Interner Bericht

The Reduction Oriented Calculus mj

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The intuitionistic calculus mj for sequents, in which no other logical symbols than those for implication and universal quantification occur, is introduced and analysed. It allows a simple backward application, called mj-reduction here, for searching for derivation trees. Terms needed in mj-reduction can be found with the unification algorithm. mj-Reduction with unification can be seen as a natural extension of SLD-resolution. mj-Derivability of the sequents considered here coincides with derivability in Johansson's minimal intuitionistic calculus LHM in [6]. Intuitionistic derivability of formulae with negation and classical derivability of formulae with all usual logical symbols can be expressed with mj-derivability and hence be verified by mj-reduction. mj-Derivations can be easily translated into LJ-derivations without "Schnitt", or into NJ-derivations in a slightly sharpened form of Prawitz' normal form.

In the first three sections, the systematic use of mj-reduction for proving in predicate logic is emphasized. Although the fourth section, the last and largest, is exclusively devoted to the mathematical analysis of the calculus mj, the first three sections may be of interest to a wider readership, including readers looking for applications of symbolic logic. Unfortunately, the mathematical analysis of the calculus mj, as the study of Gentzen's calculi, demands a large amount of technical work that obscures the natural unfolding of the argumentation. To alleviate this, definitions and theorems are completely embedded in the text to provide a fluent and balanced mathematical discourse: new concepts are indicated with **bold-face**, proofs of assertions are outlined, or omitted when it is assumed that the reader can provide them.

1 Formulae

In this section terms and formulae of the kind considered here are defined. With the help of Frege's Begriffsschrift a decomposition of formulae is introduced. Two kinds of substitution of function symbols of arity 0 with terms are introduced.

§1

Terms and formulae

In the formulae considered here, only the implication $\supset$ and the universal quantifier $\forall$ are allowed as **logical symbols**. As in [5] variables bound with quantifiers in formulae are taken from a different set than the free variables, hence there should be four disjoint classes of function symbols of arity 0 in the language: (1) the class of symbols for **constants**;

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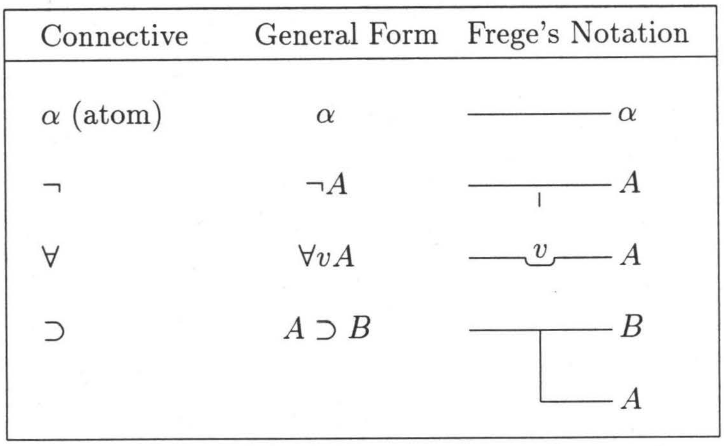


Figure 1: *Review of Frege's Begriffsschrift. Segments ending with A or B may be recursively concatenated with the left extreme of representations of formulae A or B. Examples may be seen in figure 2 and 6.*

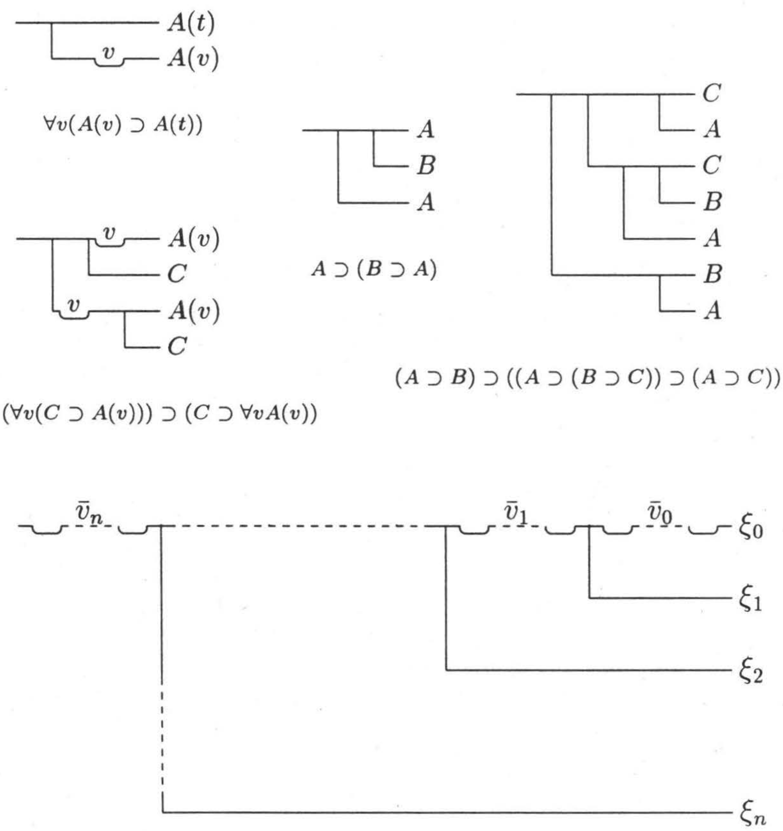
(2) the infinite class of the symbols predestinated to be **bound variables**; (3) the infinite class of symbols predestinated to be **free variables**, these symbols should be understood as representing "arbitrary constants"; (4) the infinite class of symbols predestinated to be **unknowns**, these symbols should be understood as representing unknown terms, they are necessary in section 3. Formulae should not contain any symbol for bound variables not bound with a quantifier, but an **open formula** may contain such symbols. Terms should not contain any symbol for bound variables, but **open terms** may contain such symbols. With exception of bound variables, every occurrence of a function symbol in any formula should be understood as representing the same object, for a bound variable this happens only among its occurrences in a formula, inside the scope of its quantifier.

§2
Frege's representation of formulae

The open formulae considered here are exactly those representable in **Frege's Begriffsschrift**, his graphical notation in [3], without occurrences of the **negation stroke** (Verneinungsstrich). A review of this notation can be seen in figure 1.

§3
Decomposition of a formula

The **hollows** of the horizontal **main stroke** (Höhlungen des Inhaltsstriches) containing symbols for bound variables in Frege's representation of a formula ξ (see figure 2) correspond to universal quantifiers that after a renaming could be moved to the front of the formula, outside of the scope of any implication, for getting an equivalent formula. The main stroke of the representation of ξ has at its right end an atomic open formula ξ_0 called the **head** of ξ and denoted by **Kopf**(ξ), vertical **conditional strokes** (Bedingungsstriche) connect the main stroke of the representation of ξ with the main strokes of representations of open formulae $\xi_n, \ldots, \xi_1$, which form a possibly empty list of formulae called **body** of ξ and denoted by **Rumpf**(ξ).



$$\xi \equiv \forall \bar{v}_n(\xi_n \supset \cdots \supset (\xi_2 \supset \forall \bar{v}_1(\xi_1 \supset \forall \bar{v}_0 \xi_0)))$$

Figure 2: *Some examples and the general form of* ξ *showing the blocks* $\bar{v}_n$, ..., $\bar{v}_0$, $\bar{v}_0$ *of variables in the hollows of the main stroke, the atomic head* ξ_0 *and the open formulae* ξ_n, ..., ξ_1 *of the body.*

$\xi =$	$T =$	$T * \xi =$
ξ	$R \cdot S$	$S * (R * \xi)$
ξ	$[]$	ξ
A atomic	T	A
$\eta \supset \zeta$	T	$\eta \supset (T * \zeta)$
$\forall v \eta$	$[t]$	$\eta^{(v \to t)}$

Figure 3: *The instance $T * \xi$ of ξ may be defined without Frege's Begriffsschrift, recursively on the structure of ξ and on the length of T. Here is $R \cdot S$ the concatenation of R and S.*

§4

Substitution

A **substitution** α is a function that associates a term t_k to each element f_k of a finite set $T_\alpha = \{f_1, \dots, f_m\}$ consisting of function symbols of arity 0 of the language. This α is denoted by $(f_1 \to t_1, \dots, f_m \to t_m)$. The set $\{f \in T_\alpha : \alpha(f) \neq f\}$ is called **support** of the substitution α. For an open term t one defines t^α by replacing simultaneously each f of the support of α appearing in t by $\alpha(f)$. For an open formula ζ one defines ζ^α recursively with $R(t_1, \dots, t_n)^\alpha = R(t_1^\alpha, \dots, t_n^\alpha)$, $(\eta \supset \xi)^\alpha = \eta^\alpha \supset \xi^\alpha$, and $(\forall v \chi)^\alpha = \forall v \chi^{\alpha_v}$, where $\alpha_v(v) = v$ and $\alpha_v(w) = \alpha(w)$ for $w \neq v$.

Variables on the main stroke of a formula ξ may be replaced properly by terms not containing symbols for bound variables, yielding a new formula ξ' which is implied by ξ. For a formula ξ and a list T of terms one defines the **instance** $T * \xi$ of ξ as the formula obtained by replacing the variables in the main stroke of ξ, from left to right, by the terms of T, from left to right, until all variables are replaced or until there are no more terms in T for the rest of the variables. Figure 3 shows how to define $T * \xi$ recursively. — A list T of terms with as many members as hollows in the main stroke of ξ is called **appropriate for** ξ, in this case $T * \xi$ has no hollow in the main stroke, the open formulae in Rumpf$(T * \xi)$ and Kopf$(T * \xi)$ are formulae.

The following is essential here: for a set Σ of formulae, a $\xi \in \Sigma$ and a T appropriate for ξ, Kopf$(T * \xi)$ is derivable from Σ if every formula in Rumpf$(T * \xi)$ is derivable from Σ.

2 The Calculus

In this section the calculus mj is introduced. Emphasis is given on mj-reduction for building mj-derivations. Properties of mj relevant to mj-reduction are explained.

§5

The sequents treated here

A **sequent** $\Sigma \vdash \xi$ of the kind treated here is a pair consisting of a finite set of formulae Σ and a formula ξ. The set Σ is called **antecedent** of the sequent $\Sigma \vdash \xi$, the formula ξ its **succedent**. Eventually infinite antecedents may be allowed.

§6

Calculi, reduction and deduction, composition and derivation trees

A **rule** (Schlußfigur) consist of a possibly empty list of **oversequents** and an **undersequent**. Sets of rules are used to recursively define **derivability**, a set of rules for this purpose is called a **calculus**. Of course, the concept of derivability is dependent on the calculus. A group of rules may be described with a general **schema**, and a calculus with a set of schemata. A rule of a schema or set k is called **k-rule**.

A rule r can be read in two ways, from the oversequents to the undersequent or from the undersequent to the oversequents: one says that from the **derivability** of all oversequents the derivability of the undersequent is **r-deduced**, or that the derivability of the undersequent is **r-reduced** to the derivability of the oversequents. If r is in a group of rules k or a rule given by a schema k, one can also use the terms k-deduced and k-reduced above.

A **k-composition (tree)** of a set k of rules is a tree built with k-rules, from the root to the leaves with k-reductions, or from the leaves to the root with k-deductions; a rule of k may be used more than once. Rules and sequents in a composition are called **H-rules** and **H-sequents** (H-Schlußfiguren und H-Sequenzen), they are considered different if they appear in different parts of the tree, even if they are **formally equal**. The sequent at the root is called **endsequent** (Endsequenz), and the leaves that are not undersequents of H-rules having no oversequent are called **firstsequents** (Anfangssequenzen). A **k-derivation (tree)** is a k-composition tree without firstsequents. A k-derivation whose endsequent is S is called k-derivation of S. A sequent S is **k-derivable** if there is a k-derivation of it.

In figures representing mj-derivations here, the endsequent is at the top and the firstsequents are at the bottom, this is consistent with concepts of computer sciences: the construction of an mj-derivation by mj-reduction is "top-down", the construction of an mj-derivation by mj-deduction is "bottom-up".

§7

The calculus mj

The three schemata for the rules of the calculus mj are:

$$\text{m:}\frac{\Sigma \cup \{\xi\} \vdash \text{Rumpf}(T * \xi)}{\Sigma \cup \{\xi\} \vdash \text{Kopf}(T * \xi)}, \quad \text{d:}\frac{\Sigma \cup \{\eta\} \vdash \xi}{\Sigma \vdash \eta \supset \xi}, \quad \text{g:}\frac{\Sigma \vdash [q] * \forall v \chi}{\Sigma \vdash \forall v \chi}.$$

An m-rule may have none, one or several oversequents: one for each ξ_k in Rumpf$(T * \xi)$, the expression $P \vdash$ Rumpf$(T * \xi)$ denotes the list of all sequents of the form $P \vdash \xi_k$ with ξ_k in Rumpf$(T * \xi)$. In the g-schema, q represents a symbol for free variables not appearing in $\Sigma \cup \{\forall v \chi\}$. The members of T in an m-rule and the q in a g-rule are called **auxiliary terms** of the rule, a d-rule has no auxiliary term. The free variable q is also called **proper variable** (Eigenvariable) of the g-rule. The formula ξ in the m-schema is called **determining formula** of the m-rule.

