Interner Bericht

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ABSTRACT. This report presents a generalization of tensor-product B-spline surfaces. The new scheme permits knots whose endpoints lie in the interior of the domain rectangle of a surface. This allows local refinement of the knot structure for approximation purposes as well as modeling surfaces with local tangent or curvature discontinuities. The surfaces are represented in terms of B-spline basis functions, ensuring affine invariance, local control, the convex hull property, and evaluation by de Boor's algorithm. A dimension formula for a class of generalized tensor-product spline spaces is developed.

CONTENTS

1. Introduction

Tensor-product B-spline surfaces are a powerful tool in design and approximation of free-form surfaces. Their flexibility arises from non-uniform knot spacing, which allows an adaptation of the knot lines to a particular problem. Discontinuities of the first (second, etc.) derivative can be introduced through multiple knots. One disadvantage of the traditional tensor-product approach is that the knot lines always extend across the whole domain rectangle, either from left to right or from top to bottom.

This report proposes a surface scheme in which the endpoints of knots may lie in the interior of the domain. These knots are called knot segments and knot rays, as opposed to knot lines. As in the traditional scheme, tensor-products of univariate B-splines are used as basis functions. However, these bivariate B-splines are not required to share the same, global knot vectors. Some knot sharing between basis functions whose supports overlap is necessary to obtain a nicely behaved scheme. This leads to the notion of semi-regular collections of B-splines. Semi-regularity ensures linear independence of the B-splines (thus making them a true basis instead of just a spanning set), affine invariance, and the strong convex hull property. A surface constructed from a semi-regular basis can be evaluated by multiple application of de Boor's algorithm.

The spline spaces mentioned so far are defined synthetically by a collection of spanning functions. A different, analytical approach to spline spaces chooses some partition of the domain and then considers piecewise polynomial functions that have a certain smoothness across the edges of this partition. For computations in such a space, it is important to know the dimension and a base or spanning set. A collection of B-splines induces a partition Δ, which in turn defines a spline space $S(\Delta)$. This space is possibly larger than the span of $\mathfrak{B}$. For a certain class of semi-regular bases, it can be shown that $\mathfrak{B}$ spans the whole space $S(\Delta)$. In this sense, semi-regular bases are complete.

The remainder of this report is structured as follows. Section 2 reviews univariate and bivariate, tensor-product spline spaces, generalizing the latter to allow knot segments and knot rays. Section 3 defines semi-regularity and proves some properties of semi-regular bases. Section 4 investigates the dimension of certain partition-defined spline spaces and demonstrates the above-mentioned completeness of semi-regular bases.

2. Spline Spaces

Splines are piecewise polynomial functions. The polynomial pieces are separated by **knots**. In the univariate case, the pieces are also called segments. Each segment is defined over an interval, and the knots are discrete points on the real line.

Definition 2.1. *A* ***knot vector*** *is a vector $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ that satisfies $x_1 \leq \cdots \leq x_n$. We call the numbers $x_1, \ldots, x_n$ the* ***knot values*** *of X. The* ***multiplicity*** *$\operatorname{mult}(x_i)$ of x_i is the number of times that the value x_i appears in X.*

A (univariate) ***spline function*** *or* ***spline curve*** *of degree d over knot vector X is a function $s : \mathbb{R} \to \mathbb{R}$ such that*

(1) *s restricted to each half-open interval $[x_i, x_{i+1})$ is a polynomial of degree d,*
(2) *s is $d - \operatorname{mult}(x_i)$ times continuously differentiable in each knot value x_i, and*
(3) *s vanishes outside the interval $[x_1, x_n)$.*

The (univariate) ***spline space*** *$S_d(X)$ of degree d over X is the linear space of all spline functions of degree d over X.*

We use the term "knot value" to distinguish the univariate case from the bivariate case, where the knots are line segments.

A widely-used basis for univariate spline spaces is formed by univariate B-splines. They can be defined in many ways, for example:

Definition 2.2. *Let X be a knot vector. The* ***normalized univariate B-splines*** *$N_i^d(X|x)$ of degree d over X are given by the recursion formula*

$$N_i^0(X|x) := \begin{cases} 1 & , x \in [x_i, x_{i+1}) \\ 0 & , else \end{cases} \qquad , i = 1, \ldots, n-1$$

$$N_i^d(X|x) := \frac{x - x_i}{x_{i+d} - x_i} N_i^{d-1}(X|x) + \frac{x_{i+d+1} - x}{x_{i+d+1} - x_{i+1}} N_{i+1}^{d-1}(X|x) \qquad , i = 1, \ldots, n-d-1$$

The B-splines $N_1^d(X|x), \ldots, N_{n-d-1}^d(X|x)$ form a basis of the space $S_d(X)$, provided that no knot value has multiplicity $d+2$ or more. (In that case, at least one of the B-splines is identically zero. After removing the zero functions, one obtains a basis of the space.)

The represention

$$s(x) = \sum_i b_i N_i^d(X|x)$$

of a spline function s in terms of this basis is called the B-spline representation of s. The coefficients b_i are called **control points**. The following properties, among others, have made B-splines a representation of choice in modern CAD/CAM systems (see, e.g., [Far87, HL93].)

- Local control: The support of $N_i^d(X|x)$ is contained in the interval $[x_i, x_{i+d+1})$. A change of the control point b_i affects only the part of s over this interval. This facilitates interactive design of free-form curves.
- Affine invariance: The B-splines partition unity over the interval $[x_{d+1}, x_{n-d})$. Applying an affine transformation to the image of this interval is equivalent to applying the same transformation to the control points.
- Strong convex hull: B-splines are non-negative. Together with their bounded support and partition of unity, this implies that $s(x)$ lies in the convex hull of $d+1$ consecutive control points for any fixed $x \in [x_{d+1}, x_{n-d})$.
- Numerically stable evaluation by de Boor's algorithm. This algorithm starts with $d+1$ control points and successively forms convex combinations in order to compute the function value $s(x)$. The weights that are used in the convex combinations depend only on x and on the knot vector X (cf. [dB72].)
  The above three properties of the B-spline representation can be inferred from de Boor's algorithm.

In practical applications, only the restriction of s to the interval $[x_{d+1}, x_{n-d})$ is considered. It is common to use the first and last knot values with multiplicity $d+1$ and to extend the restricted spline continuously to the right endpoint x_{n-d} $(= x_{n-d+1} = \cdots = x_n)$ of the domain interval. In this case, the first and last control points are interpolated:

$$s(x_1) = b_1 \quad \text{and} \quad s(x_n) := \lim_{\substack{x \to x_n \\ x < x_n}} s(x) = b_{n-d-1}$$

A bivariate spline function consists of polynomial pieces that are defined over polygonal regions. The pieces meet with a certain smoothness across the edges of those regions. In this report, we are particularly interested in the case when all edges are parallel to the coordinate axes.

Definition 2.3. *Let $\Omega \subset \mathbb{R}^2$ be bounded by a simple polygon. A **domain partition** or simply partition Δ of Ω consists of three finite sets C, E, V and a function* $\mathrm{mult} : E \to \mathbf{N} \cup \{\infty\}$ *such that*

(1) *C contains open, simply connected polygonal regions, called **cells**,*
(2) *E contains the polygon edges (open line segments) of all cells $c \in C$, called **knot edges***
(3) *V contains the endpoints of all edges $e \in E$, called **vertices**,*
(4) *$C \cup E \cup V$ is a partition in the point-set sense of the closure $\overline{\Omega}$ of Ω, and*
(5) $\mathrm{mult}(e) = \infty$ *if and only if e is a subset of the boundary of Ω.*

We call $\mathrm{mult}(e)$ *the **multiplicity** of edge $e \in E$. A partition Δ is called **rectangular** if each knot edge is parallel to one of the coordinate axes.*

In a rectangular partition, a vertex v can be incident on 2, 3, or 4 edges. Correspondingly, we call v a 2-, 3-, or 4-vertex.

Definition 2.4. *Let Δ be a rectangular partition of $\Omega \subset \mathbb{R}$. A function $s : \mathbb{R}^2 \to \mathbb{R}$ is called a (generalized)* ***tensor-product spline*** *of degree $d_x \times d_y$ over Δ if s satisfies:*

(1) *The restriction of s to any cell $c \in C$ is a polynomial of coordinate degree $d_x \times d_y$,*
(2) *over any vertical (horizontal) edge $e \in E$, s is $d_x - \mathrm{mult}(e)$ times ($d_y - \mathrm{mult}(e)$ times) continuously differentiable,*
(3) *s is right and top continuous, i.e.*

$$\lim_{\substack{h \to 0 \\ h>0}} s(x+h, y) = \lim_{\substack{h \to 0 \\ h>0}} s(x, y+h) = s(x,y) \quad \text{for all } (x,y) \in \mathbb{R}^2 \quad ,$$

and
(4) *s vanishes identically over the exterior of Ω, $\mathbb{R}^2 \setminus \overline{\Omega}$.*

A linear space consisting of such spline functions is called a (generalized) ***tensor-product spline space*** *of degree $d_x \times d_y$ over Δ. The space $S_{d_x \times d_y}(\Delta)$ that contains all splines of degree $d_x \times d_y$ over Δ is called the* ***maximal tensor-product spline space*** *of degree $d_x \times d_y$ over Δ.*

If an edge e has multiplicity $\mathrm{mult}(e) \geq d_x + 1$ (or $\mathrm{mult}(e) \geq d_y + 1$, respectively,) we interpret Condition (2) to mean that s may be discontinuous over e. In particular, this applies to the boundary edges of Δ, which have multiplicity ∞. It should be mentioned that one can define other, quite different classes of bivariate spline spaces. One commonly used class uses polynomial pieces of some total degree d over triangular cells, with all interior edges having the same multiplicity.

Due to their application in free-form surface design, bivariate spline functions are also called **spline surfaces**. In keeping with the terminology of traditional tensor-product B-spline surfaces, we collect collinear edges into line segments, which are then called knots.

