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mj-Reduction for Proving in Predicate Logic
—Extended Abstract—

(Internal Report 291/97 — Universität Kaiserslautern)

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Here[1] the general problem of verifying if a formula φ is derivable from a set of hypotheses Σ in intuitionistic logic[2] is considered, where the formulae in $\Sigma \cup \{\varphi\}$ may contain only the implication $\supset$ and the universal quantifier $\forall$ as logical symbols. This problem is denoted by the "sequent" $\Sigma \vdash \varphi$; the solution procedure called here "mj-reduction" is the backward application of a sequent calculus mj similar to Gentzen's LJ calculus[3] for reducing such a problem to problems of the same kind, trying to recursively reduce the original problem to only trivial ones[4]. — With the help of a predicate symbol $\Box$ of arity 0 representing contradiction the negation of a formula ξ can be expressed as $\xi \supset \Box$: without adding "axioms for $\Box$" to the antecedent Σ of the original sequent $\Sigma \vdash \varphi$ the procedure verifies derivability in the minimal calculus LHM of [6], adding the "intuitionistic axiom for contradiction" $n_R := \forall\bar{v}(\Box \supset R(\bar{v}))$ for every predicate symbol R different from $\Box$ it verifies intuitionistic derivability, adding the "axioms for contradiction" $w_R := \forall\bar{v}(((R(\bar{v}) \supset \Box) \supset \Box) \supset R(\bar{v}))$ it verifies classical derivability[5]. Having the w_R in Σ the usual logical symbols can be circumscribed with $\supset$, $\forall$ and $\Box$.

The formulae treated here can be represented in Frege's Begriffsschrift, his graphical notation in [3]. The hollows of the horizontal main stroke (Höhlungen des Inhaltsstriches) containing symbols for bound variables in Frege's representation of a formula ξ correspond to universal quantifiers that after a renaming could be moved to the front of the formula, outside of the scope of any implication, for getting an equivalent formula. A list T of terms with as many members as such hollows is called "appropriate for ξ". To each bound variable corresponding to each hollow, from left to right in the main stroke, may be associated a term, from left to right in the list T; furthermore, the symbols for each of these variables may be substituted properly with the corresponding terms for getting a formula $T * \xi$ with no hollow in the main stroke of its representation. The main stroke of the representation of $T * \xi$ has at its right end an atomic formula ξ_0 called the "head" of $T * \xi$ and denoted by Kopf$(T * \xi)$, vertical conditional strokes (Bedingungsstriche) connect

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[1]As also detailed in [10].

[2]Fundamentals of proof theory may be found in Kleene's book [7].

[3]The calculi NJ and LJ are certainly best exposed in Gentzen's original publication [5].

[4]In [4] a human oriented algorithmic proof system for intuitionistic predicate logic is provided, from the many motivating examples one can easily see that, restricted to propositional calculus for the simple formulae treated here, this system is similar to my proof procedure, but for the treatment of quantifiers it develops a "logic of Skolem functions" while I introduce constants. Important proofs for the correctness of the system are also provided in [4].

[5]The Formulae $w_R \supset n_R$ are derivable in LHM.

the main stroke of the representation of $T * \xi$ with the main strokes of representations of formulae $\xi_n, \ldots, \xi_1$, which build a possibly empty list called "body" of $T * \xi$ and denoted by Rumpf($T * \xi$). Essential to the proof procedure is the fact: if ξ is in Σ and if every formula in Rumpf($T * \xi$) is derivable from Σ, then Kopf($T * \xi$) is also derivable from Σ.

The three schemata for the rules of the mentioned calculus mj are:

$$\text{m:}\frac{\Sigma \cup \{\xi\} \vdash \text{Rumpf}(T * \xi)}{\Sigma \cup \{\xi\} \vdash \text{Kopf}(T * \xi)}, \quad \text{d:}\frac{\Sigma \cup \{\eta\} \vdash \xi}{\Sigma \vdash \eta \supset \xi}, \quad \text{g:}\frac{\Sigma \vdash [q] * \forall v \xi}{\Sigma \vdash \forall v \xi}.$$