§8

The analytic property of mj

Each sequent $P \vdash \varphi$ can be reduced only with rules of one schema which can be immediately determined by inspecting the structure of the formula φ: this is called the **analytic property** of mj. The m-schema is for atomic φ and there are as many possible m-reductions of $P \vdash \varphi$ as adequate pairs (ξ, T) with $\xi \in P$, T appropriate for ξ and Kopf$(T * \xi) = \varphi$. The d-schema is

$$\vdash (\forall v(C \supset A(v))) \supset (C \supset \forall v A(v))$$

$$\downarrow d$$

$$\forall v(C \supset A(v)) \vdash C \supset \forall v A(v)$$

$$\downarrow d$$

$$\forall v(C \supset A(v)), C \vdash \forall v A(v)$$

$$\downarrow g$$

$$\underline{\forall v(C \supset A(v))}, C \vdash A(q)$$

$$\downarrow m$$

$$\forall v(C \supset A(v)), \underline{C} \vdash C$$

$$\perp$$

$$m$$

Figure 4: *In this mj-derivation, as in the one shown in figure 5, each mj-reduction necessarily follows from the previous: for each sequent there is exactly one possible reduction. The determining formula of each m-reduction is underlined.*

for φ of the form $\eta \supset \xi$ and there is exactly one possible d-reduction of $P \vdash \varphi$. The g-schema is for a φ of the form $\forall v\chi$ and there are as many possible g-reductions of $P \vdash \varphi$ as possible selections of q, but renaming shows that all these g-reductions are essentially the same in the process of searching for an mj-derivation by mj-reduction: it is sufficient that in every g-reduction a **new** q be chosen, a q not appearing in the formulae of $P \vdash \varphi$ nor in any other sequent nor used before in a g-reduction. — Figures 4 and 5 show, how simple is the search for an mj-derivation in some cases. Two other examples, in which the search space is a little larger, are in figure 7. This simplicity is because of the analytic property.

Since in the process of searching for an mj-derivation by mj-reduction one can reach **irreducible nodes** or detect **infinite branches**, it may be necessary to consider **alternative m-reductions** of nodes reduced before. Examples may be seen in figure 9.

§9

mj-Reduction for propositional logic

In propositional logic only **propositional sequents** are considered, these are sequents containing only formulae without quantifiers. A g-reduction of a propositional sequent is not

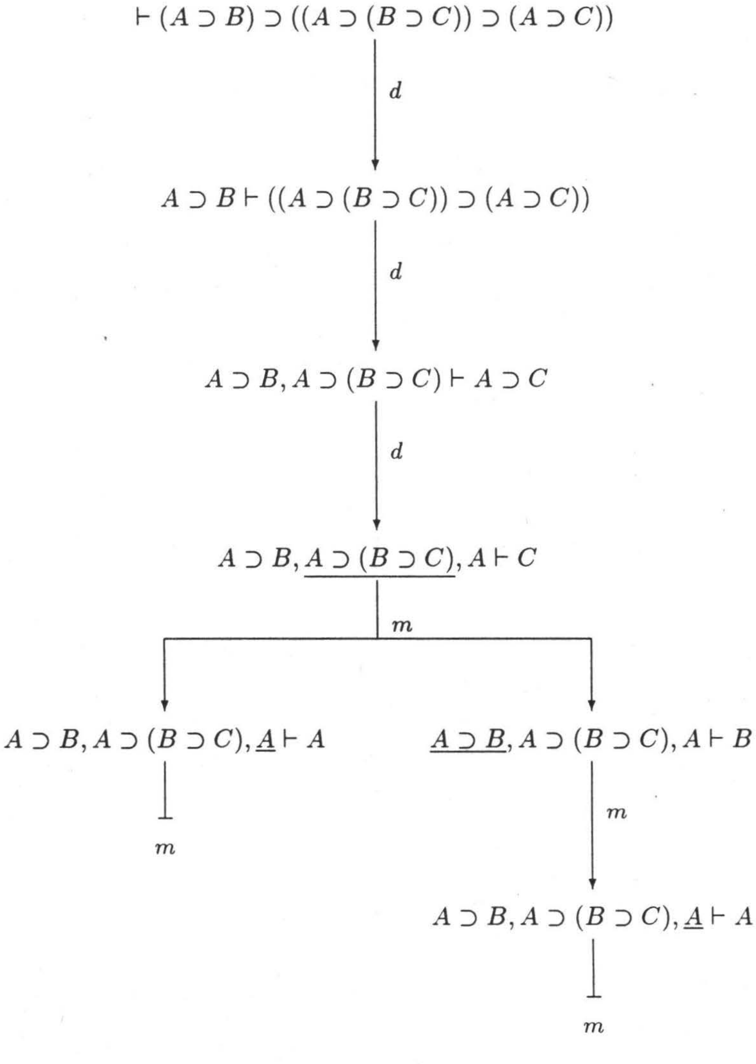


Figure 5: *mj-Derivation of a propositional sequent. Here, two m-reductions lead to no sequent, one leads to one sequent, and one leads to two. g- and d-Reductions lead always to only one sequent. g-Reductions do not appear in propositional logic. The antecedents in threads of mj-deduction trees form a growing chain: after a d-reduction a new formula appear in the antecedent, with a g- or m-reduction the antecedent remains unchanged.*

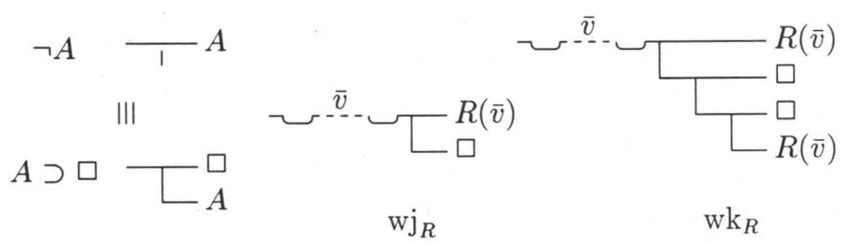


Figure 6: *Frege's negation stroke (Verneinungsstrich) may be paraphrased with the help of the new predicate symbol □ representing contradiction. For each predicate symbol R, wj_R is an intuitionistic contradiction axiom, wk_R a classical contradiction axiom.*

possible, for an m-reduction no instantiation is necessary.

An mj-reduction of a propositional sequent leads to propositional sequents built with subformulae of the original sequent. The set of sequents consisting of the subformulae of formulae appearing in a propositional sequent is finite if the antecedent of the original sequent is finite; an mj-derivation of the original sequent has only sequents of this finite set; any thread of this mj-derivation having more elements than this finite set necessarily contains repetitions and can be contracted, the subtree beginning with the first appearance of a sequent in a thread can be replaced by the smaller subtree beginning with the second appearance; thus the mj-derivation can always be converted into an mj-derivation of the same sequent not deeper than the cardinality of the set. Hence, for every propositional sequent whose antecedent is finite one can always find a number, so that the search for an mj-derivation of the sequent can be restricted to the search for an mj-derivation of depth smaller than the number. This search always ends with a positive or a negative answer, the mj-derivability of propositional sequents is hence **decidable**. mj-Reductions of propositional sequents may be found in figure 5, in the first part of figure 7, in figure 8 and figure 9.

§10

mj-Reduction for classical and intuitionistic logic

Intuitionistic derivability of the restricted formulae considered here coincides with derivability in Johansson's minimal calculus LHM, and hence with mj-derivability.

With the help of a predicate symbol □ of arity 0 representing contradiction, the negation of any formula ξ can be paraphrased as $\xi \supset \Box$. Let **wj** be the set consisting of a formula of the form $\mathbf{wj_R} \equiv \forall\bar{v}(\Box \supset R(\bar{v}))$ for each predicate symbol R, these formulae are called **intuitionistic contradiction axioms**. Let **wk** be the set consisting of a formula of the form $\mathbf{wk_R} \equiv \forall\bar{v}(((R(\bar{v}) \supset \Box) \supset \Box) \supset R(\bar{v}))$ for each relational symbol R, these formulae are called **classical contradiction axioms**.

The LHM-derivability[1] of the sequent $\Sigma \vdash \xi$, whose formulae may contain the negation symbol, coincides with the mj-derivability of the sequent $\Sigma' \vdash \xi'$ obtained by paraphrasing the negation. The intuitionistic derivability of $\Sigma \vdash \xi$ coincides with the mj-derivability of the sequent $\Sigma' \cup \mathrm{wj} \vdash \xi'$.

[1]This is the LHM-derivability of ξ from Σ.

The intuitionistic derivability of $(A_2 \supset B) \supset ((A_1 \supset B) \supset ((A_2 \vee A_1) \supset B))$ *is verified:*

1.	$\mathrm{wk}_B \vdash (A_2 \supset B) \supset ((A_1 \supset B) \supset (((A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box)) \supset B))$		$d, 2$
2.	$\mathrm{wk}_B, A_2 \supset B \vdash (A_1 \supset B) \supset (((A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box)) \supset B)$		$d, 3$
3.	$\mathrm{wk}_B, A_2 \supset B, A_1 \supset B \vdash ((A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box)) \supset B$		$d, 4$
4.	$\underline{\mathrm{wk}_B}, A_2 \supset B, A_1 \supset B, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box)$	$\vdash B$	$m, 5$
5.	$\mathrm{wk}_B, A_2 \supset B, A_1 \supset B, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box)$	$\vdash (B \supset \Box) \supset \Box$	$d, 6$
6.	$\mathrm{wk}_B, A_2 \supset B, A_1 \supset B, \underline{(A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box)}, B \supset \Box$	$\vdash \Box$	$m, 7, 11$
7.	$\mathrm{wk}_B, A_2 \supset B, A_1 \supset B, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box), B \supset \Box$	$\vdash A_2 \supset \Box$	$d, 8$
8.	$\mathrm{wk}_B, A_2 \supset B, A_1 \supset B, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box), \underline{B \supset \Box}, A_2$	$\vdash \Box$	$m, 9$
9.	$\mathrm{wk}_B, \underline{A_2 \supset B}, A_1 \supset B, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box), B \supset \Box, A_2$	$\vdash B$	$m, 10$
10.	$\mathrm{wk}_B, A_2 \supset B, A_1 \supset B, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box), B \supset \Box, \underline{A_2}$	$\vdash A_2$	m
11.	$\mathrm{wk}_B, A_2 \supset B, A_1 \supset B, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box), B \supset \Box$	$\vdash A_1 \supset \Box$	$d, 12$
12.	$\mathrm{wk}_B, A_2 \supset B, A_1 \supset B, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box), \underline{B \supset \Box}, A_1$	$\vdash \Box$	$m, 13$
13.	$\mathrm{wk}_B, A_2 \supset B, \underline{A_1 \supset B}, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box), B \supset \Box, A_1$	$\vdash B$	$m, 14$
14.	$\mathrm{wk}_B, A_2 \supset B, A_1 \supset B, (A_2 \supset \Box) \supset ((A_1 \supset \Box) \supset \Box), B \supset \Box, \underline{A_1}$	$\vdash A_1$	m

The classical derivability of $(\forall v(A(v) \supset B)) \supset (\exists v A(v) \supset B)$ *is verified:*

1.	$\mathrm{wk}_B \vdash (\forall v(A(v) \supset B)) \supset (((\forall v(A(v) \supset \Box)) \supset \Box) \supset B)$			d
2.	$\mathrm{wk}_B, \forall v(A(v) \supset B) \vdash ((\forall v(A(v) \supset \Box)) \supset \Box) \supset B$			d
3.	$\underline{\mathrm{wk}_B}, \forall v(A(v) \supset B), (\forall v(A(v) \supset \Box)) \supset \Box$	$\vdash$	B	m
4.	$\mathrm{wk}_B, \forall v(A(v) \supset B), (\forall v(A(v) \supset \Box)) \supset \Box$	$\vdash$	$(B \supset \Box) \supset \Box$	d
5.	$\mathrm{wk}_B, \forall v(A(v) \supset B), \underline{(\forall v(A(v) \supset \Box)) \supset \Box}, B \supset \Box$	$\vdash$	$\Box$	m
6.	$\mathrm{wk}_B, \forall v(A(v) \supset B), (\forall v(A(v) \supset \Box)) \supset \Box, B \supset \Box$	$\vdash$	$\forall v(A(v) \supset \Box)$	$g(q)$
7.	$\mathrm{wk}_B, \forall v(A(v) \supset B), (\forall v(A(v) \supset \Box)) \supset \Box, B \supset \Box$	$\vdash$	$A(q) \supset \Box$	d
8.	$\mathrm{wk}_B, \forall v(A(v) \supset B), (\forall v(A(v) \supset \Box)) \supset \Box, \underline{B \supset \Box}, A(q)$	$\vdash$	$\Box$	m
9.	$\mathrm{wk}_B, \underline{\forall v(A(v) \supset B)}, (\forall v(A(v) \supset \Box)) \supset \Box, B \supset \Box, A(q)$	$\vdash$	B	$m([q])$
10.	$\mathrm{wk}_B, \forall v(A(v) \supset B), (\forall v(A(v) \supset \Box)) \supset \Box, B \supset \Box, \underline{A(q)}$	$\vdash$	$A(q)$	m

