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Sergei V. Pereverzev

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Fachbereich Informatik

Universität Kaiserslautern · Postfach 3049 · D-67653 Kaiserslautern

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Universität Kaiserslautern
AG Numerische Algorithmen
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67653 Kaiserslautern

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Sergei V. Pereverzev, Kaiserslautern[1]

Abstract

Optimization of Projection Methods for Solving ill-posed Problems. In this paper we propose a modification of the projection scheme for solving ill-posed problems. We show that this modification allows to obtain the best possible order of accuracy of Tikhonov Regularization using an amount of information which is far less than for the standard projection technique.

AMS (MOS) Subject Classification: 65J10, 65J20

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Optimisierung von Projektionsverfahren für die Lösung von inkorrekt gestellten Problemen. In dieser Arbeit wird eine Modifizierung des Projektionsschemas zur Lösung inkorrekt gestellter Probleme vorgeschlagen. Wir zeigen, daß diese Modifizierung es ermöglicht, eine Genauigkeit der Tikhonov-Regularisierung von bestmöglicher Ordnung zu erhalten, wobei man eine wesentlich kleinere Menge von Informationen benutzt als beim Standard-Projektionsschema.

1 Introduction

Let $e_1, e_2, \cdots, e_m, \cdots$ be some orthonormal basis of Hilbert space X, and let P_m be the orthogonal projector on *span* $\{e_1, e_2, \cdots, e_m\}$, that is

$$P_m f = \sum_{i=1}^{m} e_i(f, e_i),$$

where $(\cdot,\cdot)$ is an inner product in the Hilbert space X and as usual, $\|f\|_X = \sqrt{(f,f)}$. We denote by $X^r, 0 < r < \infty$, the linear subspace of X which is equipped with the norm

$$\|f\|_{X^r} = \|f\|_X + \|D_r f\|_X,$$

[1]This work was done during the time as a visiting professor at the University of Kaiserslautern. Prof. Pereverzev is a member of the Institute of Mathematics of the Ukrainian Academy of Science, Tereschenkivska Str. 3, 252601 Kiev, Ukraine

where D_r is some linear (non-bounded) operator acting from X^r to X, and for any $m = 1, 2 \cdots$

$$\|I - P_m\|_{X^r \to X} \leq c_r m^{-r},$$

where I is the identity operator and the constant c_r is independent of m.
In this paper we study the problem of finite-dimensional approximation of the solution of ill-posed problems of the form

$$Ax = f, \tag{1.1}$$

where A is a linear compact operator from X into X and the free term f belongs to the *Range* $(A) := \{f : f = Ag, g \in X\}$, i. e. equation (1.1) is solvable. However, as a rule, instead of the free term f we have some approximation $f_\delta \in X$ such that $\|f_\delta - f\|_X \leq \delta$, where δ is a small positive number which is usually known .
The traditional approach to finite-dimensional approximation of the solution of (1.1) lies in the following. We choose some finite-dimensional operator $A_{N,\varepsilon}$ such that *rank* $A_{N,\varepsilon} = N$ and $\|A - A_{N,\varepsilon}\|_{X \to X} \leq \varepsilon$, where ε depends on δ. Further, as the approximate solution of (1.1) we take some minimizer x_α of the so-called Tikhonov functional

$$\Omega_\alpha(A_{N,\varepsilon}, x) = \|A_{N,\varepsilon}x - f_\delta\|_X^2 + \alpha\|x\|_X^2, \tag{1.2}$$

where α is the parameter of regularization depending on δ. We may define x_α from the Euler equation for (1.2)

$$\alpha x + A_{N,\varepsilon}^* A_{N,\varepsilon} x = A_{N,\varepsilon}^* f_\delta \tag{1.3}$$

where the star denotes the adjoint operator. Note that the solution of (1.3) belongs to the *Range* $(A_{N,\varepsilon}^*)$, *dim Range* $(A_{N,\varepsilon}^*) =$ *rank* $A_{N,\varepsilon} = N$, and finding an element x_α reduces to solving a system of N linear algebraic equations.
So-called Projection Methods for solving ill-posed problems (1.1) lie in the following (see e. g. [4], § 6.3). In the above scheme of finite-dimensional approximation we put $A_{N,\varepsilon} = P_m A P_\ell$ and define x_α from the equation

