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# Interner Bericht

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Tensor Product Bézier Volumes**

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## Fachbereich Informatik

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# Rational Tensor Product Bézier Volumes

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**Abstract.** Free form volumes in rational Bézier representation are derived via homogeneous coordinates. Some properties and constructions are presented and two applications of free form volumes are discussed: definition of solid primitives and curve and surface modelling by the way of volume deformation.

**Keywords.** Rational Bézier volumes, solids, free form deformation.

## I. Introduction

While in the past, CAGD has been mostly concerned with curves and surfaces, more recently, there has been an increasing interest in higher dimensional, trivariate objects such as free form volumes which are suitable to describe inhomogeneous solids. The two most widely used methods of representing solids are the Constructive Solid Geometry Representation (CSG-Rep.) - a solid based method - and the Boundary Representation (B-Rep.) - a surface based method (see e.g. [Cas 85]). However, free form character of both methods is not very substantial, and they also assume internal homogeneity. On the other hand, free form volumes, of which this paper is dealing with, possess per definition a very high free form characteristic and describe every interior point as well as every point on the boundary surface of the volume uniquely. No assumption on internal homogeneity or structure is done. Beside solid modelling, there are some more applications of free form volumes, for instance, the description of spatial movement or deformation of a surface, the description of physical fields, such as temperature or pressure, etc. as functions of several variables, e.g. the positional coordinates, the modification of curves and surfaces through volume deformation, etc.

Free form volumes can be defined by the *tensor (Cartesian) product definition*,

$$V(u, v, w) = \sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n V_{i,j,k} u^i v^j w^k, \quad u, v, w \in [0, 1]. \quad (1)$$

The above given definition is based on monomials, in CAGD (*Computer Aided Geometric Design*) *Bernstein polynomials*, see [Far 92], [Hos 92],

$$B_k^n(w) = \binom{n}{k} w^k (1-w)^{n-k}, \quad w \in [0, 1], \quad (2)$$

of degree  $n$  in  $w$ , and analogously for  $v$  and  $u$ , are very popular. This is, because the expansion in terms of Bernstein polynomials yields, firstly, a numerically very stable behavior of all algorithms. Secondly, a geometric relationship between subject and coefficients of its defining equation. Note, that these properties are not available in case of monomials, Lagrange or Hermite polynomials. Probably this is one of the main reasons why Bézier representations, which are based on the use of Bernstein polynomials, became the de facto industry standard in CAGD during the past years. Thus, a *tensor product Bézier volume* - briefly *TPB volume* - of degree  $(l, m, n)$  in  $(u, v, w)$  is defined by

$$V(u, v, w) = \sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n V_{i,j,k} B_i^l(u) B_j^m(v) B_k^n(w), \quad u, v, w \in [0, 1]. \quad (3)$$

The coefficients  $V_{i,j,k} \in \mathbb{R}^3$  are called *Bézier points*. They form, connected in the ordering given by their subscripts, a spatial net which is called *Bézier grid*, [Hos 92].

Because of the polynomial character of (3) only polynomials can be represented exactly, sphere (segments) for example, cannot be constructed by (3). Therefore many (primary) elements of CAD systems can not be converted exactly into this kind of free form representation. Rational Bézier representations overcome this disadvantage for the most part: primary elements like conic sections and quadrics, as well as tori and cyclids, for example, can be constructed, thus allowing exact conversion from CSG-Rep. and B-Rep. based solid primitives into rational free form representation.

This paper is concerned with free form volumes in rational Bézier representation defined via the Cartesian product. Mathematical description using homogeneous coordinates, properties and constructions such as point and derivative evaluation, degree raising, subdivision, and continuity conditions are presented in Section 2. Section 3 and 4 discuss two examples of application of free form volumes: generation of solid primitives defined by rational TPB volumes and curve and surface modelling by free form deformation.

## II. Bézier Volumes

Using homogeneous coordinates  $r^i$ , points  $\mathbf{R} = (x, y, z)^T$  of  $\mathbb{R}^3$  can be represented by points  $\mathbf{R} = (r^1, r^2, r^3, r^4)^T$  of  $\mathbb{R}^4$  via the projection of  $\mathbb{R}^4$  into the hyperplane  $r^4 = 1$  according to (cf. Figure 1)

$$\mathcal{H}(\mathbf{R}) = \begin{cases} \frac{1}{r^4} (r^1, r^2, r^3)^T & \text{for } r^4 \neq 0 \\ (r^1, r^2, r^3)^T & \text{for } r^4 = 0. \end{cases}$$

The center of projection is the origin of the 4D cartesian coordinate system. A point  $\mathbf{R} = (x, y, z)^T$  is the projection of  $\omega(x, y, z, 1)^T$ , where  $\omega \neq 0$ ,  $\omega \in \mathbb{R}$ . The real number  $\omega$  is called *weight* of the corresponding point. Note,  $\mathbf{R} \in \mathbb{R}^4$  and  $\varrho \mathbf{R} \in \mathbb{R}^4$ ,  $\varrho \neq 0$ ,  $\varrho \in \mathbb{R}$ ,

describe the same point of  $\mathbb{R}^3$ .  $\mathcal{H}(\mathbf{R})$  is unlimited for  $r^4 \rightarrow 0$ ,  $r^4 \neq 0$ . In this way infinite points of  $\mathbb{R}^3$  can be described by finite points of  $\mathbb{R}^4$  with  $r^4 = 0$ . In  $\mathbb{R}^3$  these points will be represented by direction vectors, [Pie 87].

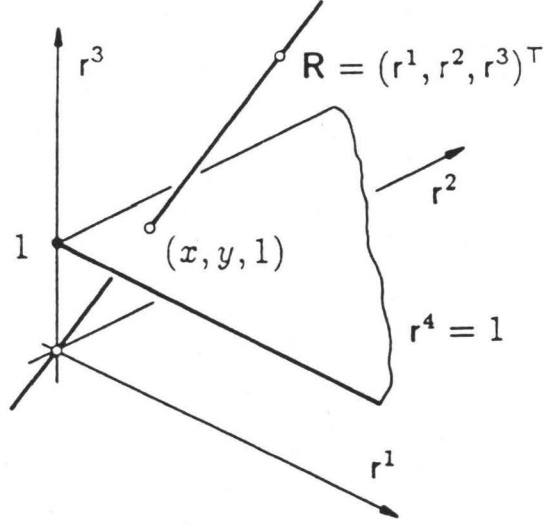


Figure 1. Introducing homogeneous coordinates via projection  $\mathcal{H}(\cdot)$  of points of  $\mathbb{R}^3$  into the plane  $r^3 = 1$ .

Now, a *rational tensor product Bézier volume* - briefly *rational TPB volume* - of degree  $(l, m, n)$  in  $(u, v, w)$  is defined by, [Kir 89], [Las 90],

$$\mathbf{V}(\mathbf{u}) = \mathcal{H}(\mathbf{V}(\mathbf{u})), \quad (4)$$

where

$$\mathbf{V}(\mathbf{u}) = \sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n \mathbf{V}_{i,j,k} B_i^l(u) B_j^m(v) B_k^n(w), \quad u, v, w \in [0, 1],$$

and  $\mathbf{u} = (u, v, w)$ .  $\mathbf{V}(\mathbf{u})$  is called homogeneous form of  $\mathbf{V}(\mathbf{u})$ . Thus, a rational TPB volume of degree  $(l, m, n)$  in  $\mathbb{R}^3$ ,  $\mathbf{V}(\mathbf{u})$ , is defined by a non-rational, polynomial TPB volume of degree  $(l, m, n)$  in  $\mathbb{R}^4$ ,  $\mathbf{V}(\mathbf{u})$ , i.e. is the projection of a polynomial TPB volume into the hyperplane  $r^4 = 1$ .

