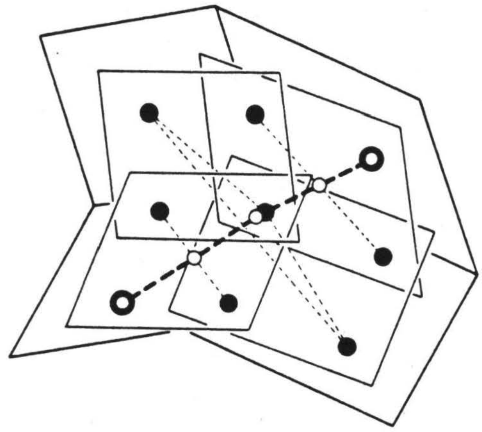

Fig. 2f. Bézier points $\mathbf{B}_R$

Auxiliary points $\mathbf{b}_{0,0}^{2,2}(u^2_{\mathbf{I}^u}, v^2_{\mathbf{I}^v})$ are at the same time Bézier points of the surface subsegment of $\mathbf{F}(u,v)$ which is defined for $u \in [u_0, u_1]$ and $v \in [v_0, v_1]$ (compare with the subdivision procedure for Bézier triangles [Gol 83], see also [Ros 88]).

III.2 Composition of TPB-surfaces and TPB-volumes

The foundation for the composition $\mathbf{V}(\mu,\nu) = \mathbf{V}(\mathbf{F}(\mu,\nu)) = \mathbf{V}(u(\mu,\nu), v(\mu,\nu), w(\mu,\nu))$ of TPB-surfaces $\mathbf{F}(\mu,\nu)$ and TPB-volumes $\mathbf{V}(u,v,w)$ is given by Theorem 2:

Theorem 2. TPB-surfaces and TPB-volumes
Let $\mathbf{F}(\mu,\nu) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a polynomial TPB-surface of degree (L, M),

$$\mathbf{F}(\mu,\nu) \;=\; \sum_{I=0}^{L}\sum_{J=0}^{M} \mathbf{f}_{I,J}\; B_I^L(\mu)\; B_J^M(\nu)\,, \qquad \mu,\nu \in [0,1]\,,$$

where $\mathbf{F}(\mu,\nu) = (u(\mu,\nu), v(\mu,\nu), w(\mu,\nu))$, and Bézier points $\mathbf{f}_{I,J} = (u_{I,J}, v_{I,J}, w_{I,J})$. And let $\mathbf{V}(u,v,w) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a polynomial TPB-volume of degree (l, m, n),

$$\mathbf{V}(u,v,w) \;=\; \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \mathbf{b}_{i,j,k}\; B_i^l(u)\; B_j^m(v)\; B_k^n(w)\,, \qquad u,v,w \in [0,1]\,,$$

where $\mathbf{V}(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$, Bézier points $\mathbf{b}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})$.
For each $r = \alpha + \beta + \gamma$ where $\alpha \in \{0,\ldots,l\}$ and $\beta \in \{0,\ldots,m\}$ and $\gamma \in \{0,\ldots,n\}$, i.e. $r \in \{0,\ldots,l+m+n\}$, we have

$$\mathbf{V}(\mu,\nu) \;=\; \mathbf{V}(\mathbf{F}(\mu,\nu)) \;=\; \sum_{R=0}^{rL}\sum_{S=0}^{rM} \mathbf{B}_{R,S}\, B_R^{rL}(\mu)\, B_S^{rM}(\nu)\,, \tag{4}$$

where

$$\mathbf{B}_{RS} \;=\; \sum_{|\mathbf{I}|=R}\sum_{|\mathbf{J}|=S} C_R^{\alpha,\beta,\gamma}(L,\mathbf{I})\, C_S^{\alpha,\beta,\gamma}(M,\mathbf{J})\, \mathbf{V}^{\alpha,\beta,\gamma} \tag{5}$$

with

$$\mathbf{V}^{\alpha,\beta,\gamma} \;=\; \sum_{i=0}^{l-\alpha}\sum_{j=0}^{m-\beta}\sum_{k=0}^{n-\gamma} \mathbf{b}_{i,j,k}^{i+\alpha,j+\beta,k+\gamma}\, B_i^{l-\alpha}(u(\mu,\nu))\, B_j^{m-\beta}(v(\mu,\nu))\, B_k^{n-\gamma}(w(\mu,\nu))\,, \tag{6}$$

where $\mathbf{b}_{i,j,k}^{i+\alpha,j+\beta,k+\gamma} = \mathbf{b}_{i,j,k}^{i+\alpha,j+\beta,k+\gamma}(u^{\alpha}_{\mathbf{I}^u,\mathbf{J}^u}, v^{\beta}_{\mathbf{I}^v,\mathbf{J}^v}, w^{\gamma}_{\mathbf{I}^w,\mathbf{J}^w})$, and with constants

$$C_R^{\alpha,\beta,\gamma}(L,\mathbf{I}) \;=\; \frac{\prod_{Q^u=1}^{\alpha}\binom{L}{I^u_{Q^u}}\;\prod_{Q^v=1}^{\beta}\binom{L}{I^v_{Q^v}}\;\prod_{Q^w=1}^{\gamma}\binom{L}{I^w_{Q^w}}}{\binom{rL}{R}}$$

and $C_S^{\alpha,\beta,\gamma}(M,\mathbf{J})$ similarly.