A m-rule may have none, one or many oversequents: one for each ξ_k in Rumpf($T * \xi$), the expression $P \vdash$ Rumpf($T * \xi$) denotes the list of all sequents of the form $P \vdash \xi_k$ with ξ_k in Rumpf($T * \xi$). The q in the g-schema represents a symbol for free variable[6] not appearing in $\Sigma \cup \{\forall v \xi\}$; the expression $[q] * \forall v \xi$ represents the formula gotten by substituting all occurrences in ξ of the symbol v bound by the first quantifier $\forall v$ with q, this free variable[7] q is called "auxiliary constant[8] of the g-rule". — A m-rule may be considered as the introduction of a ground sequent followed by Gentzen's rules FEA and AEA, or as the supposition of ξ followed by rules FB and AB. A d-rule corresponds to rules FES and FE. A g-rule corresponds to rules AES and AE. Every mj-derivation can thus be transformed into a LJ-derivation without cut-rules (Schnitt), into a NJ-derivation in Prawitz' normal form[9] whose branches have only atomic minimal formulae, and into a LHM-derivation with the help of the deduction theorem. The equivalence of mj to these calculi may be proved comparing mj with the part of LJ excluding rules containing logical symbols other than $\supset$ and $\forall$: with mj are derived exactly the sequents $\Sigma \vdash \varphi$ of the kind treated here in which φ is derivable from Σ in intuitionistic logic.

Each sequent $P \vdash \varphi$ can be reduced with only one schema which can be immediately determined inspecting the form of the formula φ: this is called "the analytic property of mj". The m-schema is for atomic φ and there are so many possible "m-reductions" as adequate pairs (ξ, T) with $\xi \in P$, T appropriate for ξ and Kopf$(T * \xi) = \varphi$. The d-schema is for φ of the form $\eta \supset \xi$ and there is exactly one possible "d-reduction". The g-schema is for φ of the form $\forall v \xi$ and there are so many possible "g-reductions" as possible selections of q, but renaming shows that all these g-reductions are essentially the same; it is enough that in every g-reduction a "new" q be chosen, that is one not appearing in the formulae of $P \vdash \varphi$ nor in any other sequent of the procedure nor used before in a g-reduction. — From the analytic property follows the equivalence of mj with the calculus having the only schema

$$\text{mj}(\Sigma \vdash \varphi, Q, \xi, T): \quad \frac{\Sigma \cup \text{Rumpf}(Q * \varphi) \vdash \text{Rumpf}(T * \xi)}{\Sigma \vdash \varphi}.$$

The expression $\Sigma \cup$ Rumpf($Q * \varphi$) denotes the expansion of the set Σ with the members of the list Rumpf($Q * \varphi$). The quadruple $(\Sigma \vdash \varphi, Q, \xi, T)$ is such that Q be an appropriate list for φ of different symbols for free variables not appearing in the arbitrary sequent $\Sigma \vdash \varphi$, such that ξ be in $\Sigma \cup$ Rumpf($Q * \varphi$) and such that T be an appropriate list of terms for ξ with Kopf$(T * \xi) =$ Kopf$(Q * \varphi)$.

[6] Free variables in terms and formulae here, as in [5], are represented by different symbols from the ones used for representing variables bound by quantifiers.

[7] With exception of symbols for bound variables, every occurrence of a function symbol in any formula necessarily represents the same object.

[8] Free variables behave as "arbitrary constants": semantically like variables, formally like constants.

[9] Introduced in Prawitz' book [9].

For a sequent $\Sigma \vdash \varphi$ with formulae containing "symbols for unknowns"[10] one can consider the generalized problem of "finding an unknown term" for each of these symbols, so that after the substitution the formula φ be derivable from the set Σ in intuitionistic logic. This problem is solved through reductions like the prior one, but after each reduction new symbols for unknowns and "constraints" to be fulfilled by the unknowns to be found may appear. The terms for the unknowns are found through these constraints: some of them are "term-equations" that may be "solved" with the unification algorithm[11], the other are "prohibitions" of the appearance of some symbols on the terms to be found. The d-schema can be applied exactly as before. The g-schema is applied as before, but keeping in mind the prohibition that the auxiliary constant should not appear in the unknowns of the sequent to be reduced. For applying the m-schema it is enough to select a $\xi \in \Sigma$ and an appropriate list T of terms[12] keeping in mind the term-equation $\text{Kopf}(T * \xi) = A$ that can be solved at any time during the proof process, perhaps together with other accumulated equations, for substituting the symbols representing the found unknown terms by these terms in all sequents containing them, if these terms don't contain forbidden auxiliary constants, that is, fulfill the constraints added by g-reductions[13]. — The purpose of considering this generalized problem with unknowns is to postpone the determination of the appropriate lists T in m-reductions for finding them later through the unification algorithm, this can be done because a "lifting lemma", like the one for SLD-resolution exposed in [8], yields: For each $\xi \in \Sigma$ there is essentially one possible "general" m-reduction, it is enough to choose an appropriate list T of different new unknowns, they may be substituted later by the terms of any list T leading to a correct derivation. Hence there are at most as many possible essentially different general m-reductions as elements of Σ, at most one d-reduction, at most one g-reduction: this strengthens the analytic property and, for example, is very helpful for proving non-derivability in some specific cases.