$$\alpha x + P_\ell A^* P_m A P_\ell x = P_\ell A^* P_m f_\delta. \tag{1.4}$$

Within the framework of projection methods the question arises as to the relationship between m and ℓ. In Section 2 we discuss this question for some class of equation (1.1). In Section 3 and 4 we construct a modification of projection scheme (1.4) which leads to more economical algorithms in the sense of complexity in comparison with the standard projection technique.
Now we define the class of equations (1.1) which will be considered in the sequel. First of all it will be assumed that the operators A have some "smoothness". Namely,

$$\begin{aligned} A \in \mathcal{H}_\gamma^r \ := \ & \{A : \|A\|_{X \to X^r} \leq \gamma_1 \ , \ \|A^*\|_{X \to X^r} =\leq \gamma_2, \\ & \|(D_r A)^*\|_{X \to X^r} \leq \gamma_3\} \ , \ \gamma = (\gamma_1, \gamma_2, \gamma_3), \end{aligned}$$

where $\|\cdot\|_{X\to X^r}$ is the usual norm in the space of all linear bounded operators from X into X^r ; A^* is such that for any $f,g\in X$ $(f,Ag)=(A^*f,g)$.
Let us illustrate these assumptions. We consider an integral equation

$$Ax(t) := \int_0^1 a(t,s)x(s)ds = f(t) \tag{1.5}$$

and as Hilbert space X we take the space L_2 of square-summable functions on (0.1) with the usual norm $\|\cdot\|_2$ and inner product. Moreover, for $r=1$ as X^r we take the Sobolev space W_2^1 of functions f having square-summable derivatives $f'\in L_2$,

$$\|f\|_{W_2^1} = \|f\|_2 + \|\frac{d}{dt}f\|_2.$$

If $e_1,e_2,\cdots,e_m,\cdots$ is the Haar orthonormal basis, where $e_1=1$, and for $m=2^{k-1}+j, k=1,2,\cdots;j=1,2\cdots,2^{k-1}$,

$$e_m(t)=\begin{cases} 2^{(k-1)/2} & , \ t\in[(j-1)/2^{k-1} \ , \ (j-\frac{1}{2})/2^{k-1}) \\ -\ 2^{(k-1)/2} & , \ t\in[(j-\frac{1}{2})/2^{k-1} \ , \ j/2^{k-1}) \\ 0 & , \ t\notin[(j-1)/2^{k-1} \ , \ j/2^{k-1}] \end{cases}$$

and P_m is the orthogonal projector on *span* $\{e_1,e_2,\cdots,e_m\}$ then it is known that

$$\|I-P_m\|_{W_2^1\to L_2}\leq c_1 m^{-1}.$$

This means that for L_2 and W_2^1 all conditions determining $X^r(r=1)$ hold. In this case $D_r=\frac{d}{dt}$. If the kernel $a(t,s)$ of the integral operator A of (1.5) has mixed partial derivatives and

$$\int_0^1\int_0^1\left[\frac{\partial^{i+j}a(t,s)}{\partial t^i\partial s^j}\right]^2 dtds<\infty \ , \ i,j=0,1,$$

then it is easy to see that $A\in\mathcal{H}_\gamma^r$ for $X=L_2, X^1=W_2^1$ and some γ.
Let us introduce some notation: If $N(b)$ and $M(b)$ are functions defined on some set B, we write

$$N(b)\prec M(b)$$

if there is a constant $c>0$ such that for all $b\in B$ $N(b)\leq cM(b)$. We write $N(b)\asymp M(b)$, if $N(b)\prec M(b)$ and $M(b)\prec N(b)$.
Now we note that in the theory of ill-posed problems (1.1) the sets

$$M_\rho(A) := \{u : u=A^*Av \ , \ \|v\|_X\leq\rho\}$$

play an important role. Namely, it is known that for $f\in AM_\rho(A) := \{f : f=Au, u\in M_\rho(A)\}$ equation (1.1) has a unique solution $x=A^{-1}f\in M_\rho(A)$. Moreover, from a theoretical result in [7], it follows that if $x_\alpha=x_\alpha(A_{N,\epsilon},f_\delta)$ is an approximation to the solution of (1.1) obtained within the framework of scheme (1.2), (1.3), then

$$\inf_{A_{N,\epsilon}} \ \sup_{f\in AM_\rho(A)} \ \sup_{\substack{f\delta: \\ \|f-f\delta\|_X\leq\delta}} \|A^{-1}f-x_\alpha(A_{N,\epsilon},f_\delta)\|_X\asymp\delta^{2/3}. \tag{1.6}$$

Therefore in the sequel we shall consider the class Φ_γ^r of equation (1.1) with free terms $f\in AM_\rho(A)$ and with operators A from $\mathcal{H}_\gamma^r$.