$\sum_{|\mathbf{I}|=R}$ has the same meaning as for Theorem 1, but with $\mathbf{I} = (\mathbf{I}^u, \mathbf{I}^v, \mathbf{I}^w)$, and $\sum_{|\mathbf{J}|=S}$ analogously. $\mathbf{b}_{i,j,k}^{i+\alpha,j+\beta,k+\gamma}$ is defined recursively by de Casteljau's construction.

According to (4), $\mathbf{V}(\mathbf{F}(\mu,\nu))$ is polynomial and can be represented as TPB-surface of degree (rL, rM). Bézier points of this representation are given as, (5), convex combinations of auxiliary points $\mathbf{V}^{\alpha,\beta,\gamma}$ which are calculated, (6), for parameter values $(u^{\alpha}_{\mathbf{I}^u,\mathbf{J}^u}, v^{\beta}_{\mathbf{I}^v,\mathbf{J}^v}, w^{\gamma}_{\mathbf{I}^w,\mathbf{J}^w})$ via the *blossoming principle* (cf. Section 1) applying de Casteljau's algorithm.

Proof of Theorem 2. Essentially like the proof for Theorem 1. The difference is that higher dimensions are involved, and to see how this will be put down into the calculations. Because of the similarity to the proof of Theorem 1, only the proof of the inductive statement for the u parameter direction is drawn out here. Thus, we assume,

$$\mathbf{V}(\mu,\nu) \;=\; \mathbf{V}(\mathbf{F}(\mu,\nu)) \;=\; \sum_{R=0}^{rL}\sum_{S=0}^{rM} \mathbf{B}_{R,S}\, B_R^{rL}(\mu)\, B_S^{rM}(\nu)\,,$$

with $r = (\alpha-1)+\beta+\gamma$ is valid, i.e.

$$\mathbf{B}_{R,S} \;=\; \sum_{|\mathbf{I}|=R}\sum_{|\mathbf{J}|=S} C_R^{\alpha-1,\beta,\gamma}(L,\mathbf{I})\, C_S^{\alpha-1,\beta,\gamma}(M,\mathbf{J})\, \mathbf{V}^{\alpha-1,\beta,\gamma}$$

with

$$\mathbf{V}^{\alpha-1,\beta,\gamma} \;=\; \sum_{i=0}^{l-\alpha+1}\sum_{j=0}^{m-\beta} \mathbf{b}_{i,j,k}^{i+\alpha-1,j+\beta,k+\gamma}\, B_i^{l-\alpha+1}(u(\mu,\nu))\, B_j^{m-\beta}(v(\mu,\nu))\, B_k^{n-\gamma}(w(\mu,\nu))\,,$$

where $\mathbf{b}_{i,j,k}^{i+\alpha-1,j+\beta,k+\gamma} = \mathbf{b}_{i,j,k}^{i+\alpha-1,j+\beta,k+\gamma}(u_{\mathbf{I}^u,\mathbf{J}^u}^{\alpha-1}, v_{\mathbf{I}^v,\mathbf{J}^v}^{\beta}, w_{\mathbf{I}^w,\mathbf{J}^w}^{\gamma})$, and $\mathbf{I} = (\mathbf{I}^u, \mathbf{I}^v, \mathbf{I}^w)$ with $\mathbf{I}^u = (I_1^u, \ldots, I_{\alpha-1}^u)$, $\mathbf{I}^v = (I_1^v, \ldots, I_\beta^v)$, $\mathbf{I}^w = (I_1^w, \ldots, I_\gamma^w)$, and $\mathbf{J} = (\mathbf{J}^u, \mathbf{J}^v, \mathbf{J}^w)$ with $\mathbf{J}^u = (J_1^u, \ldots, J_{\alpha-1}^u)$, $\mathbf{J}^v = (J_1^v, \ldots, J_\beta^v)$, $\mathbf{J}^w = (J_1^w, \ldots, J_\gamma^w)$.

As in the proof of Theorem 1, we get for the sum over i

$$\sum_{i=0}^{l-\alpha} \left[(1 - u(\mu,\nu))\, \mathbf{b}_{i,j,k}^{i+\alpha-1,j+\beta,k+\gamma} + u(\mu,\nu)\, \mathbf{b}_{i+1,j,k}^{i+\alpha,j+\beta,k+\gamma}\right] B_i^{l-\alpha}(u(\mu,\nu)),$$

where the argument of $\mathbf{b}_{***}^{***}$ is $(u_{\mathbf{I}^u,\mathbf{J}^u}^{\alpha-1}, v_{\mathbf{I}^v,\mathbf{J}^v}^{\beta}, w_{\mathbf{I}^w,\mathbf{J}^w}^{\gamma})$.