Definition 2.5. *Let Δ be a domain partition of $\Omega \in \mathbb{R}^2$. A* ***knot*** *K of Δ is a closed line segment such that*

(1) *K is the union of some edges from E and some vertices from V and*
(2) *K has maximal length.*

A knot K is said to have ***constant multiplicity*** *if all its edges have the same multiplicity, i.e. if $\mathrm{mult}(e_1) = \mathrm{mult}(e_2)$ for all $e_1, e_2 \subset K$, $e_1, e_2 \in E$. A partition Δ is said to have* ***constant multiplicities*** *if each knot of Δ has constant multiplicity.*

We will have to differentiate between three types of knots, depending on whether their endpoints lie on the boundary or in the interior of Ω. A knot is called a **knot line** if both its endpoints lie on the boundary, a **knot ray** if one endpoint lies on the boundary and the other lies in the interior, and a **knot segment** if both endpoints lie in the interior.

A traditional **tensor-product B-spline surface** of degree $d_x \times d_y$ is defined in the form

$$s(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} N_i^{d_x}(X|x) N_j^{d_y}(Y|y) \tag{2.1}$$

where the first and last knot values of X and Y have multiplicity $d_x + 1$ and $d_y + 1$, i.e.

$$x_1 = \cdots = x_{d_x+1} \qquad x_{m+1} = \cdots = x_{m+d_x+1}$$

$$y_1 = \cdots = y_{d_y+1} \qquad y_{n+1} = \cdots = y_{n+d_y+1}$$

In this scheme, $s(x,y)$ is a linear combination of the bivariate **tensor-product B-splines** $N_i(X|x)N_j(Y|y)$. The linear span of these functions is a bivariate spline space. It is called the tensor product of the univariate spaces $S_{d_x}(X)$ and $S_{d_y}(Y)$. The spanning functions are linearly independent, provided that no knot value of X (of Y) has multiplicity more than $d_x + 1$ (more than $d_y + 1$). Local control, affine invariance, and the strong convex hull property are inherited from the univariate B-spline representation. The knot vectors X and Y are global in the sense that they are shared by all basis functions. We call X the horizontal and Y the vertical knot vector.

The traditional tensor-product spline spaces form a subclass of the maximal tensor-product spline spaces of Definition 2.4. This subclass is characterized by partitions Δ in which all knots are knot lines of constant multiplicity. The knots lie on the lines defined by the equations

$$x - x_i = 0 \quad , \quad i = d_x + 1, \ldots, m + 1$$

and

$$y - y_j = 0 \quad , \quad j = d_y + 1, \ldots, n + 1$$

where x_i, y_i are the knot values of the global knot vectors.

On the other hand, any spline space $S_{d_x \times d_y}(\Delta)$ is a subspace of some traditional tensor-product spline space: Extend each knot segment and knot ray of Δ to a knot line, and assign to each edge e the maximum multiplicity that occurs along the knot on which e lies. The resulting partition $\overline{\Delta}$ has only knot lines of constant multiplicity, and $S_{d_x \times d_y}(\overline{\Delta})$ is a traditional tensor-product spline space. The transition from Δ to $\overline{\Delta}$ introduces new edges and increases multiplicities of old edges, so every spline from $S_{d_x \times d_y}(\Delta)$ is also in $S_{d_x \times d_y}(\overline{\Delta})$. One consequence of this subspace inclusion is that any generalized tensor-product spline can be represented in traditional B-spline form.

Thus, it can be stored and processed by CAD/CAM systems, and communicated using standard data exchange formats.

3. Collections of Tensor-Product B-Splines

The fact that in the traditional B-spline surface scheme all knots are knot lines is a consequence of using global knot vectors X, Y which are shared by all basis functions. This section discusses schemes in which the basis functions have individual knot vectors. In the most general case, we take some finite collection

$$\mathfrak{B} = \left\{ B_k(x,y) = N_1^{d_x}(X^{(k)}|x) N_1^{d_y}(Y^{(k)}|y) \right\}$$

of tensor-product B-splines B_k and form linear combinations of them. We will always assume that all B_k are of the same degree, $d_x \times d_y$. The knot vectors of B_k are $X^{(k)} = (x_1^{(k)}, \ldots, x_{d_x+2}^{(k)})$ and $Y^{(k)} = (y_1^{(k)}, \ldots, y_{d_y+2}^{(k)})$. For ease of notation, we discuss only the case when $d_x = d_y = d$. All results in this section generalize to the case $d_x \neq d_y$.

Our general scheme does not guarantee linear independence nor partition of unity over the domain rectangle. In fact, the proper domain may not even be rectangular. By "proper domain" we mean the support of $\mathfrak{B}$, which we define as

$$\operatorname{supp}(\mathfrak{B}) := \bigcup_{B_k \in \mathfrak{B}} \operatorname{supp}(B_k)$$

where

$$\operatorname{supp}(B_k) := \left\{ (x,y) \in \mathbb{R}^2 : B_k(x,y) \neq 0 \right\}$$

is the support of B_k. Some restrictions on $\mathfrak{B}$ are necessary to achieve a nicely behaved scheme.

We are interested in collections $\mathfrak{B}$ that have rectangular support. The support of an individual tensor-product B-spline B_k is a rectangle whose sides are parallel to the coordinate axes. If the support of the whole collection $\mathfrak{B}$ is rectangular, the sides of this rectangle, too, must be parallel to the coordinate axes.

$\mathfrak{B}$ induces a domain partition of $\operatorname{supp}(\mathfrak{B})$ in the following way: Each B_k partitions its own support by knot edges that lie on the lines $x - x_i^{(k)} = 0$ and $y - y_j^{(k)} = 0$, where $x_i^{(k)}$ and $y_j^{(k)}$ are the knot values used by B_k. We define the multiplicity of an edge to be the multiplicity with which B_k uses the corresponding knot value. The partition Δ that is induced by $\mathfrak{B}$ is obtained by superimposing the individual partitions of all B_k. In this new partition, some of the edges from the B_k may be split where they cross knots from other individual partitions, and new vertices may arise in the intersection points. The multiplicity of an interior edge e of Δ is the maximal multiplicity of all edges from the B_k that contain e. Finally, we assign multiplicity ∞ to all boundary edges of Δ. Obviously, Δ may contain knot segments and knot rays, and the knots need not have constant multiplicities.

As in the case of univariate B-splines, we want to exclude the zero function from our spanning set. Also, we will not allow any B_k to "jump" over knots which are induced by other functions from $\mathfrak{B}$. If a knot runs through the support of B_k, then we require B_k to use the corresponding knot value with the maximal possible multiplicity.

Definition 3.1. *A collection $\mathfrak{B}$ of tensor-product B-splines is called* ***non-degenerate*** *if all $B_k \in \mathfrak{B}$ satisfy the following conditions:*

(1) *B_k is not identically zero.*
(2) *Let K be any horizontal (vertical) knot of the partition Δ that is induced by $\mathfrak{B}$. Let K lie on the line $y - y_0 = 0$ $(x - x_0 = 0)$. If K divides the support of B_k into two rectangles of non-zero area, then the knot vector $Y^{(k)}$ $(X^{(k)})$ of B_k contains the knot value y_0 (x_0) with multiplicity*

$$r := \min_{\substack{e \in E \\ e \subset K \\ e \subset \mathrm{supp}(B_k)}} \mathrm{mult}(e)$$

Condition (1) is equivalent to demanding that no $B_k \in \mathfrak{B}$ uses a knot value with multiplicity $d + 2$.

The following definitions lead to collections $\mathfrak{B}$ for which the surface can be evaluated by multiple application of the univariate de Boor's algorithm. As we shall see, this implies linear independence and partition of unity.

Definition 3.2. *A non-degenerate collection $\mathfrak{B}$ of tensor-product B-spline functions is called* ***horizontally regular*** *if there exist knot vectors*

$$X = (x_1, \ldots, x_{m+d+1}) \quad \text{and}$$

$$Y^{(i)} = (y_1^{(i)}, \ldots, y_{n(i)+d+1}^{(i)}) \quad , \quad i = 1, \ldots, m + d + 1$$

where the first (last) $d+1$ entries of all $Y^{(i)}$ are the same knot value $y_{\min}$ $(y_{\max})$ with multiplicity $d + 1$, such that

$$\mathfrak{B} = \left\{ N_i(X|x) N_j(Y^{(i)}|y) \right\}_{i=1,\ldots,m;\, j=1,\ldots,n(i)} \tag{3.1}$$

Vertical regularity *is defined analogously with knot vectors $X^{(j)}$ and Y. $\mathfrak{B}$ is said to be* ***globally semi-regular*** *if it is horizontally or vertically regular.*

The partition induced by a globally semi-regular basis can have knot segments and knot rays. However, they must all run in the same direction, either vertical or horizontal. The following definition localizes the semi-regularity condition. In each cell c of the partition, it considers only those B_k that are relevant for c. By "relevant" we mean that B_k does not vanish identically over c.