Figure 7: *Sequents of a derivation or composition tree can also be written down in rows. After a reduction the original sequent is marked with the name of the schema used (m,d or g), the new sequents are written down in rows below it. The graph structure can be given after numbering the rows. The formulae* wk_R *are defined in figure 6.*

$\mathrm{wk}_A, \mathrm{wk}_B$	$\vdash$	$((A \supset B) \supset A) \supset A$	d
$\underline{\mathrm{wk}_A}, \mathrm{wk}_B, (A \supset B) \supset A$	$\vdash$	A	m
$\mathrm{wk}_A, \mathrm{wk}_B, (A \supset B) \supset A$	$\vdash$	$(A \supset \Box) \supset \Box$	d
$\mathrm{wk}_A, \mathrm{wk}_B, (A \supset B) \supset A, \underline{A \supset \Box}$	$\vdash$	$\Box$	m
$\mathrm{wk}_A, \mathrm{wk}_B, \underline{(A \supset B) \supset A}, A \supset \Box$	$\vdash$	A	m
$\mathrm{wk}_A, \mathrm{wk}_B, (A \supset B) \supset A, A \supset \Box$	$\vdash$	$A \supset B$	d
$\mathrm{wk}_A, \underline{\mathrm{wk}_B}, (A \supset B) \supset A, A \supset \Box, A$	$\vdash$	B	m
$\mathrm{wk}_A, \mathrm{wk}_B, (A \supset B) \supset A, A \supset \Box, A$	$\vdash$	$(B \supset \Box) \supset \Box$	d
$\mathrm{wk}_A, \mathrm{wk}_B, (A \supset B) \supset A, \underline{A \supset \Box}, A, B \supset \Box$	$\vdash$	$\Box$	m
$\mathrm{wk}_A, \mathrm{wk}_B, (A \supset B) \supset A, A \supset \Box, \underline{A}, B \supset \Box$	$\vdash$	A	m

Figure 8: *The formula $((A \supset B) \supset A) \supset A$ does not contain the symbol $\Box$, it is derivable in classical logic, but, as seen in figure 9, not in intuitionistic logic. One can easily see that a formula not containing $\Box$ derivable in intuitionistic logic is also mj-derivable.*

All usual logical symbols in classical logic can be paraphrased in the language considered here with the help of $\neg\xi \equiv \xi \supset \Box$, $\xi_2 \vee \xi_1 \equiv (\xi_2 \supset \Box) \supset ((\xi_1 \supset \Box) \supset \Box)$, $\xi_2 \wedge \xi_1 \equiv (\xi_2 \supset (\xi_1 \supset \Box)) \supset \Box$, $\exists v\xi \equiv (\forall v(\xi \supset \Box)) \supset \Box$. Classical derivability of a sequent $\Sigma \vdash \xi$, whose formulae may contain all usual logical symbols, coincides with mj-derivability of the sequent $\Sigma' \cup \mathrm{wk} \vdash \xi'$ obtained by paraphrasing all its logical symbols.

§11

Restart as an alternative to contradiction axioms

Among all possible reductions of a sequent $\Sigma \vdash A$ with an atomic A and Σ containing the axioms wk, there is always the one with the m-schema and the appropriate wk_R leading to $\Sigma \vdash (A \supset \Box) \supset \Box)$; this latter sequent can only be reduced to $\Sigma \cup \{A \supset \Box\} \vdash \Box$ with the d-schema; this sequent should be reduced with the m-schema, for this a $\xi \in \Sigma \cup \{A \supset \Box\}$ and an appropriate T with $\mathrm{Kopf}(T * \xi) = \Box$ are necessary, the formula $A \supset \Box$ is one of such formulae ξ, the set Σ may contain other such formulae, especially the ones of the form $B \supset \Box$ added to Σ in reductions with the m-schema using a wk_R; now it is clear what the possible reductions are, this leads to the following remark: adding the contradiction axioms wk is equivalent to the introduction of the **restart-rule**[2], this rule allows the reduction of sequents $\Sigma \vdash A$ with an atomic A to sequents $\Sigma \vdash B$ with $B = \Box$ or B being the atomic succedent of an "ancestor" sequent $\Pi \vdash B$ of the proof process, of course, reductions of $\Sigma \vdash A$ with the m-schema remain possible. — Adding the axioms wj is equivalent to the introduction of the similar rule that allows the replacing of an atomic succedent by $\Box$ in the sequent to be reduced.

3 Unknowns and Unification

In this section it is explained how to use unknowns in the process of building an mj-derivation by mj-reduction for postponing the choice of auxiliary terms, and how to use unification for choosing them. In this way, the infinite number of possible choices is reduced to a finite number.

[2]This rule appears in other proof procedures, see for example in [1] or [11].

The formula $wk_A \supset wj_A$ is mj-derivable, but not $wj_A \supset wk_A$:

1.		$\vdash$	$\mathrm{wj}_A \supset \mathrm{wk}_A$	d
2.	wj_A	$\vdash$	wk_A	d
3.	$\underline{\mathrm{wj}_A}, (A \supset \Box) \supset \Box$	$\vdash$	A	m
4.	$\mathrm{wj}_A, \underline{(A \supset \Box) \supset \Box}$	$\vdash$	$\Box$	m
5.	$\mathrm{wj}_A, (A \supset \Box) \supset \Box$	$\vdash$	$A \supset \Box$	d
6.	$\mathrm{wj}_A, \underline{(A \supset \Box) \supset \Box}, A$	$\vdash$	$\Box$	m
7.	$\mathrm{wj}_A, (A \supset \Box) \supset \Box, A$	$\vdash$	$A \supset \Box$	d
8.	$\mathrm{wj}_A, \underline{(A \supset \Box) \supset \Box}, A$	$\vdash$	$\Box$	
*	Sequents 8 and 6 are identical,			
*	only repetitions of 6, 7 and 8 can follow.			

The formula $((A \supset B) \supset A) \supset A$ is not derivable in intuitionistic logic:

1.	$\mathrm{wj}_A, \mathrm{wj}_B$	$\vdash$	$((A \supset B) \supset A) \supset A$	d
2.	$\underline{\mathrm{wj}_A}, \mathrm{wj}_B, (A \supset B) \supset A$	$\vdash$	A	m
3.	$\mathrm{wj}_A, \mathrm{wj}_B, (A \supset B) \supset A$	$\vdash$	$\Box$	
*	No possible reduction for 3, an alternative for reducing 2:			
2.	$\mathrm{wj}_A, \mathrm{wj}_B, \underline{(A \supset B) \supset A}$	$\vdash$	A	m
3.	$\mathrm{wj}_A, \mathrm{wj}_B, (A \supset B) \supset A$	$\vdash$	$A \supset B$	d
4.	$\mathrm{wj}_A, \underline{\mathrm{wj}_B}, (A \supset B) \supset A, A$	$\vdash$	B	m
5.	$\mathrm{wj}_A, \mathrm{wj}_B, (A \supset B) \supset A, A$	$\vdash$	$\Box$	
*	No possible reduction for 5.			

Figure 9: *Non-derivability proofs. A sequent s is not mj-derivable when every mj-composition tree with s as its endsequent (root) is part of an "infinite" mj-composition tree having s as its endsequent or of one having a non reducible firstsequent. In propositional logic one can always determine if a sequent is derivable or not, because all composition trees having the same endsequent s are built with sequents of a finite set dependent on s. The formulae wj_R are defined in figure 6.*

§12

Potential mj-reductions

For a sequent with formulae containing symbols for unknowns one can consider the generalized problem of "finding an unknown term" for each of these symbols, so that the sequent be after the substitution mj-derivable. This problem is solved by building **potential mj-derivations** with **potential mj-reductions**, after each potential reduction new symbols for unknowns and **constraints** to be fulfilled by the unknowns may appear, a substitution of the unknowns of a potential derivation satisfying the constraints converts the potential mj-reductions into mj-reductions, the potential mj-derivation into an mj-derivation, such a substitution is said to **actualize** the potential mj-derivation. A set of constraints is **satisfiable** if there is a substitution satisfying all of them. — The problem is thus solved by searching for a potential mj-derivation with satisfiable constraints, and then for a substitution actualizing it. The terms for the unknowns are found by these constraints: some of them are **term-equations** that may be **solved** with the **unification algorithm**[3], the other are **prohibitions** of the appearance of some symbols on the terms to be found.

Potential reductions also follow the m-, d- and g-schemata, but interpreted in a different way. A **potential d-reduction** is exactly like a d-reduction. A **potential g-reduction** is performed as a g-reduction, but keeping in mind the prohibition that the proper variable should not appear in the unknowns of the sequent to be reduced. For a **potential m-reduction** of a sequent $\Sigma \vdash A$ it is sufficient to select a $\xi \in \Sigma$ and an appropriate list T of terms[4] keeping in mind the term-equation $\mathrm{Kopf}(T * \xi) = A$. — Term-equations can be solved at any stage during the process of constructing an mj-derivation, they can be solved as soon as they arise, they can be accumulated for solving a group of them as a system of equations at a later stage in the process. By solving term-equations one obtains terms for unknowns appearing in the sequents of the partially constructed mj-derivation; these unknowns must be replaced by the found terms, if these terms do not contain forbidden proper variables, that is, if they fulfill the constraints added by potential g-reductions[5], otherwise one should search for alternative m-reductions. Examples of potential mj-derivations can be seen in figures 10, 11 and 12.

The purpose of considering this generalized problem with unknowns is to postpone the determination of the appropriate lists T in m-reductions for finding them later with the unification algorithm, this can be done because a "lifting lemma", like the one for SLD-resolution treated in [8], holds: For each $\xi \in \Sigma$ there is essentially one possible "**most general**" potential m-reduction, it is sufficient to choose an appropriate list T of different new unknowns, they may be substituted later by the terms of any list T leading to a correct mj-derivation. Hence, there are at most as many possible essentially different most general potential m-reductions as elements of Σ, at most one potential d-reduction, at most one potential g-reduction: this strengthens the analytic property and, for example, is very helpful for proving non-derivability in some specific cases.

§13

mj-Reduction for proving non derivability

In §9 it was shown that mj-derivability for propositional sequents is decidable. In figure 9 there are examples in which non-derivability of propositional sequents are proved.

[3] As for example treated in [2] or [8].

[4] Perhaps containing symbols for unknowns.

[5] If these terms contain other unknowns, new prohibitions may be necessary.