2 Projection Methods

For the class Φ_γ^r, using results of [6], we can solve the question of relation between m and ℓ in (1.4).
First of all we note that for $A \in \mathcal{H}_\gamma^r$

$$\begin{aligned} \|(I-P_m)A\|_{X\to X} \le \|I-P_m\|_{X^r\to X}\|A\|_{X\to X^r} \le \gamma_1 c_r m^{-r}, \\ \|A(I-P_\ell)\|_{X\to X} \le \|(I-P_\ell)A^*\|_{X\to X} \le \gamma_2 c_r \ell^{-r} \end{aligned} \tag{2.1}$$

Let $x_{\alpha,m,\ell}$ be a solution of (1.4), i. e.

$$x_{\alpha,m,\ell} = (\alpha I + P_\ell A^* P_m A P_\ell)^{-1} P_\ell A^* P_m f_\delta. \tag{2.2}$$

Then from [6], (Theorem 3.1) and (2.1) it follows that for

$$\alpha \asymp \delta^{2/3}\ ,\ m \asymp \delta^{-1/(3r)}\ ,\ \ell \asymp \delta^{-2/(3r)} \tag{2.3}$$

$$\sup_{A\in\mathcal{H}_\gamma^r}\ \sup_{f\in AM_\rho(A)}\ \sup_{f_\delta:\|f-f_\delta\|_X\le\delta} \|A^{-1}f - x_{\alpha,m,\ell}\|_X \asymp \delta^{2/3}. \tag{2.4}$$

When this estimation is compared with estimation (1.6) it is apparent that for the class Φ_γ^r projection method (2.2), (2.3) is best possible with respect to order of accuracy.
Now we estimate the complexity of projection method (2.2), (2.3).
First of all we assume that only the values of inner products of the following type are available:

$$(e_i, Ae_j)\ ,\ (e_i, f_\delta). \tag{2.5}$$

Let us assing to each inner product (e_i, Ae_j) a point (i,j) on the coordinate plane $(-\infty,\infty,)\times(-\infty,\infty,)$. This point is called the number of the inner product (e_i, Ae_j). As to inner product (e_i, f_δ) it has the number i.
We would like to obtain an approximate solution $x_\alpha = x_{\alpha,m,\ell}$ in the form

$$x_\alpha = \sum_{i=1}^{\ell} d_i e_i, \tag{2.6}$$

where d_i are some coefficients. But for $m < \ell$ it is more convenient to seek a solution of (1.4) in the form

$$x_{\alpha,m,\ell} = \sum_{\nu=1}^{m} x_\nu \Psi_\nu, \tag{2.7}$$

where $\Psi_\nu = P_\ell A^* e_\nu$, $\nu = 1, 2\cdots, m$, and the unknown coefficients x_ν are found from the following system of linear algebraic equations

$$\alpha x_\nu + \sum_{\mu=1}^{m} a_{\nu,\mu} x_\mu = (e_\nu, f_\delta)\ ,\ \nu = 1, 2, \cdots, m, \tag{2.8}$$

where

$$
\begin{aligned}
a_{\nu,\mu} &= (e_\nu, A\Psi_\mu) = (e_\nu, AP_\ell A^* e_\mu) = \\
&= (e_\nu, \sum_{i=1}^{\ell} Ae_i(e_i, A^*e_\mu)) = \sum_{i=1}^{\ell}(e_\nu, Ae_i)(e_i, A^*e_\mu) = \\
&= \sum_{i=1}^{\ell}(e_\nu, Ae_i)(e_\mu, Ae_i) \ , \ \nu, \mu = 1, 2, \cdots, m.
\end{aligned} \tag{2.9}
$$