Definition 3.3. *Let $\mathfrak{B}$ be a finite collection of tensor-product B-splines, and let Δ be the domain partition induced by $\mathfrak{B}$. $\mathfrak{B}$ is called* ***locally semi-regular*** *if the following conditions hold:*

(1) supp $(\mathfrak{B})$ *is rectangular.*

(2) *For each point* $(x^*, y^*) \in \operatorname{supp}(\mathfrak{B})$ *which lies in a cell of* Δ, *either*

(a) *there exist knot vectors* $X = (x_1, \ldots, x_{2d+2})$ *and* $Y^{(i)} = (y_1^{(i)}, \ldots, y_{2d+2}^{(i)})$, $i = 1, \ldots, 2d+2$, *such that*

$$\begin{aligned} x_{d+1} < x^* < x_{d+2} \quad &\textit{and} \\ y_{d+1}^{(i)} < y^* < y_{d+2}^{(i)} \quad &\textit{for all} \quad i = 1, \ldots, d+1 \end{aligned} \tag{3.2}$$

and

$$\{B \in \mathfrak{B} : (x^*, y^*) \in \operatorname{supp}(B)\} = \left\{N_i(X|x) N_j(Y^{(i)}|y)\right\}_{i,j=1,\ldots,d+1} \tag{3.3}$$

or

(b) *there exist knot vectors* $X^{(j)} = (x_1^{(j)}, \ldots, x_{2d+2}^{(j)})$, $j = 1, \ldots, 2d+2$, *and* $Y = (y_1, \ldots, y_{2d+2})$ *for which conditions analogous to* (3.2) *and* (3.3) *are satisfied.*

A B-spline $B_k \in \mathfrak{B}$ either vanishes identically over a cell, or it is non-zero over the whole cell. Therefore, the set of B-splines in Equation (3.3) is the same for all points (x^*, x^*) of a cell. Note that the "choice" between Conditions (2a) and (2b) is made individually for each cell. This allows us to combine horizontal and vertical knot segments in one partition.

Corollary 3.1. *Any globally semi-regular set* $\mathfrak{B}$ *is locally semi-regular.*

Proof. W.l.o.g., let $\mathfrak{B}$ be horizontally regular with knot vectors X, $Y^{(i)}$. First, we note that $\operatorname{supp}(\mathfrak{B}) = [x_1, x_{n+d+1}) \times [y_{\min}, y_{\max})$ is rectangular. Next, consider some point $(x^*, y^*) \in \operatorname{supp}(\mathfrak{B})$. If this point does not lie on a knot, then x^* must lie strictly between two consecutive horizontal knot values x_{i^*} and x_{i^*+1}, say. Since the extremal knot values x_1 and x_{n+d+1} of X have multiplicity $d+1$, at least $d+1$ knot values of X lie to the left of x^*, and likewise to the right. The univariate B-splines over X that are non-zero in x^* are exactly $N_{i^*-d}(X|x), \ldots, N_{i^*}(X|x)$. By defining $\tilde{X} := (x_{i^*-d}, \ldots, x_{i^*+d+1})$, we can rewrite those B-splines as $N_1(\tilde{X}|x), \ldots, N_{d+1}(\tilde{X}|x)$. It follows from the definition of $\tilde{X}$ that $\tilde{x}_{d+1} = x_{i^*} < x^* < x_{i^*+1} = \tilde{x}_{d+2}$, so $\tilde{X}$ satisfies Equation (3.2). In the same manner, we can construct vertical knot vectors $\tilde{Y}^{(1)}, \ldots, \tilde{Y}^{(d+1)}$ satisfying Equation (3.2) from the vectors $Y^{(i^*-d)}, \ldots, Y^{(i^*)}$. The tensor-product B-splines that are non-zero in (x^*, y^*) are now given as

$$\left\{N_i(\tilde{X}|x) N_j(\tilde{Y}^{(i)}|y)\right\}_{i,j=1,\ldots,d+1}$$

This shows that $\mathfrak{B}$ is locally semi-regular. □

Since local semi-regularity includes global semi-regularity as a special case, we will refer to it simply as "semi-regularity" in the sequel.

Semi-regular collections $\mathfrak{B}$ allow us to evaluate a surface by multiple applications of the univariate de Boor's algorithm. Let $\mathfrak{B}$ be a semi-regular collection of tensor-product B-splines, and let $s \in \text{span}(\mathfrak{B})$ be given by its control points as

$$s(x,y) = \sum_{B_k \in \mathfrak{B}} b_k B_k(x,y)$$

Let $(x,y) \in \text{supp}(\mathfrak{B})$ not lie on a knot edge. Suppose that case (2a) of Definition (3.3) holds for (x,y), with knot vectors X, $Y^{(i)}$. Then, after an appropriate re-indexing of the control points b_k as b_{ij}, we can write

$$\begin{aligned} s(x,y) &= \sum_{i=1}^{d+1}\sum_{j=1}^{d+1} b_{ij} N_i(X|x) N_j(Y^{(i)}|y) \\ &= \sum_{i=1}^{d+1} \left(\sum_{j=1}^{d+1} b_{ij} N_j(Y^{(i)}|y) \right) N_i(X|x) \\ &= \sum_{i=1}^{d+1} b_i(y) N_i(X|x) \end{aligned} \tag{3.4}$$

where we define

$$b_i(y) := \sum_{j=1}^{d+1} b_{ij} N_j(Y^{(i)}|y) \tag{3.5}$$

For each $i = 1, \ldots, d+1$, the expression in (3.5) can be evaluated by the univariate de Boor's algorithm. Given the $b_i(y)$, we see from the last line of Equation (3.4) that another application of the algorithm yields the value $s(x,y)$. If case (2b) of Definition 3.3 holds for (x,y), the univariate de Boor's algorithm is first applied over each $X^{(i)}$ and then over Y. We have assumed so far that the point (x,y) lies in the interior of a cell. If it lies on a knot edge, then all B_k that are non-zero in (x,y) are also non-zero in the cell c that is incident on the knot and lies above or to the right of (x,y). Because of this, and because of the continuity condition in Definition 2.4, the evaluation algorithm can treat (x,y) as a point belonging to c.

Affine invariance and the convex hull property are immediate consequences of the fact that $s(x,y)$ can be evaluated by de Boor's algorithm. Each $b_j(y)$ is a convex combination of $b_{1j}, \ldots, b_{d+1j}$ and $s(x,y)$ is a convex combination of the $b_j(y)$, so $s(x,y)$ is a convex combination of the control points

$$\{b_{ij}\}_{i,j=1,\ldots,d+1} = \{b_k : B_k(x,y) \neq 0\}$$

Setting all control points to 1 shows that the B-splines in $\mathfrak{B}$ form a partition of unity over $\text{supp}(\mathfrak{B})$. Since the values $B_k(x,y)$ do not depend on the control points, the surface scheme is affine invariant.

Proposition 3.2. *A semi-regular collection $\mathfrak{B}$ of tensor-product B-splines is linearly independent.*

Proof. Let the spline function

$$s(x,y) = \sum_{B_k \in \mathfrak{B}} b_k B_k(x,y)$$

vanish identically over supp ($\mathfrak{B}$). Consider some cell c of the partition Δ induced by $\mathfrak{B}$. W.l.o.g., let the B_k that are non-zero over c share one horizontal knot vector X and vertical knot vectors $Y^{(1)}, \ldots, Y^{(d+1)}$. The polynomial piece of $s(x,y)$ over c vanishes identically, so for arbitrary fixed y^* we have

$$\begin{aligned} 0 &= \sum_{i=1}^{d+1}\sum_{j=1}^{d+1} b_{ij} N_i(X|x) N_j(Y^{(i)}|y^*) \\ &= \sum_{i=1}^{d+1} b_i(y^*) N_i(X|x) \qquad \text{for all } x \end{aligned} \tag{3.6}$$

where the b_{ij} are the re-indexed control points and the $b_i(y^*)$ are defined as in Equation (3.5). Equation (3.6) describes one segment of a B-spline curve. It is well-known that if this segment vanishes identically, then all corresponding control points $b_1(y^*), \ldots, b_{d+1}(y^*)$ are zero. Since y^* was chosen arbitrarily, the B-spline curve segment

$$b_i(y) = \sum_{j=1}^{d+1} b_{ij} N_j(Y^{(i)}|y)$$

vanishes identically for $i = 1, \ldots, d+1$. Again, this can only be the case if all b_{ij}, for $i, j = 1, \ldots, d+1$, are zero. Looking at all cells of Δ, we see that each control point b_k of $s(x,y)$ must be zero. This proves that the set $\mathfrak{B} = \{B_k\}$ is linearly independent. □

Any knot edge e of Δ is induced by some $B_k \in \mathfrak{B}$, and it is clear that e must lie in the closure of supp (B_k). If e is a boundary knot edge, the supports of all B_k that induce e lie on one side of e. In a general collection $\mathfrak{B}$, this may also be the case for interior knot edges. The following lemma shows that if $\mathfrak{B}$ is semi-regular, each interior knot edge is induced "from both sides."

Lemma 3.3. *Let $\mathfrak{B}$ be a semi-regular collection of tensor-product B-splines and let Δ be the partition of* supp ($\mathfrak{B}$) *induced by B. Let e be a knot edge of Δ with multiplicity r, and let c be a cell of Δ such that c is incident on e. Then $\mathfrak{B}$ contains some B-spline B that induces e with multiplicity r such that B is non-zero over c.*

Proof. There exists some B-spline $\hat{B} \in \mathfrak{B}$ which induces e. If $\hat{B}$ is non-zero over c, then $B = \hat{B}$. Otherwise, $\hat{B}$ must be non-zero over the cell $\hat{c}$ that is incident on e and lies opposite c. W.l.o.g., let e be a horizontal knot edge and let $\hat{c}$ lie above e. Let e lie on the line described by the equation $y - y_0 = 0$. The local semi-regularity condition holds for the cell $\hat{c}$, so

$$\hat{B}(x, y) = N_i(\hat{X}|x)N_j(\hat{Y}|y)$$

for some indices i, j and some knot vectors $\hat{X}$, $\hat{Y}$ according to Definition 3.3. Furthermore, $\hat{Y} = (\hat{y}_1, \dots, \hat{y}_{2d+2})$ contains y_0 as a knot of multiplicity r, so

$$\hat{y}_1 \leq \dots \leq \hat{y}_{d+1-r} < \hat{y}_{d+2-r} = \dots = \hat{y}_{d+1} = y_0$$

If $r \leq d$, we see that $\hat{y}_1$ is less than y_0 and the function

$$B(x, y) = N_1(\hat{X}|x)N_j(\hat{Y}|y)$$

is non-zero over c and induces e.

