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The Analytic Blossom

Géraldine Morin, Ron Goldman

Universität Kaiserslautern

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FACHBEREICH INFORMATIK

UNIVERSITÄT KAISERSLAUTERN

Postfach 3049 · D-67653 Kaiserslautern

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Universität Kaiserslautern

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Universität Kaiserslautern
Fachbereich Informatik
AG Computergraphik
Postfach 30 49
67653 Kaiserslautern

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Abstract. Blossoming is a powerful tool for studying and computing with Bézier and B-spline curves and surfaces – that is, for the investigation and analysis of polynomials and piecewise polynomials in geometric modeling. In this paper, we define a notion of the blossom for Poisson curves. Poisson curves are to analytic functions what Bézier curves are to polynomials – a representation adapted to geometric design. As in the polynomial setting, the blossom provides a simple, powerful, elegant and computationally meaningful way to analyze Poisson curves. Here, we define the analytic blossom and interpret all the known algorithms for Poisson curves – subdivision, trimming, evaluation of the function and its derivatives, and conversion between the Taylor and the Poisson basis – in terms of this analytic blossom.

§1. Introduction

The blossom, or polar form, presented by Ramshaw [17,18,19] and de Casteljau [6] is a simple, elegant and powerful tool, very adapted to working with the Bézier representation of a polynomial. Not only can the dual functionals of the Bernstein basis be expressed very simply in terms of the blossom, but also algorithms like subdivision or change of basis can be explained and justified very easily using the multi-affinity and symmetry of the blossom [1]. The blossom is related as well to other effective tools for analyzing Bézier and B-splines curves and surfaces like the de Boor-Fix formula [1,2,5] and the Marsden identity [10,20,7]. More recently, blossoming, in various forms, has provided a method for extending Bézier curves to non-polynomial settings [3,16,8,11,12]. In this paper, we define the equivalent of the polynomial blossom for analytic functions.

Poisson curves are analytic functions expressed in the Poisson basis, a representation adapted to geometric modeling. With a blossom for analytic functions, the algorithms and tools available for Poisson curves have a new, simple and elegant interpretation, and, as in the polynomial setting, many of these algorithms follow simply from the multi-affinity and symmetry of the blossom. The purpose of this paper is to introduce the analytic blossom, and to interpret the algorithms and tools available for Poisson curves in terms of this blossom.

§2. The Polynomial Setting

In this section we shall briefly recall the classical definition of the blossom [17]. We shall also see that the blossom can be characterized by replacing the diagonal property with the dual functional property. In the next section, this result will be extended to the analytic setting.

To simplify our notation and to emphasize that a polynomial and its blossom are just different representations of the same underlying object, we will denote a function and its blossoms by the same name. For example, if P is a polynomial of degree n, then $P[x_1, \dots, x_n]$ will denote the blossom of P evaluated at $(x_i)_{i=1}^n$. More generally, $P[x_1, \dots, x_k]$ for $k \geq n$ will denote the blossom of P considered as a polynomial of degree k evaluated at $(x_i)_{i=1}^k$. (Note that when using the standard notation p for the blossom of P, the distinction between the polynomial blossoms of different degrees is also determined implicitly by the number of arguments, or domain of the function.) Later, we shall use $P[x_1, \dots, x_n, 0 \dots] = P[x_1, \dots, x_n, 0^\infty]$ to denote the analytic blossom of P.

The polynomial blossom and its relation to dual functionals

Definition 1. *The blossom of a polynomial P of degree n is the unique symmetric, multi-affine, n-ary function that satisfies the diagonal property: $P[\underbrace{x, \dots, x}_{n\ times}] = P(x)$ [17]. That is:*

- $P[x_1, \dots, x_i, \dots, x_j, \dots, x_n] = P[x_1, \dots, x_j, \dots, x_i, \dots, x_n]$,
- $P[.., (1-a)u + av, ..] = (1-a)P[.., u, ..] + aP[.., v, ..]$,
- $P[\underbrace{x, \dots, x}_{n\ times}] = P[x^n] = P[x]$.

In this paper, we use the notation $a^k b^j$ to represent $\underbrace{a, a, \dots, a}_{k\ times}, \underbrace{b, \dots, b}_{j\ times}$.

The blossom is a linear operator which is easy to compute. For example, the blossom of $P(t) = \binom{n}{k} t^k$ is $P[x_1, \dots, x_n] = \sum_{i_a \neq i_b} x_{i_1} \dots x_{i_k}$.

The dual functional property links Bézier curves to the blossom. Let $B_k^n(t)$ denote the kth Bernstein polynomial of degree n and P_k, $k = 0 \dots n$, the coefficients (or Bézier points) of the polynomial P in the Bernstein basis of degree n. Then the dual functional property asserts that

$$P[0^{n-k} 1^k] = P_k.$$

The dual functional property is known to be an easy consequence the three blossoming axioms (see definition 1). What is not generally appreciated is that this dual functional property can actually replace the diagonal property in the blossoming axioms.

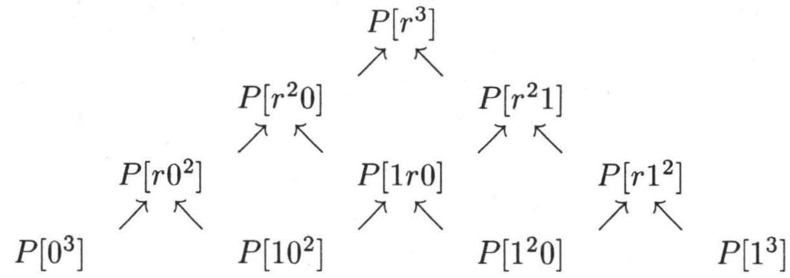

Fig. 1. The de Casteljau algorithm for a cubic Bézier curve P at a parameter r in $(0, 1)$: The de Casteljau algorithm is both an evaluation and a subdivision algorithm. The parameter r is inserted at each level using the identity $r = (1 - r)0 + r1$ and invoking the symmetry and multi-affinity property of the blossom. The input, the values of the blossom of P over the knot vector $(0^3 1^3)$, is at the base of the diagram. The value $P(r) = P[r^3]$ appears at the apex and the control points after one round of subdivision, appear along the left and right lateral edges.

Proposition 2. *The blossom of a degree n polynomial P is the unique symmetric, multi-affine, n-ary function characterized by the dual functional property:*

$$P_k = P[\underbrace{0, \ldots, 0}_{n-k \text{ times}}, \underbrace{1, \ldots, 1}_{k \text{ times}}] = P[0^{n-k} 1^k], \text{ for all } k = 0, \ldots, n,$$

where the points $P_0, \ldots, P_n$ are the Bézier control points of the polynomial P over the interval $[0, 1]$ – that is, the coefficients of P in the Bernstein basis.

