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HYPERIDENTITIES

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Introduction

The concept of a free algebra plays an essential role in universal algebra and in computer science. Manipulation of terms, calculations and the derivation of identities are performed in free algebras. Word problems, normal forms, system of reductions, unification and finite bases of identities are topics in algebra and logic as well as in computer science.

A very fruitful point of view is to consider structural properties of free algebras. A.I. Malcev initiated a thorough research of the congruences of free algebras. Henceforth congruence permutable, congruence distributive and congruence modular varieties are intensively studied. A lot of Malcev type theorems are connected to the congruence lattice of free algebras.

Here we consider free algebras as semigroups of compositions of terms and more specific as clones of terms. The properties of these semigroups and clones are adequately described by hyperidentities. Naturally a lot of theorems of "semigroup" or "clone" type can be derived.

This topic of research is still in its beginning and therefore a lot of concepts and results cannot be presented in a final and polished form. Furthermore a lot of problems and questions are open which are of importance for the further development of the theory of hyperidentities.

Dla Magdy, Dagusi i Krzysia.

Preliminaries

An identity is a pair of terms where the variables are bound by the all quantifier. Let us take the following medial identity as an example

$$\forall u \forall x \forall y \forall w \quad (u \cdot x) \cdot (y \cdot w) = (u \cdot y) \cdot (x \cdot w)$$

An identity can be considered as a notion in a first order language with equality.

Let us look at the following hyperidentity

$$\forall F \forall u \forall x \forall y \forall w \quad F(F(u,x),F(y,w)) = F(F(u,x),F(y,w))$$

A hyperidentity can be considered as a notion in a second order language with equality. A second order language allows the quantification of predicate or operator variables. We consider the operator variables F in a very specific way. Primarily all our operator variables are restricted to functions of a given arity; in our example to binary functions. Secondly we bind the interpretation of F to term functions. Therefore such kind of operator variables are called hypervariables. As it is common use we will not write quantifiers in front of identities and hyperidentities.

Let us consider the variety of distributive lattice. Then the list of all binary terms consists of

$$e_1^2(x,y) = x,\ e_2^2(x,y) = y,\ x \wedge y, x \vee y.$$

Let us replace the binary hypervariable F in the above hyperidentity by a binary term leaving the variables unchanged. For $x \wedge y$ we get the identity

$$(u \wedge x) \wedge (y \wedge w) = (u \wedge y) \wedge (x \wedge w)$$

which holds for the variety of distributive lattices. In the other three cases we get the identities

$$u = u,\ w = w \text{ and } (u \vee x) \vee (y \vee w) = (u \vee y) \vee (x \vee w)$$

which also hold for distributive lattices. We say that the above hyperidentity holds in the variety D of distributive lattices as every binary term yields an identity which holds in D. Let us take another example: the variety of abelian groups (where we write the operation as an addition). Every binary term, for instance $((x+y)+(x+x))+y$ can be written in a normal form $ax + by$, $a,b \in \mathbb{N}_0$; in our case $3x+2y$. The above hyperidentity holds for the variety of abelian groups because it is transformed to the following identity interpreting F.

$$a(au+bx) + b(ay+bw) = a(au+by) + b(ax+bw)$$

The reader will recognize that we consider only a small fragment of a second order logic. Of course all these restrictions reduce the expressive power of a second order language. But nevertheless, by hyperidentities one can express more than by identities.

Part I. Hyperidentities

§ 1.1 Hyperterms

The four sections §1.1 – §1.4 are due to [Graczynka–Schweigert]

Notations. Our nomenclature is basically the same as in [Grätzer 79]. We consider varieties of algebras of a given type. A type of algebras τ is a sequence $(n_0,n_1,\dots,n_\gamma,\dots)$ of positive integers $\gamma < 0(\tau)$, where $0(\tau)$ is an ordinal, called the order of τ. For every $\gamma < 0(\tau)$ we have a symbol f_γ for an n_γ–ary operation. Moreover, for every γ with $n_\gamma > 0$ the symbol F_γ is associated. F_γ is called an n_γ–ary hypervariable.

Definition 1.1.1. *Let τ be a given type. The n–ary hyperterms of type τ are recursively defined by:*

(1) *the variables $x_1,\ldots,x_n$ are n–ary hyperterms;*

(2) *if $T_1,\ldots,T_m$ are n–ary hyperterms and F is an m–ary hypervariable of type τ, then $F_\gamma(T_1,\ldots,T_m)$ is an n–ary hyperterm of type τ.*

$H^n(\tau)$ is the smallest set containing (1) which is closed under finite application of (2).

$H(\tau) = \bigcup(H^n(\tau)\colon n \in \mathbb{N})$ is called the set of all hyperterms of type τ (where $\mathbb{N}$ is the set of all positive integers).

A hyperidentity of type τ is a pair of hyperterms (T_1,T_2), which is also denoted by $T_1 = T_2$.

The free algebra in countably many variables of a variety V of type τ is denoted by $T(V)$ and its elements t are called terms. If V is generated by the algebra $\mathbf{A}$ we write $T(\mathbf{A})$ instead of $T(V)$.

Definition 1.1.2. *Let (T_1,T_2) be a hyperidentity of type τ and let V be a variety of type μ. If every n_γ–ary hypervariable occurring in (T_1,T_2) is replaced by an n_γ–ary term $t_\gamma \in T(V)$ leaving the variables $(x_\gamma : i \in N)$ unchanged in (T_1,T_2) then the resulting identity (t_1,t_2) is called a transformation of the hyperidentity (T_1,T_2).*

For a more formal definition consider 1.7.19

Example. Let $F\Big[F(u,x),\ F(y,v)\Big] = F\Big[F(u,y),F(x,v)\Big]$ be a hyperidentity of type (2). Let V be the variety of abelian groups $(G\colon +, -, 0)$ of type (0,1,2). Then $(u+x)+(y+v) = (u+y)+(x+v)$ is a transformation of the above hyperidentity. Let $ax+by$, $a,b \in \mathbb{Z}$ be a binary term of $T(V)$. Then

$$a(au+bx) + b(ay+bv) = a(au+by) + b(ax+bv)$$

is another example of a transformation of the above hyperidentity.

If E is a set of hyperidentities of type τ, then the set of all transformations of E for a variety V of type τ is denoted by $I_V(E)$.

Definition 1.1.3. *A variety* V *of type* μ *satisfies the hyperidentity* (T_1,T_2) *of type* τ *if the set* $I_V((T_1,T_2))$ *of all transformations of* (T_1,T_2) *is contained in the set of identities which hold for* V.

Example. The hyperidentity $F\left[F(u,x),\ F(y,v)\right] = F\left[F(u,y),F(x,v)\right]$ is satisfied by the variety of abelian groups.

Definition 1.1.4. *Let* (t_1,t_2) *be an identity which holds for a variety* V. *If every* n_γ–ary *operation symbol* f_γ *occurring in* (t_1,t_2) *is substituted by* n_γ–ary *hypervariable* F_γ *leaving the variables unchanged then the resulting hyperidentity* (T_1,T_2) *is called the transformation of* (t_1,t_2).

If Σ is a set of identities of the variety V of type μ, then $H_\mu(\Sigma)$ denotes the set of all transformations of identities in Σ.

Example. Let V be a variety of lattices of type (2,2). Let ε be the identity $x = x \vee x$. Then the transformation $H_\mu(\varepsilon)$ which equals $x = F(x,x)$ is the hyperidentity which holds for the variety of lattices. On the other hand the identity ε' which is of the form $x \vee y = y \vee x$ is transformed to the hyperidentity $H_\mu(\varepsilon')$ of the form $F(x,y) = F(y,x)$ which does not hold for a nontrivial variety V of lattices.

§ 1.2 Completeness

Definition 1.2.1. Following G. Birkhoff (comp. [Grätzer 79], [Taylor 79]), we use the following *rules of derivation for hyperidentities of a given type* τ*:*

(1) $T_1 = T_1$ *for every hyperterm* $T_1 \in H(\tau)$;

(2) $T_1 = T_2$ *implies* $T_2 = T_1$, *for hyperterms* $T_1, T_2 \in H(\tau)$;

(3) $T_1 = T_2$, $T_2 = T_3$ *implies* $T_1 = T_3$ *for hyperterms* $T_1, T_2, T_3 \in H(\tau)$;

(4) $T_i = S_i$ *for* $i = 1,\dots,m_\gamma$, *implies* $F_\gamma(T_1,\dots,T_{m_\gamma}) = F_\gamma(S_1,\dots,S_{m_\gamma})$ *for hyperterms* $T_i, S_i \in H(\tau)$ *and* m_γ–ary *hypervariables* F_γ.

(5) $T(x_1,\dots,x_n) = S(x_1,\dots,x_n)$ *implies* $T(R_1,\dots,R_n) = S(R_1,\dots,R_n)$ *for* $T, S, R_1,\dots,R_n \in H(\tau)$.

Remark 1.2.2. If one considers $H = \bigcup(H(\tau) : \tau \in \mathbb{Q})$, where $\mathbb{Q}$ is the set of all well ordered sequences then the above rules hold for hyperidentities in general.
In the sequel we shall use also an analogous rule to (5), but for hypervariables. This was the main idea of [Belousov 65] (comp. [Aczél 71], [Taylor 81]).
First, we recursively define the notion of a substitution of a hypervariable by a hyperterm.
Let T be a hyperterm of type τ. Consider a hypervariable F_{γ_1}, and a hyperterm R_1 of type τ, both of the arity m. We define the term T^*, called the substitution of the hypervariable F_{γ_1} in the term T by the hyperterm R_1, as follows:

(1^0) If T is a variable, then T^* is equal to T;

(2^0) If T has the form $F_\gamma(T_1,\dots,T_m)$ then T^* has the form:

$$R_1(\overset{*}{T}_1,\dots,\overset{*}{T}_n) \text{ if } \gamma = \gamma_1;$$

and

$$F_\gamma(\overset{*}{T}_1,\dots,\overset{*}{T}_n) \text{ if } \gamma \neq \gamma_1.$$

The rule (6) is called a hypersubstitution, and is defined in the following way:

(6) $T_1 = T_2$ *implies* $\overset{*}{T}_1 = \overset{*}{T}_2$ *for any* $T_1, T_2 \in H(\tau)$ *and any simultaneous hypersubstitution of hypervariables in* T_1 *and* T_2 *by a hyperterm of the same arity.*

Example. Consider the hyperidentity $Q(Q(x,y,z),y,z) = Q(x,y,z)$, and the hyperterm $T(x,y,z) = F(G(x,y),z)$. By rule (6) we derive $F(G(F(G(x,y),z),y),z) = F(G(x,y),z)$. The latter hyperidentity of type (2,2) is also called a hyperconsequence of type (2,2) from the former hyperidentity of type (3).

Remark 1.2.3. Note that rule (6) commutes with all rules of derivation (1)–(5) (i.e. if Σ is a set of identities closed under the rule (6) then all consequences of Σ by the rules (1)–(6) are consequences of Σ by the rules (1)–(5)).

Given a variety V of type τ, Id(V) denotes the set of all identities satisfied in V (see [Grätzer 79], p. 169, 170). $E_\tau(V)$ denotes the set of all hyperidentities of type τ which are satisfied by the variety V. Furthermore E(V) denotes the set of all hyperidentities of any type which hold for V. Obviously, $E_\tau(V) \subseteq E(V)$. Furthermore, if $V_1 \subseteq V_2$ then $E_\tau(V_2) \subseteq E_\tau(V_1)$.
$H_\tau(\Sigma)$:= set of all transformation of identites in Σ to hyperidentitites of type τ which may or may not hold in V.

Proposition 1.2.4. *Let V be a nontrivial variety of lattices of type* (2,2). *Then* $E_\tau(V) \subsetneq E(V)$.

Proof. In [Penner 81] it is proved that for any positive integer m there exists a hyperidentity (T_1, T_2) which is satisfied in V but does not follow from hyperidentities involving at most m–ary hypervariables. For m = 2 we have the statement of 1.2.4.

Definition 1.2.5. *The set* C *of all varieties* V *of type* μ *which satisfy a set* E *of hyperidentities of type* τ *is called a hypervariety* C *of type* (τ,μ)*. We say that* E *defines* C.

If $\tau = \mu$ *then* C *is called a hypervariety of type* τ.

Completeness Theorem. *A set* Σ *of hyperidentities of type* τ *can be represented in the form* $E_\tau(K)$, *for some variety* K *of type* τ, *if and only if* Σ *is closed under rules (1)–(6).*

Proof. This theorem is a slight modification of G. Birkhoff's theorem (see [Birkhoff 35]) for sets of identities. The proof is similar to that of [Grätzer 79], p. 171. Obviously the set of hyperidentities of type τ of variety K must be closed under rules (1)–(6).

Take a set Σ of hyperidentities of type τ, closed under rules (1)–(6). Consider the set $I_V(\Sigma)$ of identities of types τ for a fixed variety V of type τ. Then the set $\Sigma_1 = I_V(\Sigma)$ is closed under the rules of inference (i)–(v) of [Grätzer 79], p. 170. Consider the variety K of type τ, constructed as in [Grätzer 79], p. 171. Then Σ_1 is the set of identities of K. Moreover $\Sigma = H_\tau(\Sigma_1) = E_\tau(K)$, because of the assumption that Σ is closed under the rule (6).

§ 1.3 Solid varieties

We say that a hyperidentity is satisfied by an algebra **A**, if it is satisfied in the variety generated by **A**.
An algebra **A** is solid if every identity satisfied in **A** is transformed into a hyperidentity, which is satisfied in **A**.

Definition 1.3.1. *Let* Σ *be the set of all identities, which hold for the variety* V *of type* τ. V *is called solid if* $E_\tau(V) = H_\tau(\Sigma)$.

Theorem 1.3.2. *Let* Σ *be the set of all identities of the variety* V *of type* τ. V *is solid if and only if* $\Sigma = I_V(E_\tau(V))$.

Proof. Note, that by definition we have:

(1⁰) $E_\tau(V) \subseteq H_\tau(\Sigma)$;

(2⁰) $I_V(E_\tau(V)) \subseteq \Sigma$.

To prove the necessity, assume that V is a solid variety. Let ε be an identity from Σ. By definition 1.3.1, the transformation $H_\tau(\varepsilon)$ is a hyperidentity of type τ, satisfied in V. We conclude that $\varepsilon \in I_V(H_\tau(\varepsilon)) \subseteq I_V(E_\tau(V))$ and thus $I_V(E_\tau(V)) = \Sigma$, by (2⁰).
For sufficiency, assume that we have $I_V(E_\tau(V)) = \Sigma$. By (1⁰) we need only to prove the inclusion $H_\tau(I_V(E_\tau(V)) \subseteq E_\tau(V)$. To show this, take a hyperidentity E from the set $E_\tau(V)$ of all hyperidentities of V. Then $I_V(E)$ is contained in Σ. Now consider $H_\tau(I_V(E))$. Any element of $H_\tau(I_V(E))$ can be obtained as an element of the closure of the set $\{E\}$ by rule (6), which is contained in the set $E_\tau(V)$ – closed under (6), by the completeness theorem. Thus we conclude that V is a solid variety.