If we write  $\mathbf{V}_{i,j,k} = (\mathbf{V}_{i,j,k}^1, \mathbf{V}_{i,j,k}^2, \mathbf{V}_{i,j,k}^3, \mathbf{V}_{i,j,k}^4)^\top = (\omega_{i,j,k} \mathbf{V}_{i,j,k}^\top, \omega_{i,j,k})^\top \in \mathbb{R}^4$ , where  $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^\top \in \mathbb{R}^3$ ,  $\omega_{i,j,k} \in \mathbb{R}$ , and assume

$$\sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n \omega_{i,j,k} B_i^l(u) B_j^m(v) B_k^n(w) \neq 0, \quad u, v, w \in [0, 1],$$

then,

$$\mathbf{V}(\mathbf{u}) = \frac{\sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n \omega_{i,j,k} \mathbf{V}_{i,j,k} B_i^l(u) B_j^m(v) B_k^n(w)}{\sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n \omega_{i,j,k} B_i^l(u) B_j^m(v) B_k^n(w)}, \quad (5)$$

and 3D Bézier points  $\mathbf{V}_{i,j,k}$  are projections,  $\mathbf{V}_{i,j,k} = \mathcal{H}(\mathbf{V}_{i,j,k})$ , of 4D Bézier points  $\mathbf{V}_{i,j,k}$ . Bézier points form in their natural ordering, given by their subscripts, the vertices of the Bézier grid. Scalars  $\omega_{i,j,k} \in \mathbb{R}$  are called *weights*: if we increase one  $\omega_{i,j,k}$  the volume will be pulled towards the corresponding  $\mathbf{V}_{i,j,k}$ .

The assumption of (5) will be fulfilled if weights  $\omega_{i,j,k}$  are non-negative and all 8 corner Bézier points of the grid are no infinite control points what will be required from now on. If a control point  $\mathbf{V}_{i,j,k}$  is a *point at infinity*, i.e.  $\omega_{i,j,k} = 0$ , we replace, according to definition of  $\mathcal{H}(\cdot)$ ,  $\omega_{i,j,k} \mathbf{V}_{i,j,k}$  by  $\mathbf{V}_{i,j,k}$  in the numerator of (5). Note, infinite control points, also called *control vectors*, can be eliminated by degree elevation (cf. [Far 92] p. 260) as stated by (7).

One of the weights (e.g.  $\omega_{0,0,0}$ ) can be normalized to be of value one and on boundary curves  $u = 0$ ,  $v = 0$  and  $w = 0$ , the parametrization can be chosen so that, for instance  $\omega_{l,0,0} = \omega_{0,m,0} = \omega_{0,0,n} = 1$ . All other weights directly influence the shape of the volume. If all weights are equal, (5) yields the non-rational, polynomial TPB volume because Bernstein polynomials sum to one (see e.g. [Far 92] p. 42, [Hos 92] p. 116).

Positive weights result in volumes which have all the properties and algorithms we do know from polynomial representations. Because of definition (4) via the projection  $\mathcal{H}(\cdot)$ , properties of rational TPB volumes can be deduced from properties of the non-rational TPB volume scheme. This means that relations between Bézier grid and rational TPB volume can be deduced from those of the underlying Bézier curve scheme, and that many constructions in different parameters commute, such as degree raising, de Casteljau construction and derived constructions, segmentation for example. We list a few properties (see [Kir 89], [Las 90]).

**Convex hull property.** The rational TPB volume lies completely within the convex hull of its Bézier points.

**Parametric surfaces.** The parametric surfaces of constant  $u$  are rational TPB surfaces of degree  $(m, n)$  and analogously for parametric surfaces of constant  $v$  and  $w$ , respectively.

**Parametric lines.** Lines  $u, v = \text{constant}$ , so-called  $w$  parameter lines, are rational Bézier curves of degree  $n$  and analogously for  $v$  and for  $u$  parameter lines.

**Boundary surfaces.** The boundary surfaces of a rational TPB volume are rational TPB surfaces whose Bézier points and weights are the corresponding boundary points and weights of the Bézier grid.

**Boundary curves.** The boundary curves of a rational TPB volume are rational Bézier curves whose Bézier points and weights are the corresponding boundary points and weights of the Bézier grid.

**Vertices.** The eight vertices of a rational TPB volume coincide with the eight corresponding vertices of the Bézier grid.

**Visual approximation.** The Bézier grid gives a rough impression of the rational TPB volume.

**Derivatives.** The partial derivatives of order  $(p, q, r)$  of a rational TPB volume of degree

$(l, m, n)$ ,  $\mathbf{V}(\mathbf{u}) = \mathbf{N}(\mathbf{u})/D(\mathbf{u})$ , are given by (cf. [Far 92] p. 256)

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \mathbf{V}(\mathbf{u}) = \frac{1}{D(\mathbf{u})} \left( \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \mathbf{N}(\mathbf{u}) - \sum_{\mathbf{i} \in \mathbf{I}} \binom{p}{i} \binom{q}{j} \binom{r}{k} \frac{\partial^\beta}{\partial \mathbf{u}^\beta} D(\mathbf{u}) \frac{\partial^{\alpha-\beta}}{\partial \mathbf{u}^{\alpha-\beta}} \mathbf{N}(\mathbf{u}) \right), \quad (6)$$

where  $\mathbf{I} = \{\mathbf{i} = (i, j, k) : i = 0(1)p, j = 0(1)q, k = 0(1)r \text{ and } (i, j, k) \neq (0, 0, 0)\}$  and  $\beta = |\mathbf{i}| = i + j + k$ , using the notation

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} F(\mathbf{u}) = \frac{\partial^{|\mathbf{p}|}}{\partial \mathbf{u}^{\mathbf{p}}} F(\mathbf{u}) = \frac{\partial^{p+q+r}}{\partial u^p \partial v^q \partial w^r} F(\mathbf{u}),$$

where  $\mathbf{p} = (p, q, r)$ ,  $\alpha = |\mathbf{p}| = p + q + r$ , to indicate partial derivatives of a (vector valued) function  $F(\mathbf{u})$ . Derivative calculation of  $D(\mathbf{u})$  is according to

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} D(\mathbf{u}) = \frac{l!}{(l-p)!} \frac{m!}{(m-q)!} \frac{n!}{(n-r)!} \sum_{i=0}^{l-p} \sum_{j=0}^{m-q} \sum_{k=0}^{n-r} \Delta^{p,q,r} \omega_{i,j,k} B_i^{l-p}(u) B_j^{m-q}(v) B_k^{n-r}(w),$$

with forward differences  $\Delta^{p,q,r} \omega_{i,j,k}$ , and similarly for  $\mathbf{N}(\mathbf{u})$ , employing forward differences  $\Delta^{p,q,r}(\omega_{i,j,k} \mathbf{V}_{i,j,k})$ .

**Degree raising.** If we write a rational TPB volume, (4), of degree  $(l, m, n)$  as one of degree  $(l + \lambda, m, n)$ , the new 4D Bézier points  $\mathbf{V}_{i,j,k}^\lambda$  are given by

$$\mathbf{V}_{i,j,k}^\lambda = \sum_{I=0}^{\lambda} \lambda_I \frac{\binom{\lambda}{I} \binom{l}{i-I}}{\binom{l+\lambda}{i}} \mathbf{V}_{i-I,j,k}. \quad (7)$$

This can be seen as follows:  $\lambda(u)\mathbf{V}(\mathbf{u})$ , with an arbitrary real-valued function  $\lambda(u)$  which does not vanish on  $[0, 1]$ , and  $\mathbf{V}(\mathbf{u})$  are associated with the same rational TPB volume  $\mathbf{V}(\mathbf{u})$ . If we choose  $\lambda(u)$  to be of degree  $\lambda$ ,  $\lambda(u) = \sum_{I=0}^{\lambda} \lambda_I B_I^\lambda(u)$ , then  $\lambda(u)\mathbf{V}(\mathbf{u})$  is of degree  $l + \lambda$  in  $u$ , and equation (7) results. As a special case,  $\lambda = 1$ , the degree  $(l + 1, m, n)$  Bézier points are given by

$$\mathbf{V}_{i,j,k}^1 = \lambda_0 \left( 1 - \frac{i}{l+1} \right) \mathbf{V}_{i,j,k} + \lambda_1 \frac{i}{l+1} \mathbf{V}_{i-1,j,k}.$$

(7) means, since  $\lambda_I \in \mathbb{R}$ , that there is an infinite number of ways to represent a degree  $(l, m, n)$  volume as one of degree  $(l + \lambda, m, n)$ . The same holds, with corresponding formulae, for degree raising in the  $v$  and  $w$  variables.