Proof: By the definition of the Bézier control points,

$$P(t) = \sum_{k=0}^{n} P_k B_k^n(t). \tag{1}$$

Moreover, by the symmetry and multi-affinity of the blossom (Fig. 1 taking $r = t$)

$$P[t^n] = \sum_{k=0}^{n} P[0^{n-k} 1^k] B_k^n(t). \tag{2}$$

If we assume that the blossom satisfies the diagonal property, then from (2)

$$P(t) = \sum_{k=0}^{n} P[0^{n-k} 1^k] B_k^n(t).$$

So, from the uniqueness of the coefficients of P in the Bernstein basis,

$$P[0^{n-k} 1^k] = P_k, \text{ for all } k = 0, \ldots, n.$$

Hence the diagonal property implies the dual functional property.

Conversely, if the blossom satisfies the dual functional property, then

$$P[0^{n-k}1^k] = P_k, \text{ for all } k = 0, \dots, n.$$

Using these identities in (1) and (2), we obtain $P[t^n] = P(t)$. Hence the dual functional property implies the diagonal property. □

The knot vector $(0^n 1^n)$ corresponds to the Bernstein basis – that is, the blossom evaluated over this knot vector (at every n consecutive coordinates of the knot vector) provides the dual functionals for the Bernstein basis. The choice of the interval $[0,1]$ is arbitrary. The Bézier curve P can be parameterized over any interval $[a,b]$, in which case the dual functionals are given by $P_k = P[a^{n-k}b^k]$.

Polynomial blossom and algorithms for Bézier curves

Not only can the control points be interpreted in terms of the blossom, but so too can algorithms for Bézier curves: in particular, subdivision, conversion between monomial and Bernstein form, and evaluation of the function and its derivatives [1]. For all these algorithms, both input and output can be expressed by the blossom of n-tuples. An algorithm, a way to compute the output from the input, then follows by applying the properties of the blossom.

For example, one step of subdivision splits a Bézier curve into two curves. The control points of these new curves are computed from the control points of the original curve. From the dual functional property, the original control points $(P_0, \dots, P_n)$ over the interval $[0,1]$ are given by the blossom evaluated over the knot vector $(0^n 1^n)$ at every n consecutive coordinates of the knot vector. For subdivision at r, we compute the control points associated with the knot vectors $(0^n r^n)$ and $(r^n 1^n)$ from the control points associated with the knot vector $(0^n 1^n)$. Using the multi-affinity and the symmetry of the blossom, we can generate a dynamic programming algorithm for subdivision: this is the de Casteljau subdivision algorithm (Fig. 1).

Polynomial blossom, Marsden identity and de Boor-Fix formula

The blossom expresses Bézier control points and algorithms in a simple and elegant way. Moreover, some other powerful tools for studying Bézier and B-splines curves, such as the Marsden identity [10] and the de Boor-Fix formula [5] are closely linked to the blossom [2,20]. We shall briefly review these results and their connections here, and we will extend these identities to the analytic blossom in section 5.

Given two polynomials $P(u)$ and $Q(u)$ of degree n, define the bilinear form [7]:

$$[P(u), Q(u)]_n = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}P^{(k)}(\tau)Q^{(n-k)}(\tau).$$

Notice that the right hand side is a constant independent of τ since its derivative is zero.

The next proposition expresses evaluation of the function, its derivatives and its blossom in terms of the bracket operator.

Proposition 3.

$$P(x) = [P(u), (x-u)^n]_n \tag{3}$$

$$P^{(k)}(x) = [P(u), \frac{d^k}{dx^k}(x-u)^n]_n \tag{4}$$

$$P[x_1, \ldots, x_n] = [P(u), (X-u)^n[x_1, \ldots, x_n]]_n = [P(u), (x_1-u)\ldots(x_n-u)]_n \tag{5}$$

Proof: The explicit computation of $[P(u), \frac{d^k}{dx^k}(x-u)^n]_n = \frac{n!}{(n-k)!}[P(u), (x-u)^{n-k}]_n$ leads to the Taylor expansion of $P^{(k)}(x)$ at τ. Equation (5) holds since the right hand side satisfies the three blossoming axioms: symmetry and multi-affinity in the x_i's are immediate from the definition and the diagonal property follows from equation (3). □

Equation (5) is exactly the de Boor-Fix expression for the blossom derived by Barry [2]. This explicit expression for the blossom provides an alternative proof of the existence of the blossom.

The function $(x-u)^n$ plays a key role with respect to this bracket operator [7]. Bracketing a polynomial P with $(x-u)^n$ not only reproduces P, but also bracketing P with the derivative or the blossom of $(x-u)^n$, reproduces the derivative or the blossom of P. The expression of this same function $(x-u)^n$ in the Bernstein basis is exactly the Marsden identity for the Bernstein basis [20]

$$(x-u)^n = \sum_{k=0}^{n} (-1)^{n-k} u^{n-k} (1-u)^k B_k^n(x).$$

This identity follows directly from the dual functional property of the polynomial blossom.

§3. The Analytic Blossom

Let F be an analytic function at zero. Then F admits not only a Taylor development at zero, but also a Poisson development at zero converging on $D(0,R)$, where R is the radius of convergence of the Taylor series and $D(0,R)$ is the open disk in $\mathbb{C}$ of center 0 and radius R. That is,

$$F(t) = \sum_{k \geq 0} P_k b_k(t), \quad t \in D(0,R),$$

where $b_k(t) = e^{-t}\frac{t^k}{k!}$ [13]. The coefficients (P_k) of F in the Poisson basis $(b_k(t))$ are called the Poisson control points of the curve F. Here, for simplicity, we always consider the Poisson development at zero. This is not a real restriction, since a Poisson curve can be trimmed [14]. That is, the Poisson representation of $F(a + .)$ can be generated from the Poisson representation of F for an arbitrary parameter a in the interior of the domain of convergence.

The Poisson representation is to analytic functions what the Bézier representation is to polynomials. Many properties and algorithms generalize from

Bézier to Poisson curves, e.g., Poisson curves follow the shape of the Poisson control polygon, since the convex hull and variation diminishing properties hold in the Poisson basis. There is also a subdivision algorithm for analytic functions, based on the Poisson representation [13].

The main goals of this paper are to define a blossom for analytic functions, establish its existence and uniqueness, investigate its main properties, and apply it to the study of algorithms for Poisson curves. In the remainder of this section, we shall first define the blossom of an analytic function, using three axioms similar to the axioms of the polynomial blossom, and we shall provide as well some simple examples of blossoms of analytic functions. Second, we shall prove the existence of the analytic blossom. We then show that this analytic blossom provides the dual functionals for the Poisson basis, and we prove that, just as in the polynomial setting, this dual functional property can replace the diagonal property in the definition of the blossom. Finally, we will use the dual functional property to establish the uniqueness of the analytic blossom.