Now, the term in squared brackets can be written as

$$\sum_{I_\alpha^u=0}^{L} \sum_{J_\alpha^u=0}^{M} \left\{(1 - u_{I_\alpha^u,J_\alpha^u})\, \mathbf{b}_{i,j,k}^{i+\alpha-1,j+\beta,k+\gamma} + u_{I_\alpha^u,J_\alpha^u}\, \mathbf{b}_{i+1,j,k}^{i+\alpha,j+\beta,k+\gamma}\right\} B_{I_\alpha^u}^{L}(\mu)\, B_{J_\alpha^u}^{M}(\nu).$$

Using the substitution

$$\mathbf{b}_{i,j,k}^{i+\alpha,j+\beta,k+\gamma} = (1 - u_{I_\alpha^u,J_\alpha^u})\, \mathbf{b}_{i,j,k}^{i+\alpha-1,j+\beta,k+\gamma} + u_{I_\alpha^u,J_\alpha^u}\, \mathbf{b}_{i+1,j,k}^{i+\alpha,j+\beta,k+\gamma},$$

where the argument is $(u_{\mathbf{I}^u,\mathbf{J}^u}^{\alpha}, v_{\mathbf{I}^v,\mathbf{J}^v}^{\beta}, w_{\mathbf{I}^w,\mathbf{J}^w}^{\gamma})$ with $\mathbf{I}^u = (I_1^u, \ldots, I_{\alpha-1}^u, I_\alpha^u)$, on the left side, while it is $(u_{\mathbf{I}^u,\mathbf{J}^u}^{\alpha-1}, v_{\mathbf{I}^v,\mathbf{J}^v}^{\beta}, w_{\mathbf{I}^w,\mathbf{J}^w}^{\gamma})$ with $\mathbf{I}^u = (I_1^u, \ldots, I_{\alpha-1}^u)$, on the right side, yields

$$\sum_{i=0}^{l-\alpha} \left[\sum_{I_\alpha^u=0}^{L} \sum_{J_\alpha^u=0}^{M} \mathbf{b}_{i,j,k}^{i+\alpha,j+\beta,k+\gamma}(u_{\mathbf{I}^u,\mathbf{J}^u}^{\alpha}, v_{\mathbf{I}^v,\mathbf{J}^v}^{\beta}, w_{\mathbf{I}^w,\mathbf{J}^w}^{\gamma})\, B_{I_\alpha^u}^{N}(t)\right] B_i^{l-\alpha}(u(\mu,\nu)).$$

Therefore, the expression for $\mathbf{B}_{R,S}$ becomes

$$\mathbf{B}_{R,S} = \sum_{|\mathbf{I}|=R} \sum_{|\mathbf{J}|=S} C_R^{\alpha-1,\beta,\gamma}(L,\mathbf{I})\, C_S^{\alpha-1,\beta,\gamma}(M,\mathbf{J}) \sum_{I_\alpha^u=0}^{L} \sum_{J_\alpha^u=0}^{M} \mathbf{V}^{\alpha,\beta,\gamma}\, B_{I_\alpha^u}^{L}(\mu)\, B_{J_\alpha^u}^{M}(\nu)$$

with $\mathbf{V}^{\alpha,\beta,\gamma} = \mathbf{V}^{\alpha,\beta,\gamma}(u_{\mathbf{I}^u,\mathbf{J}^u}^{\alpha}, v_{\mathbf{I}^v,\mathbf{J}^v}^{\beta}, w_{\mathbf{I}^w,\mathbf{J}^w}^{\gamma})$, now given as in (6).

Regrouping the terms of the sums $\sum_{|\mathbf{I}|=R}$ and $\sum_{I_\alpha^u=0}^{L}$ into one sum, as well as the terms of the sums $\sum_{|\mathbf{J}|=S}$ and $\sum_{J_\alpha^u=0}^{M}$ into another sum, and considering the product formula for Bernstein polynomials for the product of $B_R^{rL}(\mu)$ and $B_{I_\alpha^u}^{L}(\mu)$, and for the product of $B_S^{rM}(\nu)$ and $B_{J_\alpha^u}^{M}(\nu)$, this completes the proof. □

Setting $\alpha = l$, $\beta = m$ and $\gamma = n$ results in:

Corollary 2. Bézier surfaces and TPB-volumes

Let $\mathbf{V}(u,v,w)$ and $\mathbf{F}(\mu,\nu)$ be given as in Theorem 2. For $\mathbf{V}(\mu,\nu) = \mathbf{V}(\mathbf{F}(\mu,\nu)) = \mathbf{V}(u(\mu,\nu), v(\mu,\nu), w(\mu,\nu))$ we have

$$\mathbf{V}(\mu,\nu) = \sum_{R=0}^{rL} \sum_{S=0}^{rM} \mathbf{B}_{R,S}\, B_R^{rL}(\mu)\, B_S^{rM}(\nu),$$

where $r = l + m + n$ and with Bézier points

$$\mathbf{B}_{R,S} = \sum_{|\mathbf{I}|=R} \sum_{|\mathbf{J}|=S} C_R^{l,m,n}(L,\mathbf{I})\, C_S^{l,m,n}(M,\mathbf{J})\, \mathbf{b}_{0,0,0}^{l,m,n}(u_{\mathbf{I}^u,\mathbf{J}^u}^{l}, v_{\mathbf{I}^v,\mathbf{J}^v}^{m}, w_{\mathbf{I}^w,\mathbf{J}^w}^{n}).$$