Thus, for the calculation of all coefficients $a_{\nu,\mu}$ we must perform N arithmetic operations on the values of inner products (2.5), where $N \asymp \ell m^2$. Moreover, solving the system (2.8), reduces to executing no fewer than m^2 arithmetic operations. If the coefficients x_ν in (2.7) are known, then we can pass from representation (2.7) to the standard representation (2.6). Namely,

$$
\begin{aligned}
x_{\alpha,m,\ell} &= \sum_{\nu=1}^{m} x_\nu \Psi_\nu = \sum_{\nu=1}^{m} x_\nu P_\ell A^* e_\nu = \sum_{\nu=1}^{m} x_\nu \sum_{i=1}^{\ell} e_i(e_i, A^*e_\nu) = \\
&= \sum_{i=1}^{\ell} d_i e_i \ , \ d_i = \sum_{\nu=1}^{m} x_\nu(Ae_i, e_\nu) \ , \ i = 1, 2, \cdots, \ell,
\end{aligned}
$$

and for this passage another $(2m-1)\ell$ arithmetic operations are required.
Let us denote respectively by $Card\ (A0)$ and $Card\ (IP)$ the numbers of arithmetic operations and inner products (2.5) required to construct an approximate solution x_α. Using above reasons for the best possible projection method (2.2), (2.3) we have

$$
Card\ (A0) \asymp \ell m^2 \asymp \delta^{-4/3r}. \tag{2.10}
$$

On the other hand it follows from (2.8), (2.9) that for the realization of scheme (2.2), (2.3) we must know the inner products $(e_1, f_\delta), (e_2, f_\delta), \cdots, (e_m, f_\delta)$ and the inner products (e_i, Ae_j) with numbers (i, j) from the rectangle $[1, m] \times [1, \ell]$. So, for the projection method (2.2), (2.3),

$$
Card\ (IP) = m + ml \asymp \delta^{-1/r} \tag{2.11}
$$

3 Modification of Projection Methods

The general idea of modification of the projection scheme is as follows. We may keep the order of accuracy of the projection method while discarding the values of inner products (e_i, Ae_j) with sufficiently large numbers (i, j). For operator equations of second kind this idea was realized by various means in [1], [2], [3], [5]. Now we invoke this idea for ill-posed problems (1.1).
Let Γ_n be the plane set of the form

$$
\Gamma_n = \{1\} \times [1, 2^{2n}] \bigcup_{k=1}^{n} (2^{k-1}, 2^k] \times [1, 2^{2n-k}].
$$

We assign to each operator $A \in \mathcal{H}^r_\gamma$ the finite dimensional operator.

$$A_n = \sum_{k=1}^{n} (P_{2^k} - P_{2^{k-1}}) A P_{2^{2n-k}} + P_1 A P_{2^{2n}}.$$

It is easy to show that

$$(e_i, A_n e_j) = \begin{cases} (e_i, Ae_j) & , \quad (i,j) \in \Gamma_n \\ 0 & , \quad (i,j) \notin \Gamma_n \end{cases}. \tag{3.1}$$

This means that for the construction of A_n we use only the values of inner products (2.5) with numbers from Γ_n. Now in the scheme of finite-dimensional approximation (1.2), (1.3) we put $A_{N,\epsilon} = A_n$ and define an approximate solution $x_{\alpha,n}$ from the equation

$$\alpha x_{\alpha,n} + A_n^* A_n x_{\alpha,n} = A_n^* f_\delta \tag{3.2}$$

Theorem 3.1 *Let* $n 2^{-2nr} \asymp \alpha \asymp \delta^{2/3}$. *Then*

$$\sup_{A \in \mathcal{H}^r_\gamma} \quad \sup_{f \in AM_\rho(A)} \quad \sup_{f_\delta : \|f - f_\delta\|_X \le \delta} \|A^{-1} f - x_{\alpha,n}\|_X \asymp \delta^{2/3}.$$

The proof of this theorem is based on the following lemmas.