Definition and examples of the analytic blossom

Definition 4. *The blossom of a function F, analytic at zero, is the unique function that takes an infinite number of arguments almost all of which are zero, is symmetric, multi-affine, and satisfies a diagonal property. Thus, we have:*

- $F[.., x_i, .., x_j, ..] = F[.., x_j, .., x_i, ..]$,
- $F[.., (1-a)u + av, ..] = (1-a)F[.., u, ..] + aF[.., v, ..]$,
- $F[\underbrace{\frac{t}{n}, \ldots, \frac{t}{n}}_{n \text{ times}}, 0, \ldots]$ *converges uniformly to $F(t)$ as n goes to ∞ on any disk $\bar{D}(0, b)$, where $b < R$, R is the radius of convergence of the Taylor expansion of F at zero, and $\bar{D}(0, b)$ is the closed disk of center 0 and radius b.*

Moreover, these three properties completely characterize the blossom – existence is proved in the next subsection and uniqueness at the end of the section.

Here are some examples of analytic blossoms:

- if $F(t) = t$, then $F[x_1, \ldots, x_n, 0, \ldots] = \sum_i x_i$,
- if $F(t) = \frac{t^2}{2}$, then $F[x_1, \ldots, x_n, 0, \ldots] = \sum_{i \neq j} x_i x_j$.

More generally

- if $F(t) = \frac{t^k}{k!}$, then $F[x_1, \ldots, x_n, 0, \ldots] = \sum_{i_a \neq i_b} x_{i_1} \ldots x_{i_k}$.

Finally, we have

- if $F(t) = e^t$, then $F[x_1, \ldots, x_n, 0, \ldots] = \sum_{k \geq 0} \sum_{i_a \neq i_b} x_{i_1} \ldots x_{i_k} = \prod_{i=1}^{n} (1 + x_i)$.

Notice that, whereas the polynomial blossom of degree n is defined on $\mathbb{C}^n$, the analytic blossom is defined on the direct sum $\bigoplus_{i=1}^{\infty} C_i$, where $C_i = \mathbb{C}$ for all i and not on the direct product $\times_{i=1}^{\infty} C_i$.

Existence of an analytic blossom

The main point of this section is to establish the existence of an analytic blossom satisfying the three axioms of the definition. This proof of existence is based on a relation between the analytic and polynomial blossoms. In particular, a polynomial P has both an analytic and polynomial blossoms.

We start with the following lemma. This result is crucial to proving both the existence and the uniqueness of the analytic blossom.

Lemma 5. *Let $F(t)$ be an analytic function and let R be the radius of convergence of its Poisson series $\sum P_k b_k(t)$ at zero. Then as $n \to \infty$*

$$\sum_{k=0}^{n} P_k B_k^n(\frac{t}{n}) \text{ converges uniformly to } \sum_{k\geq 0} P_k b_k(t) \text{ on } \bar{D}(0,b),$$

where $b < R$.

Proof: First let us prove that $B_k^n(\frac{t}{n})$ converges uniformly to $b_k(t)$ on $\bar{D}(0,b)$. By definition

$$B_k^n(\frac{t}{n}) = \binom{n}{k}(\frac{t}{n})^k(1-\frac{t}{n})^{n-k}$$

$$= \frac{n\ldots(n-k+1)}{n^k}\frac{t^k}{k!}(1-\frac{t}{n})^n(1-\frac{t}{n})^{-k}.$$

But, $\frac{n\ldots(n-k+1)}{n^k}$ converges to 1, and the functions $(1-\frac{t}{n})^n$ and $(1-\frac{t}{n})^{-k}$, for any fixed k, converge respectively to e^{-t} and 1 uniformly on $\bar{D}(0,b)$. Therefore $B_k^n(\frac{t}{n})$ converges uniformly to $b_k(t)$ on $\bar{D}(0,b)$.

Now, consider

$$\left|\sum_{k=0}^{n} P_k B_k^n(\frac{t}{n}) - \sum_{k\geq 0} P_k b_k(t)\right|$$

$$\leq \underbrace{\left|\sum_{k=0}^{m} P_k\left(B_k^n(\frac{t}{n}) - b_k(t)\right)\right|}_{A} + \underbrace{\left|\sum_{k>m} P_k b_k(t)\right|}_{B} + \underbrace{\left|\sum_{k=m+1}^{n} P_k B_k^n(\frac{t}{n})\right|}_{C}. \quad (6)$$

Because both $F(t)$ and $F(t)e^t$ are defined by their Taylor and Poisson series at 0 on $\bar{D}(0,b)$, for any ε we can choose m so large that

$$\sum_{k\geq m} |P_k \frac{t^k}{k!}| < \varepsilon \text{ and } \sum_{k\geq m} P_k |b_k(t)| < \varepsilon \quad (7)$$

for all $t \in \bar{D}(0,b)$. Thus, in particular, $B < \varepsilon$.

Moreover, since $B_k^n(\frac{t}{n}) \longrightarrow b_k(t)$ as $n \to \infty$, for any fixed $0 \leq k \leq m$, there exists n_0 such that for all $n \geq n_0$: $A < \varepsilon$.

Finally, we need to bound C. First observe that, for all $n \geq 0$, $0 \leq i \leq n$ and $t \in \bar{D}(0,b)$

$$\left|(1-\frac{t}{n})^i\right| \leq e^b. \tag{8}$$

because

$$\left|(1-\frac{t}{n})^i\right| = \left|\sum_{j=0}^{i} \binom{i}{j} (\frac{-t}{n})^j\right| = \left|\sum_{j=0}^{i} \frac{i\ldots(i-j+1)}{n^j}\frac{(-t)^j}{j!}\right|$$

$$\leq \sum_{j\geq 0} \left|\frac{(-t)^j}{j!}\right| \leq e^b.$$

Now

$$C = \left|\sum_{k=m+1}^{n} P_k \frac{t^k}{k!} \frac{n\ldots(n-k+1)}{n^k}(1-\frac{t}{n})^{n-k}\right|.$$

But $0 \leq n-k \leq n-m+1 \leq n$, so from (8) and (7)

$$C \leq e^b \sum_{m+1}^{n} \left|P_k \frac{t^k}{k!}\right| \leq e^b \varepsilon.$$

Since A, B and C can each be bounded uniformly on $\bar{D}(0,b)$, the result follows by (6). □

The next lemma relates the analytic blossom of Poisson and Taylor basis functions respectively to the polynomial blossom of the Bernstein polynomials and the Taylor monomials.

Lemma 6.

$$b_k[x_1,\ldots,x_n,0^\infty] = B_k^n[x_1,\ldots,x_n] \tag{9}$$

$$\frac{X^k}{k!}[x_1,\ldots,x_n,0^\infty] = \binom{n}{k} X^k[x_1,\ldots,x_n] \tag{10}$$

Proof: First, we need to check that the right hand sides of these two identities are well defined. That is, for $x_n = 0$, we must have

$$B_k^{n-1}[x_1,\ldots,x_{n-1}] = B_k^n[x_1,\ldots,x_{n-1},0], \tag{11}$$

$$\binom{n-1}{k} X^k[x_1,\ldots,x_{n-1}] = \binom{n}{k} X^k[x_1,\ldots,x_{n-1},0]. \tag{12}$$

By the dual functional property, equation (11) is satisfied for $[x_1,\ldots,x_{n-1}] = [0^{n-1-j}1^j]$; it is then satisfied on any $[x_1,\ldots,x_n]$ by symmetry and multi-affinity. Moreover, since

$$X^k[x_1,\ldots,x_n] = \frac{1}{\binom{n}{k}} \sum_{i_a \neq i_b} x_{i_1}\ldots x_{i_k},$$

equation (12) holds. This proves that the two identities are well defined.