A horn clause is a formula ξ having all its quantifiers as a block at the beginning and such that $\text{Rumpf}(T * \xi)$ and $\text{Kopf}(T * \xi)$ consist only of atomic formulae for every[14] appropriate T. A sequent $\Sigma \vdash \varphi$ consisting only of horn clauses, not having symbols for unknowns in its antecedent Σ, and with atomic succedent φ can only be reduced with the m-schema to sequents of the same form and with the same antecedent Σ, it is the same reduction gotten with SLD-resolution: mj-reduction hence is a generalization of SLD-Resolution giving an intuitionistic sense to it.

A mj-derivation of an arbitrary sequent of the form $\Sigma \cup \{\varphi\} \vdash \varphi$ may be recursively found because with $mj(\Sigma \cup \{\varphi\} \vdash \varphi, Q, \varphi, Q)$ one gets a list $\Sigma \cup \{\varphi\} \cup \text{Rumpf}(Q * \varphi) \vdash \text{Rumpf}(Q * \varphi)$ of sequents of the same form whose succedents has less occurrences of logical symbols. The structure of this derivation depends on the complete structure of φ, this is also the case in a derivation with Robinson's resolution[15] in which it is necessary to express the problem through disjunctions of literals, but this is not the case for traditional calculi in which the superficial structure of the formulae may be enough for more precisely describing a derivation. — One could define $T * \xi$ for arbitrary T substituting the variables

[10]The "symbols for unknowns" are function symbols of arity zero, different from the ones used for free or bound variables, semantically representing fixed, but unknown, terms; formally they can be substituted like variables with appropriate terms.

[11]As for example treated in [2] or [8].

[12]Perhaps containing symbols for unknowns.

[13]If these terms contain symbols for unknowns, new prohibitions may be necessary.

[14]It is enough that it happen for one.

[15]Treated in Chang and Lee's book [2] together with some refinements.

in the hollows from left to right by terms of T until there be no more terms in T or all variables be substituted, also many other decompositions of $T * \xi$ as Rumpf($T * \xi$) and Kopf($T * \xi$) may be allowed without demanding that the last be atomic, but so that the m-schema remain correct: hence one could consider the problem $\Sigma \cup \{\varphi\} \vdash \varphi$ as a trivial one in which Rumpf($[] * \varphi$) $= []$ and Kopf($[] * \varphi$) $= \varphi$. — With such an extension of mj one can derive exactly the same sequents as with the original mj, it may be easier for an intelligent prover, but the analytic property is lost, and with it the advantages for proving non-derivability and for automation.