**Point and derivative evaluation.** A volume point  $\mathbf{V}(\mathbf{u}_0)$ , for any  $\mathbf{u}_0 = (u_0, v_0, w_0)$ ,  $u_0, v_0, w_0 \in [0, 1]$ , can be computed by repeated *de Casteljau* steps adapted on 4D Bézier points in  $u$ ,  $v$  and  $w$ , for example in the  $u$  direction by the recursion formula

$$\mathbf{V}_{\alpha,\gamma,\epsilon}^{\beta,\delta,\xi}(\mathbf{u}_0) = (1 - u_0) \mathbf{V}_{\alpha,\gamma,\epsilon}^{\beta-1,\delta,\xi}(\mathbf{u}_0) + u_0 \mathbf{V}_{\alpha+1,\gamma,\epsilon}^{\beta,\delta,\xi}(\mathbf{u}_0) \quad (8)$$

and analogously for  $v$  and  $w$ , where

$$\mathbf{V}_{\alpha,\gamma,\epsilon}^{\alpha,\gamma,\epsilon} = \mathbf{V}_{\alpha,\gamma,\epsilon} \quad \text{and} \quad \mathbf{V}(\mathbf{u}_0) = \mathbf{V}_{0,0,0}^{l,m,n}(\mathbf{u}_0) \quad \text{and} \quad \mathbf{V}(\mathbf{u}_0) = \mathcal{H}(\mathbf{V}(\mathbf{u}_0)).$$

This result stems from the fact that in (4), the de Casteljau algorithm can be applied separately to each of the parameter directions. The de Casteljau steps in different directions commute, and the result is independent of the order. Similarly, the derivative of order  $(p, q, r)$  of a rational TPB volume of degree  $(l, m, n)$  can be found using the de Casteljau algorithm. Indeed, starting with equation (6) derivatives of  $\mathbf{N}(\mathbf{u})$  are given by

$$\frac{\partial^{p+q+r}}{\partial u^p \partial v^q \partial w^r} \mathbf{N}(\mathbf{u}) = \frac{l!}{(l-p)!} \frac{m!}{(m-q)!} \frac{n!}{(n-r)!} \Delta^{p,q,r}(\omega_{0,0,0}^{l-p,m-q,n-r} \mathbf{V}_{0,0,0}^{l-p,m-q,n-r}),$$

with forward differences  $\Delta^{p,q,r}(\omega_{\alpha,\gamma,\epsilon}^{\beta,\delta,\xi} \mathbf{V}_{\alpha,\gamma,\epsilon}^{\beta,\delta,\xi})$  which are now operating on both the subscripts and superscripts, and similarly for derivatives of  $D(\mathbf{u})$ .

**Subdivision.** A rational TPB volume of degree  $(l, m, n)$  can be subdivided into two rational TPB volumes of the same degree which join along the parametric surface corresponding to  $u = u_0$  with  $l$  continuous derivatives in  $u$  direction. The Bézier points  $\mathbf{V}_{0,j,k}^{i,j,k}$  and  $\mathbf{V}_{i,j,k}^{l,j,k}$  of the two subsegments can be found by applying the de Casteljau algorithm for  $u = u_0$  to all  $i$ -polygons of the Bézier grid. Using the parameter transformations  $\bar{u} = u/u_0$  for  $u \in [0, u_0]$  and  $\bar{u} = (u - u_0)/(1 - u_0)$  for  $u \in [u_0, 1]$ , respectively, we can reparametrize the subsegments to again be defined on the unit cube.

**Continuity conditions.** To construct  $GC^r$  or  $C^r$  continuous rational volumes,  $\mathbf{V}(\mathbf{u})$  and  $\bar{\mathbf{V}}(\bar{\mathbf{u}})$ , defined via homogeneous coordinates, i.e. as volumes  $\mathbf{V}(\mathbf{u})$  and  $\bar{\mathbf{V}}(\bar{\mathbf{u}})$ , we have to take into account that all  $\eta(\mathbf{u})\mathbf{V}(\mathbf{u})$  and  $\mathbf{V}(\mathbf{u})$  with an arbitrary real valued function  $\eta(\mathbf{u})$  which does not vanish for any  $\mathbf{u}$  of parameter domain, are associated with the same rational volume  $\mathbf{V}(\mathbf{u})$  in  $\mathbb{R}^3$ , and similarly for  $\bar{\mathbf{V}}(\bar{\mathbf{u}})$ . In view of this the two 3D volumes meet continuously along a common regular boundary surface  $\mathbf{B}$ ,  $\bar{\mathbf{V}}(\bar{\mathbf{u}})|_{\mathbf{B}} = \mathbf{V}(\mathbf{u})|_{\mathbf{B}}$ , iff the  $\mathbb{R}^4$  representations satisfy

$$\bar{\mathbf{V}}(\bar{\mathbf{u}})|_{\mathbf{B}} = \varrho(\mathbf{u}) \mathbf{V}(\mathbf{u})|_{\mathbf{B}}. \quad (9)$$

Note,  $\mathbf{V}$  and  $\bar{\mathbf{V}}$  do not have to meet continuously, for  $\mathbf{V}$  and  $\bar{\mathbf{V}}$  joining continuously.

Suppose  $\mathbf{B}$  is given by  $v = 1, \bar{v} = 0, \bar{\mathbf{V}}(\bar{u}, 0, \bar{w}) = \mathbf{V}(u, 1, w)$  for all  $\bar{u} = u, \bar{w} = w$ , then continuity condition (10) results in

$$\bar{\mathbf{V}}_{i,0,k} = \varrho_0 \mathbf{V}_{i,m,k},$$

where  $\varrho_0 = \varrho(\bar{u}, 0, \bar{w}) = \text{const.} \neq 0$  or in terms of inhomogeneous coordinates of  $\mathbb{R}^3$

$$\begin{aligned} \bar{\omega}_{i,0,k} \bar{\mathbf{V}}_{i,0,k} &= \varrho_0 \omega_{i,m,k} \mathbf{V}_{i,m,k} \\ \bar{\omega}_{i,0,k} &= \varrho_0 \omega_{i,m,k} \end{aligned}$$

end therefore,

$$\bar{\mathbf{V}}_{i,0,k} = \mathbf{V}_{i,m,k}.$$

Iff  $\varrho_0 = 1$ , then  $\bar{\omega}_{i,0,k} = \omega_{i,m,k}$  and polynomial denominators of  $\mathbf{V}(\mathbf{u})$  and  $\bar{\mathbf{V}}(\bar{\mathbf{u}})$  join continuously too.

$\mathbf{V}(\mathbf{u})$  and  $\bar{\mathbf{V}}(\bar{\mathbf{u}})$  join  $GC^r$  continuously (with contact of order  $r$ ) across  $\mathbf{B}$  iff (cf. [Hos 92])



$$\frac{\partial^\alpha}{\partial \bar{v}^\alpha} \bar{\mathbf{V}}(\bar{\mathbf{u}}) \Big|_{\mathbf{B}} = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} \left( \varrho(\bar{\mathbf{u}}) \mathbf{V}(\mathbf{u}(\bar{\mathbf{u}})) \right) \Big|_{\mathbf{B}}, \quad \alpha = 0(1)r, \quad (10)$$

where  $\mathbf{V}(\mathbf{u})$  has been reparametrized by  $\mathbf{u} \mapsto \mathbf{u}(\bar{\mathbf{u}})$ . Applying the chain rule we get the following additional condition for a  $GC^1$  continuous connection in  $v$  direction

$$\bar{\mathbf{V}}_{\bar{v}} = \varrho_0 (u_1 \mathbf{V}_u + v_1 \mathbf{V}_v + w_1 \mathbf{V}_w) + \varrho_1 \mathbf{V}, \quad (11)$$

where  $\varrho_0 > 0$ ,  $b_1 > 0$ , and for a  $GC^2$  continuous join in  $v$  direction the additional condition

$$\bar{\mathbf{V}}_{\bar{v}\bar{v}} = \varrho_0 (u_1 \ v_1 \ w_1) \begin{pmatrix} \mathbf{V}_{uu} & \mathbf{V}_{uv} & \mathbf{V}_{uw} \\ \mathbf{V}_{uv} & \mathbf{V}_{vv} & \mathbf{V}_{vw} \\ \mathbf{V}_{uw} & \mathbf{V}_{vw} & \mathbf{V}_{ww} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} + (\mathbf{V}_u \ \mathbf{V}_v \ \mathbf{V}_w) \begin{pmatrix} \varrho_0 u_2 + 2\varrho_1 u_1 \\ \varrho_0 v_2 + 2\varrho_1 v_1 \\ \varrho_0 w_2 + 2\varrho_1 w_1 \end{pmatrix} + \varrho_2 \mathbf{V}, \quad (12)$$

where we have used the abbreviations

$$u_\alpha = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} u(\bar{\mathbf{u}}) \Big|_{\mathbf{B}}, \quad v_\alpha = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} v(\bar{\mathbf{u}}) \Big|_{\mathbf{B}}, \quad w_\alpha = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} w(\bar{\mathbf{u}}) \Big|_{\mathbf{B}}, \quad \varrho_\alpha = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} \varrho(\bar{\mathbf{u}}) \Big|_{\mathbf{B}}.$$

Notice,  $\mathbf{V}$  and  $\bar{\mathbf{V}}$  do not have to meet  $GC^r$  continuously, for  $\mathbf{V}$  and  $\bar{\mathbf{V}}$  joining  $GC^r$  continuously.