Now we need to check that the right hand sides of equations (9) and (10) are symmetric, multi-affine, and satisfy the diagonal property of the analytic blossom. The symmetry and multi-affinity properties follow from the corresponding properties of the polynomial blossom.

From the diagonal property of the polynomial blossom

$$B_k^n[\left(\frac{t}{n}\right)^n] = B_k^n(\frac{t}{n}).$$

But, from lemma 5, $\lim_{n\to\infty} B_k^n(\frac{t}{n}) = b_k(t)$ and the convergence is uniform on $\bar{D}(0,b)$ for any $b > 0$. Thus the diagonal property holds for equation (9).

For the right hand side of equation (10), the diagonal property holds because

$$[\tbinom{n}{k}\, t^k][\left(\frac{t}{n}\right)^n] = \frac{n\ldots(n-k+1)}{k!}\left(\frac{t}{n}\right)^k,$$

which converges uniformly to $\frac{t^k}{k!}$ on $\bar{D}(0,b)$. □

The next lemma is the main result of this section; existence of the analytic blossom follows directly from it. Equations (13) and (14) express an analytic blossom of Poisson or Taylor series as the polynomial blossoms of Bernstein or Taylor polynomials.

Lemma 7.

$$(\sum_{k\geq 0} P_k b_k)[x_1,\ldots,x_n,0^\infty] = (\sum_{k=0}^{n} P_k B_k^n)[x_1,\ldots,x_n] \tag{13}$$

$$(\sum_{k\geq 0} A_k \frac{X^k}{k!})[x_1,\ldots,x_n,0^\infty] = \left(\sum_{k=0}^{n} A_k \tbinom{n}{k} X^k\right)[x_1,\ldots,x_n] \tag{14}$$

Proof: The proof of these two identities mimics the proof of lemma 6. From lemma 6 and by linearity the right hand sides of these identities are well defined. Moreover, from the symmetry, multi-affinity and linearity of the polynomial blossom, the right hand sides of these equations are symmetric and multi-affine.

From the diagonal property of the polynomial blossom

$$(\sum_{k=0}^{n} P_k B_k^n)[\left(\frac{t}{n}\right)^n] = \sum_{k=0}^{n} P_k B_k^n(\frac{t}{n}),$$

and from lemma 5 the sum on the right hand side converges uniformly to $\sum_{k\geq 0} P_k b_k(t)$ on $\bar{D}(0,b)$, where $b < R$ and R is the radius of convergence of the Poisson series. Thus the diagonal property for the analytic blossom is satisfied and therefore equation (13) holds.

Finally, equation (14) follows from the equation (13). From [13]

$$\sum A_k \frac{t^k}{k!} = \sum P_k b_k(t) \text{ where } P_k = \sum_{i=0}^{k} \binom{k}{i} A_k,$$

and from [9]

$$\binom{n}{i} t^i = \sum_{k=i}^{n} \binom{k}{i} B_k^n(t).$$

Thus

$$(\sum A_k \frac{X^k}{k!})[x_1, \ldots, x_n, 0^\infty] = (\sum_{k=0}^{n} \left(\sum_{i=0}^{k} \binom{k}{i} A_i \right) B_k^n)[x_1, \ldots, x_n]$$

$$= (\sum_{i=0}^{n} A_i \sum_{k=i}^{n} \binom{k}{i} B_k^n)[x_1, \ldots, x_n]$$

$$= (\sum_{i=0}^{n} A_i \binom{n}{i} X^i)[x_1, \ldots, x_n]. \square$$

Lemma 7 gives explicit expressions for the analytic blossom of Poisson and Taylor series at zero in terms of the polynomial blossom. Thus, since any analytic function F at zero admits both a Poisson and a Taylor development, an analytic blossom certainly exists.

Note that the blossom of F could as well be defined by another family of polynomials (F_n) if the following two conditions hold:

- the uniform convergence property

$$F(t) = \lim_{n \to \infty} F_n(\frac{t}{n})$$

- the compatibility property verified in the proof of lemma 6:

$$F_{n-1}[x_1, \ldots, x_{n-1}] = F_n[x_1, \ldots, x_{n-1}, 0].$$

We shall encounter another such family of functions in section 5, where we provide an alternative proof of existence by exploiting the functions $F(t) = e^{-ut}$ and $F_n(t) = (1 - ut)^n$.

The next lemma shows that the analytic blossom in n non-zero arguments is the blossom of the nth Poisson or Taylor partial sum.

Lemma 8.

$$(\sum_{k \geq 0} P_k b_k)[x_1, \ldots, x_n, 0^\infty] = (\sum_{k=0}^{n} P_k b_k)[x_1, \ldots, x_n, 0^\infty] \qquad (15)$$

$$(\sum_{k\geq 0} A_k \frac{X^k}{k!})[x_1, \ldots, x_n, 0^\infty] = \sum_{k=0}^{n} A_k (\frac{X^k}{k!})[x_1, \ldots, x_n, 0^\infty] \qquad (16)$$

Proof: These results follow immediately from lemmas 6 and 7 by linearity. □

Lemma 8 says that if we want to compute the analytic blossom of an infinite Poisson or Taylor expansion, we can always truncate to a finite sum. In effect, then, the linearity of the analytic blossom holds even for infinite series, since

$$b_k[x_1, \ldots, x_n, 0^\infty] = 0, \quad k > n,$$

$$X^k[x_1, \ldots, x_n, 0^\infty] = 0, \quad k > n.$$

The dual functional property

We will now prove that, as in the polynomial case, the diagonal property can be replaced by a dual functional property in the definition of the blossom. This second characterization of the blossom in terms of the dual functional property will be very useful for many reasons; uniqueness will follow very easily from the diagonal property and, in the analytic case, the dual functional property is simpler, more intuitive, and often easier to apply than the diagonal property. The dual functional property also allows us to avoid issues of uniform convergence over subsets of $\mathbb{C}$.

The following lemma is the first step in proving the equivalence of the two definitions of the analytic blossom.