Remark 2. Bézier surfaces and volumes

- **Tetrahedra Bézier volume of degree n**
  Parameter planes and planes of parameter space in general position map to triangle Bézier surfaces of degree n.
- **TPB-volume of degree (l,m,n)**
  Parameter planes map to tensor product Bézier surfaces of degree (l, m), (l, n) or (m, n), respectively.
  Planes of parameter space in general position map to tensor product Bézier surfaces of degree $(l + m + n, l + m + n)$.

Now, a generalization to the case of composing tensor products of arbitrary dimensions is straight forward.

IV. Composition of rational TPB-representations

IV.1 Composition of rational TPB-curves and TPB-surfaces

Theorem 3 forms the foundation for the composition of rational curves $\mathbf{K}(t)$ and rational TPB-surfaces $\mathbf{F}(u, v)$:

Theorem 3. Rational Bézier curves and TPB-surfaces
Let $\mathbf{K}(t) : \mathbb{R} \to \mathbb{R}^2$ be a planar rational Bézier curve of degree N,

$$\mathbf{K}(t) = \frac{\sum\limits_{I=0}^{N} \beta_I \mathbf{k}_I \, B_I^N(t)}{\sum\limits_{I=0}^{N} \beta_I \, B_I^N(t)}, \qquad t \in [0, 1],$$

where $\mathbf{K}(t) = (u(t), v(t))$, Bézier points $\mathbf{k}_I = (u_I, v_I)$, and weights $\beta_I \in \mathbb{R}$. And let $\mathbf{F}(u, v) : \mathbb{R}^2 \to \mathbb{R}^d, d = 2,\ 3,$ be a rational TPB-surface of degree (l, m),

$$\mathbf{F}(u, v) = \frac{\sum\limits_{i=0}^{l}\sum\limits_{j=0}^{m} \omega_{i,j} \mathbf{b}_{i,j} \, B_i^l(u) \, B_j^m(v)}{\sum\limits_{i=0}^{l}\sum\limits_{j=0}^{m} \omega_{i,j} \, B_i^l(u) \, B_j^m(v)}, \qquad u, v \in [0, 1],$$

where $\mathbf{F}(u,v) = (x(u,v), y(u,v), z(u,v))$, Bézier points $\mathbf{b}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})$, and weights $\omega_{i,j} \in \mathbb{R}$. If $d = 2$ (2D-Solid): $z_{i,j} = 0$, for all i, j.

For each $r = \alpha + \beta$ where $\alpha \in \{0, \ldots, l\}$, $\beta \in \{0, \ldots, m\}$, i.e. $r \in \{0, \ldots, l + m\}$, we have

$$\mathbf{F}(t) = \mathbf{F}(\mathbf{K}(t)) = \frac{\sum\limits_{R=0}^{rN} \Omega_R \mathbf{B}_R \, B_R^{rN}(t)}{\sum\limits_{R=0}^{rN} \Omega_R \, B_R^{rN}(t)}, \tag{7}$$

where

$$\Omega_R \mathbf{B}_R = \sum_{|\mathbf{I}|=R} B_R^{\alpha,\beta}(N,\mathbf{I})\ \mathbf{F}^{\alpha,\beta}$$
$$\Omega_R = \sum_{|\mathbf{I}|=R} B_R^{\alpha,\beta}(N,\mathbf{I})\ G^{\alpha,\beta} \tag{8}$$

with

$$\mathbf{F}^{\alpha,\beta} = \sum_{i=0}^{l-\alpha}\sum_{j=0}^{m-\beta} \omega_{i,j}^{i+\alpha,j+\beta}\mathbf{b}_{i,j}^{i+\alpha,j+\beta}\ B_i^{l-\alpha}(u(t))\ B_j^{m-\beta}(v(t))$$
$$G^{\alpha,\beta} = \sum_{i=0}^{l-\alpha}\sum_{j=0}^{m-\beta} \omega_{i,j}^{i+\alpha,j+\beta}\ B_i^{l-\alpha}(u(t))\ B_j^{m-\beta}(v(t))\,, \tag{9}$$

where $\mathbf{b}_{i,j}^{i+\alpha,j+\beta} = \mathbf{b}_{i,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^{\alpha}, v_{\mathbf{I}^v}^{\beta})$, $\omega_{i,j}^{i+\alpha,j+\beta} = \omega_{i,j}^{i+\alpha,j+\beta}(u_{\mathbf{I}^u}^{\alpha}, v_{\mathbf{I}^v}^{\beta})$, and with constants

$$B_R^{\alpha,\beta}(N,\mathbf{I}) = \frac{\prod_{Q^u=1}^{\alpha} \beta_{I_{Q^u}^u}\binom{N}{I_{Q^u}^u}\ \prod_{Q^v=1}^{\beta} \beta_{I_{Q^v}^v}\binom{N}{I_{Q^v}^v}}{\binom{rN}{R}}\,.$$