1. $\mathrm{wk}_S \vdash (\forall v(S(v) \supset \forall v S(v)) \supset \Box) \supset \Box,$
   d
2. $\mathrm{wk}_S, \underline{(\forall v(S(v) \supset \forall v S(v)) \supset \Box)} \quad \vdash \quad \Box,$
   $m([x_1])$
3. $\mathrm{wk}_S, \forall v((S(v) \supset \forall v S(v)) \supset \Box) \quad \vdash \quad S(x_1) \supset \forall v S(v),$
   d
4. $\mathrm{wk}_S, \forall v((S(v) \supset \forall v S(v)) \supset \Box), S(x_1) \quad \vdash \quad \forall v S(v),$
   $g(q_1), (q_1|x_1)$
5. $\underline{\mathrm{wk}_S}, \forall v((S(v) \supset \forall v S(v)) \supset \Box), S(x_1) \quad \vdash \quad S(q_1),$
   $m([q_1])$
6. $\mathrm{wk}_S, \forall v((S(v) \supset \forall v S(v)) \supset \Box), S(x_1) \quad \vdash \quad (S(q_1) \supset \Box) \supset \Box,$
   d
7. $\mathrm{wk}_S, \underline{\forall v((S(v) \supset \forall v S(v)) \supset \Box)}, S(x_1), S(q_1) \supset \Box \quad \vdash \quad \Box,$
   $m([x_2])$
8. $\mathrm{wk}_S, \forall v((S(v) \supset \forall v S(v)) \supset \Box), S(x_1), S(q_1) \supset \Box \quad \vdash \quad S(x_2) \supset \forall v S(v),$
   d
9. $\mathrm{wk}_S, \forall v((S(v) \supset \forall v S(v)) \supset \Box), S(x_1), S(q_1) \supset \Box, S(x_2) \quad \vdash \quad \forall v S(v),$
   $g(q_2), (q_2|x_1, x_2)$
10. $\underline{\mathrm{wk}_S}, \forall v((S(v) \supset \forall v S(v)) \supset \Box), S(x_1), S(q_1) \supset \Box, S(x_2) \quad \vdash \quad S(q_2),$
    $m([q_2])$
11. $\mathrm{wk}_S, \forall v((S(v) \supset \forall v S(v)) \supset \Box), S(x_1), S(q_1) \supset \Box, S(x_2) \quad \vdash \quad (S(q_2) \supset \Box) \supset \Box,$
    d
12. $\mathrm{wk}_S, \forall v((S(v) \supset \forall v S(v)) \supset \Box), S(x_1), \underline{S(q_1) \supset \Box}, S(x_2), S(q_2) \supset \Box \vdash \Box,$
    m
13. $\mathrm{wk}_S, \forall v((S(v) \supset \forall v S(v)) \supset \Box), S(x_1), S(q_1) \supset \Box, \underline{S(x_2)}, S(q_2) \supset \Box \vdash S(q_1),$
    $m, S(x_2) = S(q_1)$

Figure 10: *Potential mj-derivation with satisfiable constraints. The two potential m-reductions with* wk_S *are not most general, but all other potential m-reductions are, the ones reducing the sequents on rows 2 and 7 introduce the new unknowns* x_1 *and* x_2*. The constraints are the equation* $S(x_2) = S(x_1)$ *and the prohibitions* $(q_1|x_1)$*,* $(q_2|x_1, x_2)$*, where the expression* $(q|u_1, \ldots u_n)$ *means that the symbol* q *should not appear in the terms for the unknowns* u_k*. A general solution for the constraints is* $x_2 \equiv q_1$ *and any term for* x_1 *free of* q_1*,* q_2*.*

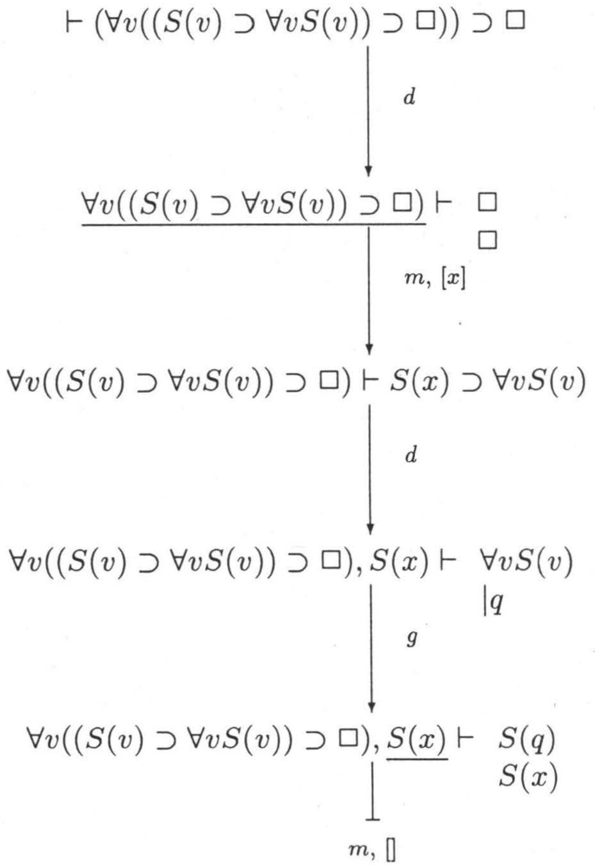


Figure 11: *Potential mj-derivation with unsatisfiable constraints. New unknowns are introduced at each application of the m-schema. An atom B below the atomic succedent A of a sequent reduced with the m-schema denotes the equation $A = B$. The expression $|q$ with a symbol q for free variables below the succedent A of a sequent $\Sigma \vdash A$ reduced with the g-schema denotes the prohibition that q appear in the terms for the unknowns of this sequent. Hence the unsatisfiable constraints here are $(q|x)$ and $x = q$. Since there are no other reduction alternatives, the original formula is not mj-derivable.*

$$\varphi_1 \equiv \forall u_1 \forall u_2 (B(u_1) \supset (B(f(u_1, u_2)) \supset B(u_2)))$$
$$\varphi_2 \equiv \forall v_1 \forall v_2 B(f(v_1, f(v_2, v_1)))$$
$$\varphi_3 \equiv \forall w_1 \forall w_2 \forall w_3 B(f(f(w_1, w_2), f(f(w_1, f(w_2, w_3)), f(w_1, w_3))))$$
$$\Sigma \equiv \{\varphi_1, \varphi_2, \varphi_3\}$$

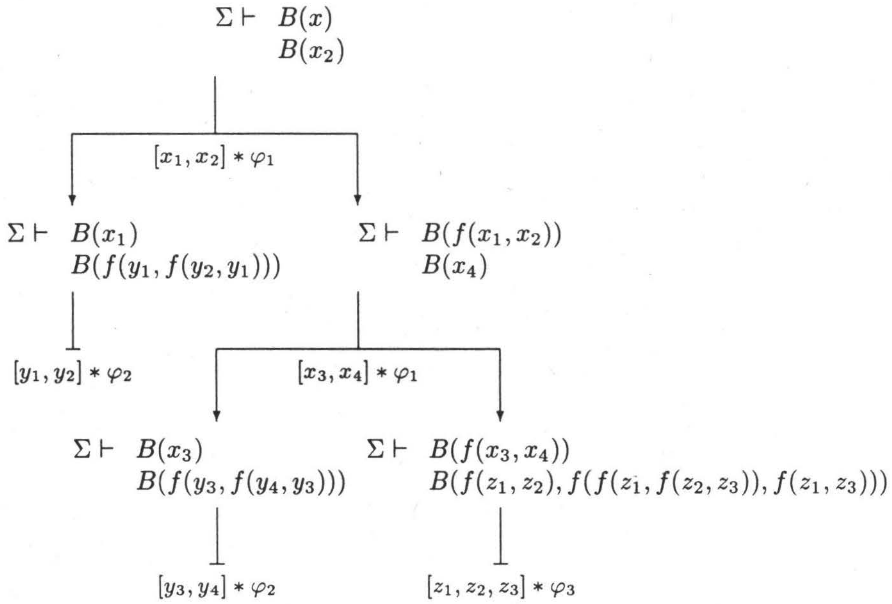


Figure 12: *A term for the unknown x was successfully found. Only reductions with the m-schema and new unknowns can be used for sequents of this kind, mj-derivations of them can be seen as SLD-resolution trees. The formulae and unknowns used in the m-reductions are indicated with smaller letters. With the constraints one can express all unknowns as terms dependent on y_3 and y_4, to which one can assign arbitrary terms; in particular $x \equiv f(y_3, y_3)$. There is an infinite number of alternative mj-derivation trees leading to different values of x.*

Potential mj-reduction can also be used for proving non-derivability in predicate logic, figure 11 shows an example. Here is considered the generalized problem of finding unknown terms for $x_1, \ldots, x_n$ so that $\Box$ be derivable from the set Σ containing only the formulae $\forall v((S(v) \supset \forall v S(v)) \supset \Box)$, $S(x_1), \ldots, S(x_n)$ and the intuitionistic contradiction axioms. A first reduction is only possible with the m-schema and the first formula, the new goal is the derivability of $S(x_{n+1}) \supset \forall v S(v)$, where x_{n+1} represents a new unknown. A second reduction is only possible with the d-schema, to the $S(x_1), \ldots, S(x_n)$ in Σ the formula $S(x_{n+1})$ is added, the new goal is the derivability of $\forall v S(v)$. A third reduction is only possible with the g-schema and a proper variable q_{n+1} not appearing in the terms to be found for the x_i, the new goal is the derivability of $S(q_{n+1})$. A fourth reduction is only possible with the m-schema and wj_S, with a $S(x_i)$ it is impossible because of the constraints imposed in the previous reduction; the new goal, the derivability of $\Box$, is as the original, only $S(x_{n+1})$ was added to the set of hypotheses. Continuing like this would only add more new formulae $S(x_i)$; since there were no other alternatives for reducing the problem, one can conclude that there is no solution to the original problem. Hence, the intuitionistic non-derivability of $(\forall v((S(v) \supset \forall v S(v)) \supset \Box)) \supset \Box$ can be concluded, and also the non-derivability of its intuitionistic implicant $\exists v(S(v) \supset \forall v S(v))$. In classical logic, both formulae are clearly equivalent; an mj-derivation of the sequent having the first formula as succedent and $\{\mathrm{wk}_S\}$ as antecedent can be seen in figure 10.

§14

SLD-Resolution as a special case of mj-Reduction

A **horn clause** is a formula ξ having all its quantifiers as a block at the beginning and such that Rumpf(ξ) and Kopf(ξ) consist only of atomic formulae. A sequent $\Sigma \vdash \varphi$ consisting only of horn clauses, not having symbols for unknowns in its antecedent Σ, and whose succedent φ is atomic, can only be reduced with the m-schema to sequents of the same form and with the same antecedent Σ, it is the same reduction obtained with SLD-resolution: mj-reduction is an extension of SLD-resolution with an intuitionistic meaning[6]. The mj-derivation in figure 12, for example, can be easily transformed into a SLD-derivation as defined in [8].

4 Proof-Theoretical Analysis of the Calculus

In this section the relation between mj and well known calculi, like Hilbert's, Gentzen's and Johansson's ones, and the relation between mj and Prawitz' natural deduction, is outlined. Familiarity with the methods of proof theory, specially with the arguments in [5], [6] and [9], and with the "lifting lemmata" in [2] and [8], is very helpful, if not necessary, for the understanding of this section.

§15

Five equivalent calculi

In figure 13 the calculi **cm**, **lm**, **nm** and **hm**, considered here together with mj, are introduced. The terms t, q, and the members of the lists T and Q appearing in figure 13

[6]In [4] a human oriented algorithmic proof system for intuitionistic predicate logic, called N-Prolog, is provided. From the many motivating examples, one can easily see that, restricted to propositional calculus for the simple formulae considered here, N-Prolog is similar to mj-reduction; but for the treatment of quantifiers the authors of [4] develop a "logic of Skolem functions" while mj-reduction introduces constants. Important proofs for the correctness of the system are also provided in [4].

The compact calculus cm:

$$\mathrm{cm}(\Sigma \vdash \xi, Q, \pi, T) : \frac{\Sigma \cup \mathrm{Rumpf}(Q * \xi) \vdash \mathrm{Rumpf}(T * \pi)}{\Sigma \vdash \xi}.$$

The logicistic calculus lm:

$$\mathrm{grnd}: \frac{}{\Sigma \cup \{\zeta\} \vdash \zeta},$$

$$\mathrm{fea}: \frac{\Sigma \vdash \eta, \Sigma \cup \{\xi\} \vdash \zeta}{\Sigma \cup \{\eta \supset \xi\} \vdash \zeta}, \quad \mathrm{fes}: \frac{\Sigma \cup \{\eta\} \vdash \xi}{\Sigma \vdash \eta \supset \xi},$$

$$\mathrm{aea}: \frac{\Sigma \cup \{[t] * \psi\} \vdash \zeta}{\Sigma \cup \{\psi\} \vdash \zeta}, \quad \mathrm{aes}: \frac{\Sigma \vdash [q] * \psi}{\Sigma \vdash \psi}.$$

The calculus nm for natural deduction:

$$\mathrm{an}: \frac{}{\Sigma \cup \{\zeta\} \vdash \zeta},$$

$$\mathrm{fb}: \frac{\Sigma \vdash \eta, \Sigma \vdash \eta \supset \xi}{\Sigma \vdash \xi}, \quad \mathrm{fe}: \frac{\Sigma \cup \{\eta\} \vdash \xi}{\Sigma \vdash \eta \supset \xi},$$

$$\mathrm{ab}: \frac{\Sigma \vdash \psi,}{\Sigma \vdash [t] * \psi}, \quad \mathrm{ae}: \frac{\Sigma \vdash [q] * \psi}{\Sigma \vdash \psi}.$$

The calculus hm for axiomatic theories:

$$\mathrm{ax}: \frac{}{\Sigma \vdash \sigma} \quad (\text{with } \sigma \in \Sigma),$$

$$\mathrm{f1}: \frac{\Sigma \vdash \eta, \Sigma \vdash \eta \supset \xi}{\Sigma \vdash \xi}, \quad \mathrm{f2}: \frac{}{\Sigma \vdash \eta \supset (\xi \supset \eta)},$$

$$\mathrm{f3}: \frac{}{\Sigma \vdash (\eta \supset \xi) \supset ((\eta \supset (\xi \supset \zeta)) \supset (\eta \supset \zeta))},$$

$$\mathrm{a1}: \frac{\Sigma \vdash [q] * \psi}{\Sigma \vdash \psi}, \quad \mathrm{a2}: \frac{}{\Sigma \vdash \psi \supset [t] * \psi},$$

$$\mathrm{a3}: \frac{}{\Sigma \vdash (\forall v(\kappa \supset \chi)) \supset (\kappa \supset \forall v \chi)}.$$

Figure 13: *Four calculi equivalent to mj. Here, as in the definition of mj, v represents a symbol for bound variables, q a symbol for free variables not appearing in the sequent $\Sigma \vdash \psi$, t a term, η, ξ, ζ, κ formulae, ψ a formula of the form $\forall v \chi$, and σ a formula in Σ. Q represents an appropriate list for ξ of different symbols for free variables not appearing in the arbitrary sequent $\Sigma \vdash \xi$, π a formula in $\Sigma \cup$ Rumpf$(Q * \xi)$, T an appropriate list of terms for π. In cm only quadruples $(\Sigma \vdash \xi, Q, \pi, T)$ with Kopf$(T * \pi) =$ Kopf$(Q * \xi)$ are allowed. The expression $\Sigma \cup$ Rumpf$(Q * \xi)$ denotes the expansion of the set Σ with the members of the list Rumpf$(Q * \xi)$.*

are called, as in the definition of mj, **auxiliary terms** of the corresponding rules, q and the members of Q **proper variables.** Some schemata appear in different calculi with different names: grnd, an and ax denote the same schema; d, fes and fe denote the same schema; fb and f_1 denote the same schema; g, aes, ae, and a_1 denote the same schema.