In the case that $r = d + 1$, we look at the local semi-regularity condition over cell c. It yields $d + 1$ B-splines of the form

$$\tilde{B}_i(x, y) = N_i(\tilde{X}^{(i)}|x)N_{d+1}(\tilde{Y}^{(i)}|y) \quad \text{for} \quad i = 1, \dots, d + 1$$

which are non-zero over c. (The notation allows different horizontal as well as different vertical knot vectors. Of course, $\tilde{X}^{(1)} = \dots = \tilde{X}^{(d+1)}$ or $\tilde{Y}^{(1)} = \dots = \tilde{Y}^{(d+1)}$ must hold.) The vertical knot vectors $\tilde{Y}^{(i)} = (\tilde{y}_1^{(i)}, \dots, \tilde{y}_{2d+2}^{(i)})$ are such that

$$y_0 \leq \tilde{y}_{d+2}^{(i)} \leq \dots \leq \tilde{y}_{2d+2}^{(i)}$$

If none of these vectors has y_0 as a knot with multiplicity $r = d+1$, then all $\tilde{y}_{2d+2}^{(i)}$ are greater than y_0 and all $\tilde{B}_i$ are non-zero over $\hat{c}$. None of the $\tilde{B}_i$ can share a vertical knot vector with $\hat{B}$, since this knot vector would have to contain y_0 with multiplicity r. Thus, the B-splines that are non-zero over $\hat{c}$ must share the common horizontal knot vector $\hat{X} = \tilde{X}^{(1)} = \dots = \tilde{X}^{(d+1)}$ and some vertical knot vectors $\hat{Y}^{(1)}, \dots, \hat{Y}^{(d+1)}$. In particular, $\hat{B}$ is based on $\hat{Y}^{(i_0)}$ for some i_0. $\tilde{B}_{i_0}$ can be represented over $\hat{Y}_{i_0}$, too, in contradiction to our observation that $\hat{B}$ cannot share a vertical knot vector with any of the $\tilde{B}_i$. Therefore, some $\tilde{Y}^{(i)}$ must contain y_0 with multiplicity r, and the corresponding $\tilde{B}_i$ is our B-spline B. □

Equation (3.3) of the local semi-regularity condition connects the supports of the basis functions B_k to the topology of the induced partition Δ. This implies certain restrictions on the kinds of topologies that Δ can assume. The following lemma illustrates one of these restrictions.

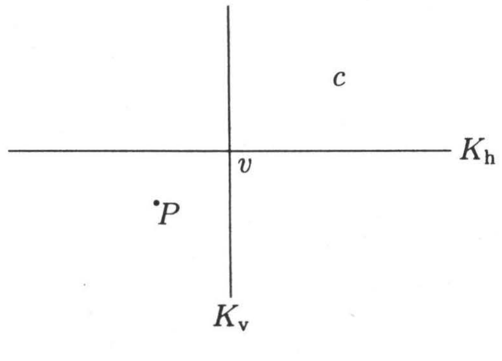
Lemma 3.4. *Let $\mathfrak{B}$ be a semi-regular collection of tensor-product B-splines and let Δ be the partition of* supp $(\mathfrak{B})$ *induced by $\mathcal{B}$. Let K_{h}, K_{v} be two knots of Δ such that K_{h} and K_{v} intersect in an interior vertex $v = (x_0, y_0)$. Then one of the following holds:*

(1) *Starting from either endpoint of K_{h}, the number of vertices on K_{h} from the endpoint up to and including v is greater than or equal to $d + 2$.*
(2) *Starting from either endpoint of K_{v}, the number of vertices on K_{v} from the endpoint up to and including v is greater than or equal to $d + 2$.*

In both cases, multiple vertices are counted with their multiplicity.

Proof. Throughout the proof, we count vertices with their multiplicities. "Vertices to the right" denotes v itself and the vertices to the right of v, likewise "to the left," "above," and "below." Assume that K_{h} is horizontal and K_{v} is vertical. It is first shown by contradiction that K_{h} must have $d+2$ vertices to the left or K_{v} must have $d+2$ vertices below. Assume that K_{h} has less than $d+2$ vertices to the left and K_{v} has less than $d+2$ vertices below. Any knot has at least $d+2$ vertices, so K_{h} must extend to the right of v and K_{v} must extend above v. Let c be the cell of Δ that lies to the right of and above v and is incident on v.

Since v is an interior vertex, there exists a point $P = (x^*, y^*) \in$ supp $(\mathfrak{B})$ which lies below and to the left of v such that no knot lies between P and v. The following diagram depicts this situation. Note that v may be an endpoint of K_{v} or K_{h}, although the two knots are drawn as extending below and to the left of v in the diagram.



W.l.o.g., let the B-splines that are non-zero in P share one horizontal knot vector X and some vertical knot vectors $Y^{(i)}$ as in Definition 3.3. Let r_{h} and r_{v} be the multiplicities of v as a vertex of K_{v} and K_{h}, respectively.

Consider the B-spline

$$B(x, y) = N_{r_{\mathrm{v}}}(X|x) N_{r_{\mathrm{h}}}(Y^{(r_{\mathrm{v}})}|y)$$

which is non-zero over P. The knot values of X and $Y^{(r_{\mathrm{v}})}$ are grouped around x^* and

y^* in the following way:

$$x_1 \leq \cdots \leq x_{r_v} \leq \cdots \leq x_{d+1} < \quad x^* \quad < x_{d+2} \leq \cdots \leq x_{r_v+d+1} \leq \cdots \leq x_{2d+2}$$

$$y_1^{(r_v)} \leq \cdots \leq y_{r_h}^{(r_v)} \leq \cdots \leq y_{d+1}^{(r_v)} < \quad y^* \quad < y_{d+2}^{(r_v)} \leq \cdots \leq y_{r_h+d+1}^{(r_v)} \leq \cdots \leq y_{2d+2}^{(r_v)}$$

Since no knot lies between P and v, we see that

$$x_0 \leq x_{d+2} \quad \text{and} \quad y_0 \leq y_{d+2}^{(r_v)}$$

If $Y^{(r_v)}$ contains y_0 as a knot value with multiplicity $\geq r_h$, then the line segment

$$\tilde{a} = [x_{r_v}, x_{r_v+d+1}] \times \{y_0\}$$

is a knot of B with multiplicity $\geq r_h$. In fact, B uses at most r_h vertical knot values equal to y_0, and $\tilde{a}$ has exactly multiplicity r_h. Now $\tilde{a}$ and K_h lie on the same line $y - y_0 = 0$, and both contain the point v, so $\tilde{a}$ is a subsegment of K_h. This shows that K_h has $d + 2 - r_v$ vertices at the points

$$(x_{r_v}, y_0), \ldots, (x_{d+1}, y_0)$$

which lie strictly left of v. Counting these points and v with its multiplicity of r_v yields $d + 2$ vertices, in contradiction to our assumption. We conclude that if B uses knot value y_0 at all, then it does so with multiplicity less than r_h. Therefore, the chain

$$y_0 \leq \underbrace{y_{d+2}^{(r_v)} \leq \cdots \leq y_{r_h+d+1}^{(r_v)}}_{r_h \text{ knots}}$$

contains a strict inequality, and the support of B extends above v. By analogy, $\operatorname{supp}(B)$ also extends to the right of v, so B is non-zero over the cell c.

Since K_v has a knot edge immediately above v, and since c is incident on this knot edge, Lemma 3.3 shows that there is some $B_v \in \mathfrak{B}$ that induces the knot edge with multiplicity r_v and is non-zero over c. B_v and B cannot be represented over the same horizontal knot vector, since this would require B to use x_0 as a knot value with multiplicity $\geq r_v$. Likewise, the knot edge of K_h immediately to the right of v is induced by some $B_h \in \mathfrak{B}$ which is non-zero over c, and B_h cannot share a vertical knot vector with B. The three B-splines B, B_v, and B_h are a contradiction to the local semi-regularity condition over c, which requires that they share a horizontal or a vertical knot vector. This shows that

(lb) K_h has $d + 2$ vertices to the left or K_v has $d + 2$ vertices below.

By symmetry,

(la) K_h has $d + 2$ vertices to the left or K_v has $d + 2$ vertices above,
(ra) K_h has $d + 2$ vertices to the right or K_v has $d + 2$ vertices above, and
(rb) K_h has $d + 2$ vertices to the right or K_v has $d + 2$ vertices below.

The statement of the lemma follows from these symmetric arguments. If K_{h} has less than $d+2$ vertices to the left, then (la) and (lb) show that K_{v} must have $d+2$ vertices above and $d+2$ vertices below. The same holds, by (ra) and (rb), if K_{h} has less than $d+2$ vertices to the right. So K_{h} or K_{v} has $d+2$ vertices to both sides of v, counting multiplicities and including v in each count. □

An immediate consequence of this lemma is that all cells of Δ are rectangular.

Corollary 3.5. *Let $\mathfrak{B}$ be a semi-regular collection of B-splines, and let Δ be the partition induced by $\mathfrak{B}$. Then Δ has no interior 2-vertices, and all cells of Δ are convex.*

Proof. Every interior vertex v is an intersection point of two knots K_{h} and K_{v}. Since $\mathfrak{B}$ is non-degenerate, v cannot have a multiplicity of more than $d+1$ on K_{h} nor K_{v}. By Lemma 3.4, one of the knots must have at least one more vertex on both sides of v, and v is incident on at least 3 knot edges.

In order to be non-convex, a cell c must have some vertex v which forms a strictly concave corner of c. This is only possible if v is a 2-vertex, so v cannot be an interior vertex of Δ. Since $\operatorname{supp}(\mathfrak{B})$ is convex, v cannot be a boundary vertex, either. We see that all cells of Δ are convex. □

4. A DIMENSION FORMULA

Spline spaces of total degree d over triangulations and other non-rectangular partitions have been studied widely in the literature (see, e.g., [DM83] and references therein.) Determining the dimension of these spaces and constructing bases for them can be extremely difficult, depending on the underlying partition.