The above results also hold if we restrict us to bases of hyperidentities and identities.

Remark 1.3.3. The completeness theorem can be reformulated in the following way:

Let Σ be a set of hyperidentities of type τ. The following conditions are equivalent:

(1) Σ is closed under the rules (1)–(6).

(2) $\Sigma = H_\tau(Id(K))$ for some solid variety K of type τ.

Theorem 1.3.4. *A variety* V *of type* τ *is solid if and only if it is closed under the condition:*

(1.3.4) *Let* **A** *be an algebra of* V, *of type* $\tau = (n_1, n_2, \ldots, n_\gamma, \ldots : \gamma < 0(\tau))$. *If* t_γ *is the realization of an* n_γ–ary *term operation of type* τ *in* **A**, *then* $\mathbf{A} = (A; t_1, t_2, \ldots, t_\gamma, \ldots : \gamma < 0(\tau))$ *is an algebra of* V.

Proof. Let V be a solid variety. Consider the algebra $\mathbf{A} = (A; t_1, t_2, \ldots, t_\gamma, \ldots : \gamma < 0(\tau))$. The identities of V are transformed into hyperidentities of V and hence hold for the term functions t_γ. Especially they hold for **A**. Hence $\mathbf{A} \in V$. Let the condition (1.3.4) hold for V. Then the identities of V hold for all term functions of the suitable arity and hence are transformed into hyperidentities, i.e. V is a solid variety.

§ 1.4 Derived Algebras

Notation 1.4.1. Let K be a class of algebras of a given type $\tau = n_0, n_1, \ldots, n_\gamma, \ldots)$. The algebra **B** is called a derived algebra of $\mathbf{A} = (A; f_0, f_1, \ldots, f_\gamma, \ldots.)$ if there exists term operations $t_0, t_1, \ldots, t_\gamma, \ldots$ of type τ such that $\mathbf{B} = (A; t_0, t_1, \ldots, t_\gamma, \ldots)$. For a class K of algebras of type τ we denote by D(K) the class of all derived algebras of type τ of K. We use the closure operator D to reformulate theorem 1.3.4.

Theorem. *Let* V *be a class of algebras of a given type* τ. V *is a solid variety if and only if* V *is closed under homomorphic images* H, *subalgebras* S, *direct products* P *and derived algebras* D, *i.e.*

$$H(V) \subseteq V; \quad S(V) \subseteq V; \quad P(V) \subseteq V; \quad D(V) \subseteq V.$$

Problem 1.4.2. *Describe the semigroup generated by the operators* H,S,P,D. Compare [Pigozzi]

Theorem 1.4.3. *Let* V *be a class of algebras of given type* τ. V *is a solid variety if and only if* V = HSPD(V).

Proof.

(a) $DP(V) \subseteq PD(V)$. For $\mathbf{B} \in DP(V)$ we have $\mathbf{B} = (A; t_0, t_1, \ldots, t_\gamma, \ldots)$ with $\mathbf{A} = (A; f_0, f_1, \ldots, f_\gamma, \ldots.)$ and $\mathbf{A} = \Pi \mathbf{A}_i$, $\mathbf{A}_i = (A_i; f_0, f_1, \ldots, f_\gamma, \ldots.)$. Consider $\mathbf{B}_i := (A_i; t_0, t_1, \ldots, t_\gamma, \ldots)$ then we have $\mathbf{B} = \Pi \mathbf{B}_i$ and hence $\mathbf{B} \in PD(V)$.

(b) $DS(V) \subseteq SD(V)$. For $\mathbf{B} = (B; t_0, t_1, \ldots, t_\gamma, \ldots) \in DS(V)$ we have $\mathbf{C} = (B; f_0, f_1, \ldots, f_\gamma, \ldots.)$ is a subalgebra of some algebra $\mathbf{A} = (A; f_0, f_1, \ldots, f_\gamma, \ldots.)$. As $(B; t_0, t_1, \ldots, t_\gamma, \ldots)$ is a subalgebra of $(A; t_0, t_1, \ldots, t_\gamma, \ldots)$ we have $\mathbf{B} \in SD(V)$.

(c) $DH(V) \subseteq HD(V)$. Let $\mathbf{B} = (B; t_0, t_1, \ldots, t_\gamma, \ldots) \in DH(V)$. Then there is a homomorphic image $f[\mathbf{A}] = (f[A]; f_0, f_1, \ldots, f_\gamma, \ldots.)$ of an algebra $\mathbf{A}$ with $f[A] = B$. But $(B; t_0, t_1, \ldots, t_\gamma, \ldots)$ is also a homomorphic image of $(A; t_0, t_1, \ldots, t_\gamma, \ldots.)$ because $f[A] = B$ and $f(t_\gamma(x_1, \ldots, x_{n_\gamma})) = t_\gamma(f(x_1), \ldots, f(x_n))$. Now we have $DHSP(V) \subseteq HSPD(V)$. (Observe that for some V we have $DS(V) \nsubseteq SD(V)$.)

Remark. In the sense of [Schweigert 87a] a derived algebra **B** from the algebra **A** has the property that T(**B**) is a surjective image of a clone homomorphism from the clone T(**A**) onto the clone T(**B**). Also weak endomorphisms [Goetz 66, Schweigert 85a] induce such clone homomorphisms.

Example 1.4.4. The variety U of semigroups of type (2) defined by the following identities, is a solid variety.

$$x \circ x = x$$

$$x \circ (y \circ z) = (x \circ y) \circ z$$

$$(u \circ x) \circ (y \circ v) = (u \circ y) \circ (x \circ v).$$

Proof. One can show that $F_2 = \left\{x, y, x \circ y, y \circ x, x \circ y \circ x, y \circ x \circ y\right\}$ is the set of all binary terms of the variety U. Furthermore, for these terms the transformed identities: $F(x,x) = x$. $F(x,F(y,z)) = F(F(x,y),z)$ and $F(F(u,x), F(y,v)) = F(F(u,y), F(x,v))$ hold as hyperidentities for U.

Remark 1.4.5. The transformation of some identities for an algebra **A** always leads to hyperidentities which hold for **A** (for example $x = x$ *or* $x \circ x = x$).

§1.5 Weak isomorphisms

The notion of weak homomorphism and weak isomorphism has been introduced by Marczewski and Goetz ([Glazek, Michalski 77], [Schweigert 84]). For these definitions we have to consider the clone T(**A**) of all term functions of an algebra **A**.

Definition 1.5.1. *Let* $\mathbf{A} = (A,\Omega_1)$ *and* $\mathbf{B} = (B,\Omega_2)$ *be algebras not necessarily of the same type and let* $h{:}A \longrightarrow B$ *be a mapping. Let* $\varphi \in T(\boldsymbol{A})$ *and* $\psi \in T(\mathbf{B})$ *be of the same arity n. Then* φ *and* ψ *are in the relation* R_h, *i.e.* $(\varphi,\psi) \in R_h$ *iff* $h(\varphi(x_1,\ldots,x_n)) = \psi(h(x_1),\ldots,h(x_n))$.

Definition 1.5.2. *Let* $\mathbf{A} = (A,\Omega_1)$ *and* $\mathbf{B} = (B,\Omega_2)$ *be algebras not necessarily of the same type. The mapping* $h{:}A \longrightarrow B$ *is called a weak homomorphism of* **A** *into* **B** *iff*

(i) *for every* $\varphi \in T(\mathbf{A})$ *there is a* $\psi \in T(\mathbf{B})$ *with* $(\varphi,\psi) \in R_h$,

(ii) *for every* $\alpha \in T(\mathbf{B})$ *there is a* $\beta \in T(\mathbf{A})$ *with* $(\beta,\alpha) \in R_h$.

Remark 1.5.3. It is easy to show that (i) and (ii) can be replaced by the weaker conditions (a) and (b).

(a) for every $\omega \in \Omega_1$ there is a $\psi \in T(\mathbf{B})$ with $(\omega,\psi) \in R_h$,

(b) for every $\eta \in \Omega_2$ there is a $\varphi \in T(\mathbf{A})$ with $(\varphi,\eta) \in R_h$.

If $h{:}A \longrightarrow B$ is a homomorphism of the algebra **A** into the algebra **B** of the same type, then h is also a weak homomorphism, because we have $h(\omega_A(x_1,\ldots,x_n)) = \omega_B(h(x_1),\ldots,h(x_n))$ for every operation $\omega_A \in \Omega_1$ and the corresponding operation $\omega_B \in \Omega_2$.

A weak homomorphism $h{:}A \longrightarrow B$ is called a weak isomorphism if h is bijective.

Definition 1.5.4. *A weak homomorphism* $h{:}A \longrightarrow B$ *is called a near isomorphism if* h *is the identity map.*

Example 1.5.5. Let $\mathbf{B} = \left[\{1,0\};\wedge,\vee,\diagdown,0,1\right]$ be the Boolean algebra on the set $\{0,1\}$. Let $\mathbf{R} = \left[\{1,0\};+,0,\cdot,1\right]$ be the commutative ring on the set $\{0,1\}$ where the addition is modulo 2. Then **B** and **R** are near isomorphic. Especially we have:

(a) 1) $x \vee y = (x+y)+x\cdot y$ 4) $0 = 0$

2) $x \wedge y = x\cdot y$ 5) $1 = 1$

3) $x^{\diagdown} = x+1$

(b) 1) $x+y = (x^{\diagdown} \wedge y) \vee (y \wedge x^{\diagdown})$ 3) $0 = 0$

2) $x\cdot y = x \wedge y$ 4) $1 = 1.$

Hence conditions (a) and (b) of 1.5.3 are fulfilled.

Lemma 1.5.6. *Let* $\mathbf{A} = (A,\Omega_1)$ *and* $\mathbf{B} = (B,\Omega_2)$ *be algebras not necessarily of the same type. If* $h{:}A \longrightarrow B$ *is a weak isomorphism then there is an isomorphism* $\alpha{:}A \longrightarrow B$ *for an algebra* $\mathbf{B}^* = (B,\Omega_1)$ *and a near isomorphism* $g{:}B \longrightarrow B$ *from* $\mathbf{B}^*$ *onto* **B** *such that* $h = g\circ\alpha$.

Proof. We define the operation ω_{B^*} of $\mathbf{B}^*$ by setting $\omega_{B^*}(b_1,\dots,b_n) = h(\omega_A(h^{-1}(b_1),\dots,h^{-1}(b_n)))$. Furthermore we define $\alpha(a) := h(a)$ for every $a \in A$. Then α is bijective. Put $b_i = h(a_i)$, $i = 1,\dots,n$. Then we have

$$\begin{aligned}\alpha(\omega_A(a_1,\dots,a_n)) &= h(\omega_A(a_1,\dots,a_n)) = h(\omega_A(h^{-1}(b_1),\dots,h^{-1}(b_n)) \\ &= \omega_{B^*}(b_1,\dots,b_n) = \omega_{B^*}(h(a_1),\dots,h(a_n)) = \omega_{B^*}(\alpha(a_1),\dots,\alpha(a_n))).\end{aligned}$$

Hence α is an isomorphism. For the identity map $g{:}B \longrightarrow B$ and the corresponding relation R_g the following holds:

(a) For $\omega_{B^*} \in \Omega_1$ we have a term function $\psi \in T(\mathbf{B})$ with $(\omega_A,\psi) \in R_h$ such that

$$h(\omega_A(a_1,\dots,a_n)) = \psi(h(a_1),\dots,h(a_n)).$$

Therefore $\qquad \alpha(\omega_A(a_1,\dots,a_n)) = \psi(h(a_1),\dots,h(a_n))$

hence $\qquad \omega_{B^*}(\alpha(a_1),\dots,\alpha(a_n)) = \psi(h(a_1),\dots,h(a_n)),$

and hence $\qquad \omega_{B^*} = \psi$, i.e. $(\omega_{B^*},\psi) \in R_g$.

(b) is proved similarly.

Remark 1.5.7. A weak isomorphism $h:A \longrightarrow B$ for the algebras $\mathbf{A} = (A,\Omega_1)$ and $\mathbf{B} = (B,\Omega_2)$ also defines a map

$$\bar{h}:T(\mathbf{A}) \longrightarrow T(\mathbf{B}) \text{ by } (\bar{h}(\psi))(b_1,\ldots,b_n) := h(\psi(h^{-1}(b_1),\ldots,h^{-1}(b_n))).$$

This map $\bar{h}$ is a clone isomorphism and $\bar{h}$ is compatible with the composition of term functions, permutation of variables and with fictitious variables. $\bar{h}$ also preserves the arity of term functions ([Glazek, Michalski 77], [Schweigert 84]).

Notations 1.5.8. Let $\mathbf{A} = (A,\Omega_1)$ and $\mathbf{B} = (A,\Omega_2)$ be two algebras not necessarily of the same type. Let $\mathbf{A}$ be near isomorphic to $\mathbf{B}$ and $\bar{h}:T(\mathbf{A}) \longrightarrow T(\mathbf{B})$ the corresponding map for the clones of term functions. If $\epsilon \equiv (\varphi = \psi)$ is an equation which holds for the algebra $\mathbf{A}$ then $\bar{h}(\epsilon) \equiv (\bar{h}\varphi = \bar{h}\psi)$ is an equation which holds for the algebra $\mathbf{B}$. $\bar{h}(\epsilon)$ is called the transformation of the equation $(\varphi = \psi)$ by $\bar{h}$.

Example 1.5.9. Let $\mathbf{B} = \left[\{0,1\};\wedge,\vee,\neg,0,1\right]$ be the Boolean algebra and $\mathbf{R} = \left[\{0,1\};+,0,\cdot,1\right]$ be the commutative ring on the set $\{0,1\}$. By 1.5.5 $\mathbf{B}$ is near isomorphic to $\mathbf{R}$. Some axioms for the Boolean algebra are transformed in the following way.

$$B1: x\wedge(y\wedge z) = (x\wedge y)\wedge z$$

$$\bar{h}(B1): x\cdot(y\cdot z) = (x\cdot y)\cdot z$$

$$B2: x\wedge y = y\wedge x$$

$$\bar{h}(B2): x\cdot y = y\cdot x$$

$$B3: x\wedge x = x$$

$$\bar{h}(B3): x\cdot x = x$$

$$B4: x\wedge(y\vee x) = x$$

$$\bar{h}(B4)\colon x\cdot((y+x)+(y\cdot x)) = x$$

$$B5\colon x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)$$

$$\bar{h}(B5)\colon x\cdot((y+z)+(y\cdot z)) = ((x\cdot y)+(x\cdot z))+((x\cdot y)\cdot(x\cdot z))$$

$$B6\colon x\wedge x^{\setminus} = 0$$

$$\bar{h}(B6)\colon x\cdot(x+1) = 0.$$

Remark 1.5.10. If Σ is an equational basis for the equational theory of **A** then the transformation $\bar{h}(\Sigma)$ of Σ by h is not an equational basis for **B** besides in special cases. Therefore it is necessary to add some further equations to $\bar{h}(\Sigma)$ to get an equational basis for **B**.