Having the axioms for contradiction w_R in the set of hypotheses Σ one can derive[16] for every formula φ the formula $W_\varphi := ((\varphi \supset \Box) \supset \Box) \supset \varphi$; after d- and g-reductions it is enough to add $((\varphi \supset \Box) \supset \Box)$ and every formula in Rumpf($Q * \varphi$) to the set of hypotheses and derive Kopf($Q * \varphi$); the last formula is atomic with a predicate symbol R and a list of arguments T, after a m-reduction considering $T * w_R = ((\text{Kopf}(Q * \varphi) \supset \Box) \supset \Box) \supset \text{Kopf}(Q * \varphi)$ it is enough to derive $(\text{Kopf}(Q * \varphi) \supset \Box) \supset \Box$; after a d-reduction it is enough to add $(\text{Kopf}(Q * \varphi) \supset \Box)$ to the set of hypotheses and derive $\Box$; after a m-reduction considering the first formula added to the set of hypotheses it is enough to derive $\varphi \supset \Box$; after a d-reduction it is enough to add φ to the set of hypotheses and derive $\Box$; after a m-reduction considering the formula $\text{Kopf}(Q * \varphi) \supset \Box$ in the set of hypotheses it is enough to derive Kopf($Q * \varphi$); after a m-reduction considering the formula φ in the set of hypotheses it is enough to derive every formula in Rumpf($Q * \varphi$); all these formulae are in the set of hypotheses, and hence derivable as shown in the paragraph above. — If one can prove every W_φ from a set of hypotheses Σ, then one can also prove every w_R. For traditional calculi the formulae W_φ are more natural axioms for contradiction than the formulae w_R; for the calculus mj, in which the complete structure of the formulae plays a fundamental rôle, the simple formulae w_R are very comprehensible, the formulae W_φ inappropriate: (1) having all possible W_φ in Σ a g-reduction would be impossible, there would be no new symbols q for free variables not appearing in Σ; this could be solved by "generalizing all symbols for free variables", considering the formulae of the form $\forall \bar{v}(((\varphi(\bar{v}) \supset \Box) \supset \Box) \supset \varphi(\bar{v}))$ without symbols for free variables instead of the W_φ, but among these formulae are the w_R; (2) for each atomic formula A there is an infinite number of pairs (W_φ, T) with $T * W_\varphi = A$, while only one of the form (w_R, T); this makes a big difference in the amount of possible m-reductions. — For the induction schema of formal arithmetic there is a similar problem as with the W_φ as axioms, but I don't have a similar solution.

Among all possible reductions of a sequent $\Sigma \vdash A$ with atomic A and Σ containing the axioms w_R there is always the one with the m-schema and the appropriate w_R leading to $\Sigma \vdash (A \supset \Box) \supset \Box)$; the last sequent can only be reduced to $\Sigma \cup \{A \supset \Box\} \vdash \Box$ with the d-schema; this sequent should be reduced with the m-schema, for that a $\xi \in \Sigma \cup \{A \supset \Box\}$ and an appropriate T with Kopf($T * \xi$) $= \Box$ are necessary, the formula $A \supset \Box$ is one of such ξ, in the set Σ may be other such formulae, especially the ones of the form $B \supset \Box$ added to Σ in reductions with the m-schema using a w_R; now is clear what the possible reductions are, this leads to the following remark: adding the axioms of contradiction w_R is equivalent to the introduction of the "restart-rule"[17], this rule allows the reduction of sequents $\Sigma \vdash A$ with atomic A to sequents $\Sigma \vdash B$ with $B = \Box$ or B being the atomic succedent of an "ancestor" sequent $\Pi \vdash B$ of the proof process, of course, reductions of $\Sigma \vdash A$ with the m-schema remain possible.

[16]The analogous result for the axioms n_R yields also.

[17]This rule appears in other procedures, see for example in [1] or [11].

As a last example is considered the generalized problem of finding unknown terms for $x_1, \ldots, x_n$ such that $\Box$ be derivable from the set Σ containing only the formulae $\forall v((S(v) \supset \forall v S(v)) \supset \Box)$, $S(x_1), \ldots, S(x_n)$ and the intuitionistic axioms for contradiction. A first reduction is only possible with the m-schema and the first formula, the new goal is the derivability of $S(x_{n+1}) \supset \forall v S(v)$, where x_{n+1} represents a new unknown. A second reduction is only possible with the d-schema, to the $S(x_1), \ldots, S(x_n)$ in Σ the formula $S(x_{n+1})$ is added, the new goal is $\forall v S(v)$. A third reduction is only possible with the g-schema and an auxiliary constant q_{n+1} not appearing in the terms to be found for the x_i, the new goal is $S(q_{n+1})$. A fourth reduction is only possible with the m-schema and n_S, with the $S(x_i)$ it is impossible because of the constraints imposed in the last reduction; the new goal $\Box$ is as the original, to the set of hypotheses was only added $S(x_{n+1})$. To continue so would only add other $S(x_i)$, since there were no other alternatives for reducing the problem one can conclude that there is no solution to the original problem. Hence the intuitionistic non-derivability of $(\forall v((S(v) \supset \forall v S(v)) \supset \Box)) \supset \Box$ can be concluded, and also the non-derivability of its intuitionistic implicant $\exists v(S(v) \supset \forall v S(v))$. In classical logic both formulae are clearly equivalent, the derivability of the first one from w_S can be verified through thirteen reductions.

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