Lemma 9. *Let* $F_n(t) = \sum_{k=0}^{n} P_k B_k^n\left(\frac{t}{n}\right)$ *and* $F(t) = \sum_{k\geq 0} P'_k b_k(t)$ *on the open disk* $D(0, R)$. *If*

$$(F_n)_{n\geq 0} \text{ converges uniformly to } F(t) \text{ on } \bar{D}(0,b),\ 0 < b < R$$

then

$$P_k = P'_k, \text{ for all } k.$$

Proof: Since we know from lemma 5 that $\sum_{k=0}^{n} P'_k B_k^n\left(\frac{t}{n}\right)$ converges uniformly to F on $\bar{D}(0,b)$, it is sufficient to prove: if $G_n(t) = \sum_{k=0}^{n}(P_k - P'_k) B_k^n\left(\frac{t}{n}\right)$ converges uniformly to $G(t) = 0$ on $\bar{D}(0,b)$, then $Q_k = P_k - P'_k = 0$ for all k.

The functions G_n are polynomials on $\mathbb{C}$ and therefore analytic on $\mathbb{C}$. The uniform convergence of a series of analytic functions (G_n) to an analytic function G (here the zero function) on $\bar{D}(0,b)$ implies the uniform convergence of the derivatives $(G_n^{(k)})$ to the derivative $G^{(k)}$ on the same domain $\bar{D}(0,b)$ [4].

Using this strong convergence property, we shall show by induction that $Q_i = 0$ for all i. First $Q_0 = 0$ because $G_n(0) = Q_0$ for any n and $G(0) = 0$.

Thus, from the convergence hypothesis of the lemma, Q_0 converge to 0 – that is, $Q_0 = 0$. Suppose that $Q_i = 0$ for $i \le m-1$. Then,

$$G_n(t) = \sum_{k=m}^{n} Q_k B_k^n\left(\frac{t}{n}\right) = \sum_{k=m}^{n} Q_k \tbinom{n}{k} (1-\frac{t}{n})^{n-k}\left(\frac{t}{n}\right)^k$$

$$= t^m Q_m \tbinom{n}{m} (1-\frac{t}{n})^{n-m}\frac{1}{n^m} + (\frac{t}{n})^{m+1} \sum_{k=m+1}^{n} Q_k \tbinom{n}{k} (1-\frac{t}{n})^{n-k}\left(\frac{t}{n}\right)^{k-(m+1)}.$$

Thus

$$G_n^{(m)}(0) = Q_m m! \tbinom{n}{m}\frac{1}{n^m} = Q_m \frac{n\ldots(n-m+1)}{n^m}.$$

Since the sequence $\left(G_n^{(m)}(0)\right)$ converges to $G^{(m)}(0) = 0$ and

$$\lim_{n\to\infty} \frac{n\ldots(n-m+1)}{n^m} = 1,$$

$\lim_{n\to\infty} Q_m = 0$. Thus $Q_m = 0$. □

Proposition 10. *The blossom of an analytic function $F(t) = \sum_{k\ge0} P_k b_k(t)$ is a symmetric, multi-affine function over an infinite number of arguments almost all of which are zero, characterized by the following dual functional property*

$$P_k = F[1^k 0^\infty] \textit{ for all } k \ge 0.$$

Proof: Let G be a symmetric, multi-affine function. Then

$$G\left[\left(\frac{t}{n}\right)^n 0^\infty\right] = \sum_{k=0}^{n} G[1^k 0^\infty] B_k^n\left(\frac{t}{n}\right). \tag{17}$$

Now suppose that G is a blossom of F. Then from the diagonal property

$$G\left[\left(\frac{t}{n}\right)^n 0^\infty\right] \text{ converges to } F(t) = \sum_{k\ge0} P_k b_k(t) \tag{18}$$

uniformly on any disk $\bar{D}(0,b)$ where $b < R$. From equations (17) and (18)

$$\lim_{n\to\infty} \sum_{k=0}^{n} G[1^k 0^\infty] B_k^n\left(\frac{t}{n}\right) = \sum_{k\ge0} P_k b_k(t). \tag{19}$$

and this convergence is uniform on any disk $\bar{D}(0,b)$ where $b < R$ and R is the radius of convergence of $\sum P_k b_k(t)$. Thus, from equation (19) and lemma 9

$$P_k = G[1^k 0^\infty], \text{ for all } k.$$

Conversely, if G satisfies the hypotheses of the proposition, we need to prove that G is a blossom of F – that is, G satisfies the diagonal property. Since $G[1^k 0^\infty] = P_k$ are the Poisson coefficients of the analytic function F, by lemma 5, $\sum_{k=0}^{n} G[1^k 0^\infty] B_k^n(\frac{t}{n})$ converges to $\sum_{k\ge0} G[1^k 0^\infty] b_k(t) = \sum_{k\ge0} P_k b_k(t) = F(t)$ uniformly on $\bar{D}(0,b)$.

Thus, $G\left[\left(\frac{t}{n}\right)^n 0^\infty\right] = \sum_{k=0}^{n} G[1^k 0^\infty] B_k^n(\frac{t}{n})$ (equation (17)) converges uniformly to $F(t)$ on $\bar{D}(0,b)$, so G satisfies the diagonal property and by definition G is a blossom of F. □

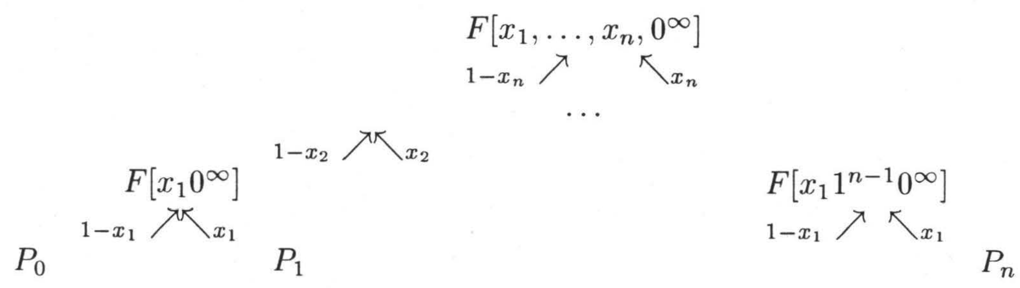

Fig. 2. The analytic blossom $F[x_1, \ldots, x_n, 0^\infty]$ can be computed from the Poisson control points of F by inserting x_i at level i and using the symmetry and multi-affinity properties.

Corollary 11.

$$b_k[1^j 0^\infty] = \delta_{jk}.$$

Uniqueness of the analytic blossom

Now we have all the necessary tools to prove one of the main results of this paper: as in the polynomial case, the analytic blossom exists and is unique.

Theorem 12. *For any analytic function $F(t) = \sum P_k b_k(t)$ there exists a unique function that takes an infinite number of arguments almost all of which are zero, is symmetric, multi-affine and satisfies the diagonal property $\lim_{n\to\infty} F\left[\left(\frac{t}{n}\right)^n 0^\infty\right] = F(t)$, (or equivalently the dual functional property $F[1^k 0^\infty] = P_k$).*

Proof: Existence has been proved already, as a direct consequence of lemma 7. Uniqueness follows from proposition 10. The control points (P_k) are uniquely defined; therefore the values of any blossom at $[1^k 0^\infty]$ are fixed. Moreover, defining the blossom at $[1^k 0^\infty]$ determines it everywhere: by the multi-affinity and symmetry properties, the values of the blossom at $[x_1 1^k 0^\infty]$ are easily computed. By induction, so are the values at $[x_1, \ldots, x_n, 1^k 0^\infty]$ and hence too at $[x_1, \ldots, x_n, 0^\infty]$ for arbitrary x_i's (see Fig. 2). Therefore, the blossom of an arbitrary analytic function F exists and is unique. □

The next two results follow from theorem 12. We shall use these results in the next section.