For these calculi, one may consider only sequents whose antecedents are finite, or allow such infinite[7] antecedents, so that an infinite number of symbols for free variables and symbols for unknowns not appearing in its formulae can be found[8]. In the latter case, one can allow a language with an infinite number of relational and function symbols[9].

It should be added to the definitions in §6 that the definition of a composition tree coincides with Gentzen's general definition of a derivation (Beweisfigur) in [5], 3.2-3.3, page 181. Since in [5] there are no rules without oversequents, Gentzen imposes that the firstsequents of his LJ-derivations be of a specific form called "groundsequent" (Grundsequenz), but this condition is here implicit in the presence of rules having the groundsequents as undersequent and no oversequent, together with the condition that a derivation have no firstsequents.

For two calculi k_1 and k_2, one writes $k_1 \subseteq k_2$ if every k_1-derivable sequent is also k_2-derivable, the two calculi are **equivalent** if and only if $k_1 \subseteq k_2$ and $k_2 \subseteq k_1$ holds. The equivalence of the four calculi with mj will be proved.

A rule is **valid** in a calculus k, or k-valid, if given k-derivations of its oversequents one can find a k-derivation of its undersequent. A schema is **valid** in k if all the rules it represents are also valid in k. The rules of a calculus are valid in it.

A rule r is a **combination (rule)** of a set k of rules (k-combination) if there is a k-composition whose endsequent is formally equal to the undersequent of r, and such that every firstsequent of it is formally equal to an oversequent of r. Combinations of k-valid rules, for example k-combinations, are always k-valid. This implies $k_1 \subseteq k_2$ if every k_1-rule is k_2-valid.

Some schemata with generic formulae look like rules, for example all the ones appearing in the definitions of lm, nm, hm and mj with the exception of the m-schema; one could find compositions of these schemata by building trees with generic formulae, and hence define combinations of schemata, for example ab is a combination of a_2 and f_1. All rules of such a "combination schema" are combination rules of rules of the involved schemata, corresponding composition trees are similar.

§16

Renaming of proper variables

For a substitution α and for a set of formulae Σ, a list of formulae T, a sequent s or a rule r, one can define the set Σ^α, the list T^α, the sequent s^α or the rule r^α in the natural way, by replacing[10] every formula κ appearing in them by κ^α. Also for a calculus k and a k-derivation H one can define the calculus k^α and the k^α-derivation H^α exactly in the same way. The derivation H^α is also a k-derivation if for each k-rule r appearing in H the rule r^α is also a k-rule. For every rule r belonging to a schema of mj, lm, nm or hm, with the exception of the schema denoted by the names g, aes, ae, a_1, the rule r^α belong to the same schema. For proving this, it is sufficient to prove first the equality $(T * \xi)^\alpha = T^\alpha * \xi^\alpha$ for arbitrary T and the equalities $(\text{Kopf}(T * \xi))^\alpha = \text{Kopf}(T^\alpha * \xi^\alpha)$, $(\text{Rumpf}(T * \xi))^\alpha = \text{Rumpf}(T^\alpha * \xi^\alpha)$

[7]Infinite sets should be given by finitary methods.
[8]Otherwise there would be a problem with g-reductions and necessary renamings.
[9]In the first case there would be a problem introducing contradiction axioms.
[10]This definition holds also if the antecedent is infinite as defined above.

for appropriate T. The rule r^α is a g-rule if r is a g-rule whose proper variable q and whose undersequent s satisfy the properties that the term q^α also be a symbol for free variables and that this q^α do not appear in the sequent s^α; for a cm-rule it is sufficient that every $q \in Q$ satisfy the two conditions, and that all members in Q^α be different.

A **renaming** for the proper variables of a k-derivation H, where k is mj, cm, lm, nm or hm, is an injective substitution $\alpha = (f_1 \to t_1, \ldots, f_m \to t_m)$, such that $T_\alpha = \{f_1, \ldots, f_m\}$ is the set[11] of proper variables of H, and such that the set $B_\alpha = \{t_1, \ldots, t_m\}$ (of the same cardinality as T_α) consists of symbols for free variables not appearing in H. From the paragraph above, it follows that H^α is also a k-deduction, its set of proper variables is B_α.

A k-derivation, where k is one of the five calculi, has **unambiguous proper variables** if proper variables of different H-rules are (formally) different and each proper variable appears at most in sequents above the undersequent of its rule (of course excluding this undersequent). — Renaming the proper variables of a derivation with unambiguous proper variables leads to a derivation with unambiguous proper variables and with the same endsequent. — If, for example, q appears as proper variable of two rules and in the endsequent, then a single renaming would replace q by a q' that would also appear in these three places. In spite of it, following 3.10, page 198 in [5], one can find for every k-derivation H a k-derivation H' with unambiguous proper variables and with the same endsequent. The k-deduction H' is found recursively on the number of nodes in H; for the oversequents of the rule r in H whose undersequent is the endsequent of H one has shorter k-derivations (subtrees of H), and hence one can find k-derivations of them with unambiguous proper variables; since these proper variables can be renamed, one can demand that proper variables appearing in different derivations also be different and do not appear in the undersequent of r (endsequent of H), these k-derivations together with r form the desired derivation H'.

§17

Thinning and substitution of function symbols of arity 0

Two important schemata are

$$\text{erw:}\frac{\Sigma \vdash \xi}{\Sigma \cup P \vdash \xi}, \quad \text{sub:}\frac{\Sigma \vdash \xi}{(\Sigma \vdash \xi)^\alpha}.$$

The validity of these schemata in each calculus treated here may be proved in a similar way: one can modify each sequent in a derivation H of the oversequent, by adding the formulae of P to its antecedent or by applying the substitution α to its formulae, in order to obtain a derivation of the undersequent. This argument works, if modifying the sequents of every rule of H in the same way yields a valid rule in the calculus used. This condition holds if no proper variable of a rule of H appears in P, or if no proper variable of a rule of H is in the support of α or appears in the image of an element in the support. Since one can always find a derivation with unambiguous proper variables and then rename the proper variables without changing the endsequent, one can always find a derivation H satisfying the above conditions.

Since antecedents of sequents in mj-, cm-, nm- and hm-compositions grow along threads, it is clear that an erw-rule whose oversequent and undersequent are different cannot be a combination of rules of these calculi. This is an example of a valid rule that is not a combination. The presence of the schema "Verdünnung im Antezedens" in Gentzen's LJ correspond to the validity of erw in the five calculi treated here.

[11] In sets formally equal elements are considered equal.

§18

mj-Validity of ab, inversion of rules, conditional validity

From the analytic property of mj it follows that for every mj-derivation of $\Sigma \vdash \psi$ with a ψ of the form $\forall v \chi$ there is an mj-derivation of $\Sigma \vdash [q] * \psi$ with a q not appearing in $\Sigma \vdash \psi$, and from the mj-validity of sub it follows that there is an mj-derivation of $\Sigma \vdash [t] * \psi$ for an arbitrary term t. This means that ab is mj-valid.

For a rule r having exactly one oversequent, one can define its **inversion** rule r^{-1} as the rule having as undersequent the oversequent of r and as oversequent the undersequent of r. A schema having only one generic oversequent can also be inverted.

The mj-validity of fe^{-1} follows immediately from the analytic property, the m-validity of ae^{-1} follows immediately from the mj-validity of ab.

In each rule of the five calculi only a finite number of formulae appearing in antecedents are "essential", the occurrence of any other could be deleted from all antecedents of sequents of the rule yielding a rule of the same schema. These essential formulae are called **special antecedent formulae.**

With a derivation H of the oversequent $\Sigma \cup P \vdash \xi$ of an erw^{-1}-rule in one of the five calculi, one finds a derivation of its undersequent $\Sigma \vdash \xi$ by deleting the formulae of $(\Sigma \cup P) - \Sigma$ from the antecedent of each sequent of the derivation, if H fulfills the condition that all the special antecedent formulae of it appear in Σ. The schema erw^{-1} is called **conditionally valid**, because for finding a derivation of the undersequent it is not sufficient to have derivations of the oversequents, these derivations should fulfill some conditions. From the conditional validity of erw^{-1}, it follows that for every derivation of $\Sigma \vdash \xi$ in one of the five calculi, there is a derivation of a sequent $\Sigma' \vdash \xi$, where Σ' is a finite subset of Σ, in which only H-sequents having finite antecedents are involved.

§19

mj-Validity of schnitt and fb

For a finite set P of formulae let $||P||$ be $\max(\{0\} \cup \{1 + ||\pi|| : \pi \in P\})$, where $||\pi||$ is the number of logical symbols appearing in π. Therefore: (a) $||P|| = 0$ if and only if $P = \emptyset$, (b) $||P'|| < ||P||$ if P' consists of formulae in Rumpf$(T * \pi)$ for a $\pi \in P$. — Given a finite set $P = \{\pi_1, \ldots, \pi_n\}$ of formulae, an mj-derivation of $\Sigma \cup P \vdash \zeta$ and a sequence of mj-derivations of $\Sigma \vdash \pi_1$, ..., $\Sigma \vdash \pi_n$, one can find an mj-derivation of $\Sigma \vdash \zeta$ by double recursion, first on $||P||$, and then on the number of nodes in the given mj-derivation of $\Sigma \cup P \vdash \zeta$. This statement is trivial for $||P|| = 0$. The endsequent $\Sigma \cup P \vdash \zeta$ of the given mj-derivation is the undersequent of an mj-rule r having oversequents of the form $\Sigma' \cup P \vdash \zeta'$ with $\Sigma \subseteq \Sigma'$. From the mj-derivation of $\Sigma \cup P \vdash \zeta$ one can extract mj-derivations of these oversequents $\Sigma' \cup P \vdash \zeta'$, they are smaller subtrees; with the mj-derivations of $\Sigma \vdash \pi_i$ and with the mj-valid schema erw one can find mj-derivations of $\Sigma' \vdash \pi_i$ for each $\pi \in P$; by the inductive hypothesis one can find an mj-derivation of $\Sigma' \vdash \zeta'$ for every oversequent $\Sigma' \cup P \vdash \zeta'$ of r. Now, three cases for the mj-rule r mentioned above are considered. (1) If r is a d-rule, then its undersequent is of the form $\Sigma \cup P \vdash \eta \supset \xi$ with $\zeta = \eta \supset \xi$ and its only oversequent is $(\Sigma \cup \{\eta\}) \cup P \vdash \xi$ $(\Sigma' = \Sigma \cup \{\eta\})$, by the remark above one finds an mj-derivation of $\Sigma \cup \{\eta\} \vdash \xi$, and with an additional d-rule the desired mj-derivation of $\Sigma \vdash \eta \supset \xi$. (2) If r is a g-rule, then the succedent ζ of its undersequent $\Sigma \cup P \vdash \zeta$ is of the form $\forall v \chi$ and its only oversequent is of the form $\Sigma \cup P \vdash [q] * \zeta$ for some q $(\Sigma' = \Sigma)$, by the remark above one finds an mj-derivation of $\Sigma \vdash [q] * \zeta$, and with an additional g-rule the desired mj-derivation of $\Sigma \vdash \zeta$. (3) If r is an m-rule, then its undersequent is of the form $\Sigma \cup P \vdash$ Kopf$(T * \pi)$

and its list of oversequents is of the form $\Sigma \cup P \vdash \text{Rumpf}(T * \pi)$ for a $\pi \in \Sigma \cup P$ and an appropriate T ($\zeta = \text{Kopf}(T * \pi)$, $\Sigma' = \Sigma$), by the same remark used above one can find mj-derivations of all sequents in $\Sigma \vdash \text{Rumpf}(T * \pi)$; π may be in Σ or in P, in the case $\pi \in \Sigma$ the desired mj-derivation of $\Sigma \vdash \text{Kopf}(T * \pi)$ may be constructed with the mj-derivations of $\Sigma \vdash \text{Rumpf}(T * \pi)$ and an additional m-rule, in the case $\pi \in P$ there is an mj-derivation of $\Sigma \vdash \pi$, with iterated applications of the mj-valid schemata fe^{-1} and ab one can find an mj-derivation of $\Sigma \cup \text{Rumpf}(T * \pi) \vdash \text{Kopf}(T * \pi)$, since the set P' containing the members of Rumpf$(T * \pi)$ satisfy $||P'|| < ||P||$ the inductive hypothesis yields the desired mj-derivation of $\Sigma \vdash \text{Kopf}(T * \pi)$. If in case (3) the list Rumpf$(T * \pi)$ is empty, one finds the desired mj-derivation without an essential application of the inductive hypothesis, in this case, and only on this case, the given mj-derivation of $\Sigma \cup P \vdash \zeta$ has only one node.