Notation 1.5.11. Let ω be an n–place operation of the algebra $\mathbf{B} = (B,\Omega_2)$ then $\bar{h}^{-1}(\omega)$ is a term function of the algebra **A** and hence may be presented by a term $\psi(x_1,\ldots,x_n)$ of the term algebra of **A**. $\bar{h}(\psi)$ is a term function of **B** and can be presented by a term $\varphi(x_1,\ldots,x_n)$. Obviously the equation $\varphi(x_1,\ldots,x_n) = \omega(x_1,\ldots,x_n)$ holds for **B**. We denote this equation by π_ω and consider the set $\{\pi_\omega \mid \omega \in \Omega_2\}$ of equations.

Example 1.5.12. Consider the operation + and · of the ring R. Then we have

$$\pi_+\colon \Big[((x+1)\cdot y)+(x\cdot(y+1))\Big] + \Big[((x+1)\cdot y)\cdot(x\cdot(y+1))\Big] = x+y$$

$$\pi_\cdot\colon x\cdot y = x\cdot y \text{ (which can be dropped because of triviality).}$$

For the following result compare also [Felscher 66] p. 148 thm4.

Theorem 1.5.13. *Let Σ_1 be an equational basis for the equational theory of the algebra $\mathbf{A} = (A,\Omega_1)$. If $\mathbf{B} = (A,\Omega_2)$ is near isomorphic to* **A** *by* h *then $\Sigma_2 = \bar{h}(\Sigma_1) \cup \{\pi_\omega \mid \omega \in \Omega_2\}$ is an equational basis for the equational theory of* **B**.

Proof. We show that any equation of $\varphi(x_1,\dots,x_n) = \psi(x_1,\dots,x_n)$ holding for **B** can be derived from Σ_2. $(h^{-1}(\varphi))(x_1,\dots,x_n) = (h^{-1}(\psi))(x_1,\dots,x_m)$ is an equation holding for **A** and hence can be derived by a sequence $(\sigma_1,\dots,\sigma_k)$ of equations from Σ_1, the properties of the equality sign and the substitution [Grätzer 79], p. 381. Transforming this sequence by $\bar{h}$ we have $(\bar{h}(\sigma_1),\dots,\bar{h}(\sigma_k))$ a sequence of equations from $\bar{h}(\Sigma_1)$ which proves the equality $(h(h^{-1}(\varphi)))(x_1,\dots,x_n) = (h(h^{-1}(\psi)))(x_1,\dots,x_n)$. By the equations from $\{\pi_\omega \mid \omega \in \Omega_2\}$ we have that $\varphi(x_1,\dots,x_n) = (h(h^{-1}(\varphi)))(x_1,\dots,x_n)$ and $\psi(x_1,\dots,x_n) = (h(h^{-1}(\psi)))(x_1,\dots,x_n)$.

Example 1.5.14. We like to show that $x+y = y+x$ holds for **R**. A proof of this equation in **B** is the following: ($B2^{\smallsetminus}$ is the dual of B2)

$$(x^{\smallsetminus}\wedge y)\vee(x\wedge y^{\smallsetminus}) \underset{(B2^{\smallsetminus})}{=} (x\wedge y^{\smallsetminus})\vee(x^{\smallsetminus}\wedge y) \underset{(B2)}{=} (y^{\smallsetminus}\wedge x)\vee(x^{\smallsetminus}\wedge y) \underset{(B2)}{=} (y^{\smallsetminus}\wedge x)\vee(y\wedge x^{\smallsetminus}).$$

By transformation we get the following

Proof.

$$\begin{aligned}
x+y &\underset{\pi_+}{=} (((x+1)\cdot y)+(x\cdot(y+1)))+(((x+1)\cdot y)\cdot(x\cdot(y+1))) \\
&\underset{\bar{h}(B2^{\smallsetminus})}{=} ((x\cdot(y+1))+((x+1)\cdot y)+(((x\cdot(y+1))\cdot((x+1)\cdot y)) \\
&\underset{\bar{h}(B2)}{=} ((y+1)\cdot x))+((x+1)\cdot y)+(((y+1)\cdot x)\cdot(x+1)\cdot y)) \\
&\underset{\bar{h}(B2)}{=} ((y+1)\cdot x)+((y\cdot(x+1))+(((y+1)\cdot x)\cdot(y\cdot(x+1))) \\
&\underset{\pi_+}{=} y+x
\end{aligned}$$

Theorem 1.5.15. *Let* $\mathbf{A} = (A,\Omega_1)$ *and* $\mathbf{B} = (B,\Omega_2)$ *be of finite type and weakly isomorphic to one another. The equational theory of* **B** *is finitely based if and only if the equational theory of* **A** *is finitely based.*

Proof. By lemma 1.5.6 we have that theorem 1.5.13 holds also in the case of a weak isomorphism. Furthermore if Σ_1 is finite and the type of **B** is finite then $\bar{h}(\Sigma_1) \cup \{\pi_\omega \mid \omega \in \Omega_2\}$ is a finite basis for the equational theory of **B**.

Definition 1.5.16. *A weak homomorphism* h:A $\longrightarrow$ B *is called type-preserving if the algebras* **A**, **B** *are of the same type.*

Notation 1.5.17. Let K be class of algebras of type τ. Let W(K) be the class of all algebras which are images from algebras of K by type–preserving weak isomorphisms.

Problem. *Describe solid varieties with the class operators* W. *Can* W *replace the class operator* D?

Definition 1.5.18. *A congruence relation θ of an algebra* **A** *is totally invariant if* (a,b) $\in$ θ *implies* (h(a),h(b)) $\in$ θ *for every type–preserving weak endomorphism* h *of* **A** *and every* a,b $\in$ A.

One observes that $\Sigma \subseteq \mathrm{Id}\ \tau$ is a solid theory if and only if Σ is a totally invariant congruence of $F_\tau(X)$ [Schweigert 89].

The closures with respect to the deduction rules (1) – (6) correspond to the properties of Σ as follows.

Σ is closed under (1) – (3) $\Leftrightarrow$ Σ is an equivalence relation on $F_\tau(x)$.

Σ is closed under (1) – (4) $\Leftrightarrow$ Σ is a congruence relation

Σ is closed under (1) – (5) $\Leftrightarrow$ Σ is a fully invariant conruence

Σ is closed under (1) – (6) $\Leftrightarrow$ Σ is a totally invariant congruence.

§1.6 Types

Let us consider two weakly isomorphic algebras $\mathbf{A} = (A, \Omega)$ and $\mathbf{B} = (A, \Omega)$. They have isomorphic clones $T(\mathbf{A})$, $T(\mathbf{B})$ on the set A. But these clones may be generated by different fundamental operations. The fundamental operations determine the identities. Therefore, the identities of $\mathbf{A}$ may appear very different from the identities of $\mathbf{B}$. In this respect the type of a variety plays an essential role.

Remark 1.6.1. Let V be a variety of type τ. Let the type τ be contained in the type μ. Then the set $H_\tau(V)$ of hyperidentities of type τ is contained in the set $H_\mu(V)$ of hyperidentities of type μ. For the variety of semilattices SL one can present an increasing sequence of types

$$\mu_1 \subsetneq \mu_2 \subsetneq \ldots \subsetneq \mu_{i-1} \subsetneq \mu_i \subsetneq \ldots$$

such that there are hyperidentities in $H_{\mu_i}(SL)$ which cannot be implied by $H_{\mu_{i-1}}(SL)$ [Penner 81].

Notation 1.6.2. Let $Q = T$ be a hyperidentity of type μ. (*) Assume that we have hyperterms R_i, $i \in I$, of type τ available such that any hypervariable of type μ in $Q = T$ can be hypersubstituted by hyperterms R_i of type τ. The result of such a hypersubstitution (rule 6) is called hyperconsequence of type τ from the hyperidentity $Q = T$ of type μ.

Example. $(Q = T) : T(T(x,y,z),y,z) = T(x,y,z)$ is a hyperidentity of type $\langle 3 \rangle$. Consider $R = F(G(x,y),z)$ a hyperterm of type $\langle 2,2 \rangle$. Then

$$(M = N) : F(G(F(G(x,y),z),y),z) = F(G(x,y),z)$$

is a hyperconsequence of type τ from the hyperidentity $Q = T$ of type μ.

Notation 1.6.3. If the assumption (*) holds we say that type τ is compatible with type μ. As the reader will observe this assumption it is usually fulfilled.

Example. Consider the variety D of distributive lattices of type $\langle 2,2 \rangle$. Then the hyperidentity $F(F(x,y),y,z) = F(x,y,z)$ is of type $\langle 3 \rangle$ and holds for all ternary terms of D. Obviously also every hyperconsequence $M = N$ of type $\langle 2 \rangle$ holds in D.

Theorem 1.6.4. *Let* V *be a variety of type* τ *and let* τ *be compatible with the type* μ. *A hyperidentity* $P = Q$ *of type* μ *holds for* V *if an only if every hyperconsequence* $M = N$ *of type* τ *holds for* V.

Proof. Let $P = Q$ of type μ hold for V and let the hyperidentity $M = N$ of type τ be a hyperconsequence of $P = Q$. Then any hypersubstitution of the hypervariable yields an identity of type τ which is also implied by $P = Q$ and hence hold for V. $M = N$ holds for V. Let every hyperconsequence of type τ hold for V and let $p = q$ be the result of hypersubstituting the hypervariables in $P = Q$ by the appropriate terms. Transform $p = q$ into a hyperidenty $M = N$ of type τ. Obviously $M = N$ is hyperconsequence of $P = Q$ and holds for V by hypothesis. Hence $p = q$ is an identity for V. Hence $P = Q$ is a hyperidentity for V.

Remark. The above result shows that for variety of type τ it is sufficient to consider hyperidentities of type τ. To consider a hyperidentity of other types for V may be useful as this hyperidentity may stand for an huge set of hyperconsequences of type τ. Hence such a hyperidentity of type μ is a short notation for a set of hyperidentities of type τ. Again a hyperidentity of type τ may be considered as a short notation for a possibly infinite set of identities of type τ. This is one of the essential properties of hyperidentities.

Example. Consider the solid variety B of regular bands defined by

$$xo(yoz) = (xoy)oz$$

$$xox = x$$

$$(uox)\,o(yow) = (uoy)o(xow)$$

B is of type $\langle 2\rangle$. We add as an additional fundamental operation $\square$ with $x \square y = x$ and have $\bar{B}$ of type $\langle 2,2\rangle$. This operation reflects the projection $e_1^2, e_1^2(x,y) = x$, and is contained in T(B) anyhow. Let $\bar{B}$ denote this variety of type $\langle 2,2\rangle$ with $T(B) = T(\bar{B})$
Statement. $\bar{B}$ is not solid.

If $\bar{B}$ would be solid then $F(x,y) = x$ would be a hyperidentity. Now $e_2^2(x,y) = y$ would imply $x = y$. $\bar{B}$ is an example of a variety which is equivalent to solid variety but is not solid itself. On the other hand a reduct of a solid variety is solid. We call two varieties V,W equivalent if they can be generated by weakly isomorphic algebras.

Problem 1.6.5. *Let* V *be a variety. Under which conditions is* V *equivalent to a solid variety* W?

This problem can be considered as well from a syntactic point of view as from a semantic one. For the semantic point of view we have

Theorem 1.6.6. *If* V *is equivalent to a solid variety* W *then every subdirectly irreducible derived algebra* **A** *from* V *is weakly isomorphic to a subdirectly irreducible algebra* **C** *of* V.

Proof. Let $\mathbf{A} = (A, t_1,\dots,t_{nj},\dots)$ be a subdirectly irreducible derived algebra from V. **A** must be constructed from a algebra $\bar{\mathbf{A}}$ of V. Then there is an algebra $\bar{\mathbf{B}} = (B_i\ f_1,\dots,f_{n\mu},\dots)$ in W with a weak isomorphism h from $\bar{\mathbf{A}}$ to $\bar{\mathbf{B}}$. We have a derived algebra $\mathbf{B} = (B, t_1^h, \ldots, t_{n\mu}^h, \ldots)$ where h describes the transformation of the operations into $t_1, \ldots, t_{n\mu}, \ldots,$. We have that **B** is weakly isomorphic to **A**. As W is solid **B** is a subdirectly irreducible algebra of W. Then there exist a subdirectly irreducible algebra **C** in V such **B** is weakly isomorphic to **C** because W is equivalent to V. Hence **A** is weakly isomorphic to **C**.

Definition 1.6.5. *The solid envelope* s(V) *of a variety V is the smallest solid variety of the same type containing* V.
Another definition would be s(V) = HSPD(V). It follows that s(V) is generated by the subdirectly irreducible derived algebras from V. From 1.6.4 we get

Theorem 1.6.5. V *of type* τ *is equivalent to a solid variety* W *of some type* μ *if and only if* V *is equivalent to its solid envelope* s(V).

§1.7 Transformations

Definition 1.7.1. *Let* τ *be a given type and* $H(\tau)$ *be the set of all hyperterms of type* τ. *Let* **V** *be a variety of type* μ *and* $W(\mu)$ *be set of all terms of type* μ. *The mapping* $\sigma: H(\tau) \longrightarrow W(\mu)$ *is defined recursively*

$$\sigma(x_i) = x_i \textit{ for every variable } x_i$$

$$\sigma(F_\delta(x_1, \ldots, x_n)) = t_\delta(x_1, \ldots, x_n)$$

where to every n_δ–ary *hypervariable* F_δ *of type* τ *an* n_δ–ary *term* t_δ *of type* μ *is assigned.* σ *is then extended by the construction of hyperterms to* $H(\tau)$.

Example. Let F(F(x,y,z),y,z) be a hyperterm of type 3 and let t(x,y,z) $= x \wedge (y \vee z)$ be a term of type (2,2) of the variety of lattices. Then

$$\sigma(F(F(x,y,z),y,z) = (x \wedge (y \vee z)) \wedge (y \vee z).$$

If we consider the hyperidentity $F(F(x,y,z),y,z) = F(x,y,z)$ then by σ we get the identity

$$(x \wedge (y \vee z)) \wedge (y \vee z) = x \wedge (y \vee z).$$

Notation 1.7.2. The mapping σ in 1.7 is called a (τ,μ)–transformation.

Definition 1.7.3. *A hyperidentity of type τ holds in a variety of type μ of every (τ,μ)–transformation of the hyperidentity yields an identity which holds for* V. (For $\tau = \mu$ compare Def. 1.1.2)

Remark. For a given variety V we like to consider all hyperidentities of any type which hold for V. How has one to choose the hypervariables F_δ? A rough estimate would be to have a fundamental hyperterm $F_\delta(x_1,\ldots,x_{n_\delta})$ for every term $t_\delta(x_1,\ldots,x_{n_\delta})$ for the variety. With this crude construction we may get a type which furthers us with all hyperidentities for V. An alternative is to consider a type which is the set of all ordered sequences.

Definition 1.7.4. *A type ν is called a general type for a variety* V *if the set all hyperidentities for* V *is equivalent to the set of all hyperidentities of type ν which holds for* V.