Suppose  $\mathbf{V}(\mathbf{u})$ , of degree  $(l, m, n)$ , and  $\bar{\mathbf{V}}(\bar{\mathbf{u}})$ , of degree  $(l, \bar{m}, n)$ , join continuously along boundary surface  $\mathbf{B}$  given as above, then,  $GC^1$  continuity condition (11) results in

$$\begin{aligned} \bar{m} \Delta^{0,1,0} \bar{\mathbf{V}}_{i,0,k} &= \varrho_1 \mathbf{V}_{i,m,k} \\ &+ \varrho_0 \left( (l-i) a_0 \Delta^{1,0,0} \mathbf{V}_{i,m,k} + i a_1 \Delta^{1,0,0} \mathbf{V}_{i-1,m,k} + m b_0 \Delta^{0,1,0} \mathbf{V}_{i,m-1,k} \right. \\ &\quad \left. + (n-k) c_0 \Delta^{0,0,1} \mathbf{V}_{i,m,k} + k c_1 \Delta^{0,0,1} \mathbf{V}_{i,m,k-1} \right), \end{aligned} \quad (*)$$

where, for the purpose of solving (11) by comparing coefficients and to keep calculations easy, we assumed functions  $\varrho_0$ ,  $\varrho_1$  and  $b_0 \equiv v_1$  to be constants, such that  $\varrho_0 b_0 > 0$ , function  $u_1$  to be linear in  $u$ ,  $u_1 = (1-u)a_0 + ua_1$ ,  $a_0, a_1 \in \mathbb{R}$ , and function  $w_1$  to be linear in  $w$ ,  $w_1 = (1-w)c_0 + wc_1$ ,  $c_0, c_1 \in \mathbb{R}$ . (\*) represents four conditions: the fourth equation of (\*) is a condition on volume weights and looks exactly like (\*) but with weights  $\omega_{i,j,k}$  and  $\bar{\omega}_{i,j,k}$ , the first three equations of (\*) are conditions on the products of weights and 3D Bézier points. If we solve the  $\bar{\omega}_{i,j,k}$  equation for  $\varrho_1 \omega_{i,m,k}$  and insert this expression in the condition for  $\bar{\omega}_{i,1,k} \bar{\mathbf{V}}_{i,1,k}$  a formula for 3D Bézier points  $\bar{\mathbf{V}}_{i,1,k}$ , very similar to (\*) can be derived:

$$\begin{aligned} \bar{m} \Delta^{0,1,0} \bar{\mathbf{V}}_{i,0,k} &= \frac{\varrho_0}{\bar{\omega}_{i,1,k}} \left( (l-i) a_0 \omega_{i+1,m,k} \Delta^{1,0,0} \mathbf{V}_{i,m,k} + i a_1 \omega_{i-1,m,k} \Delta^{1,0,0} \mathbf{V}_{i-1,m,k} \right. \\ &\quad \left. + m b_0 \omega_{i,m-1,k} \Delta^{0,1,0} \mathbf{V}_{i,m-1,k} + (n-k) c_0 \omega_{i,m,k+1} \Delta^{0,0,1} \mathbf{V}_{i,m,k} \right. \\ &\quad \left. + k c_1 \omega_{i,m,k-1} \Delta^{0,0,1} \mathbf{V}_{i,m,k-1} \right). \end{aligned}$$

To get, for  $GC^1$  continuity, necessary and sufficient conditions for Bézier points and weights, more general factors  $\varrho_0$ ,  $\varrho_1$  and  $u_1$ ,  $v_1$ ,  $w_1$  have to be allowed. However, calcula-

tions become very technical, extensive and difficult to view. We omit them, but want to point out that [Was 91] gives a very detailed treatment of  $GC^r$  continuous Bézier surfaces what can be extended to volume representations.

### III. Solid Primitives

Compared with solid definitions such as CSG or B-Rep, TPB volumes possess a very high free form characteristic, describe every interior point as well as every point on the boundary surface of the volume uniquely, and no assumptions are done on internal homogeneity or structure. On the other side, they also allow the exact representation of solid primitives. Bézier volumes can be constructed in various ways. First, by specification of all control points (and weights), second, by interpolation or approximation of digitized points, and third, by performing sweep, spin, loft, etc. transformations on *profile surfaces*, [Cas 85], [Sai 87]. Rational volumes are in particular very useful for giving exact descriptions of solid primitives such as sphere, cylinder, torus, etc. Following the idea of [Faro 85], we calculate Bézier points of volumes  $\mathbf{V}(u)$  by performing a continuum of geometric transformations  $\mathbf{M}(w)$  on a profile surface  $\mathbf{F}(u, v)$ . To be able to generate solid primitives as well as free form volumes we base our considerations on rational representations and homogeneous coordinates. Therefore,  $\mathbf{M}(w)$  might be given by the following  $4 \times 4$  matrix

$$\mathbf{M}(w) = \begin{pmatrix} m^{1,1} & m^{1,2} & m^{1,3} & t^1 \\ m^{2,1} & m^{2,2} & m^{2,3} & t^2 \\ m^{3,1} & m^{3,2} & m^{3,3} & t^3 \\ p^1 & p^2 & p^3 & s \end{pmatrix},$$

where  $m^{\alpha,\beta}$  indicates rotation, reflection, scaling and shear,  $t^\beta$  indicates translation,  $p^\alpha$  perspective projection, and  $s$  an overall scaling. In the following we demonstrate the construction of solid primitives applying this method. We start by looking at *Sweep volumes* which result by moving a given surface  $\mathbf{F}(u, v)$  along a prescribed curve  $\mathbf{K}(w)$  which is sometimes referred to as *directrix*. Translation volumes are special sweep volumes and we are going to discuss them first. *Translation volumes* are specified by

**Theorem 1.** Let  $\mathbf{K}(w)$  be homogeneous form of a rational Bézier curve  $\mathbf{K}(w)$  of degree  $n$  with 4D Bézier points  $\mathbf{K}_k = (\mathbf{K}_k^1, \mathbf{K}_k^2, \mathbf{K}_k^3, \mathbf{K}_k^4)^\top = (\beta_k \mathbf{K}_k^\top, \beta_k)^\top$ , where  $\mathbf{K}_k = (x_k, y_k, z_k)^\top$  and weights  $\beta_k$  (w.l.o.g.  $\beta_0 = \beta_n = 1$ ).

Let  $\mathbf{F}(u, v)$  be homogeneous form of a rational TPB surface  $\mathbf{F}(u, v)$  of degree  $(l, m)$  with 4D Bézier points  $\mathbf{F}_{i,j} = (\mathbf{F}_{i,j}^1, \mathbf{F}_{i,j}^2, \mathbf{F}_{i,j}^3, \mathbf{F}_{i,j}^4)^\top = (\beta_{i,j} \mathbf{F}_{i,j}^\top, \beta_{i,j})^\top$ , where  $\mathbf{F}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})^\top$  and weights  $\beta_{i,j}$  (w.l.o.g.  $\beta_{0,0} = \beta_{l,0} = \beta_{0,m} = 1$ ).