Corollary 13. *Let $G(x) = F(rx)$, then:*

$$G[x_1, \ldots, x_n, 0^\infty] = F[rx_1, \ldots, rx_n, 0^\infty]. \tag{20}$$

Proof: By uniqueness, it is sufficient to verify that the right hand side of equation (20) satisfies the three properties characterizing the blossom. The symmetry and multi-affinity properties of G follow from those of F. Moreover, for x in $D(0, R)$

$$G[\left(\frac{x}{n}\right)^n 0^\infty] = F[\left(\frac{rx}{n}\right)^n 0^\infty]$$

which by the diagonal property of F converges uniformly to $F(rx) = G(x)$ on any closed disk $\bar{D}(0, b)$, $\frac{b}{r} < R$. □

Corollary 14. *If $(P_k(r))$ denotes the set of control points of the function $G(t) = F(rt)$, then*

$$P_k(r) = F[r^k 0^\infty], \text{ for all } k.$$

Proof: $P_k(r) = G[1^k 0^\infty] = F[(r1)^k 0^\infty]$ by corollary 13. □

As in the polynomial case, one can define a knot vector corresponding to the Poisson basis: $(\ldots 0\ 1 \ldots)$, the infinite vector containing first an infinite number of 0's, and then an infinite number of 1's. The Poisson control points are generated by the blossom evaluated over this knot vector; the first point is the blossom evaluated over all the zeros, and the following points are given successively by moving one position to the right which adds each time one more 1 to the set of arguments. Similarly, the points $(P_k(r))$ correspond to the knot vector $(\ldots 0\ r \ldots)$, which corresponds to the basis $b_k(\frac{t}{r})$. Indeed by definition

$$F(rt) = \sum P_k(r) b_k(t)$$

so by corollary 14, we have

$$F(t) = \sum F(r^k 0^\infty) b_k(\frac{t}{r}).$$

This change of basis will be treated in the subdivision section. Other change of basis algorithms are treated in the subsequent sections.

§4. Algorithms for Poisson Curves

As in the polynomial setting, many of the algorithms acting on a Poisson curve can be expressed in terms of blossoming. When both the input and the output of the algorithm are blossom expressions, then the algorithm itself often follows from the multi-affinity and the symmetry of the blossom. Some convergence results, but not all, follow from the diagonal property.

Subdivision

The subdivision algorithm for Poisson curves given in [13] has a very elegant blossoming interpretation. At the first step of subdivision, the control points $(P_k(r))_{k\geq 0}$ of $F(rx)$, where $0 < r < 1$, are generated from the initial Poisson control points $(P_k)_{k\geq 0}$ of F; similarly, at the next stage of subdivision, the points $(P_k(\rho r))_{k\geq 0}$ are computed from the points $(P_k(r))_{k\geq 0}$, for any r and ρ in $(0,1)$. Since $P_k(r) = F[r^k 0^\infty]$ (Corollary 14), the first step of subdivision consists of getting from the knot vector $(\ldots 0\ 1 \ldots)$ to the knot vector $(\ldots 0\ r \ldots)$ – that is, from the control points of F in the Poisson basis $(b_k(t))$, to the control points of F in the basis $\left(b_k(\frac{t}{r})\right)_{k\geq 0}$; the next step goes from the knot vector $(\ldots 0\ r \ldots)$ to the knot vector $(\ldots 0\ \rho r \ldots)$. These steps can be computed easily using the multi-affinity and symmetry properties of the blossom (Fig. 3). This computation is exactly the algorithm proposed in [13] which corresponds, in the polynomial setting, to the left hand side of the de Casteljau algorithm (Fig. 1). It is shown in [13] that on any finite

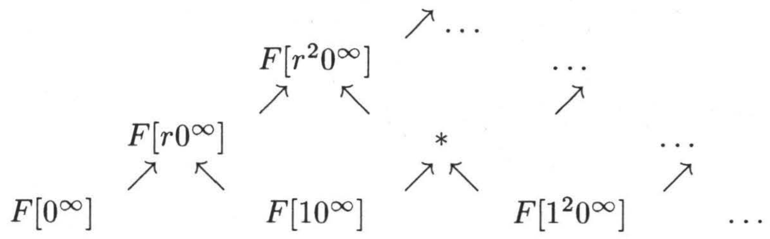

Fig. 3. The subdivision algorithm for Poisson curves. The Poisson control points are placed at the base, and the new control points, after one step of Poisson subdivision, appear along the left lateral edge of the diagram. This process can be iterated to execute further subdivision steps.

interval $[0, b]$, $b < R$, and R is the radius of convergence of F at 0, the control polygons defined by the control points $(P_k(r))_{k\geq 0}$ converge uniformly to the Poisson curve F as r goes to zero, although, as in the polynomial case, the convergence of the subdivision process does not follow from the blossom interpretation.

Trimming

The trimming algorithm for Poisson curves proposed in [14] also has a very nice blossom interpretation, from which, using the multi-affine and symmetry properties of the blossom, the trimming algorithm is retrieved. The aim of the trimming algorithm is to compute the Poisson control points $(P_0(a), P_1(a), \ldots)$ of the function $F(a+.)$, where a is a fixed but arbitrary number in $[0, R)$ [14]. Each control point $P_k(a)$ is the limit of a family of control points $(P_k^n(a))_{n\geq 0} = (F[1^k(\frac{a}{n})^n 0^\infty])_{n\geq 0}$. The control points $(P_k^n(a))_{k\geq 0}$ not only converge to the control points of $F(a+.)$, but also characterize a collection of functions F_a^n that converge uniformly to $F(a+.)$ on any compact subinterval of the domain as n goes to infinity. The convergence of the points follows from the blossom interpretation. First, from the diagonal property:

$$\lim_{n\to\infty} P_0^n(a) = \lim F[(\frac{a}{n})^n 0^\infty] = F(a) = F(a+0) = P_0(a).$$

Second, $(P_1, P_2, \ldots)$ and $(P_1(a), P_2(a), \ldots)$ are the control points respectively of the functions $F + F'$ and $F(a+.) + [F(a+.)]'$ [14]. Since, by proposition 10, $P_k = F[1^k 0^\infty] = (F+F')[1^{k-1}0^\infty]$, the blossom of $F + F'$ is

$$(F+F')[x_1, \ldots, x_n, 0^\infty] = F[1, x_1, \ldots, x_n, 0^\infty].$$

Thus, from the diagonal property

$$\lim_{n\to\infty} P_1^n(a) = \lim F[1(\frac{a}{n})^n 0^\infty] = \lim (F+F')[(\frac{a}{n})^n 0^\infty] = (F+F')(a) = P_1(a).$$

Finally, by induction, $\lim_{n\to\infty} P_k^n(a) = P_k(a)$. The convergence of the functions F_a^n, however, does not follow from the blossom interpretation. Since $P_k^n(a) = F[1^k \left(\frac{a}{n}\right)^n 0^\infty]$, the knot vector corresponding to the function F_a^n approximating $F(a+.)$ is $(\ldots 0 \ \left(\frac{a}{n}\right)^n \ 1 \ldots)$. The algorithm for computing the control points of F_a^n is illustrated by figure 4.