$\sum_{|\mathbf{I}|=R}$ and $\mathbf{I} = (\mathbf{I}^u, \mathbf{I}^v)$ have the same meaning as for Theorem 1. $\omega_{i,j}^{i+\alpha,j+\beta}\mathbf{b}_{i,j}^{i+\alpha,j+\beta}$ and $\mathbf{b}_{i,j}^{i+\alpha,j+\beta}$ are defined recursively by de Casteljau's construction, analogously to Theorem 1.

Proof of Theorem 3. Essentially like the foregoing proofs. The difference is that rational representations are now involved. Therefore, only the proof of the statement for the u parameter direction is drawn out briefly: We assume, (7) is valid, with

$$\Omega_R \mathbf{B}_R = \sum_{|\mathbf{I}|=R} B_R^{\alpha-1,\beta}(N,\mathbf{I})\ \mathbf{F}^{\alpha-1,\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta})$$
$$\Omega_R = \sum_{|\mathbf{I}|=R} B_R^{\alpha-1,\beta}(N,\mathbf{I})\ G^{\alpha-1,\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta})$$

and

$$\mathbf{F}^{\alpha-1,\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta}) = \sum_{i=0}^{l-\alpha+1}\sum_{j=0}^{m-\beta} \omega_{i,j}^{i+\alpha-1,j+\beta}\mathbf{b}_{i,j}^{i+\alpha-1,j+\beta}\ B_i^{l-\alpha+1}(u(t))\ B_j^{m-\beta}(v(t))$$
$$G^{\alpha-1,\beta}(u_{\mathbf{I}^u}^{\alpha-1}, v_{\mathbf{I}^v}^{\beta}) = \sum_{i=0}^{l-\alpha+1}\sum_{j=0}^{m-\beta} \omega_{i,j}^{i+\alpha-1,j+\beta}\ B_i^{l-\alpha+1}(u(t))\ B_j^{m-\beta}(v(t))\,,$$

where $\mathbf{I} = (\mathbf{I}^u, \mathbf{I}^v)$ and $\mathbf{I}^u = (I_1^u, \ldots, I_{\alpha-1}^u)$, $\mathbf{I}^v = (I_1^v, \ldots, I_\beta^v)$. Then, as above, we first get

$$\sum_{i=0}^{l-\alpha} \left[(1-u(t))\,\omega_{i,j}^{i+\alpha-1,j+\beta}\mathbf{b}_{i,j}^{i+\alpha-1,j+\beta} + u(t)\,\omega_{i+1,j}^{i+\alpha,j+\beta}\mathbf{b}_{i+1,j}^{i+\alpha,j+\beta}\right] B_i^{l-\alpha}(u(t))$$
$$\sum_{i=0}^{l-\alpha} \left[(1-u(t))\,\omega_{i,j}^{i+\alpha-1,j+\beta} + u(t)\,\omega_{i+1,j}^{i+\alpha,j+\beta}\right] B_i^{l-\alpha}(u(t))\,,$$

and in the following

$$\sum_{i=0}^{l-\alpha}\left[\sum_{I^u_\alpha=0}^{N} \omega_{i,j}^{i+\alpha,j+\beta}\mathbf{b}_{i,j}^{i+\alpha,j+\beta}\, B^N_{I^u_\alpha}(t)\right] B_i^{l-\alpha}(u(t)) \Big/ \left(\sum_{I=0}^{N}\beta_I\, B^N_I(t)\right)^\alpha$$

$$\sum_{i=0}^{l-\alpha}\left[\sum_{I^u_\alpha=0}^{N} \omega_{i,j}^{i+\alpha,j+\beta}\, B^N_{I^u_\alpha}(t)\right] B_i^{l-\alpha}(u(t)) \Big/ \left(\sum_{I=0}^{N}\beta_I\, B^N_I(t)\right)^\alpha ,$$

where we have used the substitutions

$$\begin{aligned} \omega_{i,j}^{i+\alpha,j+\beta}\mathbf{b}_{i,j}^{i+\alpha,j+\beta} &= (1-u_{I^u_\alpha})\,\omega_{i,j}^{i+\alpha-1,j+\beta}\mathbf{b}_{i,j}^{i+\alpha-1,j+\beta} + u_{I^u_\alpha}\,\omega_{i+1,j}^{i+\alpha,j+\beta}\mathbf{b}_{i+1,j}^{i+\alpha,j+\beta} \\ \omega_{i,j}^{i+\alpha,j+\beta} &= (1-u_{I^u_\alpha})\,\omega_{i,j}^{i+\alpha-1,j+\beta} + u_{I^u_\alpha}\,\omega_{i+1,j}^{i+\alpha,j+\beta} . \end{aligned}$$