(Of course, two sets of hyperidentities are equivalent to one another if they can be derived by (1)–(6) from one another).

The general type can be used to present a completeness theorem which holds for hyperidentities of any type. In order to stress the semantic aspect one can use the following.

Notation 1.7.5. $D_\nu(V)$ is the set of all algebras $(A;(t_i)i\epsilon I)$ of type ν where t_i are term operations of V of type ν. $D_\nu(V)$ is called the derived variety of V of type ν. One can say that a hyperidentity ϵ of type ν holds for V if and only if the set of transformations of ϵ yields identities of $D_\nu(V)$.

It is up to the reader to reformulate and generalize some of the results from the section before.

Part II Iterative hyperidentities

§2.1 Iterative hyperidentities

One can define operations on the set H(τ) of all hyperterms of type τ (compare def. 1.1.1) by the hypervariables F in the usual way. We call this algebra HT(X) of type τ a hyperterm algebra generated by the variables $x \in X$.

As an example consider the hyperterm algebra HT(2) in 2 variables x,y and a binary hypervariable F for the variety of semilattices x,y

F(x,y),F(y,x)	(We have F(x,x) = x, F(y,y) = y)
F(F(x,y),x), F(F(y,x)y)	(We have F(F(x,y),y) = F(x,y),...).

Here we use the hyperidentities as given in example 1.4.4.

Some bit more formally we use the following

Notations 2.1.1. Let V be a variety of type τ. Let F be an n–ary hypervariable with respect to τ. Then $F(x_1,...,x_n)$ is called a fundamental hyperterm. The hyperterm algebra HT(V) is the set of all hyperterm for type τ closed under the application of all fundamental hyperterms as operations on the hyperterms.

Let us denote the set of all fundamental hyperterms by FT(V). Obviously we have that for every map

α: FT(V) $\longrightarrow$ T(V)	there exists an extension
β:(HT(V) $\longrightarrow$ T(V)	where β is surjective.

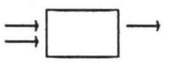
Remark 2.1.2. If V is a solid variety then free algebra coincides with the hyperterm algebra of V. If V is not solid and if s(V) is the solid envelope of V then the free algebra of s(V) coincide with the hyperterm algebra of V.

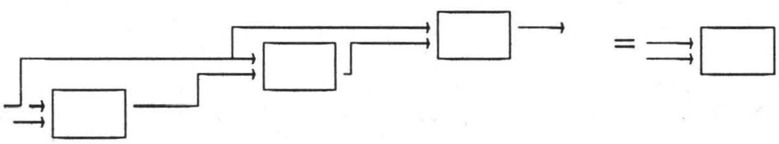
Problem 2.1.3. Determine the hyperterm algebra for the variety of your choice.

Problem 2.1.4. Determine the hyperterm algebra in three generators x,y,z and the binary hypervariable F,G for the variety of modular lattices.

Problem 2.1.5 If one can decide the equality of two hyperterms in a finite number of steps in the variety V, we say that the "hyperword problem" of V is solvable. Let V be a variety with a solvable hyperword problem. Under which condition is the word problem of V solvable?

Problem 2.1.6 Consider the reverse problem to 2.1.5.

Example 2.1.7. Let  be a symbol for an arbitary switching circuit realizing some Boolean function f: $\{0,1\}^2 \longrightarrow \{0,1\}$



We have the hyperidentity F(F(F(x,y),y),y) = F(x,y), which we will write in the short form

$$F^3(x,y) = F(x,y)$$

If we like to prove this in a syntactic way we have to list up all 16 Boolean terms

$$x, y, x^{\backprime}, y^{\backprime}, x \wedge y, x \vee y, x^{\backprime} \wedge x, x \vee x^{\backprime},$$

and check every term, for instance $x \wedge y$

$$((x \wedge y) \wedge y) \wedge y = x \wedge y$$

If we like to prove this in a semantic way one can proceed as follows. Consider the semigroup of polynomial functions of the algebra $\mathbf{B} = \{0,1\}; \wedge, \vee, {}^{\backprime}, 0, 1)$. This is the semigroup T_2 of all transformations on $\{0,1\}$. It fulfills the semigroup identity

$$p \circ p \circ p = p \text{ or written in another way}$$

$$p^3 = p$$

(Transformations: $f_1(x) = x$, $f_2(x) = 0$, $f_3(x) = 1$, $f_4(x) = x^{\backprime}$.

We have $f^3{}_i(x) = f_i(x)$, $i = 1,2,3,4$.)

In the following we will prove that any identity of the semigroup of polynomial functions of an algebra A yields a set of hyperidentities for A. For instance for the Boolean algebra we would have

$$F_1{}^3(x) = F_1(x)$$

$$F_2{}^3(x,y) = F_2(x,y)$$

$$F_3{}^3(x,y,z) = F_3(x,y,z)$$

where F_i is a hypervariable of arity i, $i = 1,2,3,\ldots$.

§ 2.2 Iterations of functions

Proposition 2.2.1. *Let* $f{:}A \longrightarrow A$ *be a function,* $|A| = n$. *There exists a least number* $\lambda(f)$, *(the index of* f*) such that* $f^{\lambda(f)+1}[A] = f^{\lambda(f)}[A]$.

Proposition 2.2.2. *Let* $f: A \longrightarrow A$ *be a function,* $|A| = n$. *Then there exists a least number* $\pi(f)$ *(the period of* f*) such that* $f^{\lambda(f)+\pi(f)} = f^{\lambda(f)}$.

Proposition 2.2.1 and Proposition 2.2.2 date back to Frobenius [Frobenius 1895].

Proposition 2.2.3. *Let* S *be a semigroup of functions on* A, $|A| = n$. *Let* $f,g \in S$ *such that* $f^{\lambda(f)+\pi(f)} = f^{\lambda(f)}$ *and* $g^{\lambda(g)+\pi(g)} = g^{\lambda(g)}$. *Then we have*

$$h^{\max(\lambda(f),\lambda(g))+\mathrm{lcm}(\pi(f),\pi(g))} = h^{\max\{\lambda(f),\lambda(g)\}}$$

for every $h \in \{f,g\}$.

Here $\mathrm{lcm}\ (\pi(f),\pi(g))$ *denotes the least common multiple of the integers* $\pi(f),\pi(g)$.

Definition 2.2.4. *Let* S *be a semigroup of functions on* A, $|A| = n$.

$\lambda_S := \max\ \{\lambda(f) \mid f \in S\}$ *is called the index of* S.

$\pi_S : = \mathrm{lcm}\ \{\pi(f) \mid f \in S\}$ *is called the period of* S.

We denote by F,G,H variables which stand for the functions f,g,h,... in S. Obviously the functions in S fulfill the equation

$$F^{\lambda_S+\pi_S} = F^{\lambda_S}.$$

Definition 2.2.5. *An equation* $F^r = F^s$, $r < s$ *for the semigroup* S *is called irreducible if for every equation* $F^k = F^l$, $k < l$, *which hold for* S *we have* $r \leq k$ *and* $s \leq l$.

Obviously $F^{\lambda_S+\Pi_S} = F^{\lambda_S}$ is an irreducible equation for S.

Proposition 2.2.6. *Let* U *be a subsemigroup of a semigroup* S *of functions on* A, $|A| = n$. *For the equations* $F^{\lambda_s+\pi_s} = F^{\lambda_s}$ and $F^{\lambda_u+\pi_u} = F^{\lambda_u}$ *which are irreducible for* S *respectively* U *we have that*

$$\lambda_u \leq \lambda_s \quad \textit{and} \quad \pi_u \textit{ divides } \pi_s.$$

Proof by definition of the index and the period.

Proposition 2.2.7. *Let* S *be a semigroup of functions on* A, $|A| = n$, *let* $f \in S$ *such that* $\pi(f) = rt$. *Then there exists* $g \in S$ *with* $\lambda(g) \leq \lambda(f)$ *and* $\pi(g) = t$.

Proof. We consider the cyclic group of permutations on $A_f := f^{\lambda(f)}[A]$ which is generated by $f/_{A_f}$ and we put $g := f^r$. Now we have

$$g^{\lambda(f)+t} = f^{r(\lambda(f)+t)} = f^{r\lambda(f)+rt} = f^{r\lambda(f)+\pi(f)}$$

$$= f^{\lambda(f)+\pi(f)+(r-1)\lambda(f)} = f^{\lambda(f)+(r-1)\lambda(f)} = f^{r\lambda(f)} = g^{\lambda(f)}.$$

Proposition 2.2.8. *Let* S *be a semigroup of functions on* A, $|A| = n$ *with the equation* $F^{\lambda_s+\Pi_s} = F^{\lambda_s}$. *For every prime power* p^m *which divides* π_s *there exists* $g \in S$ *such that* $g^{\lambda(g)+p^m} = g^{\lambda(g)}$.

Proof. There exists a set $\{f_1,...,f_k\}$ of functions such that $\pi_s = \operatorname{lcm}\{\pi(f_i) \mid i = 1,...,k\}$. Because of the definition of lcm we have that p^m divides $\pi(f_j)$ for some $j \in \{1,...,k\}$. Now we apply proposition 2.2.7 to the function f_j with $\pi(f_j) = p^m \cdot s$.

Notation. $\gamma(n) := \max(\operatorname{lcm}(x_1,...,x_m))$ denotes the maximum of the least common multiple of $x_1,...,x_m$ of all partitions of n, $n = x_1+...+x_m$, $m = 1,...,n$. For $n = 1,...,301$ the values of $\gamma(n)$ can be found in the table of [Nicolas], p. 187.

Theorem 2.2.9. *Let* $f:A \longrightarrow A$, $|A| = n$. *Then we have*

$$\gamma(f) \le \sigma(n-\lambda(f)).$$

Proof. The order of the permutation $f/_{A_f}$ where $A_f := f^{\lambda(f)}[A]$ is the least common multiple of the length $x_1,\ldots,x_m$ of m disjoint cycles representing the permutation [Hall 59]. On $|A_f|$ numbers every partition $|A_f| = x_1+\ldots+x_m$ represents some permutation. The maximal order of these permutations is $\gamma(|A_f|)$ and hence we have $\gamma(n-\lambda(f)) \ge \gamma(A_f) \ge \pi(f)$.

Corollary 2.2.10. $\lambda(f) + \pi(f) \le n-1 + \gamma(n-\lambda(f))$.

The above formula gives an useful estimate for the size of powers in a finite semigroup of functions. Indeed we have $\lambda(f) + \pi(f) \sim \gamma(n-\lambda(f))$ for large numbers.

<u>The Transformation Semigroup of a Finite set.</u>

For $n \ge 1$ let $\langle n \rangle$ be the set $\{0,1,\ldots,n-1\}$. By T_n we shall denote the set of all transformations of $\langle n \rangle$,

$$T_n = \{f \mid f:\langle n \rangle \longrightarrow \langle n \rangle\}.$$

T_n is a monoid with the composition of transformations; its unit element is the identical transformation of $\langle n \rangle$.

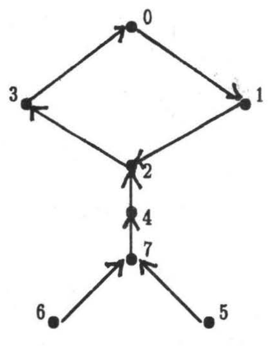
Let $\kappa(n)$ be the least common multiple of $\{1,2,\ldots,n\}$.

Theorem 2.2.11. *Let* k,l *be two natural members,* $k > l > 0$. *Then* $f^k = f^l$ *for all transformations* $f \in T_n$ *if and only if* $l \ge n-1$ *and* $k \equiv l(\mathrm{mod}\ \kappa(n))$.

Proof. (Reischer, Simovici) Let f be a fixed element of T_n. We shall consider the directed graph $G_f = (\langle n \rangle, E_f)$ having $\langle n \rangle$ as set of vertices; the set of edges E_f is given by $E_f = \{(x,f(x)) \mid x \in \langle n \rangle\}$. Since the out–degree of each vertex $x \in \langle n \rangle$ is 1, it is clear that G_f consists of oriented cycles to which trees may be attached by their roots. For instance the graph of the transformation $f \in T_8$ given by the table

x	0	1	2	3	4	5	6	7
f(x)	1	2	3	0	2	7	7	4

is presented here. Let b be the length of the longest attached branch. If $l \geq b$, for any $x \in \langle n \rangle$ b, $f^l(x)$ will be a vertex on a directed cycle. Therefore, if $\kappa(n)$ is the least common multiple of the cycle lengths we shall have $f^k(x) = f^l(x)$, for all $x \in \langle n \rangle$, if $k \equiv l (\mathrm{mod}\ \kappa)$. Varying the transformations we get the necessity of the statement. The proof of sufficiency is similar.



§ 2.3 Monoids of polynomial functions.

The following results are essentially from [Schweigert 79]

Theorem 2.3.1. *Let V be a variety generated by the algebras* $\{B_i \mid i \in I\}$. *Let* M(V) *be the variety of monoids generated by* $\{P_1(B_i) \mid i \in I\}$.
If B *is an algebra* V *then* $P_1(B)$ *is a monoid of* M(V).

Proof. Let A be an algebra of V and $f: A \longrightarrow B$ be a surjective homorphism. Then f can be extended to surjective homorphism $g: P_1(A) \longrightarrow P_1(B)$. [Lausch, Nöbauer (3.3.1)].

Let $A = \prod_{i \in I} A_i$ be a direct product then $P_1(A)$ is isomorphic to a subdirect product of $P_1(A_i)$, $i \in I$ [Lausch, Nöbauer (3.4.1)]. If B is a subalgebra of A we consider the subsemigroup U of $P_1(A) U = \{\psi \in P_1(A) \mid \psi(x_1) \in B[x_1]\}$ where $B[x_1]$ is the polynomial algebra of B in the indeterminate x_1. Then $U \longrightarrow P_1(B)$ is a surjective semigroup homorphism. Therefore, $P_1(B)$ is a isomorphic to a homomorphic image of a subsemigroup of $P_1(A)$

Under which condition on V does the reverse direction of the theorem 2.3.1. hold?

Example 2.3.2. A lattice L is distributive if and only if $P_1(L)$ is idempotent [Schweigert 75].

Example 2.3.3. Given the distributive lattice $D = (\{0,1\}; \wedge, \vee)$ we have the following equational base for $P_1(D)$

$$p^2 = p$$

$$p \circ q \circ p = p \circ q$$

From these equations we can derive the following hyperidentities for the variety of distributive lattices.

$$F(F(x_1,\dots,x_k),x_2,\dots,x_k) = F(x_1,\dots,x_k)$$

$$F(G(F(x_1,\dots,x_k),x_2,\dots,x_l),x_2,\dots,x_k) = F(G(x_1,\dots,x_l),x_2,\dots,x_k)$$

for every $k,l \in \mathbb{N}$. If we consider hyperidentities of a fixed type (2.2) then we have $k = l = 2$.

Example 2.3.4. An equational base of the semigroup T_2 of all transformations on the set $\{0,1\}$ is given by

$$x^3 = x$$

$$xyx^2 = xy$$

$$xy^2 = yxyx$$

Theorem 2.3.5. [Volkov] *The semigroup* T_n *of all transformation on an* n*–element set has no finite basis of identities for* $n \geq 3$.