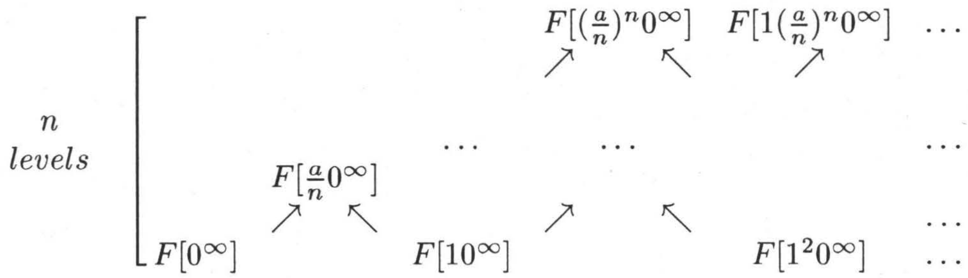

Fig. 4. The trimming algorithm for Poisson curves. Here, just like for subdivision, the computation is justified by the multi-affinity and symmetry of the blossom.

Evaluation algorithms for functions and derivatives

The blossom can also characterize the control points of the derivatives $F^{(m)}$ of F, as in the polynomial setting [1]. We shall see, though, that the corresponding equalities from the polynomial setting differ by constant factors.

The analytic blossom is a polynomial in the non-zero parameters, so we can homogenize the analytic blossom; indeed the homogenization of the analytic blossom is similar to the homogenization in the polynomial setting. (Detailed descriptions of the homogeneous blossom can be found in [17,1].) Formally, the homogeneous blossom of an analytic function F is the unique symmetric, multi-linear function that when dehomogenized reduces to the multi-affine blossom of F, that is,

$$F[(x_1, 1), \ldots, (x_n, 1), (0, 1) \ldots] = F[x_1, \ldots, x_n, 0 \ldots].$$

Let (P'_k) denote the Poisson control points of F'. Since $b'_k = b_{k-1} - b_k$,

$$\begin{aligned} P'_k &= P_{k+1} - P_k \\ &= F[1^{k+1}0^\infty] - F[1^k 0^\infty] \end{aligned} \tag{21}$$

From equation (21) and the linearity of the homogenized analytic blossom

$$P'_k = F[\delta 1^k 0^\infty] \text{ where } (1, 1) - (0, 1) = (1, 0) = \delta.$$

By induction, if $(P_k^{(m)})_{k \geq 0}$ denotes the control points of $F^{(m)}$, then

$$P_k^{(m)} = F[\delta^m 1^k 0^\infty]. \tag{22}$$

Thus, more generally

$$F^{(m)}[x_1, \ldots, x_n, 0^\infty] = F[\delta^m, x_1, \ldots, x_n, 0^\infty]. \tag{23}$$

Using equation (22) and the linearity of the homogeneous blossom, we can compute the control points of a derivative of any order of F from the Poisson control points of F (Fig. 5).

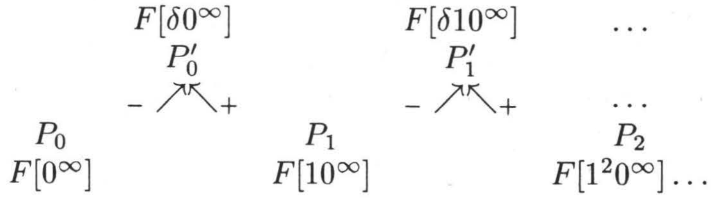

Fig. 5. The control points of the first derivative (upper level) are generated from the original control points (lower level). This process can be iterated to generate the control points of arbitrary derivatives of F.

By first finding the control points of $F^{(m)}$ and then using the trimming algorithm, the sequence of points $F[\delta^m \left(\frac{t}{n}\right)^n 0^\infty]$ is generated. From equation (23) and the diagonal property

$$F^{(m)}(t) = \lim_{n\to\infty} F[\delta^m \left(\frac{t}{n}\right)^n 0^\infty]. \tag{24}$$

Thus we have a de Casteljau like algorithm to approximate F and any of its derivatives.

Thanks to the blossom expression for the derivative, the blossom characterizes the degree of continuity of two analytic curves meeting at 0, as in the polynomial setting [11]. That is:

Proposition 15. *If G and F are two analytic functions at 0, then the following statements are equivalent:*

- $F^{(k)}(0) = G^{(k)}(0)$ *for all* $0 \le k \le j$,
- $F[x_1, \ldots, x_j, 0^\infty] = G[x_1, \ldots, x_j, 0^\infty]$.

Proof: By equation (23), the first statement is equivalent to: $F[\delta^k 0^\infty] = G[\delta^k 0^\infty]$, for all $0 \le k \le j$. By the linearity of the homogenized blossom, this equality is equivalent to $F[1^k 0^\infty] = G[1^k 0^\infty]$, for all $0 \le k \le j$, since $\delta + 0 = 1$. This last equality is a particular case of and induces the second statement, since j new parameters can be introduced using the symmetry and multi-affinity of the blossom as in the proof of theorem 12. □

Conversion between Poisson and Taylor basis

In the polynomial setting, algorithms to convert from the Bernstein basis to the monomial basis and back are given in [1]. As in the polynomial setting, a scaled monomial basis, the Taylor basis $(\frac{x^k}{k!})$ corresponds to the knot vector $(\ldots 0\ \delta \ldots)$. Indeed, from equation (24), taking $t = 0$: $F^{(k)}(0) = F[\delta^k 0^\infty]$. Thus, if we denote by T_k the k-th Taylor coefficient of F, $T_k = F[\delta^k 0^\infty]$. Note that the scaling of the monomial basis in the analytic setting differs from the scaling in the polynomial setting, where the knot vector $(0^n\ \delta^n)$ corresponds to the basis $(\binom{n}{k} x^k)$.

We can now apply the linearity and the symmetry of the homogenized blossom to compute the Taylor coefficients from the Poisson coefficients, and conversely, the Poisson coefficients from the Taylor coefficients (Fig. 6). Figure 6(a) is exactly the algorithm proposed in [13] to compute Poisson coefficients from Taylor coefficients.