Lemma 3.2 *For* $A \in \mathcal{H}^r_\gamma$

$$\|A_n A^* - P_{2^n} A A^*\|_{X \to X} \prec 2^{-3nr}$$

Proof: First of all we note that for $A \in \mathcal{H}^r_\gamma$

$$\begin{aligned} &\|A(I - P_\mu)\|_{X \to X^r} \le \|A(I - P_\mu)\|_{X \to X} + \|D_r A(I - P_\mu)\|_{X \to X} = \\ =\ & \|(I - P_\mu)A^*\|_{X \to X} + \|(I - P_\mu)(D_r A)^*\|_{X \to X} \le \\ \le\ & c_r \mu^{-r} (\|A^*\|_{X \to X^r} + \|(D_r A)^*\|_{X \to X^r}) \le c_r(\gamma_2 + \gamma_3)\mu^{-r} \end{aligned}$$

Using this inequality, we obtain

$$\begin{aligned} \|(I - P_\nu)A(I - P_\mu)\|_{X \to X} &\le \|I - P_\nu\|_{X^r \to X} \|A(I - P_\mu)\|_{X \to X^r} \le \\ &\le c_r^2(\gamma_2 + \gamma_3)(\nu\mu)^{-r}. \end{aligned} \tag{3.3}$$

Now from the definition of operator A_n we find

$$\begin{aligned} A_n A^* - P_{2^n} A A^* = \sum_{k=1}^{n} &(P_{2^k} - P_{2^{k-1}})(A P_{2^{2n-k}} A^* - A A^*) + \\ &+ P_1 (A P_{2^{2n}} A^* - A A^*) \end{aligned} \tag{3.4}$$

Moreover, from (3.3) we have

$$\begin{aligned} &\|(P_{2^k} - P_{2^{k-1}})(A P_{2^{2n-k}} A^* - A A^*)\|_{X \to X} \le \\ \le &\|(I - P_{2^k}) A (I - P_{2^{2n-k}})(A^* - P_{2^{2n-k}} A^*)\|_{X \to X} + \\ + &\|(I - P_{2^{k-1}}) A (I - P_{2^{2n-k}})(A^* - P_{2^{2n-k}} A^*)\|_{X \to X} \le \\ \le &c_r^2(\gamma_2 + \gamma_3)(2^{-2nr} + 2^{-(2n-1)r}) \|I - P_{2^{2n-k}}\|_{X^r \to X} \|A^*\|_{X \to X^r} \prec \\ \prec &2^{-(4n-k)r}, \end{aligned} \tag{3.5}$$

$$\|P_1(AP_{2^{2n}}A^* - AA^*)\|_{X\to X} \leq \|A(I - P_{2^{2n}})(A^* - P_{2^{2n}}A^*)\|_{X\to X} \prec \prec 2^{-4nr}. \tag{3.6}$$

Uniting (3.4) - (3.6) we obtain

$$\|A_nA^* - P_{2^n}AA^*\|_{X\to X} \prec \sum_{k=0}^{n} 2^{-(4n-k)r} = = 2^{-4nr}\sum_{k=0}^{n} 2^{kr} \asymp 2^{-3nr}.$$

The lemma is proved.

Lemma 3.3 *For $A \in \mathcal{H}_\gamma^r$*

$$\|A^*A - A_n^*A_n\|_{X\to X} \prec n2^{-2nr}$$

Proof: It is easy to see that

$$\begin{aligned}\|A^*A - A_n^*A_n\|_{X\to X} &\leq \|A^*(I - P_{2^n})(I - P_{2^n})A\|_{X\to X} + \\ +\|A^*P_{2^n}A - A_n^*A_n\|_{X\to X} &\leq \|I - P_{2^n}\|^2_{X^r\to X}\|A\|^2_{X\to X^r} + \\ +\|A^*P_{2^n}A - A_n^*A_n\|_{X\to X} &\leq c_r^2\gamma_1^2 2^{-2nr} + \\ +\|A^*P_{2^n}A - A_n^*A_n\|_{X\to X}\end{aligned} \tag{3.7}$$

On the other hand,

$$A_n^* = \sum_{k=1}^{n} P_{2^{2n-k}}A^*(P_{2^k} - P_{2^{k-1}}) + P_{2^{2n}}A^*P_1 \tag{3.8}$$

and

$$\begin{aligned}A_n^*A_n &= \sum_{k=1}^{n} P_{2^{2n-k}}A^*(P_{2^k} - P_{2^{k-1}})AP_{2^{2n-k}} + \\ &+ P_{2^{2n}}A^*P_1AP_{2^{2n}}.\end{aligned} \tag{3.9}$$