Example 2.3.6. The semigroup T_n ($n \geq 3$) fulfills the identity

$$x^{n-1}\, y\, x^{n-2} = x^{n-1}\, y\, x^{n-2+\kappa(n)}$$

Theorem 2.3.7. *The following are equivalent for an algebra* A.

1) *The monoid equation*

$$p_1^{k_1} \circ \ldots \circ p_n^{k_n} = q_1^{h_1} \circ \ldots \circ q_m^{h_m}$$

holds for $P_1(A)$

2) *The hyperidentity*

$$T_1^{k_1} \circ \ldots \circ T_n^{k_n} = S_1^{h_1} \circ \ldots \circ S_m^{h_m}$$

holds for the variety HSP(A).

Proof. Consider the monoid equation

$$p_1^{k_1} \circ \ldots \circ p_n^{k_n} = p_{n+1}^{h_1} \circ \ldots \circ p_{n+m}^{h_m}.$$

It follows that for every polynomial function $p_i(x)$ which has some representation as a word $p_i(x) = w(x,a_{i1},\ldots,a_{ik})$ $a_{ij} \in A$, $j = 1,\ldots,k$ the above monoid equation holds. As the element a_{ij} can be selected arbitrarely from A_{ij} we may substitute these elements formally by a variable x_{ij} and the above monoid equations holds for every term function t_i of any arity. Hence, we conclude that the hyperidentity

$$T_1^{k_1} \circ \ldots \circ T_n^{k_n} = T_{n+1}^{h_1} \circ \ldots \circ T_{n+m}^{h_m}$$

holds for the variety HSP(A). Obviously also the reverse direction holds.

Notation 2.3.8. A hyperidentity is called iterative if it is constructed by the iteration of hyperterms in a fixed variable x_i. Iterative hyperidentities are connected to semigroups. It is not important in which variable x_i the iteration is executed but one cannot change the variable during the steps of iterations.

Remark 2.3.9. The set of all iterative hyperidentities for a variety V of type τ is closed under the rules (1), (2), (3), (6) and

(4`) $T = S$ implies $F(x_1,\ldots,x_{i-1},T,x_{i+1},\ldots,x_m) = F(x_1,\ldots,x_{i-1},S,x_{i+1},\ldots,x_m)$ for hyperterms $T,S \in H(\tau)$ and an n–ary hypervariable F, $i = 1,\ldots,m$

(5`) $T(x_1,\ldots,x_n) = S(x_1,\ldots,x_n)$ implies $T(x_{\Pi(1)},\ldots,x_{\Pi(n)}) = S(x_{\Pi(1)},\ldots,x_{\Pi(n)})$ for every permutation Π on $\{1,\ldots,n\}$.

A set Σ of iterative hyperidentities of type τ is called a basis for the set of all iterative hyperidentities for a variety V if every iterative hyperidentity of type τ of V can be derived by (1), (2), (3), (4`), (5`), (6) from Σ consider the example 2.3.3.

§ 2.4 Lattices and Abelian Groups

We are considering the symmetric semigroup S of the 1–place functions on an n–element set. The order of an element $f \in S$ is the least number k such that the elements of the cyclic subsemigroup $\{f,f^2,\ldots,f^k\}$ are different. We have $f^{k+1} = f^m$ for some m with $0 < m \leq k$ and we put $p = k+1-m$. The least common multiple of two numbers u, v is denoted by lcm(u,v) and the maximum of two numbers u, v by max(u,v).

The following results are contained in [Schweigert 85]. In this section we consider T(**A**) as the semigroup of term function of A with respect to composition in the first variable x_1.

Lemma 2.4.1. *The semigroup equation* $\varphi^3 = \varphi^5$ *holds for the semigroup* $T(N_5)$ *of all 1–place monotone functions of the lattice* N_5.

Proof. We consider the cases for all monotone functions f where the image of f consists of $|\text{Im f}| = n$ elements. If $|\text{Im f}| = 1$ we have $f^2 = f$, if $|\text{Im f}| = 2$ we can have $f^3 = f^2$, if $|\text{Im f}| = 3$ we have $f^4 = f^3$ and if $|\text{Im f}| = 5$ we have again $f^2 = f$. In the case $|\text{Im f}| = 4$ we consider two subcases. If Im f is a chain then we can have $f^4 = f^3$. If Im f is not a chain then there are monotone functions f with $f^3 = f$. By Proposition 2.2.3 we have altogether that $f^3 = f^5$. One notices that this also holds for congruence preserving monotone functions.

Lemma 2.4.2. *The semigroup equation* $\varphi^3 = \varphi^9$ *holds for the semigroup* $T(M_3)$ *of all 1–place monotone functions of the lattice* M_3.

Proof. If $|\text{Im f}| = 1$ we have $f^2 = f$, if $|\text{Im f}| = 2$ we can have $f^3 = f^2$, if $|\text{Im f}| = 3$ we can have $f^3 = f^2$ and if $|\text{Im f}| = 5$ we have the group of lattice automorphism with $f^7 = f$. In the case $|\text{Im f}| = 4$ there are functions with $f^4 = f^3$. Altogether we have $\varphi^3 = \varphi^9$ by Proposition 2.2.3.

Theorem 2.4.3. *The equation* $\varphi^3 = \varphi^5$ *holds for the semigroup* $T(N_5)$ *of the term functions of the lattice* N_5. *The clone equation* $\varphi^3 = \varphi^9$ *holds for the semigroup* $T(M_3)$ *of the term functions of the lattice* M_3.

Proof. A function $f{:}N_5 \longrightarrow N_5$ is a polynomial function of N_5 if and only if f is congruence preserving and monotone [Wille]. Hence for the semigroup of 1–place polynomial function the equation $\varphi^3 = \varphi^9$ holds. By Lemma 4.1 in [Schweigert 83] this holds also for every term function of N_5 and hence for the clone of term functions of N_5. A function $f{:}M_3 \longrightarrow M_3$ is a polynomial function of M_3 if and only if f is monotone [Schweigert, 74]. Then by the same arguments the equation $\varphi^3 = \varphi^9$ holds for the variety $T(M_3)$.

Remark. Let T be a variety of semigroups with an equation $\varphi^m = \varphi^{m+k}$ and S a subvariety of T with an equation $\varphi^n = \varphi^{n+s}$ with $n \leq m$, $s \leq k$. By Proposition 2.2.3 we conclude that s divides k. This consequence can be used to study varieties containing $T(N_5)$ or $T(M_3)$. Especially neither $T(N_5)$ generates a subvariety of $HSP(T(M_3))$ nor $HSP(T(M_3))$ a subvariety of $HSP(T(M_5))$.

In [Schweigert 83] we have shown that a lattice L is distributive if and only if the variety T(L) of semigroups is idempotent, i.e. $\varphi^2 = \varphi$.

Theorem 2.4.4. *Let* V *be a non trivial variety such that in the variety* HSP(T(V)) *of semigroups the equation* $\varphi^2 = \varphi$ *holds. Then* V *is not congruence permutable.*

Proof. We assume that V is congruence permutable and consider the term p(x,y,z) with p(x,z,z) = p(z,z,x) = x. For φ(x,y,z) = p(y,x,z) we have p(y,p(y,x,z),z) = p(y,x,z) and therefore y = p(y,z,z) = p(y,p(y,z,z),z) = p(y,y,z)= z, a contradiction.

Let $\kappa(k)$ denote the least common multiple of $\{1,\ldots,k\}$.

Theorem 2.4.5. *Let* V *be an arithmetical variety such that* $\varphi^n = \varphi^m$ $(m > n > 0)$ *holds for the variety* HSP(T(V)) *of semigroups. If* **A** *is a simple algebra of* V *then* **A** *is finite,* $|A| \leq n+1$ *and* $\kappa(|A|)$ *is a divisor of* $m-n$.

Proof. **A** is local polynomially complete [Penner 81] and hence every permutation π of the carrier set A is a local polynomial function. π cannot be of order greater than $m-n$. Therefore **A** is finite and the order of π divides $m-n$. For $A = \{a_1,\dots,a_k\}$ we consider the polynomial function ψ defined by $\psi(a_i) = a_{i-1}$, $i = 2,\dots,k$ and $\psi(a_1) = a_1$. We have $\psi^{k-1} = \psi^k$. From this identity we have $\varphi^n = \varphi^m$ only in the case $n \geq k-1$.

Theorem 2.4.6. *Let* V *be a congruence permutable variety such that* $\varphi^n = \varphi^m$ $(m > n > 0)$ *holds for the semigroup* T(V). *If* **A** *is a finite simple algebra and p is a prime number which divides* $|A|$ *then* p *is a divisor of* $m-n$.

Proof. By a theorem of R. McKenzie **A** is either polynomially complete or affine [Pixley 77], p. 602. In the first case the theorem follows from 2.4.5, in the second case we know that **A** is polynomially equivalent to a module with $p \cdot x = 0$. We consider the polynomial function $\psi(x) = x+1$ and have $\psi^p(x) = x$, hence p divides $m-n$.

Theorem 2.4.7. *If* **G** *is a finite subdirectly irreducible abelian group then the equation* $\varphi^n = \varphi^{n+p^n(p-1)}$ *holds for* T(G) *with* $|G| = p^n$, p *a prime number.*

Proof. Let ψ be a 1–place polynomial function of **G**. Then ψ is of the form $\psi(x) = a^t x^k$ where $G = \langle a \rangle$ and $0 \leq t, k \leq p^n-1$. We have $\psi^m(x) = a^{t(1+k+\cdots+k^{m-1})} x^{k^m}$.

Now we consider the following cases. If $k = 0$ then $\psi(x) = a^t$ and we have $\psi^2 = \psi$. If $k = 1$ then $\psi^m(x) = a^{tm}x$. For $m = p^n+1$ we have $(a^t)^{p^n+1} = a^t$ and hence $\psi^{p^n+1} = \psi$. If $k > 1$ we consider two subcases. If $(k,p) = 1$ then we have by Fermat $k^{\varphi(p^n)} = 1 \bmod p^n$ for the Euler function φ. We have also

$$\frac{k^{\varphi(p^n)-1}}{k-1} = 1+k+\cdots+k^{\varphi(p^n)-1} = 0 \bmod p^n.$$

We conclude that

$$\psi^{\varphi(p^n)}(x) = a^{t(1+k+\cdots+k^{\varphi(p^n)-1})}x^{k^{\varphi(p^n)}} = a^{t\cdot 0}x = x.$$

Hence we have $\psi^{\varphi(p^n)+1} = \psi$. For the subcase $k = p^s\cdot r$ with $1 \le s < n$ we have

$$\psi^m(x) = a^{t(1+p^sr+\cdots+(p^sr)^{m-1})}x^{(p^sr)^m}.$$

Here we have $(p^sr)^m = 0 \bmod p^n$ for $m \geq n-s$. If we put $m = n$ then we have $\psi^n = \psi^{n+1}$. Altogether we have $\psi^1 = \psi^{1+1}$, $\psi^1 = \psi^{1+p^n}$, $\psi^1 = \psi^{1+\varphi(p^n)}$ and $\psi^n = \psi^{n+1}$. By Proposition 2.2.3 it follows that $\psi^n = \psi^{n+p^n(p-1)}$.

Corollary 2.4.8. *If* G *is a simple group of order* p, p *a prime number, then* $\varphi = \varphi^{p^2-p+1}$ *holds for* T(G).

Theorem 2.4.9. *Let* V *be a finitely generated variety of groups.* V *is a variety of abelian groups generated by simple groups if and only if* $\varphi^n = \varphi$ *holds for* T(V) *for some* $N \in \mathbb{N}$, $n > 1$.

Proof. One direction is implied by Corollary 2.4.8. On the other hand if $\varphi^n = \varphi$ holds for T(V) then we have the equation $[x,y,\ldots,y] = [x,y]$ because of the term function $\psi(x,y) = x^{-1}y^{-1}xy = [x,y]$. We show that every finite group of V is abelian. Let **G** be a minimal counter example. If **G** is simple then $\varphi^n = \varphi$ holds only in the case that **G** is abelian, otherwise **G** would be polynomially complete [Lausch, Nöbauer 73], p.41. If **N** is a non trivial normal subgroup of **G** then by hypothesis

G/**N** and **N** are abelian and hence by [Hall 59], Cor. 9.2.1 (p. 141), **G** is solvable. There are elements $b \in N$ and $a \in G$ such that $[a,b] \neq e$. Because **G** is solvable we have $[a,b] \in N$ ([Hall 59], Th. 9.2.1 (p. 138)). As **N** is abelian it follows that $e = [[a,b],b] = [a,b,b] = [a,b,\ldots,b] = [a,b]$ a contradiction. We have still to show that V is generated by simple groups, but this follows from the proof of Theorem 2.4.7.

Theorem 2.4.10. *Let* $(G;+)$ *be a finite elementary abelian* p–group *and let* End(**G**) *be the endomorphism ring of* **G**. *The equation* $\varphi^n = \varphi$ *for some* $n \in \mathbb{N}$, $n > 1$ *holds for* End(**G**) *if and only if* $|G| = p$.

Proof. If $|G| = p$ then every endomorphism ψ is of the form $\psi(x) = kx$, $k = 0,\ldots,p-1$ as **G** is a cyclic group and $\psi(0) = 0$. If $k = 0$, we have $\psi^2 = \psi$ and if $k = 1$, we have $\psi^2 = \psi$. In all other cases ψ is an automorphism of order p, hence $\psi^{p+1} = \psi$. On the other hand assume that $\psi^n = \psi$ holds and $|G| = p^n$ with $n > 1$. Consider $G = \langle g_1\rangle + \langle g_2\rangle + \ldots + \langle g_k\rangle$ as a direct product of simple p–groups $\langle g_i\rangle$. The map $\bar{f}(g_1) = g_2$ and $\bar{f}(g_i) = e$, $i = 2,\ldots,k$ can be extended to an endomorphism $f{:}G \longrightarrow G$. We have $f^3 = f^2$. Therefore an equation $\varphi^n = \varphi$ cannot hold for any $n \in \mathbb{N}$, $n > 1$.

In the following we use the notation of [Schweigert 79].

Theorem 2.4.11. *Let* $(G;+)$ *be a group of order* p, p *a prime number, and* R *be a subring of* End(G). *Then the* R–module G *is prepolynomially complete.*

Proof. The polynomial functions $\psi{:}G^n \longrightarrow G$ are of the form $\psi(x_1,\ldots,x_n) = a_1x_1+\ldots+a_nx_n + a_{n+1}$, where $a_i \in G = \{0,1,\ldots,p-1\}$. On the other hand every function of this form is a polynomial function. We conclude that the clone P(**G**) of the polynomial functions of **G** is the clone of all quasilinear functions on G = {0,...,p–1}. Hence P(**G**) is maximal and therefore **G** is prepolynomially complete as well as a group and a R–module.