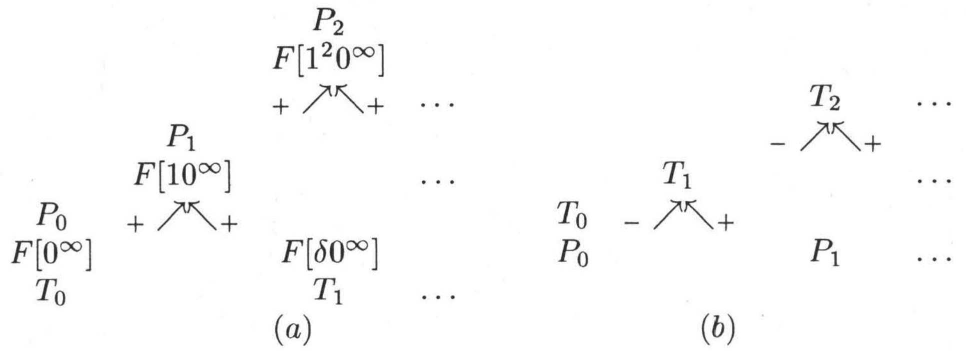

Fig. 6. To the left (a), the change of basis from Taylor coefficients (T_k) to Poisson coefficients (P_k), to the right (b) the change of basis from Poisson coefficients (P_k) to Taylor coefficients (T_k). The coefficients are computed using the linearity of the blossom because $\delta + 0 = 1$ i.e. $(1,0) + (0,1) = (1,1)$ and $1 - 0 = \delta$.

§5. De Boor-Fix Formula, Marsden Identity and Blossoming for Analytic Functions

In [9] the authors present a Marsden identity and a bilinear bracket operator characterizing de Boor-Fix dual functionals for analytic functions. Here we will derive expressions for an analytic function and its derivatives, as well as for the blossom in term of this bracket operator.

Definition 16. *Let F be an analytic function at zero, with radius of convergence R, and let $G(u) = P(u)e^{-xu}$, where P is a polynomial and x is in $[0, R)$. Then*

$$[F(u), G(u)] = \sum_{k \geq 0} \frac{(-1)^k}{k!} F^{(k)}(0) G^{(k)}(0).$$

Proposition 18 will establish that the bracket operator on such functions is well defined – that is, that the right hand side always converges.

The following proposition introduces a Boor-Fix like expression for the analytic blossom. As in the polynomial case, this new expression provides an alternative proof of the existence of the analytic blossom.

Proposition 17.

$$F[x_1, \ldots, x_n, 0^\infty] = [F(u), \prod_{i=1}^{n} (1 - x_i u)] \tag{25}$$

Proof: First, note that the bracket operator between F, an analytic function at 0, and a polynomial P is well defined, since $[F(u), P(u)]$ is a finite sum.

To prove equation (25), we check the three blossoming axioms: symmetry, multi-affinity and the dual functional property. The expression $\prod_{i=1}^{n}(1 - x_i u)$ is certainly symmetric and multi-affine in the x_i's. Thus, so is the expression

$[F(u), \prod_{i=1}^{n}(1 - x_i u)]$ by the linearity of the bracket operator. For the dual functional property, we need to prove

$$P_k = [F(u), (1-u)^k]. \tag{26}$$

In [13], an explicit expression for the Poisson control points in terms of the Taylor coefficients is given

$$P_k = \sum_{j=0}^{k} \binom{k}{j} F^{(j)}(0). \tag{27}$$

But

$$[F(u), (1-u)^k] = \sum_{j=0}^{k} \frac{(-1)^j}{j!} \frac{(-1)^j k!}{(k-j)!} F^{(j)}(0) = \sum_{j=0}^{k} \binom{k}{j} F^{(j)}(0). \tag{28}$$

Equation (26) follows from equations (27) and (28). □

From the blossom examples in section 3, or by direct verification of the blossom axioms,

$$e^{-Xu}[x_1, \ldots, x_n, 0^\infty] = \prod_{i=1}^{n}(1 - ux_i). \tag{29}$$

Thus, the function e^{-xu} plays the same role for the analytic bracket operator $[.,.]$ as the function $(x-u)^n$ plays for the polynomial bracket operator $[.,.]_n$. The next proposition states this result more precisely.

Proposition 18. *If F is an analytic function with radius of convergence R at 0, then for any x in $[0, R)$*

- $F(x) = [F(u), e^{-xu}]$,
- $F^{(p)}(x) = [F(u), \frac{d^p}{dx^p}(e^{-xu})]$,
- $F[x_1, \ldots, x_n, 0^\infty] = [F(u), e^{-Xu}[x_1, \ldots, x_n, 0^\infty]]$.

Proof: The right hand side of the first two equations is exactly the Taylor development of $F(x)$ and $F^{(p)}(x)$ at 0. The last equation follows immediately from (25) in proposition 17 and (29). □

By linearity and from proposition 18 the series defining $[F(u), G(u)]$ for $G(u) = P(u)e^{-xu}$ converges, so the bracket operator is well defined.

Similar to the polynomial setting, the expansion of the reproducing function e^{-xu} in the Poisson basis generates a Marsden identity for analytic functions [9]

$$e^{-xu} = \sum_{k \geq 0} (1-u)^k b_k(x).$$

This Marsden identity follows easily from the dual functional property of the analytic blossom, since $e^{-xu} = \sum_{k \geq 0} E_k b_k(x)$, where $E_k = (e^{-Xu})[1^k 0^\infty] = (1-u)^k$ from (29).

§6. Conclusions and Open Questions

In this paper we have defined the analytic blossom and established its existence and uniqueness. Like the polynomial blossom for Bézier curves, the analytic blossom is a simple, powerful and elegant tool for analyzing Poisson curves. All the algorithms presented in section 4 had already been developed in previous work, but each required a different approach. From the blossoming interpretation given here, these algorithms all follow directly from the multi-affinity and the symmetry of the blossom (or the linearity of the homogenized blossom).

Nevertheless, our definition of the blossom, and in particular the diagonal property, requires us to consider the domain $\mathbb{C}$, since the diagonal property requires uniform convergence on $\bar{D}(0,b)$. This convergence is used to prove the uniqueness of the blossom. However, proving directly that

$$\lim_{n\to\infty}\sum_{k=0}^{n} P_k B_k^n(\frac{t}{n}) = 0 \text{ (uniformly) on } [0,b], b>0 \Leftrightarrow P_k = 0 \text{ for all } k$$

would be sufficient to establish uniqueness and avoid variables in $\mathbb{C}$. We would like to know if this result is true.

We hope that the analytic blossom will help in developing new algorithms and novel tools for analyzing Poisson curves – that is, for investigating analytic functions. By considering the analytic blossom over more general knot vectors, it may even be possible to generate an analogue of B-splines for analytic functions. In order to generalize B-splines to the analytic setting, we may need first to extend the domain of the analytic blossom from the direct sum $\bigoplus_{i=1}^{\infty} C_i$, where $C_i = \mathbb{C}$, to the direct product $\times_{i=1}^{\infty} C_i$. We hope to address this extension of the analytic blossom in a future paper.

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