Then

$$\|A^*P_{2^n}A - A_n^*A_n\|_{X\to X} \leq \|A^*P_1A - P_{2^{2n}}A^*P_1AP_{2^{2n}}\|_{X\to X} + + \sum_{k=1}^{n} \|A^*(P_{2^k} - P_{2^{k-1}})A - P_{2^{2n-k}}A^*(P_{2^k} - P_{2^{k-1}})AP_{2^{2n-k}}\|_{X\to X}. \tag{3.10}$$

Now we note that the operator $A^*(P_{2^k} - P_{2^{k-1}})A$ is self-adjoint and

$$\begin{aligned}&\|A^*(P_{2^k} - P_{2^{k-1}})A\|_{X\to X^r} \leq \\ \leq\ &\|A^*\|_{X\to X^r}(\|I - P_{2^k}\|_{X^r\to X} + \|I - P_{2^{k-1}}\|_{X^r\to X})\|A\|_{X\to X^r} \leq \\ \leq\ &c_r\gamma_1\gamma_2(2^{-kr} + 2^{-(k-1)r}) \prec 2^{-kr}.\end{aligned} \tag{3.11}$$

Moreover, for any self-adjoint operator B from X into X^r

$$\begin{aligned}\|B - P_\nu B P_\nu\|_{X\to X} &\leq \|B - P_\nu B\|_{X\to X} + \|P_\nu(B - BP_\nu)\|_{X\to X} \leq \\ \leq & \;\|I - P_\nu\|_{X^r\to X}(\|B\|_{X\to X^r} + \|B^*\|_{X\to X^r}) \leq 2c_r\nu^{-r}\|B\|_{X\to X^r}.\end{aligned}$$

Thus, from (3.11) we have

$$\begin{aligned}&\|A^*(P_{2^k} - P_{2^{k-1}})A - P_{2^{2n-k}}A^*(P_{2^k} - P_{2^{k-1}})AP_{2^{2n-k}}\|_{X\to X} \leq \\ &\leq 2c_r 2^{-(2n-k)r}\|A^*(P_{2^k} - P_{2^{k-1}})A\|_{X\to X^r} \prec 2^{-2nr}\end{aligned}$$

Consequently (see (3.10))

$$\|A^*P_{2^n}A - A_n^*A_n\|_{X\to X} \prec \sum_{k=1}^{n} 2^{-2nr} = n2^{-2nr} \tag{3.12}$$

The assertion of the lemma follows from (3.7) and (3.12)

Proof of the Theorem: Following [6], we consider the operators

$$R_\alpha(B) = (\alpha I + B^*B)^{-1}B^*\;,\; S_\alpha(B) = I - R_\alpha(B)B.$$

From [6], one sees that for any linear bounded operator B from X to X

$$\|R_\alpha(B)\|_{X\to X} \leq c_0\alpha^{-1/2}\;,\; \|S_\alpha(B)\|_{X\to X} \leq c_1, \tag{3.13}$$

$$\|S_\alpha(B)B^*B\|_{X\to X} \leq c_2\alpha, \tag{3.14}$$

where c_0, c_1, c_2 are independent of α and B. We put $R_{\alpha,n} = R_\alpha(A_n), S_{\alpha,n} = S_\alpha(A_n)$. From the definition of $x_{\alpha,n}$ we find

$$x_{\alpha,n} = R_{\alpha,n}f_\delta,$$

$$A^{-1}f - x_{\alpha,n} = R_{\alpha,n}(f - f_\delta) + S_{\alpha,n}A^{-1}f + R_{\alpha,n}(A_n - A)A^{-1}f. \tag{3.15}$$

Taking into account (3.13) for $\alpha \asymp \delta^{2/3}$ and $B = A_n$ we have

$$\|R_{\alpha,n}(f - f_\delta)\| \leq \|R_\alpha(A_n)\|_{X\to X}\|f - f_\delta\|_X \leq c_0\alpha^{-1/2}\delta \asymp \delta^{2/3}. \tag{3.16}$$