4D Bézier points  $\mathbf{V}_{i,j,k} = (\mathbf{V}_{i,j,k}^1, \mathbf{V}_{i,j,k}^2, \mathbf{V}_{i,j,k}^3, \mathbf{V}_{i,j,k}^4)^\top = (\omega_{i,j,k} \mathbf{V}_{i,j,k}^\top, \omega_{i,j,k})^\top$ , with weights  $\omega_{i,j,k}$  and  $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^\top$  of a rational Bézier solid,  $\mathbf{V}(u)$ , represented in homogeneous form  $\mathbf{V}(u)$ , and generated by moving of  $\mathbf{F}(u, v)$  along  $\mathbf{K}(w)$ , such that  $\mathbf{V}(u, v, 0) = \mathbf{F}(u, v)$ ,  $\mathbf{V}(0, 0, w) = \mathbf{K}(w)$ , are given by

$$\mathbf{V}_{i,j,k} = \mathbf{M}_k \mathbf{F}_{i,j}, \quad (13)$$

where  $\mathbf{M}_0 = \mathbf{I}_4$  and

$$\mathbf{M}_k = \frac{1}{(K_0^4)^2} \begin{pmatrix} K_0^4 K_k^4 & 0 & 0 & K_0^4 K_k^1 - K_0^1 K_k^4 \\ 0 & K_0^4 K_k^4 & 0 & K_0^4 K_k^2 - K_0^2 K_k^4 \\ 0 & 0 & K_0^4 K_k^4 & K_0^4 K_k^3 - K_0^3 K_k^4 \\ 0 & 0 & 0 & K_0^4 K_k^4 \end{pmatrix}$$

**Proof.** Since all  $\eta(w)\mathbf{V}(u)$ , with non-vanishing real-valued function  $\eta(w)$ , are associated with the same sweep volume of coordinate space,  $\mathbf{V}(u)$  has to be defined by

$$\mathbf{V}(u) = \eta(w) \mathbf{M}(w) \mathbf{F}(u, v) \quad (14)$$

with

$$\mathbf{M}(w) = \begin{pmatrix} & \mathbf{I}_3 & & \mathbf{T} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $\mathbf{I}_3$  is a  $3 \times 3$  unite matrix and  $\mathbf{T} = \mathbf{T}(w) = (t^1(w), t^2(w), t^3(w))^T$  describes moving of  $\mathbf{F}(u, v)$  along  $\mathbf{K}(w)$ .

Since  $\mathbf{V}(0, 0, w) = \mathbf{K}(w)$  and  $\mathbf{V}(u, v, 0) = \mathbf{F}(u, v)$ , (14) implies

$$\mathbf{K}(w) = \eta(w) \mathbf{M}(w) \mathbf{K}_0.$$

Solving this  $4 \times 4$  system of linear equations for the components of  $\eta(w) \mathbf{M}(w)$  yields

$$\eta(w) \mathbf{M}(w) = \frac{1}{(K_0^4)^2} \begin{pmatrix} K_0^4 K^4(w) & 0 & 0 & K_0^4 K^1(w) - K_0^1 K^4(w) \\ 0 & K_0^4 K^4(w) & 0 & K_0^4 K^2(w) - K_0^2 K^4(w) \\ 0 & 0 & K_0^4 K^4(w) & K_0^4 K^3(w) - K_0^3 K^4(w) \\ 0 & 0 & 0 & K_0^4 K^4(w) \end{pmatrix}.$$

Now,  $\mathbf{K}(w) = (K^1(w), K^2(w), K^3(w), K^4(w))^T$  is given in Bézier representation, therefore we have

$$\eta(w) \mathbf{M}(w) = \sum_{k=0}^n \mathbf{M}_k B_k^n(w) \quad (*)$$

with  $\mathbf{M}_k$  as given in Theorem 1. Substituting the Bézier representations of  $\mathbf{F}(u, v)$  and of  $\mathbf{V}(u)$  as well as (\*) into (14) yields by comparing coefficients (13).  $\square$

Note, while 3D Bézier points  $\mathbf{V}_{i,j,k} \in \mathbb{R}^3$  result by moving the surface net defined by  $\mathbf{F}_{i,j}$  according to the control points of directrix  $\mathbf{K}(w)$ , this is not true for volume weights  $\omega_{i,j,k} \in \mathbb{R}^3$ , according to Theorem 1 they are given by

$$\omega_{i,j,k} = \frac{\beta_{i,j} \beta_k}{\beta_{0,0}}.$$

Also notice, although  $\mathbf{V}(u)$  was generated by moving a surface along a  $w$  line, parametric surfaces of constant  $u$  are congruent as well and the same holds for isoparametric surfaces of constant  $v$ .

For  $n = 1$  the trivariate analogy of a cylindrical surface is created for which Bézier points in  $w$  direction define parallel lines,

$$\mathbf{V}_{i,j,0} = \mathbf{F}_{i,j}, \quad \mathbf{V}_{i,j,1} = \mathbf{F}_{i,j} + \mathbf{T},$$

where  $\mathbf{T} = \mathbf{K}_1 - \mathbf{K}_0$  is translation vector and  $\omega_{i,j,k} = \beta_{i,j}$ , for all  $i, j$  and  $k = 0, 1$ .

A generalization of the translation volume is the *blending volume* (*loft volume*, [Sai 87]): here two non-congruent surfaces  $\mathbf{F}^0(u, v)$  and  $\mathbf{F}^n(u, v)$  defined by Bézier points  $\mathbf{F}_{i,j}^0$  and  $\mathbf{F}_{i,j}^n$  are connected appropriately. For polynomial blending,  $n$  has to be specified and, in case of  $n > 1$ , Bézier points of *intermediate nets*  $\mathbf{F}_{i,j}^k$ ,  $k = 1(1)n - 1$ . Then,

$$\mathbf{V}_{i,j,k} = \mathbf{F}_{i,j}^k. \quad (15)$$

Bézier points  $\mathbf{F}_{i,j}^k$  of intermediate nets can be provided in different ways. For example, via interpolation of  $n - 1$  intermediate surfaces (*skinning*, *lofting*), via continuity construction (s. Section II), if  $\mathbf{F}_{i,j}^0$  and  $\mathbf{F}_{i,j}^n$  are supposed to be boundary surfaces of connecting volume segments or, via interactive input procedures.

$n = 1$  gives a very special blending volume, the *linear blending volume*. It is the trivariate generalization of a ruled surface and thus is sometimes called *ruled volume* (see e.g. [Cas 85]). Defined by taking a convex combination,

$$\mathbf{V}(u) = (1 - w)\mathbf{F}^0(u, v) + w\mathbf{F}^n(u, v), \quad w \in [0, 1],$$

of surfaces  $\mathbf{F}^0(u, v)$  and  $\mathbf{F}^n(u, v)$ ,  $n = 1$ , Bézier points are given by (15).

*Volumes of revolution* also referred to as *spin volumes* are sweep volumes as well and can be created easily as stated by, [Kir 89], [Las 90],

**Theorem 2.** Let  $\mathbf{F}(u, v)$  be the homogeneous form of a rational TPB surface  $\mathbf{F}(u, v)$  of degree  $(l, m)$  with 4D Bézier points  $\mathbf{F}_{i,j} = (\mathbf{F}_{i,j}^1, \mathbf{F}_{i,j}^2, \mathbf{F}_{i,j}^3, \mathbf{F}_{i,j}^4)^\top = (\beta_{i,j}\mathbf{F}_{i,j}^\top, \beta_{i,j})^\top$ , where  $\mathbf{F}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})^\top$  and weights  $\beta_{i,j}$  (w.l.o.g.  $\beta_{0,0} = \beta_{l,0} = \beta_{0,m} = 1$ ).

The rational TPB volume  $\mathbf{V}(u)$  of degree  $(l, m, n)$ ,  $n = 2$ , with 4D Bézier points

$$\mathbf{V}_{i,j,k} = \mathbf{M}_k \mathbf{F}_{i,j}, \quad (16)$$

where  $\mathbf{M}_0 = \mathbf{I}_4$ ,

$$\mathbf{M}_1 = \begin{pmatrix} \cos \frac{\alpha}{2} & -\sin \frac{\alpha}{2} & 0 & 0 \\ \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\alpha}{2} & 0 \\ 0 & 0 & 0 & \cos \frac{\alpha}{2} \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

represents a rotation of  $\mathbf{F}(u, v)$  through the arc  $\alpha$ , with  $|\alpha| \leq 180^\circ$ , round the  $z$ -axis such that  $\mathbf{V}(u, v, 0) = \mathbf{F}(u, v)$ .