Regrouping the terms of the sums in numerator and denominator results in

$$\sum_{|\mathbf{I}|=R} B_R^{\alpha-1,\beta}(N,\mathbf{I}) \sum_{I^u_\alpha=0}^{N} \mathbf{F}^{\alpha,\beta}(u^\alpha_{\mathbf{I}^u}, v^\beta_{\mathbf{I}^v})\, B^N_{I^u_\alpha}(t) \Big/ \left(\sum_{I=0}^{N}\beta_I\, B^N_I(t)\right)^\alpha$$

$$\sum_{|\mathbf{I}|=R} B_R^{\alpha-1,\beta}(N,\mathbf{I}) \sum_{I^u_\alpha=0}^{N} G^{\alpha,\beta}(u^\alpha_{\mathbf{I}^u}, v^\beta_{\mathbf{I}^v})\, B^N_{I^u_\alpha}(t) \Big/ \left(\sum_{I=0}^{N}\beta_I\, B^N_I(t)\right)^\alpha$$

with $\mathbf{F}^{\alpha,\beta} = \mathbf{F}^{\alpha,\beta}(u^\alpha_{\mathbf{I}^u}, v^\beta_{\mathbf{I}^v})$ and $\mathbf{G}^{\alpha,\beta} = \mathbf{G}^{\alpha,\beta}(u^\alpha_{\mathbf{I}^u}, v^\beta_{\mathbf{I}^v})$, now given like in (9).

Regrouping the terms of the sums $\sum_{|\mathbf{I}|=R}$ and $\sum_{I^u_\alpha=0}^{N}$ into one sum, and considering the product formula for Bernstein polynomials for the product of $B_R^{rN}(t)$ and $B^N_{I^u_\alpha}(t)$ completes the proof, because the term $\left(\sum_{I=0}^{N}\beta_I\, B^N_I(t)\right)^\alpha$ cancels out when forming the ratio of $\mathbf{F}(\mathbf{K}(t))$. □

Setting $\alpha = l$ and $\beta = m$ results in:

Corollary 3. Rational Bézier curves and TPB-surfaces
Let $\mathbf{F}(u,v)$ and $\mathbf{K}(t)$ be given as in Theorem 3. For $\mathbf{F}(t) = \mathbf{F}(\mathbf{K}(t)) = \mathbf{F}(u(t), v(t))$ we have

$$\mathbf{F}(t) = \frac{\sum_{R=0}^{rN} \Omega_R \mathbf{B}_R\, B_R^{rN}(t)}{\sum_{R=0}^{rN} \Omega_R\, B_R^{rN}(t)} ,$$

where $r = l + m$ and weights are given by

$$\Omega_R = \sum_{|\mathbf{I}|=R} B_R^{l,m}(N,\mathbf{I})\, \omega_{0,0}^{l,m}(u^l_{\mathbf{I}^u}, v^m_{\mathbf{I}^v}) ,$$

and Bézier points are given by $\mathbf{B}_R = \frac{\Omega_R \mathbf{B}_R}{\Omega_R}$, where

$$\Omega_R \mathbf{B}_R = \sum_{|\mathbf{I}|=R} B_R^{l,m}(N,\mathbf{I})\, \omega_{0,0}^{l,m}(u^l_{\mathbf{I}^u}, v^m_{\mathbf{I}^v})\mathbf{b}_{0,0}^{l,m}(u^l_{\mathbf{I}^u}, v^m_{\mathbf{I}^v}) .$$

Remark 3. Rational Bézier curves and surfaces

- **Rational triangle Bézier surface of degree n**
  Parameter lines and lines of parameter space in general position map to rational Bézier curves of degree n.

- **Rational TPB-surface of degree (l,m)**
  Parameter lines map to rational Bézier curves of degree l or m, respectively. Lines of parameter space in general position map to rational Bézier curves of degree $l+m$.

Note, that three important special cases are included in Theorem 3 and Corollary 3:

- If $\mathbf{K}(t)$ is polynomial (i.e. $\beta_I = 1$, for all I) and $\mathbf{F}(u,v)$ is polynomial (i.e. $\omega_{i,j} = 1$, for all i,j):
  The statement of Theorem 1 results.

- If $\mathbf{K}(t)$ is polynomial (i.e. $\beta_I = 1$, for all I) and $\mathbf{F}(u,v)$ is rational:
  Constants $B_R^{\alpha,\beta}(N,\mathbf{I})$ reduce to constants $C_R^{\alpha,\beta}(N,\mathbf{I})$ of Theorem 1, but $\mathbf{F}(\mathbf{K}(t))$ is (still) rational.