Furthermore, for $f \in AM_\rho(A)$ $A^{-1}f = A^*Au$ and $\|u\|_X \leq \rho$. Then, using Lemma 3.3 and (3.13), (3.14), we obtain

$$\begin{aligned}&\|S_{\alpha,n}A^{-1}f\|_X = \|S_{\alpha,n}A^*Au\|_X \leq \|S_{\alpha,n}A_n^*A_nu\|_X + \\ +\;& \|S_{\alpha,n}(A^*A - A_n^*A_n)u\|_X \leq c_2\alpha\|u\|_X + \\ +\;& c_1\|A^*A - A_n^*A_n\|_{X\to X}\|u\|_X \leq c_2\rho\alpha + c_1\rho n2^{-2nr} \asymp \delta^{2/3}\end{aligned} \tag{3.17}$$

Let us estimate the last term of (3.15). From (3.8) it follows that $A_n^* P_{2^n} = A_n^*$ and

$$\begin{aligned} & R_{\alpha,n}(P_{2^n}A - A)A^{-1}f = \\ = \; & (\alpha I + A_n^* A_n)^{-1}(A_n^* P_{2^n} A - A_n^* A)A^{-1} f = 0. \end{aligned}$$

Then, using Lemma 3.2 and (3.13) for $f \in AM_\rho(A)$ we find

$$\begin{aligned} & \|R_{\alpha,n}(A_n - A)A^{-1}f\|_X \leq \|R_{\alpha,n}(A_n - P_{2^n}A)A^{-1}f\|_X + \\ + \; & \|R_{\alpha,n}(P_{2^n}A - A)A^{-1}f\|_X \leq \\ \leq \; & \|R_{\alpha,n}\|_{X\to X}\|A_nA^* - P_{2^n}AA^*\|_{X\to X}\|Au\|_X \prec \\ \prec \; & \alpha^{-1/2}2^{-3nr} \prec \alpha^{-1/2}n^{-3/2}\delta \prec \delta^{2/3} \end{aligned} \tag{3.18}$$

The assertion of the Theorem follows from (3.15) - (3.18) and (1.6).

4 Complexity of the Modification Scheme

Let us estimate the number of arithmetic operations ($A0$) on the values of inner products (2.5) required to construct an approximate solution $x_{\alpha,n}$. Taking into account (3.8), (3.9), we seek a solution (3.2) in the form

$$x_{\alpha,n} = \sum_{i=1}^{2^n} x_i \varphi_i, \tag{4.1}$$

where

$$\varphi_i = \begin{cases} P_{2^n}A^*e_1 & , \; i = 1, \\ P_{2^{2n-\nu}}A^*e_i & , \; i \in (2^{\nu-1}, 2^\nu] \end{cases} \quad , \quad \nu = 1, 2, \cdots, n$$

Let $T_0 = \{1\}, T_m = \{2^{m-1}+1, 2^{m-1}+2, \cdots, 2^m\}, m = 1, 2, \cdots$, $card\, T_0 = 1$, $card\, T_m = 2^{m-1}$.
It is easy to show that

$$A_n^* A_n \varphi_j = \sum_{i=1}^{2^n} \bar{a}_{ij}\varphi_i,$$

where $\bar{a}_{ij} = \bar{a}_{ji}$, $i, j = 1, 2, \cdots, 2^n$, and for

$$\begin{aligned} & i \in T_k, j \in T_\nu \; , \; k, \nu = 0, 1, \cdots, n, \quad k \leq \nu \\ & \bar{a}_{ij} = (e_i, AP_{2^{2n-\nu}}A^*e_j). \end{aligned} \tag{4.2}$$

Moreover,

$$A_n^* f_\delta = \sum_{i=1}^{2^n}(e_i, f_\delta)\varphi_i.$$

If solution of (3.2) is sought in the form (4.1) then the unknown coefficients x_i will be found from the following system of linear equations

$$\alpha x_i + \sum_{j=1}^{2^n} \bar{a}_{ij} x_j = (e_i, f_\delta), \quad i = 1, 2, \cdots, 2^n. \tag{4.3}$$