**Proof.** For a rotation of  $F(u, v)$  about the  $z$ -axis by the angle  $\alpha$ ,  $M$  of (14) now has to be given by

$$M = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

With the same arguments as in proof of Theorem 1 we result in equation (\*) but with  $M_k$  now given by

$$M_k = \begin{pmatrix} \frac{K_k^1 K_0^1 + K_k^2 K_0^2}{(K_0^1)^2 + (K_0^2)^2} & -\frac{K_k^2 K_0^1 - K_k^1 K_0^2}{(K_0^1)^2 + (K_0^2)^2} & 0 & 0 \\ \frac{K_k^2 K_0^1 - K_k^1 K_0^2}{(K_0^1)^2 + (K_0^2)^2} & \frac{K_k^1 K_0^1 + K_k^2 K_0^2}{(K_0^1)^2 + (K_0^2)^2} & 0 & 0 \\ 0 & 0 & \frac{K_k^3}{K_0^3} & 0 \\ 0 & 0 & 0 & \frac{K_k^4}{K_0^4} \end{pmatrix}. \quad (*)$$

Note,  $k = 0$  yields  $M_0 = I_4$ . Since  $M_k$  is supposed to represent a rotation, entries of  $M_k$ ,  $k = 1, 2$  can be specified further:

Rotations (about the  $z$ -axis by the angle  $\alpha$ ) can be described exactly by rational quadratics defined by Bézier points  $K_k = (K_k^1, K_k^2, K_k^3, K_k^4)^T = (\beta_k K_k^T, \beta_k)^T$ ,  $k = 0, 1, 2$ , where  $\beta_0 = 1$ ,  $\beta_1 = \cos \frac{\alpha}{2}$ ,  $\beta_2 = 1$  and rotation of  $K_0$  by  $\alpha$ ,  $R(\alpha)$ , yields  $K_2$  while rotation of  $K_0$  by  $\frac{\alpha}{2}$ ,  $R(\frac{\alpha}{2})$ , followed by a radial scaling with  $1/\cos \frac{\alpha}{2}$ ,  $S(1/\cos \frac{\alpha}{2})$ , gives  $K_1$ , see [Hos92] p. 152.

Now, solving of  $K_2 = \eta R(\alpha) K_0$  and of  $K_1 = \eta S(1/\cos \frac{\alpha}{2}) R(\frac{\alpha}{2}) K_0$  for the components of  $\eta R(\alpha)$  and of  $\eta S(1/\cos \frac{\alpha}{2}) R(\frac{\alpha}{2})$  proves that  $\eta R(\alpha)$  is equal  $M_2$  and  $\eta S(1/\cos \frac{\alpha}{2}) R(\frac{\alpha}{2})$  is equal  $M_1$  in (\*) and that components are given as stated in Theorem 2.  $\square$

Note, Bézier points  $V_{i,j,1} = M_1 F_{i,j}$  become infinite control points for  $|\alpha| = 180^\circ$ . Figures 2 and 3 show two examples of volumes of revolution, in both cases solid definition involves several infinite control points.

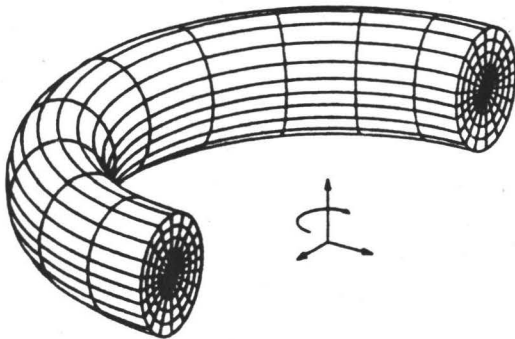


Figure 2. Solid half-torus described by a rational TPB volume of degree (1, 2, 2).

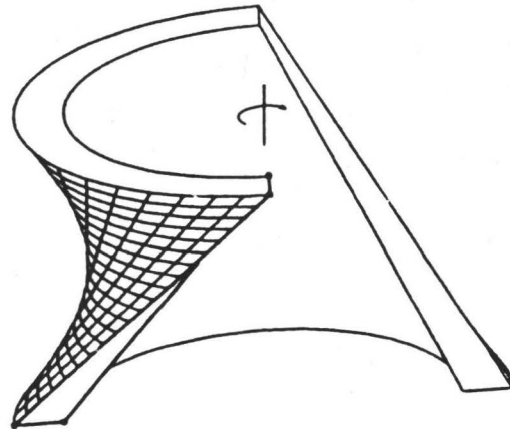


Figure 3. Part of a solid hyperboloid of revolution of one sheet described by a rational TPB volume of degree (1, 1, 2).

Rotations about the  $x$ -axis and the  $y$ -axis can be described analogously.

Since matrices  $\mathbf{M}_k$  (and *start point*  $\mathbf{K}_0$ ) define directrix  $\mathbf{K}(w)$ , different curves can be described by changing these matrices and all  $w$  parameter lines of  $\mathbf{V}(u)$  will adapt the geometry introduced by the  $\mathbf{M}_k$ . For example, if we choose parameter  $s_2$  in

$$\mathbf{K}(w) = \eta(w)\mathbf{M}(w)\mathbf{K}_0 = \sum_{k=0}^2 \hat{\mathbf{M}}_k \mathbf{K}_0 B_k^2(w) \quad (17)$$

with  $\hat{\mathbf{M}}_k = s_k \mathbf{M}_k$ ,  $\mathbf{M}_k$  according to (16),  $s_0 = s_2 = 1$ , appropriately, we obtain an arc of an ellipse, parabola or hyperbola, [Kir 89], [Las 90]: a *parabola* results for  $s_2 = s^* \equiv 1/\cos \frac{\alpha}{2}$ , because in that case  $\mathbf{K}(w)$  is a non-rational, polynomial quadratic Bézier curve (all weights are equal!), for  $s_2 > s^*$  a *hyperbola* results, for  $s_2 < s^*$  an *ellipse* and, for  $s_2 = 1$  we have Theorem 2, i.e.  $\mathbf{K}(w)$  defines a circle, see Figure 4.

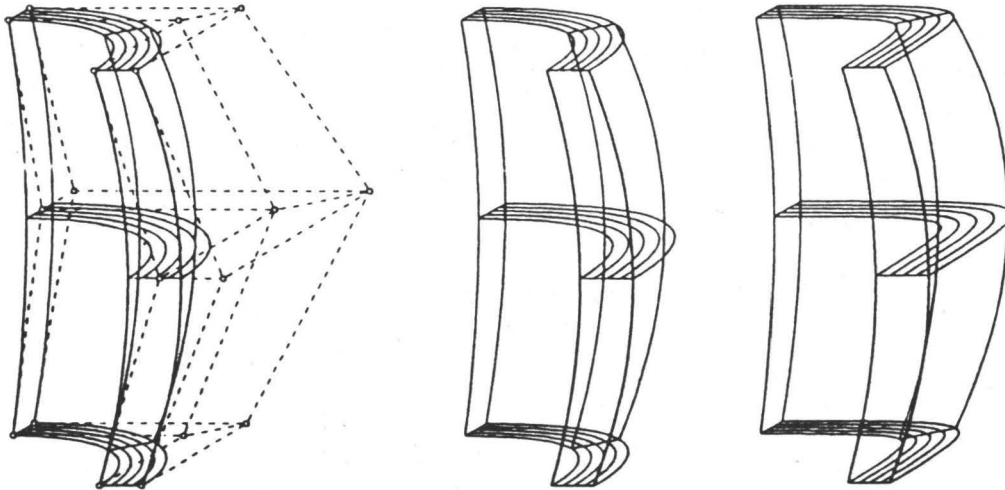


Figure 4. Three rational TPB volumes with arcs of circles ( $s_2 = 1$ , left), parabolas ( $s_2 = s^*$ , middle) and hyperbolas ( $s_2 = 3s^*$ , right) as  $w$  parameter lines.