The tensor-product spline spaces in Section 3 were defined through basis functions, so the dimension and a basis are trivially known. We want to compare them to the maximal spline spaces $S_{d_x \times d_y}(\Delta)$ defined by rectangular partitions Δ. As in Section 3, only the case $d_x = d_y = d$ is discussed. The results generalize to $d_x \neq d_y$.

If the partition Δ is induced by a collection $\mathfrak{B}$ of tensor-product splines of degree $d \times d$, then all $B_k \in \mathfrak{B}$ are splines over Δ, and span $(\mathfrak{B})$ is a subspace of $S_{d\times d}(\Delta)$. It is an interesting question whether this subspace inclusion is proper, or whether the two spaces are equal. In the latter case, $\mathfrak{B}$ is a basis of $S_{d\times d}(\Delta)$. In the former case, span $(\mathfrak{B})$ does not make full use of the flexibility that the partition allows, and one might wish to use the maximal space $S_{d\times d}(\Delta)$ instead.

This section derives an upper bound on the dimension of spline spaces $S_{d\times d}(\Delta)$ over a certain class of rectangular partitions. If Δ is induced by a semi-regular collection $\mathfrak{B}$ and if all its interior knot edges have multiplicity 1, then Δ belongs to this class. It is shown that in this case the upper bound is equal to the number of B-splines in the collection. This means that the upper bound is the exact dimension of $S_{d\times d}(\Delta)$, and that $\mathfrak{B}$ is a basis of this space.

The upper bound is developed with the technique of smoothing cofactors and conformality conditions. This technique was developed independently by Wang [Wan75] and by Schumaker [Sch79]. A short revision of the technique is given here. The interested reader is referred to Chui and Wang [CW83] for a more detailed discussion and proofs.

Let $p_1(x,y)$ and $p_2(x,y)$ be two polynomials over $\mathbb{R}^2$ that meet with continuity C^r over some line, and let that line be described by the equation $\ell(x,y) = 0$ where $\ell(x,y)$ is a polynomial of degree 1. Then the polynomial ℓ^{r+1} divides the difference of p_1 and p_2, i.e.

$$p_1 - p_2 = \ell^{r+1} q \tag{4.1}$$

for some polynomial $q(x,y)$ over $\mathbb{R}^2$. Chui and Wang call ℓ^{r+1} the **smoothing factor** and q the **smoothing cofactor** of $p_1 - p_2$. Note that the knot edge determines $\ell(x,y)$ only up to a constant factor. The smoothing cofactor depends on the particular choice of ℓ. The sign of q depends on the order of p_1 and p_2:

$$p_2 - p_1 = \ell^{r+1}(-q)$$

In order to determine the sign uniquely, we will always choose p_1 to lie on that side of the knot edge where $\ell(x,y)$ is negative.

Consider a bivariate spline $s(x, y)$, and suppose that some fixed $\ell(x, y)$ has been chosen for each knot edge. Then $s(x, y)$ is determined completely by the smoothing cofactors of all knot edges and by one of the polynomial pieces, say p_0.

Let v be an interior vertex of the underlying partition Δ. Let $p_1, \ldots, p_n$ be the polynomial pieces that are incident on v, and let the line $\ell_i(x, y) = 0$ contain the knot edge that separates p_i from p_{i+1} for $i = 1, \ldots, n$. The indices are understood in a cyclic fashion, such that $p_{n+1} = p_1$. Let $q_1, \ldots, q_n$ be the corresponding cofactors, then by going once around v we see that

$$\sum_{i=1}^{n} \ell_i^{r+1} q_i = \sum_{i=1}^{n} p_i - p_{i+1} = \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} p_i = 0 \tag{4.2}$$

Equation (4.2) is called the **conformality condition** corresponding to vertex v. Note that it depends on the required order r of smoothness across the i^{th} knot edge. This order is assumed to be the same for all knot edges.

The set of smoothing cofactors of a given spline satisfies the conformality conditions of all interior vertices. On the converse, if we choose p_0 and a set of smoothing cofactors which solve the system of conformality conditions, the underlying spline space contains a unique function $s(x, y)$ that has these smoothing cofactors and is identical to p_0 over a certain cell. Thus, the dimension of the spline space is equal to the dimension of the solution space of the conformality conditions plus the dimension of the polynomial space from which p_0 is taken. The system of conformality conditions can be transformed into a linear system of equations whose variables are the monomial coefficients of the smoothing cofactors. A lower bound on the rank of this system will yield an upper bound on the dimension of the solution space.

Smoothing cofactors and conformality conditions are now employed to derive an upper bound on the dimension of certain tensor-product spline spaces. We consider only spaces $S_{d \times d}(\Delta)$ where Δ is a rectangular partition whose interior edges have multiplicity 1, so adjacent polynomial pieces always meet with smoothness C^{d-1}. Each knot segment must have at least $d + 2$ vertices. Furthermore, we require that statement (1) or (2) of Lemma 3.4 hold for each pair of knots that intersect in an interior vertex. As we know from Section 3, this condition implies that Δ cannot have any interior 2-vertices. Note that partitions induced by semi-regular bases fulfil all of these conditions.

Let p_1 and p_2 be two adjacent polynomial pieces of $s(x, y) \in S$ that are separated by a vertical knot edge. We shall choose all smoothing factors in the form $\ell^d(x, y) = (x - x_0)^d$ (or $\ell^d(x, y) = (y - y_0)^d$ for horizontal edges.) Then

$$p_1(x, y) - p_2(x, y) = (x - x_0)^d q(x, y)$$

is a polynomial of coordinate degree $d \times d$, so $q(x, y)$ must be constant in y. Likewise, q is constant in x if the knot edge is horizontal.

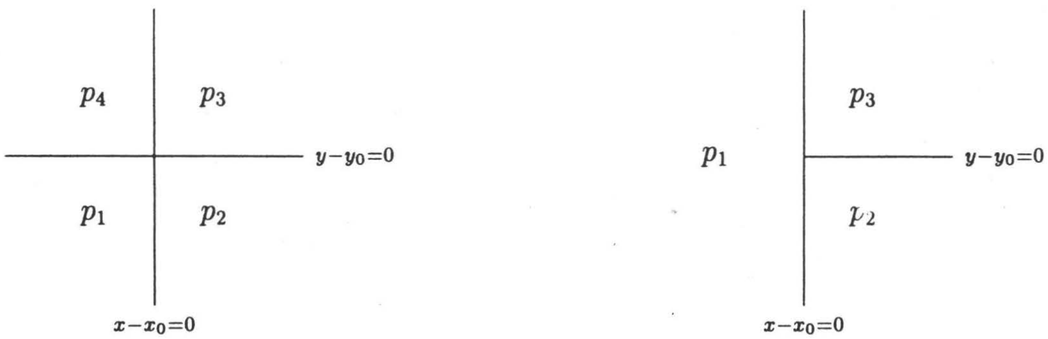


FIGURE 4.1. polynomial pieces around a vertex

Consider now a 4-vertex $v = (x_0, y_0)$ of Δ with polynomial pieces p_1, p_2, p_3, and p_4 as on the left-hand side of Figure 4.1. Due to our global ordering of adjacent polynomial pieces, Equation (4.1) becomes

$$\begin{aligned} p_1 - p_2 &= (x - x_0)^d q_1(y) \\ p_2 - p_3 &= (y - y_0)^d q_2(x) \\ p_3 - p_4 &= -(x - x_0)^d q_3(y) \\ p_4 - p_1 &= -(y - y_0)^d q_4(x) \end{aligned}$$

and the conformality condition is

$$(x - x_0)^d (q_1(y) - q_3(y)) + (y - y_0)^d (q_2(x) - q_4(x)) = 0$$

Expressing the smoothing cofactors q_i in terms of their monomial coefficients, which are denoted by $q_{i,\mu}$, and expanding the smoothing factors into a sum yields

$$\begin{aligned} 0 &= (x - x_0)^d (q_1(y) - q_3(y)) + (y - y_0)^d (q_2(x) - q_4(x)) \\ &= \left(\sum_{\mu=0}^{d} \binom{d}{\mu} x^\mu (-x_0)^{d-\mu}\right) \left(\sum_{\nu=0}^{d} (q_{1,\nu} - q_{3,\nu}) y^\nu\right) \\ &+ \left(\sum_{\nu=0}^{d} \binom{d}{\nu} y^\nu (-y_0)^{d-\nu}\right) \left(\sum_{\mu=0}^{d} (q_{2,\mu} - q_{4,\mu}) x^\mu\right) \end{aligned}$$

$$= \sum_{\mu,\nu=0}^{d} \binom{d}{\mu} (-x_0)^{d-\mu} (q_{1,\nu} - q_{3,\nu}) x^\mu y^\nu$$

$$+ \sum_{\mu,\nu=0}^{d} \binom{d}{\nu} (-y_0)^{d-\nu} (q_{2,\mu} - q_{4,\mu}) x^\mu y^\nu$$

$$= \sum_{\mu,\nu=0}^{d} \left(\binom{d}{\mu} (-x_0)^{d-\mu} (q_{1,\nu} - q_{3,\nu}) + \binom{d}{\nu} (-y_0)^{d-\nu} (q_{2,\mu} - q_{4,\mu}) \right) x^\mu y^\nu$$

Since this equation must hold for all x and y, we have

$$\binom{d}{\mu} (-x_0)^{d-\mu} (q_{1,\nu} - q_{3,\nu}) + \binom{d}{\nu} (-y_0)^{d-\nu} (q_{2,\mu} - q_{4,\mu}) = 0 \tag{4.3}$$

for all $\mu, \nu = 0, \ldots, d$. Considering only the cases where $\nu = d$ or $\mu = d$, we obtain the $2d + 1$ homogeneous linear equations

$$\binom{d}{\mu} (-x_0)^{d-\mu} (q_{1,d} - q_{3,d}) + (q_{2,\mu} - q_{4,\mu}) = 0 \quad , \quad \mu = 0, \ldots, d-1 \tag{4.4}$$

$$(q_{1,\nu} - q_{3,\nu}) + \binom{d}{\nu} (-y_0)^{d-\nu} (q_{2,d} - q_{4,d}) = 0 \quad , \quad \nu = 0, \ldots, d-1 \tag{4.5}$$

$$(q_{1,d} - q_{3,d}) + (q_{2,d} - q_{4,d}) = 0 \tag{4.6}$$

If v is a 3-vertex, we obtain similar equations. The difference is that the non-existent smoothing cofactor and its coefficients are missing. For the vertex shown on the right-hand side of Figure 4.1, this cofactor is q_4. The following discussion will not distinguish between 3-vertices and 4-vertices when this is convenient. Equations will be written in the form which they take for 4-vertices, with the understanding that some of the variables may be zero in the case of 3-vertices.