To solve this system, for example, by Gaussian elimination it is necessary to perform $N_1 \asymp 2^{3n}$ arithmetic operations on the coefficients $\bar{a}_{ij}$ and inner products (e_i, f_δ). Now we estimate the number of operations required to calculate the coefficients $\bar{a}_{ij}$.
Note that

$$\bar{a}_{ij} = (e_i, AP_{2^{2n-\nu}} A^* e_j) = \sum_{s=1}^{2^{2n-\nu}} (e_i, Ae_s)(e_j, Ae_s).$$

Thus, to define the coefficient $\bar{a}_{ij}$ we must perform $2^{2n-\nu+1} - 1$ arithmetic operations on the values of inner products (2.5). For fixed k and ν the execution of $card\ T_k \cdot card\ T_\nu \cdot (2^{2n-\nu+1} - 1)$ operations is required in order to find all coefficients $\bar{a}_{ij}$ with i and j satisfying the conditions (4.2). Furthermore, for all k and ν such that $k, \nu = 0, 1, \cdots, n, k \leq \nu$, to define the coefficients $\bar{a}_{ij}$ with indices (4.2) it suffices to perform no more than N_2 arithmetic operations on the values of inner products (2.5), where

$$\begin{aligned} N_2 &= \sum_{\nu=0}^{n} \sum_{k=0}^{\nu} card\ T_k \cdot card\ T_\nu \cdot (2^{2n-\nu+1} - 1) \asymp \\ &\asymp 2^{2n} \sum_{\nu=0}^{n} \sum_{k=0}^{\nu} 2^k \asymp 2^{2n} \sum_{\nu=0}^{n} 2^\nu \asymp 2^{3n}. \end{aligned}$$

If the coefficients x_i in (4.1) are known, then we can pass from representation (4.1) to the standard representation (2.6). Namely

$$\begin{aligned} x_{\alpha,n} &= \sum_{i=1}^{2^n} x_i \varphi_i = \sum_{\nu=1}^{n} \sum_{i \in T_\nu} x_i P_{2^{2n-\nu}} A^* e_i + \\ &+ x_1 P_{2^{2n}} A^* e_1 = \sum_{\nu=1}^{n} \sum_{s=1}^{2^{2n-\nu}} e_s \sum_{i \in T_\nu} x_i (e_i, Ae_s) + \\ &+ \sum_{s=1}^{2^{2n}} x_1 e_s (e_1, Ae_s) = \sum_{p=1}^{2^{2n}} d_p e_p, \\ d_p &= \sum_{i=1}^{2^{\nu(p)}} x_i (e_i, Ae_p), \end{aligned}$$

where $\nu(p)$ is the largest integer number such that

$$2^{\nu(p)} \leq \min\{2^n, 2^{2n}/p\}.$$

Thus, to calculate all coefficients d_p $p = 1, 2, \cdots, 2^{2n}$ it suffices to perform no more than N_3 arithmetic operations, where

$$N_3 \asymp \sum_{p=1}^{2^{2n}} 2^{\nu(p)} \leq 2^{2n} \sum_{p=1}^{2^{2n}} \frac{1}{p} \asymp 2^{2n} n.$$

From the Theorem it follows that within the framework of our modification of projection scheme (3.2) we can guarantee on the class Φ_γ^r the optimal order of accuracy $\delta^{2/3}$ in the case when $n2^{-2nr} \asymp \delta^{2/3}$. Using the notations of Section 2, we conclude that in this case for our modification scheme

$$card\ (A0) = N_1 + N_2 + N_3 \asymp 2^{3n} \asymp \delta^{-1/r} (\log \delta^{-1})^{3/(2r)} \tag{4.4}$$

Moreover, from (3.1) it follows that for the realization of our scheme we must know inner products $(e_i, f_\delta), i = 1, 2, \cdots, 2^n$, and inner products (e_i, Ae_j) with numbers (i, j) from Γ_n. It means that for our scheme

$$card\ (IP) \asymp 2^n + n2^{2n} \asymp \delta^{-2/(3r)} (\log \delta^{-1})^{1+1/r} \tag{4.5}$$

When (4.4), (4.5) are compared with estimations (2.10), (2.11) it is apparent that for the class Φ_γ^r the modification scheme (3.2) is more economical than usual projection methods.

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S. V. Pereverzev
Universität Kaiserslautern
Fachbereich Informatik
AG Numerische Algorithmen
Postfach 30 49
67653 Kaiserslautern