- If $\mathbf{K}(t)$ is rational and $\mathbf{F}(u,v)$ is polynomial (i.e. $\omega_{i,j} = 1$, for all i,j):
  $G^{\alpha,\beta} = 1$, for all α,β, $\omega_{0,0}^{l,m} = 1$, and $\mathbf{F}^{\alpha,\beta}$ is given as in Theorem 1, but $\mathbf{F}(\mathbf{K}(t))$ is (still) rational.

IV.2 Composition of rational TPB-surfaces and TPB-volumes

Theorem 4. Rational TPB-surfaces and TPB-volumes

Let $\mathbf{F}(\mu,\nu) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a rational TPB-surface of degree (L,M),

$$\mathbf{F}(\mu,\nu) = \frac{\sum_{I=0}^{L}\sum_{J=0}^{M} \beta_{I,J}\mathbf{f}_{I,J}\, B_I^L(\mu)\, B_J^M(\nu)}{\sum_{I=0}^{L}\sum_{J=0}^{M} \beta_{I,J}\, B_I^L(\mu)\, B_J^M(\nu)}, \qquad \mu,\nu \in [0,1],$$

where $\mathbf{F}(\mu,\nu) = (u(\mu,\nu), v(\mu,\nu), w(\mu,\nu))$, Bézier points $\mathbf{f}_{I,J} = (u_{I,J}, v_{I,J}, w_{I,J})$, and weights $\beta_{I,J} \in \mathbb{R}$.

Let $\mathbf{V}(u,v,w) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a rational TPB-volume of degree (l,m,n),

$$\mathbf{V}(u,v,w) = \frac{\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \omega_{i,j,k}\mathbf{b}_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w)}{\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \omega_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w)}, \qquad u,v,w \in [0,1],$$

where $\mathbf{V}(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$, Bézier points $\mathbf{b}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})$ and weights $\omega_{i,j,k} \in \mathbb{R}$.

For each $r = \alpha + \beta + \gamma$ where $\alpha \in \{0,\ldots,l\}$ and $\beta \in \{0,\ldots,m\}$ and $\gamma \in \{0,\ldots,n\}$, i.e. $r \in \{0,\ldots,l+m+n\}$, we have

$$\mathbf{V}(\mu,\nu) \;=\; \mathbf{V}(\mathbf{F}(\mu,\nu)) \;=\; \frac{\sum_{R=0}^{rL}\sum_{S=0}^{rM} \Omega_{R,S}\mathbf{B}_{R,S}\, B_R^{rL}(\mu)\, B_S^{rM}(\nu)}{\sum_{R=0}^{rL}\sum_{S=0}^{rM} \Omega_{R,S}\, B_R^{rL}(\mu)\, B_S^{rM}(\nu)}\,, \tag{10}$$

where

$$\begin{aligned} \Omega_{R,S}\mathbf{B}_{R,S} &= \sum_{|\mathbf{I}|=R}\sum_{|\mathbf{J}|=S} B_R^{\alpha,\beta,\gamma}(L,\mathbf{I})\, B_S^{\alpha,\beta,\gamma}(M,\mathbf{J})\, \mathbf{V}^{\alpha,\beta,\gamma} \\ \Omega_{R,S} &= \sum_{|\mathbf{I}|=R}\sum_{|\mathbf{J}|=S} B_R^{\alpha,\beta,\gamma}(L,\mathbf{I})\, B_S^{\alpha,\beta,\gamma}(M,\mathbf{J})\, G^{\alpha,\beta,\gamma} \end{aligned} \tag{11}$$

with

$$\begin{aligned} \mathbf{V}^{\alpha,\beta,\gamma} &= \sum_{i=0}^{l-\alpha}\sum_{j=0}^{m-\beta}\sum_{k=0}^{n-\gamma} \omega_{i,j,k}^{i+\alpha,j+\beta,k+\gamma}\mathbf{b}_{i,j,k}^{i+\alpha,j+\beta,k+\gamma}\, B_i^{l-\alpha}(u(\mu,\nu))\, B_j^{m-\beta}(v(\mu,\nu))\, B_k^{n-\gamma}(w(\mu,\nu)) \\ G^{\alpha,\beta,\gamma} &= \sum_{i=0}^{l-\alpha}\sum_{j=0}^{m-\beta}\sum_{k=0}^{n-\gamma} \omega_{i,j,k}^{i+\alpha,j+\beta,k+\gamma}\, B_i^{l-\alpha}(u(\mu,\nu))\, B_j^{m-\beta}(v(\mu,\nu))\, B_k^{n-\gamma}(w(\mu,\nu))\,, \end{aligned} \tag{12}$$

and with constants

$$B_R^{\alpha,\beta,\gamma}(L,\mathbf{I}) \;=\; \frac{\prod_{Q^u=1}^{\alpha} \beta_{I^u_{Q^u}}\binom{L}{I^u_{Q^u}} \prod_{Q^v=1}^{\beta} \beta_{I^v_{Q^v}}\binom{L}{I^v_{Q^v}} \prod_{Q^w=1}^{\gamma} \beta_{I^w_{Q^w}}\binom{L}{I^w_{Q^w}}}{\binom{rL}{R}}\,.$$

and $B_S^{\alpha,\beta,\gamma}(M,\mathbf{J})$ similarly.