We shall examine the homogeneous linear system of equations $\mathcal{A}$ that contains all equations of types (4.5), (4.4), and (4.6) from all interior vertices of Δ. The variables of $\mathcal{A}$ are the coefficients $q_{i,\mu}$ of the smoothing cofactors. $\mathcal{A}$ is a subsystem of the system that is obtained by collecting all equations of type (4.3). Therefore, its rank is a lower bound on the rank of the larger system.

The following argument will be used in determining the rank of $\mathcal{A}$. If A is a block-triangular matrix

$$A = \begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_M \end{pmatrix}$$

where each A_i is a rectangular block, then the rank of A can be estimated by

$$\operatorname{rank} A \geq \sum_{i=1}^{n} \operatorname{rank} A_i$$

If each block A_i has full row rank, equality holds and A has full row rank, too.

Let Q be the set of variables of $\mathcal{A}$. Suppose that we can partition $\mathcal{A}$ into non-empty subsystems $\mathcal{A}_1, \ldots, \mathcal{A}_M$ and Q into non-empty subsets $Q_1, \ldots, Q_n$ such that for all $j > i$, the equations in $\mathcal{A}_j$ do not use any variables from Q_i. If we sort the equations and variables such that $\mathcal{A}_i$ comes before $\mathcal{A}_{i+1}$ and Q_i comes before Q_{i+1}, then the corresponding matrix A is block-triangular, with block A_i corresponding to $\mathcal{A}_i$ and Q_i.

We will sort the equations and the variables of our system $\mathcal{A}$ in such a way that the corresponding matrix is block-triangular. Some of the equations are transformed by adding multiples of other equations. In particular, we will identify subsystems $\mathcal{A}_i$ for which the variables in Q_i do not appear in any other equation. It is clear that the corresponding block A_i can take any position in the ordered sequence $A_1, \ldots, A_M$.

To determine the subsystems $\mathcal{A}_i$, the equations and variables are first sorted such that they become grouped by knots and by degrees μ, ν. Let K be an interior horizontal knot that consists of m knot edges. We denote the smoothing cofactors of these knot edges by $q_1, \ldots, q_m$, numbered left-to-right. The number of conformality conditions along K is $m-1$, m, or $m+1$, depending on whether K is a knot line, knot ray, or knot segment. These conformality conditions involve $q_1, \ldots, q_m$ as well as the smoothing cofactors of the vertical knot edges that meet K. We number the vertices of K from left to right, starting with 0 for the left endpoint and ending with m for the right endpoint. In the i^{th} vertex, the smoothing cofactors of the knot edges that meet this vertex from below and above are denoted by $\underline{q}_i$ and $\overline{q}_i$, respectively.

For each vertex of K, we consider the linear equations of type (4.4). They contain coefficients of degree d from vertical knot edges and coefficients of some lower degree μ from the knot edges of K. Let x_i denote the x-coordinate of the i^{th} vertex, then these equations take the form

$$(q_{i+1,\mu} - q_{i,\mu}) + \binom{d}{\mu}(-x_i)^{d-\mu}(\underline{q}_{i,d} - \overline{q}_{i,d}) = 0 \quad , \quad \mu = 0, \ldots, d-1 \quad ,$$

$$i = \begin{cases} 1, \ldots, m-1 & \text{if } K \text{ is a knot line,} \\ 0, \ldots, m-1 \text{ or } 1, \ldots, m & \text{if } K \text{ is a knot ray,} \\ 0, \ldots, m & \text{if } K \text{ is a knot segment} \end{cases} \tag{4.7}$$

(Some $\underline{q}_{i,\mu}$ or $\overline{q}_{i,\mu}$ may be missing if K has 3-vertices. Also, $q_{0,\mu}$ and $q_{m+1,\mu}$ will be missing for $i = 0$ and $i = m$.) If we group these equations and their variables by the degree μ, then the subsystem of $\mathcal{A}$ corresponding to one group has a matrix of the form

$$\begin{pmatrix} & 1 & & & & & \\ & -1 & 1 & & 0 & & \\ & & -1 & 1 & & & \\ \cdots & & & \ddots & \ddots & & \cdots \\ & & 0 & & -1 & 1 & \\ & & & & & -1 & \end{pmatrix} \begin{matrix} \text{left endpoint} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{vertices } 1,\ldots,m-1 \\ \text{right endpoint} \end{matrix} \qquad (4.8)$$

m variables $q_{1,\mu},\ldots,q_{m,\mu}$

The first (last) row of this matrix is missing if the left (right) endpoint of K is a boundary vertex. Let A_i' denote the block shown in (4.8). For a knot line or a knot ray, A_i' has full row rank $m-1$ or m, respectively. We use $A_i := A_i'$ as a block for our block-triangular matrix. If K is a knot segment, the first m rows of A_i' have full rank. However, the whole block cannot have full row rank, since it has $m+1$ rows and only m variables. In this case, we replace the last equation by the sum of all $m+1$ equations. The variables $q_{i,\mu}$ cancel out in this sum, and the resulting equation contains only coefficients of type $\underline{q}_{i,d}$ and $\overline{q}_{i,d}$:

$$\sum_{i=0}^{m} \binom{d}{\mu} (-x_i)^{d-\mu} (\underline{q}_{i,d} - \overline{q}_{i,d}) = 0$$

The first m equations of A_i' form a block of full rank m. We let A_i be that block. The new equation in the last row of A_i' is sorted seperately from A_i and will be treated later.

In this way, we obtain a block A_i which uses the variables $q_{1,\mu},\ldots,q_{m,\mu}$. No other equation of the system $\mathcal{A}$ contains any of those variables. Therefore, our grouping leads to a block-triangular structure for the horizontal variables of degree $\mu < d$. For each horizontal knot segment, we have d equations

$$\sum_{i=0}^{m} \binom{d}{\mu} (-x_i)^{d-\mu} (\underline{q}_{i,d} - \overline{q}_{i,d}) = 0 \quad , \quad \mu = 0,\ldots,d-1$$

$$\Leftrightarrow \qquad \sum_{i=0}^{m} (-x_i)^{\mu} (\underline{q}_{i,d} - \overline{q}_{i,d}) = 0 \quad , \quad \mu = 1,\ldots,d \qquad (4.9)$$

left over.

For each vertical interior knot K of Δ, Equation (4.5) yields subsystems of the form (4.8). For each degree ν, the block A_i' has row rank $m-1$ in case of a knot line and m otherwise. We determine the block A_i in the same way as above, and add the m first equations to the last equation in case of a knot segment. As before, the variables $q_{1,\nu},\ldots,q_{m,\nu}$, which are now associated with vertical knot edges, are exclusive to A_i.

Collecting the $(m+1)^{\text{st}}$ equations that correspond to one knot segment and all degrees ν, we obtain the following analogon of Equation (4.9):

$$\sum_{i=0}^{m}(-y_i)^\nu(\overleftarrow{q}_{i,d} - \overrightarrow{q}_{i,d}) = 0 \quad , \quad \nu = 1, \ldots, d \tag{4.10}$$

where y_i is the y-coordinate of the i^{th} vertex of K and the coefficients $\overleftarrow{q}_{i,d}$ and $\overrightarrow{q}_{i,d}$ are associated with the knot edges to the left and right, respectively, of this vertex. We transform Equation (4.10) further, using the equations of type (4.6) that are associated with the vertices of K. Multiplying those equations with the factors $(-y_i)^\nu$ yields

$$\begin{aligned}
(-y_0)^\nu\,(-q_{1,d} + (\overleftarrow{q}_{0,d} - \overrightarrow{q}_{0,d})) &= 0 \\
(-y_i)^\nu\,((q_{i,d} - q_{i+1,d}) + (\overleftarrow{q}_{i,d} - \overrightarrow{q}_{i,d})) &= 0 \quad , \quad i = 1, \ldots, m-1 \\
(-y_m)^\nu\,(q_{m,d} + (\overleftarrow{q}_{m,d} - \overrightarrow{q}_{m,d})) &= 0
\end{aligned}$$

for all $\nu = 1, \ldots, d$. (Note that the first and last equations do exist, since K is a knot segment.) Adding these equations to (4.10) we obtain

$$(-y_0)^\nu q_{1,d} - \sum_{i=1}^{m-1}(-y_i)^\nu(q_{i,d} - q_{i+1,d}) - (-y_m)^\nu q_{m,d} = 0 \quad , \quad \nu = 1, \ldots, d \tag{4.11}$$

Note that both Equations (4.9) and (4.11) contain only coefficients belonging to vertical knot edges.

Now we group the equations of type (4.6) by horizontal knots to obtain a matrix of the form (4.8). Again, we have a block A_i of full row rank $m-1$ or m for each knot, and for each knot segment one additional equation of type (4.9), with $\mu = 0$. The blocks use variables from horizontal knot edges, so they do not share any variables with the equations of types (4.9) and (4.11). They are also variable-disjoint with each other, since each block corresponds to a different knot.