## IV. Curve and Surface Modelling

Free form volumes can be interpreted as deformation of the associated parameter domain, i.e. the unit cube, under the mapping induced by their defining equation. In general, the position of every point in the interior as well as those on the boundary surfaces is altered. It is of particular interest for designers to be able to embed curves, surfaces, and volumes in the parameter domain of a free form volume, since this allows concentrating on well-defined subsets of the parameter space. In simple cases, these subsets are in fact isoparametric lines and planes, which under the deformation are mapped to isoparametric curves and surfaces of the free form volume. The more interesting subsets, however, are non-isoparametric subsets, which might be given by any kind of representation. In context of these more general non-isoparametric subsets, the term FFD (*Free Form Deformation*) is commonly used, [Sed 86], [Cha 89], [Coq 90, 91].

In order to carry out an FFD modelling process, for every point of the object we must first find the associated parameter value in the parameter space of the volume. For this, we

define a box-shaped local coordinate system, the deformation domain, which includes all parts of the object to be deformed. It might be subdivided into sub-boxes corresponding to a segmented TPB volume. By comparing coordinates, we find the sub-box  $Q_0$ , defined by  $P_0$ ,  $U$ ,  $V$ , and  $W$  as in Figure 5 which contains the given point  $P$  of the object. We have

$$\mathbf{P} = \mathbf{P}_0 + u\mathbf{U} + v\mathbf{V} + w\mathbf{W}, \quad (18)$$

where local coordinates  $u, v, w$  are calculated by

$$u = \frac{\mathbf{V} \times \mathbf{W} \cdot (\mathbf{P} - \mathbf{P}_0)}{\mathbf{V} \times \mathbf{W} \cdot \mathbf{U}}, \quad v = \frac{\mathbf{U} \times \mathbf{W} \cdot (\mathbf{P} - \mathbf{P}_0)}{\mathbf{U} \times \mathbf{W} \cdot \mathbf{V}}, \quad w = \frac{\mathbf{U} \times \mathbf{V} \cdot (\mathbf{P} - \mathbf{P}_0)}{\mathbf{U} \times \mathbf{V} \cdot \mathbf{W}}.$$

Now we have to prescribe the polynomial degree, corresponding weights and a control point grid which covers the deformation domain. Figure 6 illustrates this process in case of a volume segment of degree  $(3, 2, 2)$ . The control points of the grid are given by

$$\mathbf{V}_{i,j,k} = \mathbf{P}_0 + \frac{i}{l}\mathbf{U} + \frac{j}{m}\mathbf{V} + \frac{k}{n}\mathbf{W}, \quad (19)$$

and initially weights  $\omega_{i,j,k}$  are chosen to be equal to one.

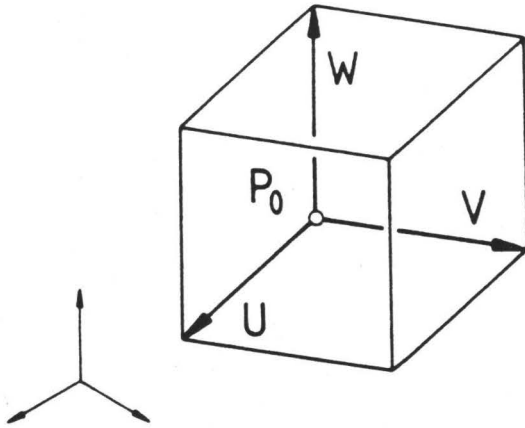


Figure 5. Local coordinate system.

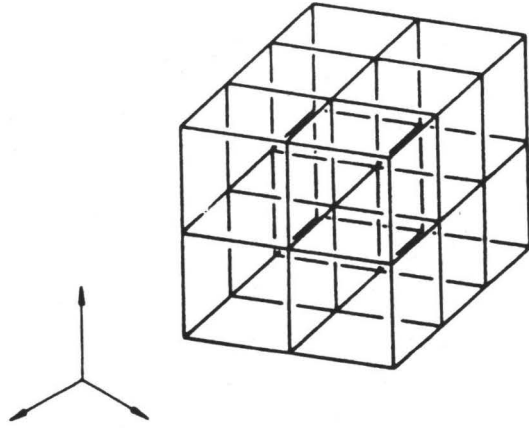


Figure 6. Control point grid.

The actual deformation of the object involves translation of points,  $\mathbf{V}_{i,j,k} \mapsto \bar{\mathbf{V}}_{i,j,k}$ , changing of weights,  $\omega_{i,j,k} \mapsto \bar{\omega}_{i,j,k}$ , and evaluation of the FFD defining equation with coefficients  $\bar{\mathbf{V}}_{i,j,k}$ , weights  $\bar{\omega}_{i,j,k}$ , and parameter values  $u, v, w$  of the object point  $\mathbf{P}$  calculated above.

From a mathematical point of view, the FFD construction involves composing two mappings, the mappings of the object and of the volume definition (cf. Figure 7) and has been investigated in [DeR 88, 93] and [Las 93]. For the case of modelling a rational Bézier curve using a rational TPB volume the mathematical foundation for exactly and explicitly describing the FFD is given by

**Theorem 3.** Let  $\mathbf{K}(t) = (u(t), v(t), w(t))^T$  be a rational Bézier curve of degree  $N$  with Bézier points  $\mathbf{K}_I = (u_I, v_I, w_I)^T$  and weights  $\beta_I$ . Let  $\mathbf{V}(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))^T$  be a rational TPB volume of degree  $(l, m, n)$  with Bézier points  $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^T$  and weights  $\omega_{i,j,k}$ .

$\mathbf{V}(t) = \mathbf{V}(\mathbf{K}(t)) = \mathbf{V}(u(t), v(t), w(t))$  is rational and can be represented as rational Bézier curve of degree  $rN$ ,  $r = l + m + n$ . We have, in terms of homogeneous coordinates,

$$\mathbf{V}(t) = \mathbf{V}(\mathbf{K}(t)) = \sum_{R=0}^{rN} \mathbf{V}_R B_R^{rN}(t), \quad (20)$$

where 4D Bézier points  $\mathbf{V}_R = (V_R^1, V_R^2, V_R^3, V_R^4)^T = (\Omega_R \mathbf{V}_R^T, \Omega_R)^T$  are given by <sup>1</sup>

$$\mathbf{V}_R = \sum_{|\mathbf{I}|=R} B^{l,m,n}(\mathbf{I}) C_R^{l,m,n}(N, \mathbf{I}) \mathbf{V}_{0,0,0}^{l,m,n}(u_{\mathbf{I}^u}^l, v_{\mathbf{I}^v}^m, w_{\mathbf{I}^w}^n), \quad (21)$$

with combinatorial constants

$$C_R^{l,m,n}(N, \mathbf{I}) = \frac{\prod_{Q^u=1}^l \binom{N}{I_{Q^u}^u} \prod_{Q^v=1}^m \binom{N}{I_{Q^v}^v} \prod_{Q^w=1}^n \binom{N}{I_{Q^w}^w}}{\binom{rN}{R}}, \quad (22)$$

and,

$$B^{l,m,n}(\mathbf{I}) = \prod_{Q^u=1}^l \beta_{I_{Q^u}^u} \prod_{Q^v=1}^m \beta_{I_{Q^v}^v} \prod_{Q^w=1}^n \beta_{I_{Q^w}^w}. \quad (23)$$

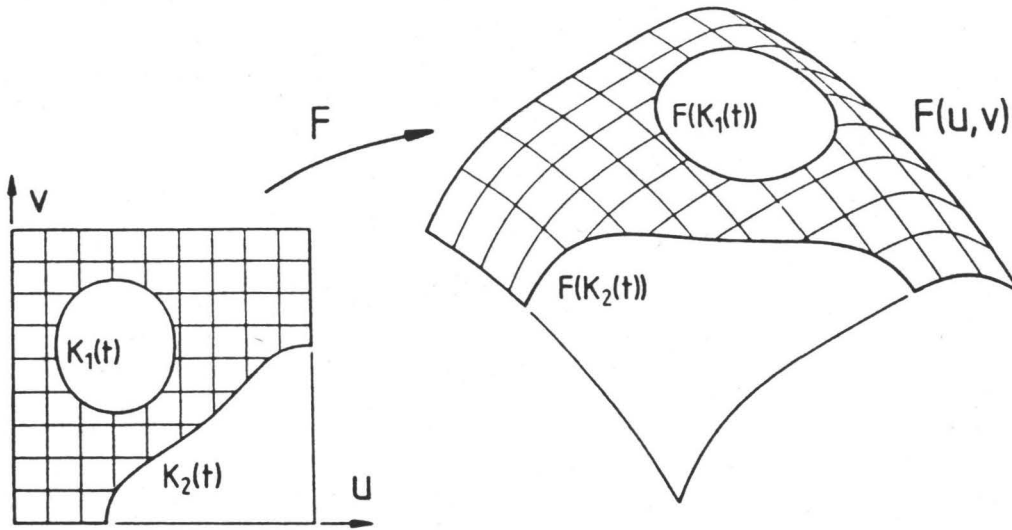


Figure 7. FFD as a composition of two mappings: Deformation of planar curves  $\mathbf{K}_1(t)$  and  $\mathbf{K}_2(t)$  by a surface mapping  $\mathbf{F}(u, v): \mathbb{R}^2 \mapsto \mathbb{R}^3$ .