§ 2.5. A criterion of primality

Two semigroups **A**, **B** can be defined by different set Id **A**, Id **B** of semigroup identities if and only if **A** and **B** generate different varieties of semigroups. In this case one can find identities which hold for HSP(**A**) but not for HSP (**B**) and vice versa. We call these identities separating identities (compare also 3.5). For an example consider [Schweigert 75].

The aim of this section is to find separating identities for the full transformation monoid and their maximal submonoids. These identities yield hyperidentities by which we can characterize primal algebras. A lot of results of this section are due to [Denecke–Pöschel 1988 a, b.]

Theorem 2.5.1. *Let* A *be a finite set and let* H *be a proper subsemigroup of the full transformation semigroup* H_A *on* A. *Then*

$$\mathrm{Var}(H) \subsetneq \mathrm{Var}(H_A).$$

Proof. We give a sketch of the proof which in its main part is due to P.P.Palfy. We start with some known facts which can be proved by group theoretic methods. Let n = |A| and let **H**,**S**,**P**, resp., denote the operator of taking homorphic images, subgroups and direct products, respectively.

Fact 1. Let H be a semigroup, G a group (both finite). If $G \in Var(H)$, then G belongs to the variety generated by the subgroups (i.e. subsemigroups which are groups) of H.

Fact 2. If $S_n \in \mathbf{HS}(G_1 \times G_2)$ (G_1, G_2 finite groups) then $S_n \in \mathbf{HS}(G_1)$ or $S_n \in \mathbf{HS}(G_2)$. Consequently, if $S_n \in Var\{G_i \mid i \in I\}$ $(= \mathbf{HSP}\{G_i \mid i \in I\})$, where $G_i (i \in I)$ are finite groups and I is finite, then $S_n \in \mathbf{HS}(G_i)$ for some $i \in I$.

Fact 3. The subgroups G of H_A are of the following form

$$G \leq \{f \in H_A \,|\, \mathrm{Im}\, f = \mathrm{Im}\, e,\ \ker f = \ker e\},$$

where $e \in H_A$ is the identity element of the group G, in particular, e is idempotent (here and in the following e does not necessarily denote the identical function on A). The mapping $f \mapsto f \mid m$ is an embedding of G into $S_{\mathrm{Im} f}$. In particular we have

$$|G| \leq |\mathrm{Im} f|! \leq n!.$$

Now we can prove the theorem. Suppose H is a proper subsemigroup of H_A such that $Var(H) = Var(H_A)$. Without loss of generality we can assume that H is maximal in H_A. Let $\{G_i \mid i \in I\}$ be the set of the subgroups of H. Since $S_A \in Var(H_A) = Var(H)$ we get from Fact 1 and Fact 2 that $S_A \in \mathbf{HS}(G_i)$ for some $i \in I$. Thus $n! = |S_A| \leq |G_i|$ and by Fact 3 ($|G_i| \leq n!$) we have $|S_A| = |G_i|$. Consequently, $S_A = G_i$, i.e. H contains S_A. Since every $f \in H_A$ with $|\mathrm{Im} f| = n$ together with S_A generates all elements of H_A, there is single maximal subsemigroup of H_A containing S_A, namely

$$H = S_A \cup \{f \in H_A \mid |\mathrm{Im} f| \leq n-2\}.$$

By 2.3, $\lambda(f) \leq n-2$ for all $f \in H$. Thus, by 2.12a, H but not H_A satisfies the hyperidentity $\psi^{n-2}(x) = \psi^{n-2+x(n)}(x)$, i.e. the semigroup identity $x^{n-2} = x^{n-2+x(n)}$, in contradiction to $Var(H) = Var(H_A)$.

Before we proceed to the general case we will point out the methods. These methods were developed in [Reischer, Schweigert, Simovici] for the case $A = \{0,1,2\}$.

Our starting point is the Slupecki criterion and Iablonskiǐ's list of the 18 maximal clones. Ultimately we wish to describe the maximal submonoids of unary operations by hyperidentities. For this it is enough to consider relations up to isomorphisms.

Let $(A;\rho_1)$ and $(B;\rho_2)$ be relational systems of the same type. A bijective map $\alpha: A \longrightarrow B$ is called an isomorphism if

$$(a_1,\ldots,a_n) \in \rho \quad \text{iff} \quad (\alpha(a_1),\ldots,\alpha(a_1)) \in \rho_2.$$

Let Endρ denote the set of unary operations preserving ρ. Let α be an isomorphism.

Proposition 2.5.2. *The monoid* $<\mathrm{End}_{\rho 1},\circ>$ *is isomorphic to the monoid* $<\mathrm{End}\rho_2,\circ>$.

Proof. For $f: A \to A$ put $f^{(\alpha)}(x) = \alpha(f(\alpha^{-1}(x)$ for all $x \in B$. It is almost immediate that $f \to f^{(\alpha)}$ is the required isomorphism. (This may be extended to 0_A and to clones preserving relations but we need only the particular form).

Put in a form of Slupecki criterion:

Theorem 2.5.3. *Let* A *be a finite set and* $|A| > 2$. *The set* X *of operation on* A *is complete if and only if*

(i) X *contains an essentially at least binary surjective operation and*

(ii) $[X]^{(1)} = 0^{(1)}$ *(i.e.* X *generates all unary operations on* A^A.*)*

Actually, Slupecki proved only sufficiency for f binary. There are several proofs and it has been generalized in several directions (the condition (ii) may be weakend to: $[X]^{(1)} = M$, where M may be the alternating group etc.). Four our purposes first we replace $[X]^{(1)} = 0_A^{(1)}$ by "$[X]^{(1)}$ is not included in a maximal submonoid of $<0^{(1)};\circ>$", and then describe the maximal submonoids functionally. For the description of the maximal submonoids we use Iablonskiĭ's list of maximal clones. It is clear that the maximal submonoids are among the $C^{(1)}$ where C is one of the 18 maximal clones. We eliminate 5 of them. The list is the following:

Lemma 2.5.4. *Let* $A = (0,1,2)$. *There are 13 maximal submonoids of* $<0_A^{(1)},\circ>$:

1) *The monoid* $\{ax + b,\ a,b \in A\}$ *of linear functions*

2) *The 3 monoids* End $\{i,j\}$ *where* $0 \leq i < j \leq 2$

3) *The 3 monoids* End $(\leq)$ *where* $\leq$ *is a chain (= linear order) on* A

4) *The 3 monoids* End θ *is a non–trivial equivalence relation on* A

5) *The 3 monoids* $\mathrm{End}\rho_{ij}$ *where* $\rho_{ij} = A^2\ \{(i,j),(j,i)\}$ $(0 \leq i \leq j \leq 2)$

(so called central relation).

Proof. The monoids $C^{(1)}$ with C maximal clone not on the list are: End$\{i\}$ $(i \in A)$ and the monoid of self–dual maps (i.e. maps satisfying $f(x+1) = f(x)+1$). The clone C of essentially unary or non–surjective operations has $C^{(1)} = 0_A^{(1)}$ and so may be omitted. Let $i \in A$ then $f \in \mathrm{End}\{i\}$ iff $f(i) = i$. It is easy to see that $f \in \mathrm{End}\rho_{jk}$ where $\{i,j,k\} = A$ (indeed, if $(a,b) \in \rho_{jk}$ and $a \neq b$, we have $a = i$ or $b = i$ and so $f(a) = i$ or $f(b) = i$, proving $(f(a),f(b)) \in \rho_{jk}$ hence $\mathrm{End}\{i\} \subsetneq \mathrm{End}\rho_{jk}$). Here $\mathrm{End}\rho_{jk}$ contains the constant j and so the inclusion is proper. Similarly, we prove that each self–dual f is linear. Clearly $f(1)+1 = f(0)+2$ hence $f(x) = x+f(0)$ for all $x \in \{0,1,2\}$, i.e. f is linear. The inclusion is again proper. We give a hyperidentity for all the above monoids. The monids of linear functions satisfies $F^2 = F$. This can be verified directly (as for $f(x) = ax + b$ we have $f^{(n)}(x) = a^n x+b(1+a+\ldots+a^{n-1})$ where $a^7 = a$ and $(1+a+\ldots+a^6 = 1)$, and follows also from a more general result in [Schweigert 83].

Note that a selfmap f of A does not satify $f^4 = f^2$ exactly if $f(x) \in \{x+1, x+2\}$. Indeed if the diagraph of f is not a cyclic permutation, then it has at most a cycle of length 2. Suppose that it has a cycle of length 2, say $\{0,1\}$. On the cycle clearly f^4 and f^2 agree. If $f(2) \neq 2$ then $f^2(2) \in \{0,1\}$ and again $f^4(2) = f^2(2)$. If f has no cycle then $f^2(2)$ is a fixed point of f for each $x \in A$ and so $f^4 = f^2$. Now it suffices to verify that neither of $x+1$ and $x+2$ belong to the monoids listed above in 2) – 5)(which is almost immediate). Thus we are led to the following

Theorem 2.5.5. *Let $|A| = 3$ and let X be a set of operations on A. Then X is complete if and only if*

(i) X *contains an essentially at least binary surjective operation and*

(ii) *the fundament* $[X]^{(1)}$ *satisfies neither* $F^4 = F$ *nor* $F^4 = F^2$.

Now we proceed to the general case. By e we denote the idenity map.

Proposition 2.5.6. *Let* $H \leq H_A$ *be a subsemigroup such that the algebra* $\langle A;H\rangle$ *has a proper subalgebra with carrier* $B \subset A$. *Let* $1 = \max\{\lambda(f) \mid f \in H\}$ *and* $n_B = \max\{|B|,|A\backslash B|\}$. *Then*

$$H \models \varphi^{l} = \varphi^{l+\kappa(n_B)}.$$

Moreover

$$H \models \varphi^{n-1} = \varphi^{n-1+\kappa(n)},\ n = |A|.$$

Proof. First consider the permutations $f \in H \cap S_A$. Since every such f preserves the subset B (and consequently also $A\backslash B$), the cycles of f have a length which belongs to $\{1,2,\ldots,B\}$ or $\{1,2,\ldots,|A\backslash B|\}$. Thus $f^{\kappa}(B) = e$, consequently $f^1 = f^{1+\kappa(B)}$. Now let $f \in H\backslash S_A$ (i.e. $\lambda(f) \geq 1$). Then the permutation $f = f|\mathrm{Im}\, f^{\lambda(f)}$ on $I = \mathrm{Im}\, f^{\lambda(f)}$ preserves the subsets $B \cap I$ and $(A\backslash B) \cap I$. Thus

$$g^{\kappa(n')} = e \ (n = \max\{|B \cap I|, |A \setminus B) \cap I|\}),$$

and we get

$$f^{\lambda(f)} = g^{\kappa(n')}(f^{\lambda(f)}) = f^{\lambda(f)+\kappa(n')}.$$

Because of $n' \leq n_B$ we get $f^1 = f^{1+\kappa(n_B)}$ by Lemma 2.1. Since $1 \leq n-1$, $n_B \leq n-1$, again by 2.1. it follows that also $\varphi^{n-1} = \varphi^{n-1+\kappa(n-1)}$ holds in H (this however directly follows from 2.4(a), too).

Proposition 2.5.7. *Let* $H \leq H_A$ *be a subsemigroup such that the algebra* $\langle A;H\rangle$ *has a non–identical automorphism* $s \in S_A$ *which consists of* r *cycles of length* $p \geq 2$ ($n = |A| = pr$. $1 \leq r \leq n-1$). *Then*

$$H \models \varphi^{r-1} = \varphi^{r-1+\kappa(n)}.$$

Moreover

$$H \models \varphi^{n-2} = \varphi^{n-2+\kappa(n)}.$$

Proof. If $f \in H \cap S_A$ then we have $f^{\kappa(n)} = id$ consequently $f^{r-1} = f^{r-1+\kappa(n)}$. Now let $f \in H\setminus S_A$. We will show that $\lambda(f) \leq r-1$. At first we note that f maps every cycle of the permutation s onto a cycle of s. In fact, $b = s^i(a)$ implies $f(b) = f(s^i(a)) = s^i(f(a)) \neq f(a)$ for $1 \leq i \leq p-1$. Since s has only r cycles (of equal length p), we get $\mathrm{Im}\, f^{r-1} = \mathrm{Im}\, f^r$. Consequently $\lambda(f) \leq r-1$. By 2.4(a) and 2.1 we get $f^{r-1} = f^{r-1+\kappa(n)}$. Finally, since $r-1 \leq n-2$, we get $f^{r-2} = f^{r-2+\kappa(n)}$ for all $f \in H$.

Proposition 2.5.8. *Let* H *be a semigroup of* H_A ($n = |A|$) *such that the algebra* $\langle A;H\rangle$ *has a non–trivial congruence relation* θ. *For* $B \subset A$ *with* $2 \leq |B| \leq n-1$ *let* θ_B *be the equivalence relation with blocks (congruence classes)* $B,\{c\}$ ($c \in A\setminus B$) *(i.e.* B *is the only non–trivial block). Then we have:*

(1) *If θ is not of the form θ_B (B ⊂ A) then*

$$H \models \varphi^{n-2} = \varphi^{n-2+\kappa(n)};$$

(2) *If $\theta = \theta_B$ for some $B \subset A$, $2 \leq |B| \leq n-1$. then*

$$H \models \varphi^{n-1} = \varphi^{n-1+\kappa(n-1)}.$$

Proof. (2): For $f \in H \setminus S_A$ the indicated identity is fulfilled trivially. For $f \in H \cap S_A$, however, f preserves B since every permutation in H must map any block onto a block; thus $H \cap S_A$ satisfies the identity in (2).

(1): For $f \in H$ with $\lambda(f) \leq n-2$ (in particular for $f \in H \cap S_A$) the identity in (1) is satisfied. If there were some function $f \in H$ with $\lambda(f) = n-1$, then f should have the following form. There is an $a \in A$ such that a, f(a), $f^2(a), \ldots, f^{n-1}(a) = f^n(a)$ are all the elements of A. Since $f \in H$ preserves θ, every non–trivial block of θ must be of the form $\{f^i(a), f^{i+1}(a), \ldots, f^{n-1}(a)\}$ $(i \in \{1,2,\ldots,n-2\})$. Consequently, θ is of the form θ_B in contradiction to the assumption in case (1).

We need the following result due to G. Rousseau.

Theorem. *A function $f \in O_A$ is Sheffer iff the algebra $\langle A;f\rangle$ has no proper subalgebras, no non–identical automorphism and is simple.*

Theorem 2.5.9. (Denecke, Pöschel) *The algebra $\langle A;f\rangle$ ($f \in O_A$) of prime power cardinality n is primal iff it satisfies none of the following hyperidentities:*

(i) $$\varphi^{n-1} = \varphi^{n-1+\kappa(n-1)},$$

(ii) $$\varphi^{n-2} = \varphi^{n-2+\kappa(n)},$$

(φ unary operation symbol). It is easy to see that $n = |A|$ is a prime power iff $\alpha(n-1) \neq \kappa(n)$.