Finally, we gather the equations of types (4.9) and (4.11) in one large block. The matrix A of $\mathcal{A}$ now has the form shown in Figure 4.2. The last block A_M contains $d+1$ equations of type (4.9) for each horizontal knot segment and d equations of type (4.11) for each vertical knot segment. In order to determine the rank of A_M, we perform a change of variables. The variables used in A_M are the degree-d coefficients $q_{i,d}$ that belong to interior vertical knot edges. Along each vertical knot with cofactors $q_1, \ldots, q_m$, let

$$\begin{aligned}
\gamma_m &:= q_{m,d} \quad \text{and} \\
\gamma_i &:= q_{i,d} - q_{i+1,d} \quad , \quad i = 1, \ldots, m-1
\end{aligned}$$

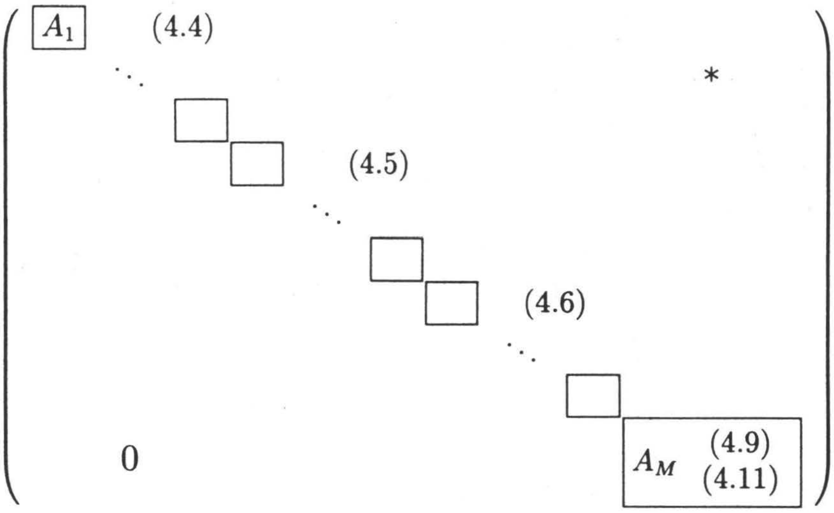


FIGURE 4.2. Subsystem $\mathcal{A}$ of conformality conditions after sorting

Each γ_i is the difference of the degree-d coefficients from the two vertical knot edges that meet in the i^{th} vertex. Note that the lower endpoint, vertex number 0, has no γ-variable associated with it. Thus, each interior vertex corresponds to a γ-variable unless it is the lower endpoint of a knot. The transition from the $q_{i,d}$ to the γ_i does not change the rank of the block.

Let $\tilde{A}_M$ be A_M expressed with respect to the new variables. Equation (4.9) becomes

$$\sum_{i=0}^{m}(-x_i)^{\mu}\gamma_i = 0 \quad , \quad \mu = 0, \ldots, d$$

provided that none of the vertices of the associated horizontal knot K_{h} is the lower endpoint of a vertical knot K_{v}. Otherwise, we will regard the corresponding γ_i as an abbreviation for the sum of all γ-variables that occur along K_{v}. The corresponding matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ (-x_0) & \cdots & (-x_m) \\ \vdots & & \vdots \\ (-x_0)^d & \cdots & (-x_m)^d \end{pmatrix} \tag{4.12}$$

has $m + 1$ columns. We have required that $m \geq d + 1$ for each knot segment. The first $d+1$ columns of (4.12) form the transpose of a $(d+1)$-by-$(d+1)$ Vandermonde's matrix, which is well-known to have full rank. The variables $\gamma_0, \ldots, \gamma_d$ that correspond to the first $d + 1$ columns are associated with the leftmost $d + 1$ vertices of K_{h}. Since statements (1) and (2) of Lemma 3.4 hold, these vertices cannot be lower endpoints of any vertical knots, so $\gamma_0, \ldots, \gamma_d$ are truly variables, not abbreviations.

This shows that the square block formed by the first $d+1$ columns of the matrix has full rank. Equation (4.11) becomes

$$\begin{aligned} 0 &= (-y_0)^\mu \sum_{i=1}^{m} \gamma_i - \sum_{i=1}^{m} (-y_i)^\mu \gamma_i \\ &= \sum_{i=1}^{m} \left((-y_0)^\mu - (-y_i)^\mu \right) \gamma_i \quad , \quad \mu = 1, \ldots, d \end{aligned}$$

with matrix

$$\begin{pmatrix} (-y_0) - (-y_1) & \cdots & (-y_0) - (-y_m) \\ \vdots & & \vdots \\ (-y_0)^d - (-y_1)^d & \cdots & (-y_0)^d - (-y_m)^d \end{pmatrix} \tag{4.13}$$

We take the first d columns of this matrix and consider their determinant

$$\begin{aligned} &\det \begin{pmatrix} (-y_0) - (-y_1) & \cdots & (-y_0) - (-y_d) \\ \vdots & & \vdots \\ (-y_0)^d - (-y_1)^d & \cdots & (-y_0)^d - (-y_d)^d \end{pmatrix} \\ &= (-1)^d \det \begin{pmatrix} (-y_1) - (-y_0) & \cdots & (-y_d) - (-y_0) \\ \vdots & & \vdots \\ (-y_1)^d - (-y_0)^d & \cdots & (-y_d)^d - (-y_0)^d \end{pmatrix} \\ &= (-1)^d \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ (y_0) & (-y_1) - (-y_0) & \cdots & (-y_d) - (-y_0) \\ \vdots & \vdots & & \vdots \\ (y_0)^d & (-y_1)^d - (-y_0)^d & \cdots & (-y_d)^d - (-y_0)^d \end{pmatrix} \\ &= (-1)^d \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ (y_0) & (-y_1) & \cdots & (-y_d) \\ \vdots & \vdots & & \vdots \\ (y_0)^d & (-y_1)^d & \cdots & (-y_d)^d \end{pmatrix} \\ &\neq 0 \end{aligned}$$

If we expand the third determinant by its first row, we can see that it is equal to the second determinant. The last matrix in the sequence is the transpose of a Vandermonde's matrix, whose determinant is non-zero. This means that the first d columns of matrix (4.13), which form a square block, have full rank. Their variables $\gamma_0, \ldots, \gamma_{d-1}$ are associated with the topmost d vertices of a vertical knot segment.

In order to show that the block $\tilde{A}_M$ has full row rank, we arrange it in block-triangular fashion. The sub-blocks in this arrangement will consist of the first $d+1$ or d columns of matrices of type (4.12) or (4.13), respectively. First, we construct an ordering $K_1, \ldots, K_N$ of all knot segments such that the leftmost $d+1$ (or topmost d) vertices of K_i are not intersected by any K_j with $j > i$. We choose K_1 in the following way: Let K_{top} be a horizontal knot segment such that no other horizontal knot segment lies at a higher y-coordinate. If the leftmost $d+1$ vertices of K_{top} are not intersected by vertical knot segments, let $K_1 := K_{\text{top}}$. Otherwise, let K_{left} be the leftmost of the intersecting knot segments. By statements (1) and (2) of Lemma 3.4, K_{left} has $d+1$ vertices above K_{top}. The choice of K_{top} ensures that those vertices cannot be intersected by any knot segments. We let $K_1 := K_{\text{left}}$. In the same way, we choose K_2 from the remaining knot segments and so on, until the ordering of all knot segments is determined.

There is a one-to-one and onto correspondance between our sub-blocks and the knot segments of Δ. If we sort the sub-blocks (i.e., the rows and columns of $\tilde{A}_M$ in which the sub-blocks lie) according to the ordering $K_1, \ldots, K_N$ of the knot segments, we can see the following: If a γ-variable of the i^{th} sub-block appears in one of the equations that contain the j^{th} sub-block, then K_i and K_j intersect, and the intersection point is one of the first $d+1$ vertices of K_i. Now the ordering is such that this cannot happen for $j > i$, so we have in fact arranged $\tilde{A}_M$ in a block-triangular fashion. Since each sub-block has full row rank, so has $\tilde{A}_M$.

The above shows that the system $\mathcal{A}$ has full (row) rank. For each interior vertex, $\mathcal{A}$ has $2d+1$ equations, so

$$\operatorname{rank} \mathcal{A} = (2d+1)\,|V_I|$$

where $|V_I|$ denotes the number of interior vertices. This rank is a lower bound on the rank of the complete system of all equations of type (4.3). The variables of this system are the coefficients of all smoothing cofactors. If $|E_I|$ is the number of interior knot edges of Δ, then the complete system has $(d+1)\,|E_I|$ variables. The dimension of its solution space is bounded above by the number

$$(d+1)\;|E_I| - \operatorname{rank} \mathcal{A} = (d+1)\;|E_I| - (2d+1)\;|V_I|$$

We obtain the following upper bound on the dimension of the spline space $S_{d\times d}(\Delta)$:

Proposition 4.1. *Let Δ be a rectangular partition whose interior edges have multiplicity* 1, *let each knot segment have at least $d+2$ vertices, and let statement* (1) *or* (2) *of Lemma* 3.4 *hold for any two knots of Δ that intersect in an interior vertex. Then the dimension of the maximal spline space $S_{d\times d}(\Delta)$ over Δ is bounded by*

$$\dim S_{d\times d}(\Delta) \leq (d+1)^2 + (d+1)\;|E_I| - (2d+1)\;|V_I| \tag{4.14}$$

If the partition Δ is induced by some semi-regular basis $\mathfrak{B}$, then span $(\mathfrak{B})$ is a subspace of $S_{d\times d}(\Delta)$, and the number $|\mathfrak{B}|$ of B-splines in $\mathfrak{B}$ yields a lower bound on the dimension of $S_{d\times d}(\Delta)$. We now proceed to determine $|\mathfrak{B}|$ from the topology of Δ.