Proof of Theorem 3, analogously like the proofs presented in [Las 93] (cf. [DeR 93]).

$\sum_{|\mathbf{I}|=R}$  has the meaning of summation over all  $\mathbf{I} = (\mathbf{I}^u, \mathbf{I}^v, \mathbf{I}^w)$ , where  $\mathbf{I}^u = (I_1^u, \dots, I_l^u)$ ,  $\mathbf{I}^v = (I_1^v, \dots, I_m^v)$ ,  $\mathbf{I}^w = (I_1^w, \dots, I_n^w)$  and where  $0 \leq I_1^u, \dots, I_l^u \leq N$ ,  $0 \leq I_1^v, \dots, I_m^v \leq N$ ,  $0 \leq I_1^w, \dots, I_n^w \leq N$ ,  $|\mathbf{I}| = |\mathbf{I}^u| + |\mathbf{I}^v| + |\mathbf{I}^w| = I_1^u + \dots + I_l^u + I_1^v + \dots + I_m^v + I_1^w + \dots + I_n^w = R$ .

<sup>1</sup>Please note, notation varies from the one used in [Las 93].



$\mathbf{V}_{0,0,0}^{l,m,n}$  is defined recursively by de Casteljau's construction. The argument  $(u_{I^u}^l, v_{I^v}^m, w_{I^w}^n)$  indicates that  $\mathbf{V}_{l,m,n}^{0,0,0}$  has to be calculated by performing  $l$  de Casteljau constructions in  $u$  direction for the  $u$  parameter values given by the indices  $\mathbf{I}^u = (I_1^u, \dots, I_l^u)$ , i.e. for the parameter values  $u_{I_1^u}, \dots, u_{I_l^u}$ ,  $m$  de Casteljau constructions in  $v$  direction for the  $v$  parameter values given by the indices  $\mathbf{I}^v = (I_1^v, \dots, I_m^v)$ , i.e. for the parameter values  $v_{I_1^v}, \dots, v_{I_m^v}$  and  $n$  de Casteljau constructions in  $w$  direction for the  $w$  parameter values given by the indices  $\mathbf{I}^w = (I_1^w, \dots, I_n^w)$ , i.e. for the parameter values  $w_{I_1^w}, \dots, w_{I_n^w}$ . Calculations for different parameter values commute, and the order in which we carry out calculations has no influence on the final result.

Note, isoparametric lines of  $(u, v, w)$  domain space should not be handled by Theorem 3 which, in that case, would yield degree raised isoparametric Bézier curves of  $\mathbf{V}(u, v, w)$ .

Spline curves are processed using Theorem 3 one curve segment at a time. Modelling by spline volumes requires the determination of intersections of  $\mathbf{K}(t)$  and boundary planes of volume segments in domain space and splitting of  $\mathbf{K}(t)$  at these intersection points by adding new knots applying de Casteljau's curve subdivision algorithm. Then, for all segments of  $\mathbf{V}(\mathbf{u})$  Theorem 3 can be adapted. Smoothness of  $\mathbf{V}(\mathbf{K}(t))$  results from application of the chain rule: if  $\mathbf{K}(t)$  is  $C^a$ -continuous in  $t = t^*$ , and  $\mathbf{V}(\mathbf{u})$  is  $C^b$ -continuous in the corresponding point  $\mathbf{K}(t) = \mathbf{K}(t^*)$  of  $(u, v, w)$  domain space,  $\mathbf{V}(\mathbf{K}(t))$  is  $C^e$ -continuous in  $\mathbf{V}(\mathbf{K}(t^*))$ , with  $e = \min\{a, b\}$ .

Generalizing Theorem 3, the mathematical foundation for the FFD approach to rational surface design via rational volume modelling is provided by, [Las 93] (cf. [DeR 93]),

**Theorem 4.** Let  $\mathbf{F}(\mu, \nu) = (u(\mu, \nu), v(\mu, \nu), w(\mu, \nu))^T$  be a rational TPB surface of degree  $(L, M)$  with Bézier points  $\mathbf{F}_{I,J} = (u_{I,J}, v_{I,J}, w_{I,J})^T$  and weights  $\beta_{I,J}$ . Let  $\mathbf{V}(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))^T$  be a rational TPB volume of degree  $(l, m, n)$  with Bézier points  $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^T$  and weights  $\omega_{i,j,k}$ .  $\mathbf{V}(\mu, \nu) = \mathbf{V}(\mathbf{F}(\mu, \nu)) = \mathbf{V}(u(\mu, \nu), v(\mu, \nu), w(\mu, \nu))$  is rational and can be represented as rational TPB surface of degree  $(rL, rM)$ ,  $r = l + m + n$ . We have, in terms of homogeneous coordinates,

$$\mathbf{V}(\mu, \nu) = \mathbf{V}(\mathbf{F}(\mu, \nu)) = \sum_{R=0}^{rL} \sum_{S=0}^{rM} \mathbf{V}_{R,S} B_R^{rL}(\mu) B_S^{rM}(\nu), \quad (24)$$

where 4D Bézier points  $\mathbf{V}_{R,S} = (V_{R,S}^1, V_{R,S}^2, V_{R,S}^3, V_{R,S}^4)^T = (\Omega_{R,S} \mathbf{V}_{R,S}^T, \Omega_{R,S})^T$  are given by <sup>2</sup>

$$\mathbf{V}_{R,S} = \sum_{|\mathbf{I}|=R} \sum_{|\mathbf{J}|=S} B^{l,m,n}(\mathbf{I}, \mathbf{J}) C_R^{l,m,n}(L, \mathbf{I}) C_S^{l,m,n}(M, \mathbf{J}) \mathbf{V}_{0,0,0}^{l,m,n}(u_{\mathbf{I}^u}^l, v_{\mathbf{I}^v}^m, w_{\mathbf{I}^w}^n). \quad (25)$$

$\sum_{|\mathbf{I}|=R}$  with  $\mathbf{I} = (\mathbf{I}^u, \mathbf{I}^v, \mathbf{I}^w)$  has the same meaning as in Theorem 3,  $\sum_{|\mathbf{J}|=S}$  with  $\mathbf{J} = (\mathbf{J}^u, \mathbf{J}^v, \mathbf{J}^w)$  similarly. Constants  $C_R^{l,m,n}(L, \mathbf{I})$  and  $C_S^{l,m,n}(M, \mathbf{J})$  are calculated according to (22) and  $B^{l,m,n}(\mathbf{I}, \mathbf{J})$  is defined as in (23) but now using surface weights  $\beta_{I,J}$ .  $\mathbf{V}_{0,0,0}^{l,m,n}$  is specified by de Casteljau's construction.

<sup>2</sup>Please note, calculation of combinatorial constants is wrong in [Las 93] Section IV.2.

Like for curves, remarks concerning isoparametric planes of domain space of  $V(u)$  as well as treatment of spline surfaces prove to be right again.

We observe that because FFDs involve functional composition, the degree of FFDs can grow very quickly. High degrees might cause problems in succeeding interrogation actions such as intersection calculations and also might not be supported by some CAD systems. Therefore, the exact description of FFDs will not always be the best approach. Suitable approximation techniques taking advantage of the knowledge of the exact and explicit FFD representation to describe  $V(K(t))$  and  $V(F(u,v))$  by lower polynomial degrees are under investigation.

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