Proof. If $\mathbf{A} = \langle A;f \rangle$ is primal then $H_A = T(\mathbf{A})$. However, H_A does not satisfy neither (i) nor (ii) (since $\kappa(n-1) \neq \kappa(n)$). Converesly, if **A** is not primal, then, by 3.1, **A** has a proper subalgebra – and therefore satisfies (i) by 2.5 – or **A** has a non–trivial congruence – and therefore satisfies (i) or (ii) by 2.7 – or **A** has a non–trivial automorphism, say s. If s consists of cycles of equal length, then, by 2.6, **A** satisfies (ii). Otherwise some power of s has fixed points. Since the fixed points of an automorphism constitute a subalgebra of **A**, **A** satisfies (i) by 2.5. Consequently, if **A** is not primal then (i) or (ii) are satisfied.

With computations of the same kind the following can be shown

Theorem 2.5.10. *An algebra* $\mathbf{A} = \langle A;f \rangle$ ($f \in 0_A$, $|A| \geq 2$) *is primal iff it does not satisfy the following unary hyperidentity:*

$$\varphi_2\varphi_1^{n-2}\varphi_2\varphi_1^{n-1}(x) = \varphi_2\varphi_1^{n-2+\kappa(n)}\varphi_2\varphi_1^{n-1}(x).$$

Theorem 2.5.11. (Denecke,Pöschel) *Let* $\mathbf{A} = \langle A;f \rangle$ *be a finite algebra* ($|A| \geq 2$). *Then* **A** *is primal iff it does not satisfy the binary hyperidentiy*

$$\psi^{n-2-2}\psi^{\top}\psi^{n2-1}(x_2,x_1) = \psi^{n2-2+\kappa(n2)}\psi^{\top}\psi^{n-1}(x_1,x_2).$$

where

$$\psi(x_1,x_2) = (\varphi(x_1,x_2),\varphi'x_1,x_2)),\ \psi^{\top}(x_1,x_2) = \varphi'(x_1,x_2)).$$

With the following concept these results can be seen from another point of view [Schweigert 89].

Definition 2.5.12. *An equation* $F^r = F^s$, $r < s$ *for the semigroup* S *is called irreducible if for every equation* $F^k = F^l$, $k < l$, *which hold for* S *we have* $r \leq k$ *and* $s \leq l$.

Notation 2.5.13. We call a hyperidentity of the form $F^r = F^s$, $r < s$ for the algebra $\mathbf{A} = (A,\Omega)$ irreducible if the equation $F^r = F^s$ irreducible holds for the semigroup $T_1(\mathbf{A})$ of one–place term functions. In the following $\kappa(n)$ is the least common multiple of the numbers $1,\ldots,n$; $\kappa(n) = \text{lcm}\{1,\ldots,n\}$.

Theorem 2.5.14. *Let* $\mathbf{A} = (A,\Omega)$ *be an algebra with an essentially at least binary surjective operation,* $|A| = n = p^m$, p^m *a prime power,* $\mathbf{A}$ *is primal if and only if*

$$(*)\quad F^{n-1} = F^{n-1+\kappa(n)}$$

is an irreducible hyperidentity for $\mathbf{A}$.

Proof. (1) Let $\mathbf{A}$ be primal, $A = \{1,\ldots,n\}$ and $g{:}A \longrightarrow A$ defined by $g(x) = x-1$ for $x \neq 1$ and $g(1) = 1$. Then we have $\lambda(g) = n-1$ and hence $\lambda(T_1(A)) = n-1$. The permutation group on A has the exponent $k(n)$ [Hall] p. 54 and hence $\pi(T_1(A))\ \kappa(n)$. We have that (*) is an irreducible hyperidentity.
(2) Let (*) hold as an irreducible hyperidentity. Hence the hyperidentities

$$(**)\quad F^{n-2+\kappa(n)} = F^{n-2} \text{ and } F^{n-1+\kappa(n-1)} = F^{n-1}$$

do not hold as $\kappa(n-1) < \kappa(n)$ for n a prime power. By the above results of Denecke and Pöschel $\mathbf{A}$ is primal.

Remark. The results of Denecke and Pöschel are proved by Rosenberg's completeness theorem. One may ask whether one can find an elementary proof. Indeed this is the case for $n = 2$.

Proposition 2.5.15. *If (*) is irreducible then* $T_1(\mathbf{A})$ *contains a cyclic permutation of order* p^m.

Proof. Because of (2.2) we have function $f_1,\ldots,f_k$ such that $\kappa(n) = \mathrm{lcm}\{\pi(f_i)$ $i = 1,\ldots,k\}$. For the prime number $n = p^m$ we have a function f with $\pi(f) = p^m \cdot s$ and hence by proposition 2.2.8 a function g with $\pi(g) = p^m$. Now g is a permutation on $A_g := g^{\lambda(g)}[A]$ and therefore consists of disjunct cycles such that the lcm of the length of these cycles is $\pi(g) = p^m$. We conclude that g is a cyclic permutation on A consisting of a single cycle of length p^m.

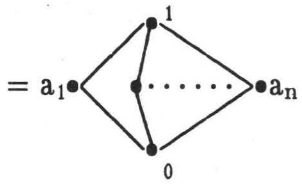
Proof for the case $n = 2$. By the above proposition 2.5.15 all permutation of A, $|A| = 2$, are in $T_1(A)$. Furthermore because of $\lambda(T_1(A)) = 1$ at least one constant function is in $T_1(A)$ and hence both of them $T_1(A)$ contain all one–place functions and **A** is primal. One should mention that this also includes a proof for $n = 2$ of the result of Denecke and Pöschel as theorem 2.5.14 is equivalent to their results.

§2.6 Algebraic monoids

Most results of this chapter are due to [Reichel, Schweigert]. We consider the following

Representation–Problem: Let M be a monoid and V be a variety. Is there an algebra $\mathbf{A} \in V$ such that M is isomorphic to the monoid $P_1(\mathbf{A})$ of 1–place polynomial functions of **A**?

Proposition 2.6.1. *Every finite monoid is isomorphic to a submonoid of the monoid* $P_1(L)$ *of the modular lattice* M_n *for some* n, $n \geq 3$.

$M_n =$ 

Proof. Consider $M_n = \{0,1,a_1,\ldots,a_n\}$ and the monoid $K = \{f \mid f{:}M_n \longrightarrow M_n,$ $f(0),f(1) = 1\}$ of self–maps on the set a: Every such self–map is order–preserving and the monoid K is the symmetric monoid S_n.

For $n \geq 3$ this lattice M_n is order–polynomially complete [Schweigert 74] and hence S_n is isomorphic to a submonoid of $P_1(M_n)$. Every finite monoid is isomorphic to some submonoid of the symmetric monoid S_n for some n.

The construction of proposition 2.6.1 is some bit crude. The monoid M may be a very small submonoid of $P_1(M_n)$. In the following we present a construction which is rather tight. To prepare the proof of this result we need some constructions on graphs.

Notation. End G denotes the monoid of endomorphisms of the graph G. End L is the monoid of order–endomorphisms of the lattices L. An order–endomorphism of L is a monotone function $f{:}L \longrightarrow L$. An order–endomorphism which preserves 0 and 1 is called $\{0,1\}$–endomorphism of L. End$\{0,1\}$ is the monoid of the $\{0,1\}$–endomorphism.

Proposition 2.6.2. *The monoid* End G *of a simple graph* G *is isomorphic to the monoid* $\mathrm{End}_{\{0,1\}}$ L *of* $\{1,0\}$–endomorphism *of a lattice* L *constructed from* G.

Proof. According to the above construction we define an embedding $\varphi: G \longrightarrow L$ by

$$\varphi(P_j) = A_j \qquad j = 1,\dots,p$$

$$\varphi(a_i) = B_i \qquad i = 1,\dots,q$$

Let $f \in \mathrm{End}\, G$. Then we define a monotone function $f^{\frown}$ on L by

$$f^{\frown}(A_j) := \varphi(f(P_j)) \qquad j = 1,\dots,p$$

$$f^{\frown}(B_i) := \varphi(f(a_i)) \qquad i = 1,\dots,q$$

$$f^{\frown}(1) := 1,$$

$$f^{\frown}(0) := 0$$

Obviously $f^{\frown}$ is monotone and preserves 0 and 1. On the other hand for every $\{0,1\}$–endomorphism L we have an endomorphism of G and hence $\mathrm{End}\, G \simeq \mathrm{End}_{\{0,1\}} L$.

Proposition 2.6.3. *Every finite monoid is isomorphic to the monoid* $\mathrm{End}_{\{0,1\}} L$ *of* $\{0,1\}$–endomorphism *of finite atomistic lattice* L *of height* 2.

Proof. According to [Hredlin Pultr] for every finite monoid M there exists a finite simple, undirected graph G such that the monoid End G is isomorphic to M.

Theorem 2.6.4. *Every finite monoid is isomorphic to a submonoid of the monoid* $P_1(L)$ *of polynomial functions of finite atomistic lattice* L *of height* 2.

Proof. We consider the lattice L constructed above and add two special elements a,b to L such that for a,b the lower neighbor is 0 and the upper neighbor is 1. According to [Schweigert 74] L is an order polynomially complete lattice and hence every orderendomorphism is an unary polynomial function of L. Hence $\mathrm{End}_{\{0,1\}} L$ is a submonoid of $P_1(L)$. We consider the connected simple graph G presented in [Frucht 50] with q vertices $P_1,\dots,P_q$ and p edges $a_1,\dots,a_p$. To this graph G corresponds a lattice L with $p + q + 2$ elements $1, 0, A_1,\dots,A_p, B_1,\dots,B_q$. The vertices are considered as the atoms $A_1,\dots,A_p$ of L and the edges are the upper neighbors $B_1,\dots,B_q$ of the atoms. An element B_i is the join of at least two atoms namely the vertices incident with this edge. We add a greatest element 1 and a least element 0 to have an atomistic lattice L of height 3.

Definition 2.6.5. *A monoid* M *is called algebraic if there exists an algebra* A *such that* $P_1(A) \simeq M$. *A variety* S *of monoids is algebraic if there exists a variety* V *such that* $S = HSP(T_1(V))$.

Example 2.6.6. The variety $W = \{p^2 = p;\ p \circ q = p\}$ of monoids is not algebraic.

Proof. Assume that V is a variety with $W = HSP(T_1(V))$. Let A be a non—trivial algebra from V. Then $p_1(x) = x$, $p_2(x) = a$ are two polynomial function of A. Obviously $p \circ q = p$ does not hold for $p = p_1$ and $q = p_2$. But if $p \circ q = p$ holds for $T_1(V)$ it also holds for $P_1(A)$ according to theorem 2.3.2.

Problem 2.6.7. *Characterize the varieties of monoids which are algebraic.*

Problem 2.6.8. *Given the variety* V *of your choice. Which are algebraic monoids for* V*?*

§ 2.7 The k–ary monoids of term operations

For the algebra **A** the k–ary monoid $\mathrm{Clone}_k(\mathbf{A})$ is defined in the following way. $\mathrm{Clone}_k(\mathbf{A})$ consists of all maps $F: A^k \longrightarrow A^k$ such that each component of F is defined by some k–ary terms of the algebra **A**.

Especially $\mathrm{Clone}_k(\mathbf{A})$ contains as components the distinguished maps p_j^k, $p_j^k(x_1,\dots,x_k) = x_j$ and hence the identity function $E: A^k \longrightarrow A^k$. Furthermore $\mathrm{Clone}_k(\mathbf{A})$ is closed under composition.

$\mathrm{Clone}_k(\mathbf{A})$ is a submonoid of End $\mathbf{A}^k$ the monoid of all endomorphism. This section reports on results of Hyndman, McKenzie and Taylor on this topic.

If **A** is finite, with $|A| = n$, then the monoid equation $u^{N-1} = u^{N-1+P}$ holds in $\mathrm{Clone}_k(\mathbf{A})$ where $N = n^k$, and $P =$ the least common multiple of all positive integers $\leq n^k$.

In the following we search for equations of this form (see [Hyndman, McKenzie, Taylor] and present values for some varieties.

	k=2	k=3	k=4
sets	N=1 P=2	N=2 P=6	N=3 P=12
semilattices	N=2 P=2	N=5 P=6	N=10 P=12
distributive lattices	N=2 P=2	N=5 P=6	N=10 P=60
modular lattices	N=2 P=2	N=5 P=6	?
lattices	N=2 P=2	∞	∞
$\mathbb{Z}_2$–modules	N=2 P=6	N=3 P=4·3·7	N=3 P=4·3·5·7
Sets	N=3 P=12	N=5 P=60	N=7 P=8·3·5·7
Boolean algebras	N=3 P=12	N=7 P=8·3·5·7	N=15 $P=2^4\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13$

It is conceptually useful to have an alternative description of $\mathrm{Clone}_k(\mathbf{A})$.

Lemma 2.7.1. *For any algebra* $\mathbf{A}$ *and any* k, *the monoid* $\mathrm{Clone}_k(\mathbf{A})$ *is dually isomorphic to the monoid* $\mathrm{End}\ \mathrm{F}_\nu(\mathrm{k})$ *of all endomorphisms of the free algebra* $\mathrm{F}_\nu(\mathrm{k})$ *on* k *generators, for* $\mathrm{V} = \mathrm{HSP}\mathbf{A}$.

Proof. If an element F of $\mathrm{Clone}_k(\mathbf{A})$ is defined by the k–tuple of terms $(q_0,\ldots,q_{k-1})$, then there exists a unique endomorphism σ of $\mathrm{F}_\nu(\mathrm{k})$ that maps each free generator x_i to the term q_i (considered as an element of the free algebra $\mathrm{F}_\nu(\mathrm{k})$). We leave it to the reader to check that this is a bijective correspondence that reverses multiplication.

In some of the subsequent sections, we will fix an algebra $\mathbf{A}$ and look for monoid equations w = w' and positive integers k such that $\mathrm{Clone}_k(\mathbf{A})$ satisfies w = w'. According to the next corollary, we cannot – except in trivial cases – expect this relation to hold for fixed $\mathbf{A}$, w and w' as $k \to \infty$. Here End k denotes the monoid of all self–maps of a k–element set.

Corollary 2.7.2. If $\mathbf{A}$ is any algebra of more than one element, and $k \geq 1$, then End k is dually isomorphic to a submonoid of $\mathrm{Clone}_k(\mathbf{A})$.

Corollary 2.7.3. Let the algebra $\mathbf{A}$ have more than one element. For any monoid equation w = w' and any $k > 0$, if $\mathrm{Clone}_k(\mathbf{A})$ satisfies w = w', then the dual of End k satisfies w = w'. Thus, if $\mathrm{Clone}_k(\mathbf{A})$ satisfies w = w' for all k, then w = w'.

Corollary 2.7.4. If $\mathbf{A}$ is an algebra of more than one element and $\mathrm{Clone}_k(\mathbf{A})$ satisfies $u^M = u^N$, then N–M is a common multiple of all the positive integers $\leq$ k.

Proof. Immediate from Corollary 2.7.3 and the fact that End k contains permutations of every order $\leq$ k.

Corollary 2.7.5. If **A** is an algebra of more than one element and $\mathrm{Clone}_k(\mathbf{A})$ satisfies $u^M = u^N$, then M = N, or N $\geq$ k–1 and M $\geq$ k–1.