The paragraph above shows the mj-validity of the schema

$$\text{schnitt} : \frac{\Sigma \vdash \eta, \Sigma \cup \{\eta\} \vdash \xi}{\Sigma \vdash \xi},$$

this schema corresponds to Gentzen's Schnitt-schema, its lm-validity corresponds to Gentzen's Hauptsatz in [5], schnitt is a combination of fb and fe and hence nm-valid. The schema fb is a combination of the mj-valid schemata fe^{-1} and schnitt, and hence also mj-valid.

Proving the lm-validity of schnitt also the lm-validity of fb and ab follows: fb is a combination of grnd, fea, and schnitt; ab is a combination of grnd, aea, schnitt. Proving the lm-validity of schnitt directly is not necessary after proving the equivalence of all these calculi: validity in a calculus implies validity in all calculi equivalent to it.

§20

Equivalence of cm and mj, mj-validity of an

The schema

$$\text{es}(\Sigma \vdash \xi, Q) : \frac{\Sigma \cup \text{Rumpf}(Q * \xi) \vdash \text{Kopf}(Q * \xi)}{\Sigma \vdash \xi}$$

with Q and ξ as in figure 13 is an mj-combination, the composition tree has as many d-rules as members in Rumpf$(Q * \xi)$, as many g-rules as members in Q. The undersequent of $\text{es}(\Sigma \cup \{\eta\} \vdash \xi, Q)$ coincides with the oversequent of d, they form a composition tree showing that $\text{es}(\Sigma \vdash \eta \supset \xi, Q)$ is a combination of them. The undersequent of $\text{es}(\Sigma \vdash [q] * \psi, Q)$ with a ψ of the form $\forall v \chi$ and a q appearing[12] in $[q] * \psi$ coincides with the oversequent of g, they form a composition tree showing that $\text{es}(\Sigma \vdash \psi, [q] \cdot Q)$ is a combination of them[13]. The oversequent of a rule of the form $\text{es}(\Sigma \vdash \xi, Q)$ is the undersequent of an m-rule having $\Sigma \cup \text{Rumpf}(Q * \xi) \vdash \text{Rumpf}(T * \pi)$ as its list of oversequents, if $\text{Kopf}(T * \pi) = \text{Kopf}(Q * \xi)$ holds: cm-rules are combinations of m- and es-rules, and hence mj-combinations. Every m-rule is also a cm-rule and hence cm-valid, d- and g-rules are cm-valid because of the properties of es explained above. The calculus cm is hence equivalent to mj.

A cm-derivation of an arbitrary sequent of the form $\Sigma \cup \{\zeta\} \vdash \zeta$ may be recursively found, because with $\text{cm}(\Sigma \cup \{\zeta\} \vdash \zeta, Q, \zeta, Q)$ one gets a list $\Sigma \cup \{\zeta\} \cup \text{Rumpf}(Q * \zeta) \vdash \text{Rumpf}(Q * \zeta)$ of sequents of the same form whose succedents contain fewer occurrences of logical symbols. The structure of this derivation depends on the complete structure of ζ. From this it follows that each an-rule is a cm-, and hence mj-combination, although the an-schema is not a combination of the schemata of these calculi.

[12]Hence q is not in Q

[13]If q does not appear in $[q] * \psi$, one can change the q by any other symbol for free variables not appearing in Q nor in $\Sigma \vdash \psi$ without altering the g-rule.

§21

Equivalence of lm, nm and mj

By now, the mj-validity of each nm-schema, and hence nm $\subseteq$ mj, has been proved. An m-rule is an lm-combination, a composition tree may be constructed with one grnd-rule, as many fea-rules as members in Rumpf($T * \xi$), and as many aea-rules as members in T; this implies mj $\subseteq$ lm. The schema fea is a combination of an, fb, fe, erw; the schema aea a combination of an, fb, fe, ab, erw; hence all lm-schemata are nm-valid and lm $\subseteq$ nm. All this implies nm $\subseteq$ mj $\subseteq$ lm $\subseteq$ nm, and hence the equivalence of nm, mj and lm.

§22

Extension mj′ of mj, nm-validity of f_2, f_3, a_2 and a_3

The rules of the schema

$$\text{bsub:}\frac{\Sigma \vdash \xi}{\Sigma \vdash T * \xi}$$

are combinations of fe^{-1}-, ab- and fe-rules, the composition trees consist of alternated fe^{-1}- and ab-rules followed by fe-rules. The schema bsub is hence nm-valid.

For a list of formulae $\xi_n, \ldots, \xi_1, \xi_0$ let $\xi_n \& \ldots \& \xi_1 \rightarrow \xi_0$ denote the formula $\xi_n \supset (\xi_{n-1} \supset (\cdots(\xi_1 \supset \xi_0)\cdots))$, the first conditional strokes on the main stroke of its representation in Frege's Begriffsschrift are connected to $\xi_n, \ldots, \xi_1$, the main stroke continues with the representation of ξ_0.

The rules of the schema

$$\text{m}' : \frac{\Sigma \vdash \pi_n, \ldots, \Sigma \vdash \pi_1}{\Sigma \vdash \pi_0},$$

where there is a $\pi \in \Sigma$ and a T such that $T * \pi = \pi_n \& \ldots \& \pi_1 \rightarrow \pi_0$, are combinations of an-, bsub-, and fb-rules, and hence nm-combinations.

The m-rules are also m′-rules, the calculus mj′ defined by the m′-, d- and g-schema is hence equivalent to mj.

One can find an mj-derivation of $\Sigma \cup \{\zeta\} \vdash \zeta$ with only one m-reduction when ζ is atomic, for a non-atomic ζ one is compelled to use first a d- or g-reduction, but a similar mj′-derivation of $\Sigma \cup \{\zeta\} \vdash \zeta$ with one m′-reduction is always possible. The same is the case with the undersequents of the schemata f_2, f_3, a_2, a_3: when the formulae involved are atomic, one can easily find an mj-derivation by mj-reduction, but one can always find a similar mj′-derivation by mj′-reduction. Every hm-schema is clearly nm- or mj′-valid.

Building mj′-derivations may be in many cases simpler than building mj-derivations, but the analytic property is lost in mj′, and with it many advantages, specially for verifying non-derivability.

§23

Equivalence of hm and nm, hm and calculi for axiomatic theories

In order to prove the equivalence of hm and nm, it suffices now to prove the hm-validity of fe. In an hm-derivation all sequents have the same antecedent Σ, deleting the antecedents Σ in all sequents appearing in hm-schemata yields a calculus for deriving formulae instead of sequents, its rules indicate how to deduce a formula of the theory described with the axioms Σ, that are given by the ax-schema, from such formulae. The hm-validity of fe is equivalent to the deduction theorem for this calculus, the deduction theorem can be proved as usual for these kind of calculi, see for example [7].

§24

Contradiction Axioms

The following two schemata are mj-valid:

$$\text{cnj}: \overline{\Sigma \cup \{\text{wj}_{R,\Box}\} \vdash \text{WJ}_{\xi,\Box}}, \quad \text{cnk}: \overline{\Sigma \cup \{\text{wk}_{R,\Box}\} \vdash \text{WK}_{\xi,\Box}},$$

where R is a predicate symbol, ξ a formula with Kopf(ξ) containing R as predicate symbol, $\Box$ any formula, $\text{WJ}_{\xi,\Box} \equiv \Box \supset \xi$, $\text{WK}_{\xi,\Box} \equiv ((\xi \supset \Box) \supset \Box) \supset \xi$, $\text{wj}_{R,\Box} \equiv \forall\bar{v}(\Box \supset R(\bar{v}))$, $\text{wk}_{R,\Box} \equiv \forall\bar{v}(((R(\bar{v}) \supset \Box) \supset \Box) \supset R(\bar{v}))$. The formula $\text{WJ}_{\xi,\Box}$ is derived by mj$'$-reduction: after d- and g-reductions it is sufficient to add $\Box$ and every formula in Rumpf($Q * \xi$) to the set of hypotheses and derive Kopf($Q*\xi$); the latter formula is atomic with a predicate symbol R and a list of arguments T, after an m$'$-reduction considering $T * \text{wj}_{R,\Box} = \Box \supset \text{Kopf}(Q * \xi)$ it is sufficient to derive $\Box$; this formula is in the set of hypotheses, and hence derivable with one m$'$-reduction. Similarly the formula $\text{WK}_{\xi,\Box}$ is derived by mj$'$-reduction: after d- and g-reductions it is sufficient to add $(\xi \supset \Box) \supset \Box$ and every formula in Rumpf($Q * \xi$) to the set of hypotheses and derive Kopf($Q * \xi$); the latter formula is atomic with a predicate symbol R and a list of arguments T, after an m$'$-reduction considering $T * \text{wk}_{R,\Box} = ((\text{Kopf}(Q * \xi) \supset \Box) \supset \Box) \supset \text{Kopf}(Q*\xi)$ it is sufficient to derive $(\text{Kopf}(Q*\xi) \supset \Box) \supset \Box$; after a d-reduction it is sufficient to add $\text{Kopf}(Q*\xi) \supset \Box$ to the set of hypotheses and derive $\Box$; after an m$'$-reduction considering the first formula added to the set of hypotheses it is sufficient to derive $\xi \supset \Box$; after a d-reduction it is sufficient to add ξ to the set of hypotheses and derive $\Box$; after an m$'$-reduction considering the formula $\text{Kopf}(Q * \xi) \supset \Box$ in the set of hypotheses it is sufficient to derive Kopf($Q*\xi$); after an m$'$-reduction considering the formula ξ in the set of hypotheses it is sufficient to derive every formula in Rumpf($Q * \xi$); all these formulae are in the set of hypotheses, an m$'$-reduction for each of them confirms their derivability.

If for a fixed $\Box$ every formula $\text{wk}_{R,\Box}$ is in Σ, then for every formula ξ the sequent $\Sigma \vdash \text{WK}_{\xi,\Box}$ is mj-derivable. One can add the formulae $\text{wk}_{R,\Box}$ to a set of hypotheses, but not all the formulae $\text{WK}_{\xi,\Box}$, because the latter contain all possible symbols for free variables, this is why the first are selected as contradiction axioms. One could "generalize" all symbols $\bar{q}$ for free variables in each $\text{WK}_{\xi,\Box}$ not appearing in $\Box$, substitute them with variables $\bar{v}$ and put a block of universal quantifiers in front of $\text{WK}_{\xi,\Box}$ yielding $\forall\bar{v}(\text{WK}_{\xi,\Box})^{\bar{q}\to\bar{v}}$, but since $[q_n, \ldots, q_1] * \text{wk}_{R,\Box} = \text{WK}_{R(q_n,\ldots,q_1),\Box}$ holds, among these formulae are the $\text{wk}_{R,\Box}$. Another advantage of the $\text{wk}_{R,\Box}$ over the $\text{WK}_{\xi,\Box}$ is that, while for each atomic formula A there is an infinite number of pairs $(W_{\varphi,\Box}, T)$ with $T * W_{\varphi,\Box} = A$, there is only one of the form $(\text{wk}_{R,\Box}, T)$; this makes a big difference in the amount of possible m-reductions. — Analogous results hold for $\text{wj}_{R,\Box}$ and $\text{WJ}_{\xi,\Box}$. For the induction schema of formal arithmetic there is a similar problem as with the contradiction axioms, but I do not have a similar solution.