Since $\mathfrak{B}$ is semi-regular, each cell of Δ supports exactly $(d+1)^2$ B-splines. On the other hand, each $B_k \in \mathfrak{B}$ that is sufficiently remote from the boundary is non-zero over at least $(d+1)^2$ cells. (We are still considering the case where all interior knot edges have multiplicity 1.) Functions that use the boundary knots with multiplicity ≥ 2 are in general non-zero over less than $(d+1)^2$ cells. In order to treat these cases, we expand each boundary knot edge into d pseudo-cells and assume that after the expansion all knot edges have multiplicity 1. Likewise, the four corner vertices are expanded into $d \times d$ pseudo-cells. Figure 4.3 shows pseudo-cells and the numbers of

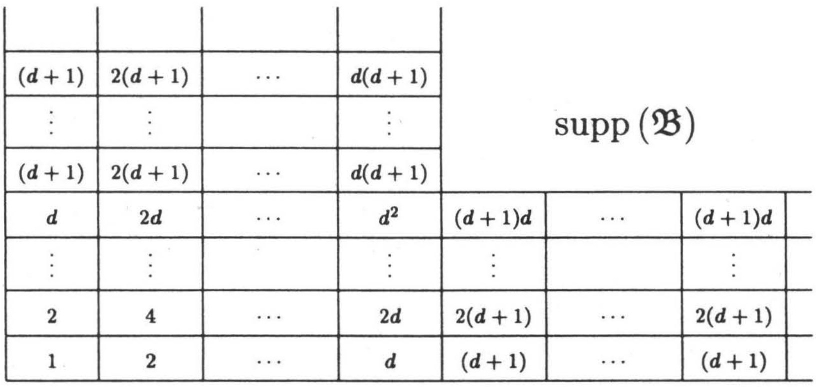


FIGURE 4.3. Pseudo-cells associated with boundary edges and a corner vertex

B-splines that we count for each of them. Each boundary knot edge expands into d pseudo-cells with a B-spline count of

$$\sum_{i=1}^{d} i(d+1) = \tfrac{1}{2}d(d+1)^2$$

For the d^2 pseudo-cells of a corner vertex, we count

$$\sum_{i=1}^{d} i \sum_{j=1}^{d} j = \tfrac{1}{4}d^2(d+1)^2$$

B-splines. If we include the pseudo-cells in our count, then each B-spline is non-zero over at least $(d+1)^2$ cells.

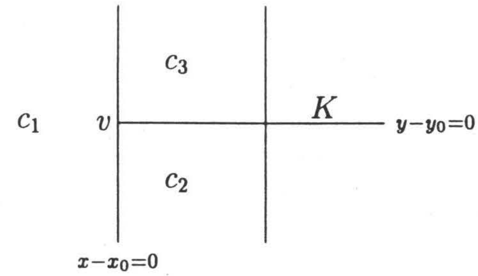


FIGURE 4.4. An interior 3-vertex

Interior 3-vertices increase our cell count. Consider the left endpoint $v = (x_0, y_0)$ of knot K in Figure 4.4. The B-splines that are non-zero over c_2 must share one horizontal knot vector $X = (x_1, \dots, x_{2d+2})$, so they are given as

$$\left\{B_{ij}(x, y) = N_i(X|x)N_j(Y^{(i)}|y)\right\}_{i,j=1,\dots,d+1}$$

where $X = (x_1, \dots, x_{2d+2})$ is such that $x_{d+1} = x_0$. Out of the vertical knot vectors $Y^{(i)}$, only $Y^{(d+1)}$ has y_0 as a knot value. For $i = 1$, the B-splines B_{1j} do not induce the knot edge between c_2 and c_3. When we said that B_{1j} is non-zero over $\geq (d+1)^2$ cells, c_2 and c_3 were counted as one single cell, so we need to increase the count by 1. The knot edge that bounds c_2 to the right in Figure 4.4 must lie on the line $x - x_{d+2} = 0$, since otherwise it would have to be induced by some B-spline that is non-zero over c_2 and does not share the knot vector X. Thus supp (B_{1j}) does not extend to the right of c_2, and K increases the cell count of B_{1j} by exactly 1. Analogous considerations for $i = 2, \dots, d$ (but not $i = d + 1$) show that K increases the cell count of B_{ij} by exactly i, for all $j = 1, \dots, d + 1$, so in total we get

$$(d+1)\sum_{i=1}^{d} i = \tfrac{1}{2}d(d+1)^2$$

more cells per interior 3-vertex.

Whenever some B_k is non-zero over more than $(d+1)^2$ (pseudo-)cells, there exists a knot K such that K intersects the interior of supp (B_k) and the intersecting part of K is not induced by B_k. Because of Condition (2) of Definition 3.1 (non-degeneracy,) this can only be the case if an endpoint of K lies in the interior of supp (B_k). This endpoint is an interior 3-vertex, and we have increased the cell count appropriately.

We have counted for each B_k the number of (pseudo-)cells over which it is non-zero, and for each (pseudo-)cell c the number of B-splines that are non-zero over c. The two counts must equal, so we have

$$\begin{aligned}
&(d+1)^2\,|\mathfrak{B}| + \tfrac{1}{2}d(d+1)^2\,|V_3| \\
&\quad = (d+1)^2\,|C| + 4\tfrac{1}{4}d^2(d+1)^2 + \tfrac{1}{2}d(d+1)^2\,|E_B| \\
\Leftrightarrow\quad & |\mathfrak{B}| = |C| + d^2 + \tfrac{1}{2}d(|E_B| - |V_3|) \qquad (4.15)
\end{aligned}$$

where $|C|$ denotes the number of genuine cells of Δ, $|E_B|$ the number of boundary edges, and $|V_3|$ the number of interior 3-vertices.

Each cell is incident on at least 4 knot edges. Each interior 3-vertex increases the edge count of one particular cell by 1. On the other hand, each interior knot edge is incident on two cells, each boundary edge on one cell. Therefore,

$$4\,|C| + |V_3| = |E_B| + 2\,|E_I|$$

where $|E_I|$ is the number of interior knot edges. We can use this to eliminate $|C|$ from Equation (4.15):

$$\begin{aligned}
|\mathfrak{B}| &= \tfrac{1}{4}(|E_B| + 2\,|E_I| - |V_3|) + d^2 + \tfrac{1}{2}d(|E_B| - |V_3|) \\
&= \tfrac{1}{2}\,|E_I| + d^2 + \tfrac{1}{4}(2d+1)(|E_B| - |V_3|) \qquad (4.16)
\end{aligned}$$

Each edge is incident on 2 vertices. There are as many boundary vertices as there are boundary edges. Four of the boundary vertices are 2-vertices, the others are 3-vertices. Thus, we have

$$\begin{aligned}
& 2(|E_I| + |E_B|) = 4\,|V_4| + 3\,|V_3| + 8 + 3(|E_B| - 4) \\
\Leftrightarrow\quad & 2\,|E_I| = 4\,|V_4| + 3\,|V_3| + |E_B| - 4
\end{aligned}$$

where $|V_4|$ is the number of 4-vertices. Solving for $|E_B|$ and substituting into (4.16) yields

$$\begin{aligned}
|\mathfrak{B}| &= \tfrac{1}{2}\,|E_I| + d^2 + \tfrac{1}{4}(2d+1)(2\,|E_I| + 4 - 4\,|V_4| - 3\,|V_3| - |V_3|) \\
&= (d+1)\,|E_I| + d^2 + 2d + 1 - (2d+1)(|V_4| + |V_3|) \\
&= (d+1)\,|E_I| + (d+1)^2 - (2d+1)\,|V_I| \qquad (4.17)
\end{aligned}$$

This lower bound on the dimension of $S_{d\times d}(\Delta)$ is equal to the upper bound derived in Equation (4.14).

Proposition 4.2. *Let $\mathfrak{B}$ be a semi-regular collection of B-splines of degree $d \times d$. If all interior edges of the domain partition Δ induced by $\mathfrak{B}$ have multiplicity 1, then $\mathfrak{B}$ is a basis of the maximal tensor-product spline space $S_{d\times d}(\Delta)$ over this partition, and*

$$\dim S_{d\times d}(\Delta) = (d+1)^2 + (d+1)\,|E_I| - (2d+1)\,|V_I|$$

where $|E_I|$ *is the number of interior edges and* $|V_I|$ *is the number of interior vertices of* Δ.

Proof. Since $\mathfrak{B}$ is semi-regular, both the upper bound in Equation (4.14) and the lower bound in Equation (4.17) hold for $S_{d\times d}(\Delta)$. The two bounds are equal, so they give the exact dimension of the space. Furthermore, $\dim S_{d\times d}(\Delta)$ equals the dimension $|\mathfrak{B}|$ of the subspace span $(\mathfrak{B})$. The two spaces must therefore be identical, and $\mathfrak{B}$ is a basis for them. □

5. Conclusion

Semi-regular collections of B-splines lead to a generalization of the traditional tensor-product spline spaces. They consist of individual tensor-product B-splines, all of the same degree $d_x \times d_y$, but using individual knot vectors. A certain amount of knot sharing between the B-splines is described by the semi-regularity condition. As a result, the collections are linearly independent and possess some favorable properties of the traditional B-spline surface scheme. A spline function represented over a semi-regular basis can be evaluated by de Boor's algorithm. The representation is affine invariant, and the strong convex hull property holds. Conversion to the traditional B-spline representation is possible for purposes of data exchange and for compatibility with existing CAD/CAM systems.

The main advantage of semi-regular bases lies in the increased flexibility of the knot structure. Knot segments and knot rays allow a truly local knot refinement. It is possible to model derivative discontinuities which are locally confined by virtue of the chosen basis.

Semi-regularity imposes some topological restrictions on the resulting domain partition Δ. In particular, all cells must be rectangular. If all interior knot edges have multiplicity 1, then the semi-regular basis spans all possible spline functions over Δ. In this case, the dimension of the spline space can be determined from the topology of the partition.

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