Proof of Theorem 4. Essentially like the foregoing proofs.

Setting $\alpha = l$, $\beta = m$ and $\gamma = n$ results in:

Corollary 4. Rational Bézier surfaces and TPB-volumes
Let $\mathbf{V}(u,v,w)$ and $\mathbf{F}(\mu,\nu)$ be given as in Theorem 4. For $\mathbf{V}(\mu,\nu) = \mathbf{V}(\mathbf{F}(\mu,\nu)) = \mathbf{V}(u(\mu,\nu),v(\mu,\nu),w(\mu,\nu))$ we have

$$\mathbf{V}(\mu,\nu) \;=\; \frac{\sum_{R=0}^{rL}\sum_{S=0}^{rM} \Omega_{R,S}\mathbf{B}_{R,S}\, B_R^{rL}(\mu)\, B_S^{rM}(\nu)}{\sum_{R=0}^{rL}\sum_{S=0}^{rM} \Omega_{R,S}\, B_R^{rL}(\mu)\, B_S^{rM}(\nu)}\,,$$

where $r = l+m+n$ and weights are given by

$$\Omega_{R,S} \;=\; \sum_{|\mathbf{I}|=R}\sum_{|\mathbf{J}|=S} B_R^{l,m,n}(L,\mathbf{I})\, B_S^{l,m,n}(M,\mathbf{J})\, \omega_{0,0,0}^{l,m,n}\,,$$

and Bézier points are given by $\mathbf{B}_{R,S} = \frac{\Omega_{R,S}\mathbf{B}_{R,S}}{\Omega_{R,S}}$, where

$$\Omega_{R,S}\mathbf{B}_{R,S} \;=\; \sum_{|\mathbf{I}|=R}\sum_{|\mathbf{J}|=S} B_R^{l,m,n}(L,\mathbf{I})\, B_S^{l,m,n}(M,\mathbf{J})\, \omega_{0,0,0}^{l,m,n}\mathbf{b}_{0,0,0}^{l,m,n}\,.$$

Remark 4. Rational Bézier surfaces and volumes

- **Rational tetrahedra Bézier volume of degree n**
  Parameter planes and planes of parameter space in general position map to rational triangle Bézier surfaces of degree n.
- **Rational TPB-volume of degree (l,m,n)**
  Parameter planes map to rational TPB-surfaces of degree (l, m), (l, n) or (m, n), respectively.
  Planes of parameter space in general position map to rational TPB-surfaces of degree $(l + m + n, l + m + n)$.

Theorem 4 and Corollary 4 also include the three special cases of $\mathbf{K}(t)$ and $\mathbf{F}(u, v)$ both being polynomial, of $\mathbf{K}(t)$ being polynomial and $\mathbf{F}(u, v)$ being rational, and of $\mathbf{K}(t)$ being rational and $\mathbf{F}(u, v)$ being polynomial (cf. remark *rational curves on surfaces* given above in Section IV.1).

An extension to arbitrary dimensions can be done similarly.

V. Spline representations

V.1 Bézier spline representations

It is possible to build up complex Bézier splines from a number of Bézier segments. The conditions for C^r-continuity of adjacent segments can be found in [Far 90] and in [Hos 91]. As an example we formulate:

Theorem 5. Bézier spline curves and surfaces

Let $\mathbf{K}(t)$ be a C^a-continuous planar polynomial Bézier spline curve of degree N. And let $\mathbf{F}(u, v)$ be a C^b-continuous polynomial Bézier surface of degree (l, m):

$\mathbf{F}(\mathbf{K}(t))$ is a Bézier subspline curve of degree $N(l + m)$ and is of smoothness C^e with $e = \min\{a, b\}$.

Proof of Theorem 5. $\mathbf{F}(\mathbf{K}(t))$ can be calculated using Theorem 1 for each Bézier curve segment. Thus, segments are of degree $N(l + m)$.
C^e-continuity results by applying the chain rule to $\mathbf{F}(\mathbf{K}(t))$. □

V.2 B-spline representations

Using the results of the foregoing sections we are able to prove the following Theorem 6:

Theorem 6. B-spline curves and surfaces

Let $\mathbf{K}(t)$ be a C^a-continuous planar polynomial B-spline curve of order $N + 1$, i.e. of degree N. And let $\mathbf{F}(u, v)$ be a C^b-continuous polynomial B-spline surface of order $(l + 1, m + 1)$, i.e. of degree (l, m):

$\mathbf{F}(\mathbf{K}(t))$ is a B-spline curve of order $N(l + m) + 1$, i.e. of degree $N(l + m)$. $\mathbf{F}(\mathbf{K}(t))$ is C^e-continuous with $e = \min\{a, b\}$. Knots of $\mathbf{F}(\mathbf{K}(t))$ have multiplicity $\mu = N(l+m) - e$.