§25

hm and classical logic

Following the remarks in §23, one can compare hm with a typical calculus for an axiomatic theory Σ in classical logic, like the one in [7], page 82. If wk is included in Σ, the cnk-schema provides the schema (postulate) 8° in [7]; and by paraphrasing other logical symbols than $\supset$ and $\forall$, one can express every rule of such a calculus as a combination of hm-rules, mj$'$ can be helpful here. Hence, if wk is included in Σ, the sequent $\Sigma \vdash \xi$ is hm-derivable if and only if ξ is derivable in classical logic from Σ.

For intuitionistic logic this argument does not hold, not all usual logical symbols can be paraphrased with $\supset$, $\forall$ and $\Box$ in intuitionistic logic, schemata containing other logical symbols than $\supset$, $\forall$ and $\Box$ may appear in a derivation of a formula restricted to these symbols: it should be proved that such schemata can be avoided.

§26

lm, LM and LJ

The important property of Gentzen's sequent calculi LK and LJ, as of Johansson's LM and the lm introduced here, is that, with the exception of the redundant Schnitt-rules, deduction rules introduce logical symbols or only rearrange the formulae in a sequent, without removing any logical symbol, at most identifying some of them appearing in identical formulae ("Zusammenziehung"). Every logical symbol appearing in the endsequent of a derivation without Schnitt-rules is introduced by a rule or appears in a firstsequent; such a derivation, whose endsequent does not contain a specific logical symbol, cannot contain a rule introducing this symbol or a firstsequent containing it. Exactly because of this, if one is interested only in derivable sequents not containing some logical symbols, one can restrict the calculus not allowing rules introducing such symbols nor Schnitt, not allowing rules containing formulae with these symbols, and the derivable restricted formulae in the original calculus will remain derivable in the restricted calculus.

Gentzen's calculi LJ and LK are for sequents of the form $\Gamma \rightarrow \Delta$, where Γ and Δ are lists of formulae; in LJ the succedents Δ may be empty or have only one formula, but they should have of no more than one formula. As remarked in [6], page 133, one can obtain a calculus LJ$'$ equivalent to LJ by paraphrasing the negation with the help of $\Box$, restricting LJ to sequents having exactly one formula in the succedent, representing empty succedents with succedents having only the formula $\Box$, removing the schemata NES and NEA, substituting the schema "Verdünnung im Sukzedens" with the schema

$$\text{NEGJ} : \frac{\Gamma \rightarrow \Box}{\Gamma \rightarrow \varphi}.$$

By removing this schema NEGJ one gets Johansson's calculus LM.

Since only the logical symbols $\supset$ and $\forall$ are allowed in formulae here, one considers only the three schemata for structural-rules in antecedent (Schemata für Struktur-Schlußfiguren im Antezedens), and the five schemata FES, FEA, AES, AEA, NEGJ restricted to these formulae, the latter schema is excluded in LM.

For a list Γ of formulae of the kind considered here let Γ^* denote the set containing the members of Γ, for a sequent S of the form $\Gamma \rightarrow \varphi$ treated in [6] and built only with formulae containing $\supset$ and $\forall$ as logical symbols let S^* denote the sequent $\Gamma^* \vdash \varphi$ of the form treated here, for a rule r of the form treated in [6] let r^* denote the rule of the form treated here obtained by replacing all sequents S appearing in it by S^*. The transformation r^* of r is lm-valid, if r is a rule of one of the LM-schemata mentioned above: rules of FES, AES, AEA and thinning (Verdünnung) are transformed into rules of fes, aes, aea and erw; a FEA-rule into a combination of an fea-rule and two erw-rules; a contraction (Zusammenziehung) or an exchange (Vertauschung) is transformed into a rule whose only oversequent is identical to its undersequent. A groundsequent (Grundsequenz) S is transformed into a sequent S^* that is the undersequent of a grnd-rule. Replacing every sequent S in an LM-derivation by S^* yields a composition of lm-valid rules whose firstsequents are lm-derivable (with only a grnd-rule): if a sequent S built only with formulae containing $\supset$ and $\forall$ as logical symbols is LM-derivable, then S^* is lm-derivable.

The rules of the schema

$$\mathrm{STR}: \frac{\Gamma \to \varphi}{\Delta \to \varphi}$$

with $\Gamma^* \subseteq \Delta^*$ are compositions of structural rules in antecedent, STR is hence LM- and LJ′-valid. With exception of grnd-rules, every lm-rule r is of the form s^* for an LM-rule s. — For every lm-derivation H of $\Sigma \vdash \varphi$ and every list of formulae Γ with $\Gamma^* = \Sigma$ one can find an LM-derivation of $\Gamma \to \varphi$ recursively on the structure of H, from the endsequent to the leaves, with the help of the two remarks above. — The sequent S built only with formulae containing $\supset$ and $\forall$ as logical symbols is LM-derivable if and only if S^* is lm-derivable.

Let lj be the calculus consisting of the lm-schemata and the schema

$$\mathrm{negj}: \frac{\Sigma \vdash \Box}{\Sigma \vdash \varphi}.$$

One can prove in the same way as above, that $\Gamma \to \varphi$ is LJ′-derivable if and only if $\Gamma^* \vdash \varphi$ is lj-derivable. — A sequent $\Sigma \vdash \varphi$ is lj-derivable if and only if $\Sigma \cup \mathrm{wj} \vdash \varphi$ is lm-derivable: with Gentzen's Hauptsatz the lj-validity of schnitt can be proved, with the definition of lj the lj-derivability of $\Sigma \vdash \mathrm{wj}_R$ for all Σ and $\mathrm{wj}_R \in \mathrm{wj}$ can be verified, and all this implies that $\Sigma \vdash \varphi$ is lj-derivable if $\Sigma \cup \mathrm{wj} \vdash \varphi$ is lm-derivable; as usual one can prove the lj-validity of erw, the lm-validity of cnj implies the lm-validity of negj if Σ contains wj, and all this implies that $\Sigma \cup \mathrm{wj} \vdash \varphi$ is lm-derivable if $\Sigma \vdash \varphi$ is lj-derivable.

§27

cm, nm, NM and Prawitz' normal form

It is not difficult to transform an nm-derivation of a sequent of the form $\vdash \xi$ into a derivation in Gentzen's NJ (or Johansson's NM) having ξ as its endformula (Endformel) and having only FB-, FE-, AB-, AE-rules. And such an NJ-derivation can also be easily transformed into an nm-derivation of $\vdash \xi$. Formulae in antecedents Σ of sequents $\Sigma \vdash \eta$ appearing in nm-derivations play the rôle of the suppositions (Annahmen) in NJ-derivations.

For every nm-rule having oversequents one can distinguish one, called the **distinguished oversequent** of the nm-rule. The distinguished oversequent of an fe-, ab- or ae-rule is its only oversequent, the distinguished oversequent of an fb-rule is the one corresponding to $\Sigma \vdash \eta \supset \xi$ in the fb-schema introduced in figure 13, an an-rule has no oversequent and hence no distinguished one.

A **thread** in a composition tree is a sequence A_0, A_1, $\ldots A_n$ of H-sequents, such that every A_{i+1} is an oversequent of an H-rule having A_i as undersequent; the sequents of the thread are to be considered as part of the composition tree, and not as an independent list of sequents. The **H-rules of the thread** are those of which the A_{i+1} are oversequents, the sequent A_0 is called the **endsequent of the thread** and A_n the **firstsequent** of it, they do not need to coincide with a firstsequent of the composition tree or with its endsequent. Here the definition of a thread differs from the ones in [5] and [9]. If the firstsequent of a thread A coincides with the endsequent of B, the **composition** AB of A and B is the thread obtained by considering all the H-Sequents of A and B together.

Sequents in an nm-composition that are distinguished oversequents of an H-rule are called **d-sequents**, the other ones are the endsequent of the nm-composition or non-distinguished oversequents of a rule, the latter are called **r-sequents**. A **d-thread** in an nm-composition is a thread whose sequents, possibly with the exception of its endsequent, are d-sequents. A sequence consisting of a single H-sequent is hence always a d-thread. The **adjacent** r-sequents of a d-thread are the non-distinguished oversequents of the H-rules of this d-thread, they are not in the d-thread, another d-thread having one of these adjacent r-sequents as endsequent is called a **ramification** of the original d-thread. A **branch** is a maximal d-thread, this is a d-thread whose endsequent is an r-sequent or the endsequent of the composition tree and whose firstsequent is the undersequent of an an-rule of the tree or a firstsequent of the tree. Every d-thread is contained in a branch, two different branches have no common H-rule.

By comparing the succedents of the distinguished oversequent and of the undersequent, one can classify the nm-rules having oversequents as follows: fe- and ae-rules are called **e-rules** or **introducing rules**, fb- and ab-rules are called **b-rules** or **eliminating rules**. An H-sequent of an nm-composition is called **detachable** if it is the undersequent of an e-rule of the tree and the distinguished oversequent of a b-rule of the tree; one can easily see that there are only two possibilities for this pair of H-rules: the e-rule is an fe-rule and the b-rule an fb-rule, in which case the detachable sequent is called **f-detachable**, or the e-rule is an ae-rule and the b-rule an ab-rule, in which case the detachable sequent is called **a-detachable**.

Now, d-threads can be classified according to their rules: an **e-thread** is a d-thread whose rules are e-rules, a **b-thread** is a d-thread whose rules are b-rules. A d-thread consisting of only one H-sequent is both an e- and a b-thread. A branch of an nm-composition contains no detachable sequent if and only if it is the composition EB of an e-thread E and a b-thread B. A branch is called **normal** if it does not contain detachable sequents and its firstsequent is the undersequent of an an-rule of the tree, because of the remark above it has always the form EB, the firstsequent of E and endsequent of B is called the **turning sequent** of it. An nm-derivation is called **normal** if it contains no detachable sequent, since such a sequent is always in a branch, an nm-derivation is normal if and only if all its branches are normal. This definition of normal derivation coincides with the one in [9], page 39.

For every m-rule there is a composition tree consisting of a b-thread and its adjacent r-sequents, the firstsequent of the b-thread being the undersequent of an an-rule of the tree, the adjacent r-sequents corresponding to the oversequents of the m-rule, the endsequent to the undersequent of the m-rule. It is also easy to see that for every es-rule introduced in §20 there is a composition tree consisting of an e-thread (and hence having no adjacent r-sequents) whose firstsequent is the oversequent of the es-rule and whose endsequent its undersequent. The undersequent of the cm-rule $\mathrm{cm}(\Sigma \vdash \xi, Q, \pi, T)$ is the undersequent of the rule $\mathrm{es}(\Sigma \vdash \xi, Q)$ whose oversequent is $\Sigma \cup \mathrm{Rumpf}(Q * \xi) \vdash \mathrm{Kopf}(Q * \xi)$, this oversequent is the undersequent of an m-rule whose oversequents are the oversequents of the cm-rule. Hence, concatenating the e-thread for this es-rule and the b-thread for this m-rule yields a nm-composition tree for the cm-rule consisting of a normal branch whose turning sequent has atomic succedent, and of the adjacent r-sequents of this normal branch. One can also see that a nm-composition tree consisting of such a normal branch together with its adjacent r-sequents corresponds to a nm-composition tree for a cm-rule. — Therefore, one can build a one-to-one correspondence between cm-derivations and normal nm-derivations whose branches have turning sequents with atomic succedents, so that corresponding derivations have the same endsequent: the image of a cm-derivation is obtained by replacing every cm-rule of it by the nm-composition tree described above, the pre-image of a normal nm-deduction can be obtained recursively, the branch containing the endsequent together with its adjacent r-sequents can be replaced by a cm-rule, the subtrees whose endsequents are these adjacent r-sequents are normal nm-deductions with less nodes than the original. — Theorem 2 in page 40 of [9] follows as a corollary: every nm-derivable sequent has a normal nm-derivation. Prawitz' elegant, original proof can easily be translated into the formalism used here.

§28

Calculi for potential derivations

So far, in all considerations about the calculi mj, cm, lm, nm or hm one can assume that no sequent, no formula and no term has a symbol for unknowns in it. If one allows them to appear, one can first assume that they represent fixed, unknown objects, as the constants do. From the validity of the schema sub it follows that, if a sequent is derivable, then all sequents obtained by replacing its unknowns with arbitrary terms are also derivable: hence, unknowns could also be seen as representing arbitrary, but fixed terms. In the calculi for potential derivations considered here, unknowns represent fixed terms, but not necessarily arbitrary ones: deductions and reductions are admissible if and only if the unknowns fulfill some conditions.

Two substitutions α and β are considered equal, if for all function symbols f of arity 0 the equality $f^\alpha = f^\beta$ holds, even if the domains of definition of α and β do not coincide. The identity ι is the substitution whose support is empty. The associative product $\alpha\beta$ of two substitutions α and β is the only substitution satisfying $f^{\alpha\beta} = (f^\alpha)^\beta$ for all function symbols f of arity 0, and hence, also for all open terms and open formulae f. Let **SU** denote the set of substitutions whose supports consist only of unknowns, SU contains ι and the products of substitutions in it: SU is a monoid.

A **potential rule** $\bar{r}$ consists of a (common) rule r and a condition C on substitutions in SU; $\bar{r}$ has a possibly empty list of oversequents and an undersequent: the ones of r. An $\alpha \in$ SU satisfying the condition is said to **actualize** the potential rule $\bar{r}$, and the rule r^α is the **α-actualization** of the potential rule.

A potential rule $\bar{r}_1$ is **more general** than a potential rule $\bar{r}_2$ if every actualization of $\bar{r}_2$ is also an actualization of $\bar{r}_1$. For a $\gamma \in$ SU and a potential rule $\bar{r}$ consisting of the rule r and the condition C, one defines the condition C^γ as the one fulfilled by α if and only if $\gamma\alpha$ is fulfilled by C, and the potential rule $\bar{r}^\gamma$ as the one with the rule r^γ and the condition C^γ. An α-actualization of $\bar{r}^\gamma$ is the $\gamma\alpha$-actualization of $\bar{r}$: the potential rule $\bar{r}$ is more general than the potential rule $\bar{r}^\gamma$. If γ actualizes $\bar{r}$, then the potential rule $\bar{r}^\gamma$ has as rule the γ-actualization of $\bar{r}$ and its condition is fulfilled by the identity $\iota \in$ SU; the γ-actualization of $\bar{r}$ can thus be identified with the potential rule $\bar{r}^\gamma$.

For a set $\bar{k}$ of potential rules, one defines $\bar{k}^\gamma$ as the set containing the rules of the form $\bar{r}^\gamma$ with $\bar{r}$ from $\bar{k}$; the set $\bar{k}$ is called **SU-stable**, if it includes $\bar{k}^\gamma$ for all $\gamma \in$ SU.

The elements of a set $\bar{k}$ of potential rules or a schema $\bar{k}$ describing it are called **potential $\bar{k}$-rules**, α-actualizations of potential $\bar{k}$-rules with all possible α are called **actual $\bar{k}$-rules**. The set of actual $\bar{k}$-rules of a SU-stable set $\bar{k}$ can be identified with the subset of $\bar{k}$ consisting of the rules that are actualized by the identity.

Since a potential rule $\bar{r}$ has also oversequents and an undersequent, the ones of its rule r, one can build compositions and derivations as usual with potential $\bar{k}$-rules of a set $\bar{k}$, these compositions and derivations are called **potential $\bar{k}$-compositions** and **potential $\bar{k}$-derivations**. For a potential $\bar{k}$-composition H, one can define the potential $\bar{k}^\gamma$-composition H^γ by replacing each potential rule $\bar{r}$ of H with $\bar{r}^\gamma$. — An $\alpha \in$ SU is said to **actualize** a potential $\bar{k}$-composition H if it actualizes each of its potential rules, that is, if it satisfies the conditions of all these rules; in this case H^α is a $\bar{k}^\alpha$-composition built with potential $\bar{k}^\alpha$-rules that are actualized by the identity and that can be considered as actual $\bar{k}$-rules if one forgets the condition, H^α as a tree built with actual $\bar{k}$-rules is called **α-actualization of** H (or actualization of H with α).

A **calculus $\bar{k}$ for potential derivations** consists of a set of potential rules, such that every derivation built with actual $\bar{k}$-rules (perhaps actualizations with different substitutions) is the actualization of a potential derivation with some substitution. The set k of actual $\bar{k}$-rules is called **actualization** of $\bar{k}$, with this k the calculus $\bar{k}$ can also be called a **calculus for potential k-derivations**. A SU-stable set $\bar{k}$ is a calculus for potential derivations. Two calculi for potential derivations are **equivalent**, if their actualizations are equivalent; a calculus for potential derivations is equivalent to a calculus, if the actualization of the former is equivalent to the latter. A **generator** of a calculus $\bar{k}$ for potential derivations is a subset $\bar{b}$ of $\bar{k}$, such that for every $\bar{k}$-composition H there is a $\bar{b}$-composition B and an $\alpha \in$ SU with $H = B^\alpha$. A generator of $\bar{k}$ is also a calculus for potential derivations, it has the same actualization as $\bar{k}$, and hence it is equivalent to $\bar{k}$.

§29

The calculus $\overline{\text{mj}}$ for potential mj-derivations

Term-equations and prohibitions are two kinds of conditions for the potential rules considered here. A **term-equation** C is a pair of terms (t_1, t_2) denoted by $t_1 = t_2$; as a condition, it is fulfilled by α if the terms t_1^α and t_1^α are identical; for a $\gamma \in$ SU the condition C^γ is the term-equation $t_1^\gamma = t_2^\gamma$. A **prohibition** C is a pair denoted by $(q|\Sigma)$ consisting of a symbol q for free variables and a set Σ of formulae; C, as a condition, is fulfilled when q does not appear in Σ^α; for a $\gamma \in$ SU the condition C^γ is the prohibition $(q|\Sigma^\gamma)$.

For a sequent $\Sigma \vdash \zeta$ with an atomic ζ, an element π of its antecedent Σ and a list T of terms appropriate for π, the expression $\bar{m}(\Sigma \vdash \zeta, \pi, T)$ denotes the potential rule whose undersequent is $\Sigma \vdash \zeta$, whose list of oversequents is Rumpf$(T * \pi)$ and whose condition is Rumpf$(T * \pi) = \zeta$. The elements of T are its auxiliary terms, the special antecedent formula π is called its determining formula. The actualizations of these rules are exactly the m-rules of mj. — For a sequent of the form $\Sigma \vdash \eta \supset \xi$ the expression $\bar{d}(\Sigma \vdash \eta \supset \xi)$ denotes the potential rule whose undersequent is $\Sigma \vdash \eta \supset \xi$, whose only oversequent is $\Sigma \cup \{\eta\} \vdash \xi$ and whose condition is the one fulfilled by all substitutions. The actualizations of these rules are exactly the d-rules of mj. — For a sequent $\Sigma \vdash \psi$ with ψ of the form $\forall v \chi$ and for a symbol q for free variables the expression $\bar{g}(\Sigma \vdash \psi, q)$ denotes the potential rule whose undersequent is $\Sigma \vdash \psi$, whose only oversequent is $\Sigma \vdash [q] * \psi$ and whose condition is $(q|\Sigma \cup \{\psi\})$. The q is its auxiliary term, also called its proper variable. This potential rule has no actualization if q appears in $\Sigma \cup \{\psi\}$, the actualizations of these rules are exactly the g-rules of mj. — The set $\overline{\text{mj}}$ consists of all potential rules of the forms $\bar{m}(S, \pi, T)$, $\bar{d}(S)$ and $\bar{g}(S, q)$.

For a rule $\bar{r}$ of the form $\bar{m}(S, \pi, T)$ or $\bar{d}(S)$ and a substitution γ, one defines $\bar{r}^\gamma$ as the rule $\bar{m}(S^\gamma, \pi^\gamma, T^\gamma)$ or $\bar{d}(S^\gamma)$. for a rule $\bar{r}$ of the form $\bar{g}(S, q)$ and a substitution γ, one defines $\bar{r}^\gamma$ as $\bar{g}(S^\gamma, q^\gamma)$ if q^γ is a symbol for free variables, otherwise $\bar{r}^\gamma$ is undefined. For $\gamma \in$ SU these definitions of $\bar{r}^\gamma$ coincide with the one in §28. Hence, $\overline{\text{mj}}$ is SU-stable and a calculus for potential mj-derivations.

From the results in §16, it follows that for every actual $\overline{\text{mj}}$-derivation there is an actual $\overline{\text{mj}}$-derivation with unambiguous proper variables and with the same endsequent, this concept of a derivation with unambiguous proper variables can also be extended to potential $\overline{\text{mj}}$-derivations.

For a substitution α and an $\overline{\text{mj}}$-rule $\bar{r}$ whose undersequent is of the form S^α, one wants conditions for the existence of an $\overline{\text{mj}}$-rule $\bar{s}$ having S as its undersequent and satisfying $\bar{r} = \bar{s}^\alpha$. If $\bar{r}$ is an $\bar{m}$-rule, then one can find a π in the antecedent of S such that π^α is the determining formula of $\bar{r}$; if additionally the list of auxiliary terms of $\bar{r}$ is of the form T^α, then $\bar{r} = \bar{s}^\alpha$

holds for $\bar{s} = \bar{m}(S, \pi, T)$. If $\bar{r}$ is a $\bar{d}$-rule, then $\bar{r} = \bar{s}^\alpha$ holds for $\bar{s} = \bar{d}(S)$. If $\bar{r}$ is a $\bar{g}$-rule and its proper variable is of the form q^α for a free variable q, then $\bar{r} = \bar{s}^\alpha$ holds for $\bar{s} = \bar{g}(S, q)$.

In a potential $\overline{\mathrm{mj}}$-derivation H whose endsequent is of the form S^α and whose rules has only auxiliary terms of the form t^α, t being a symbol for free variables if this auxiliary term is a proper variable, one can replace each potential rule of H by a new one with the corresponding auxiliary terms t instead of t^α, for obtaining a potential $\overline{\mathrm{mj}}$-derivation I having S as its endsequent and satisfying $I^\alpha = H$. This I is called an **α-pre-image** of H with endsequent S and auxiliary terms t. For finding I, it is sufficient to find the appropriate undersequent for each new rule and the determining formulae π for each new potential $\bar{m}$-rule, this is done recursively, beginning with the rule whose undersequent is the endsequent S^α of H, that should be replaced by S, and ending with the rules having no oversequents: after finding the new undersequent for a rule, one can find the appropriate π in the antecedent of this undersequent if it is an m-rule, and the new oversequents, that are the new undersequents of the next rules to be replaced. The α-pre-image is dependent on the auxiliary terms t, that are selected before beginning the recursive process, and on the determining formulae π of the $\bar{m}$-rules, that are selected during the recursive process.

Let $\overline{\mathrm{mj}}^*$ be the set containing all $\bar{d}$-rules, all $\bar{g}$-rules, and all $\bar{m}$-rules having a list of different unknowns as list of auxiliary terms. For $\bar{m}$-rules $\bar{r}_1 = \mathrm{m}(S_1, \pi_1, T_1)$, $\ldots \bar{r}_n = \mathrm{m}(S_n, \pi_n, T_n)$ one can select disjoint lists of different unknowns U_1, ..., U_n and an $\alpha \in \mathrm{SU}$ such that for all of them $U_i^\alpha = T_i$ hold; for every $\overline{\mathrm{mj}}^*$-rule $\bar{s}_i = \mathrm{m}(S_i, \pi_i, U_i)$ holds $\bar{s}_i^\alpha = \bar{r}_i$. From this and the above paragraph, it follows that for every $\overline{\mathrm{mj}}$-derivation H, there is an $\overline{\mathrm{mj}}^*$-derivation I with the same endsequent whose auxiliary terms for $\bar{m}$-rules are pairwise different symbols for unknowns not appearing in the endsequent, and a substitution α for these unknowns such that $I^\alpha = H$ holds. Therefore, it follows that $\overline{\mathrm{mj}}^*$ is a generator of $\overline{\mathrm{mj}}$, and hence also a calculus for potential mj-derivations.

A $\overline{\mathrm{mj}}^*$-derivation is called **normal** if auxiliary terms of different H-rules are (formally) different and each auxiliary term appears at most in sequents above the undersequent of its rule, excluding this undersequent. Every mj-derivable sequent S is the endsequent of a normal $\overline{\mathrm{mj}}^*$-derivation actualized by a substitution $\alpha \in \mathrm{SU}$ satisfying $S^\alpha = S$.

If there is a substitution actualizing an $\overline{\mathrm{mj}}$-derivation H, then a most general unifier of the conditions of the $\bar{m}$-rules appearing in the derivation also actualizes it, such a substitution is called a **most general substitution actualizing** H. If α is a most general substitution actualizing a normal $\overline{\mathrm{mj}}^*$-derivation I, then I^α is an actual $\overline{\mathrm{mj}}$-derivation with unambiguous proper variables; this can be proved by structural induction on the tree I considering properties of most general unifiers.

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