Proof. Immediate from Corollary 2.7.3 and the fact that End k contains the function f such that f(0) = 1, f(1) = 2,...,f(k–2) = f(k–1) = k–1.

Let L be a language. The terms of L are defined in the usual recursive manner as follows:

(i) Each v_i (variable of L) is a term.

(ii) If $\alpha_0,\dots,\alpha_{k-1}$ are terms, then ${}_jF_i^k\,\alpha_{k-1}$ is a term for every i < k and every j $\geq$ 0 where ${}_jF_i^k$ is any k–ary operation symbol.

In a context where j takes on only the value 0, we can simplify notation by writing F_i^k for ${}_0F_i^k$.

$[w]_i^k$ is now defined by recursion on the length of w as follows:

(iii) $[e]_i^k = v_i$.

(iv) $[u_jw]_i^k = {}_jF_i^k[w]_0^k\dots[w]_{k-1}^k$.

Notice that, upon taking w to be e, $[u_j]_i^k$ is defined by condition (iv) to be ${}_jF_i^kv_0\dots v_{k-1}$. If, as in much of our paper, our monoid words w contain only the variable $u = u_0$, then $[w]_i^k$ will contain only the operation symbols ${}_0F_i^k = F_i^k$. In this case, we can simplify the recursive definition, as follows:

(v) $[e]_i^k = v_i$.

(vi) $[uw]_i^k = F_i^k[w]_0^k\dots[w]_{k-1}^k$.

Lemma 2.7.6 is a generalization of the defining condition (iv), in the sense that if we take $v = u_j$ in the lemma, we come back to condition (iv). In general, if β is a term whose variables are among $v_0,\dots,v_{k-1}$, and $\alpha_0,\dots,\alpha_{k-1}$ are any terms, then $\beta(\alpha_0,\dots,\alpha_{k-1})$

denotes the result of simultaneously substituting α_i for v_i in β ($i = 1,\ldots,k-1$). The following two properties may be regarded as a definition of substitution (by recursion in the length of β).

(vii) $v_i(\alpha_0,\ldots,\alpha_{k-1}) = \alpha_i$ $(i < k)$.

(viii) For any m–ary operation symbol F we have

$$(F\beta_0\ldots\beta_{m-1})(\alpha_0,\ldots,\alpha_{k-1}) = F\beta_0(\alpha_0,\ldots,\alpha_{k-1})\ldots\beta_{m-1}(\alpha_0,\ldots,\alpha_{k-1}).$$

Lemma 2.6.6. *For any monoid–theoretic terms* v *and* w,

$$[vw]^k_i = [v]^k_i\Big[[w]^k_0,\ldots,[w]^k_{k-1}\Big].$$

Proof. By induction on the length of v. If v is a single letter, then the proof may be left to the reader (it reduces to condition (iv), as we said above). Alternatively, $v = u_jv'$, and we may calculate

$$w[vw]^k_i = [u_jv'x]^k_i = {}_jF^k_i[v'w]^k_0\ldots[v'w]^k_{k-1}$$

(2) $$= {}_jF^k_i[v']^k_0\Big[[w]^k_0,..,[w]^k_{k-1}\Big]..[v']^k_{k-1}\Big[[w]^k_0,..,[w]^k_{k-1}\Big]$$

(3) $$= \Big[{}_jF^k_i[v']^k_0\ldots[v']^k_{k-1}\Big]\Big[[w]^k_0,\ldots,[w]^k_{k-1}$$

(4) $$= [u_jv']^k_i\Big[[w]^k_0\ldots,[w]^i_{k-1}\Big]$$

(5) $$= [v]^k_i\Big[[w]^k_0,\ldots,[w]^i_{k-1}\Big]$$

Here equation (2) holds by induction, equation (3) holds by condition (viii) in the definition of substitution, equation (4) holds by condition (iv) of the recursive definition of $[w]^k_i$, and equation (5) holds by the given factorization of v.

The next lemma concerns words w, w' in two letters f and g, and self–maps $\bar{f}$ and $\bar{g}$ of a set A. If $w = h_0h_1h_2\ldots$ (with each h_i either f or g), then $\bar{w}$ will denote the self–map $\overline{h_0}\cdot\overline{h_1}\cdot\overline{h_3}\ldots$ defined by composition of functions. (If w is the empty word, then $\bar{w}$ denotes the identity function.) The lemma may be well known.

Lemma 2.7.7. *Suppose that* A *is an infinite set, and that* $\bar{f}, \bar{g}:A \longrightarrow A$ *are injective maps, with* $F = \bar{f}(A) \cap \bar{g}(A)$ *a finite set. Then for any words* w *and* w', *if* $\bar{w}$ *and* $\bar{w}$' *have equal ranges, i.e.,* if $\bar{w}(A) = \bar{w}'(A)$, *then* w = w'. *Consequently, the maps* $\bar{f}$ *and* $\bar{g}$ *are free generators of a free monoid of self–maps of* A.

Let V be a variety and $F_\nu(k)$ denotes the free V–algebra on k generators and $C_k(V)$ the clone of $F_\nu(k)$.

Theorem 2.7.8. *Suppose that* **V** *has only finitely many constant operations, and that* $F = F_\nu(k)$ *contains a subalgebra isomorphic to* $F = F_\nu(2k)$. *Then* $C_k(V)$ *satisfies no nontrivial monoid equation.*

Proof. By Lemma 2.6.1., we may prove the theorem for the monoid End F instead of $C_k(\mathbf{V})$. As we remarked above, it will suffice to find two functions $\bar{f}$ and $\bar{g}$ in this monoid that satisfy the conditions of Lemma 2.6.7. Let **B** be a subalgebra of F that is isomorphic to $F_\nu(2k)$. We let $\bar{f}$ map the k free generators of F bijectively to the first k free generators of **B**, and let $\bar{g}$ map the k free generators of f bijectively to the last k free generators of **B**. It is clear that an element in the ranges of both $\bar{f}$ and $\bar{g}$ must be a constant; since there are only finitely many constants, and since $\bar{f}$ and $\bar{g}$ are clearly injective, we have satisfied the hypotheses of Lemma 14. Therefore End F contains a generic monoid.

Let A_k denote the (for our purposes, unique) primal algebra of k elements. In the next theorem we will see that $\mathrm{Clone}_k(A_k)$ and $\mathrm{Clone}_k(A_{k+1})$ do not satisfy the same equations (and hence that the varieties $HSPA_k$ and $HSPA_{k+1}$ do not satisfy the same hyperidentities of the type $[w \approx w']^k$).

Theorem 2.7.9. *For each positive* k*, define* N *and* P *via* $N = k^k$ *and* $P =$ *the least common multiple of all positive integers* $\leq k^k$.
Then $\mathrm{Clone}_k(A_k)$ *satisfies* $u^{N-1} = uu^{N-1+P}$, *but* $\mathrm{Clone}_k(A_{k+1})$ *does not.*

Corollary 2.7.10. *If A is a finite algebra of n elements, then* $\mathrm{Clone}_k(A)$ *satisfies* $u_0^{N-1}u_1u_0^{N-2} = u_0^{N-1}u_1u_0^{N-2+P}$ *for* $N = k^k$*, and* $P =$ *the least common multiple of all positive integers* $\leq n$.

In [Taylor 81] is proved that the variety $\mathrm{Sets}^{[m]}$ can be separated from $\mathrm{Sets}^{[m+1]}$ by a hyperidentity. We establish this same result with these methods. $(V^{[m]})^{[k]}$ is equivalent to $V^{[mk]}$ for any variety V, and so this last monoid is isomorphic to $C_1(\mathrm{Sets}^{[mk]})$. We see that this monoid is dually isomorphic to the monoid of all self–maps of a set of mk elements.

Theorem 2.7.11. $\mathit{Sets}^{[m]}$ *satisfies the hyperidentity* $[u^{N-1} = u^{N+P-1}]^k$, *where* $N = mk$*, and* $P =$ *the least common multiple of all positive integers* $\leq mk$.

Theorem 2.6.12. *If* A *is a non–trivial vector space over a field* K*, and if* $k \geq 1$*, then*

$$\mathrm{Clone}_k(A) \simeq M_k(K),$$

the monoid of $k \times k$ *matrices over* K.

It was proved that the order P of any $k \times k$ Boolean matrix T divides the least common multiple of the integers 2,3,...,k. It was proved that the index N of T is $\leq (k-1)^2+1$. From these results, we have

Theorem 2.7.13. *For any positive integer* k, *let* P *denote the least common multiple of all positive integers* $\leq$ k, *and let* $N = (k-1)^2+1$. *For the monoid* M_k *of Boolean matrices, we have*

$$M_k \textit{ satisfies } u^n \approx u^{N+P}.$$

At last we give results about some lattice varieties:

Theorem 2.7.14. *The variety of modular lattices* M *satisfies the hyperidentity* $[u^5 \approx u^{11}]^3$.

Theorem 2.7.15. *The variety of distributive lattices D satisfies the hyperidentity* $[u^{10} \approx u^{70}]^4$.

Part III Hyperidentitities and clone equations

§3.1 Clone of functions

Most algebraic structures are connected to certain sets of functions. Bijective functions give rise to the concept of permutation groups on A. If one abstracts from permutation groups one gets the concept of an abstract group. From semigroups of functions one proceeds to the concept of abstract semigroups.

This conceptual development has not fully reached the sets of operations on set A. We consider the algebraic structure of sets of operations (i.e. function in several variable on A) concerning composition and manipulation of variables and use the following definition of a clone (closed set) of operations.

Definition. *Let* H *be a set of functions on* A. *The clone* $\mathbf{H} = (H,*,\zeta,\tau,\Delta,e)$ *is an algebra of type* (2,1,1,1,0) *where the operations are defined in the following way:*

(1) $(f^*g)(x_1,\ldots,x_m,x_{m+1},\ldots,x_{m+n-1}) = f(g(x_1,\ldots,x_m), x_{m+1},\ldots,x_{m+n-1})$ *for an* n–ary *function* f *and an* m–ary *function* g:

(2) $(\zeta f)(x_1,\ldots,x_n) = f(x_2,\ldots,x_n,x_1)$ *for an n–ary function* f, $n > 1$, $(\zeta f)(x_1) = f(x_1)$ *for any 1–ary function* f:

(3) $(\tau f) = (x_1,\ldots,x_n) = f(x_2,x_1,x_3,\ldots,x_n)$ *for an* n–ary *function* f, $n > 1$, $(\tau f)(x_1)$ - $= f(x_1)$;

(4) $(\Delta f)(x_1,\ldots,x_{n-1}) = f(x_1,x_1,x_2,\ldots,x_{n-1})$ *for an n–ary function* f, $n > 1$, $(\Delta f)(x_1) = f(x_1)$;

(5) $e(x_1,x_2) = x_1$.

We like to remark that any projection e_i^n, $e_i^n(x_1,...,x_n) = x_i$ is generated and hence contained in any clone. The clone of all functions on the set A is denoted by O_A.

Definition 3.1.2. *Let* $\mathbf{A} = (A,\Omega)$ *be an algebra. The clone* $T(\mathbf{A})$ *of the term functions of* A *is the subclone of* O_A *is generated by the operations of* $\mathbf{A}$. *The clone* $P(\mathbf{A})$ *of polynomial functions of* A *is the subclone of* O_A *which is generated by the operations of* $\mathbf{A}$ *and the constant functions*

$$c_a^n,\ a \in A,\ n \in \mathbb{N},\ c_a(x_1,...,x_n) = a.$$

There are several approaches to define "abstract" clones without relying on operations and give a presentation by equations. This includes I. G. Rosenberg, Malcev's preiterative algebra, Preprint Montreal 1976, I. G. Rosenberg, Malcev algebras for universal algebra term, Preprint Montreal 1989 and the work of Trkhimenko 1979. Furthermore a solution by W. Taylor is fulfilling all requirements within the frame work of categories.

Additional remark. In some application, especially in other branches of mathematics, it is very useful to have the concept of a n–clone.

Definition 3.1.3. *Let* H *be a set of* n*–place functions on set* A. *Then* n*–clone* $\mathbf{H} = (H,\circ,\zeta,\tau,\Delta,e)$ *is an algebra of type* $(2,1,1,1,0)$ *where the operations are defined in the following way.*

(1) $(f\circ g)(x_1,...,x_n) = f(g(x_1,...,x_n),x_2,...,x_n)$

(2) $(\zeta f)(x_1,...,x_n) = f(x_2,...,x_n,x_1)$

(3) $(\tau f)(x_1,...,x_n) = f(x_2,x_1,x_3,...,x_n)$

(4) $(\Delta f)(x_1,...,x_n) = f(x_1,x_1,x_2,...,x_{n-1})$

(5) $e(x_1,...,x_n) = x_1$

§3.2 Clone equations and hyperidentities

In our approach to the notion of an "abstract" clone we introduce the "clone with arity".

Definition 3.2.1. *Let* H *be a set of functions on* A. *The algebra* $H = (H;*;\xi,\tau,\Delta,e,\Box_n\ (n \in \mathbb{N}))$ *of type* (2,1,1,1,0,1,...) *is called a clone with arity where the operations* ,ξ, τ,Δ,e *are defined as in 1.1 and the operations* $\Box_n$ *are defined by*

6) $(\Box_n f)(x_1,\ldots,x_n) = f(x_1,\ldots,x_k)$ *if f is a k–ary function with* $k \leq n$ *and* $(\Box_n f)(x_1,\ldots,x_n) = f(x_1,\ldots,x_1 x_2,\ldots,x_n)$ *if* f *is a k–ary function with* $k > n$.

The def 3.2.1 is equivalent to the def 3.1.1 in the sense that every function $(\Box_n f)$ can be generated by $*,\xi,\tau$,e. If f is a k–ary function with $k \leq n$ we consider $e_1^m * f$ which gives $(e_1^m * f)(x_1,\ldots,x_{m-k+1}) = e_1^m(f((x_1,\ldots,x_k),x_{k+1},\ldots,x_{m-k+1})$. For $m = n+k-1$ we have $\Box_n = (e_1^m * f)$. If f is a k–ary function with $k > n$ we apply Δ(k–n) times and we get $\Box_n f = (\Delta^{k-n} f)$.

Equations which hold for variety of clones are called clone equations. We denote the variables in these equations by $X,Y,Z,X_1,X_2,X_3,\ldots,$. We are considering the following example $\Delta(\Box_2 X) = e$. Obviously this clone equation holds for every term function of a lattice.

This clone equation $\Delta(\Box_2 X) = e$ yields the hyperidentity $F(x,x) = x$ which holds for any lattice. On the other hand we get from the hyperidentity

$$G(G(x,y,z),y,z) = G(x,y,z)$$

the clone equation

$$\Box_3 X * \Box_3 X = \Box_3 X.$$

3.2.2 Principle of clone equations

1) Every clone equation for a clone T(V) of term functions can be translated into a hyperidentity or a set of hyperidentities for V.

2) Every hyperidentity for V can be translated into a clone equation arity operators $\square_n$.

There is an important step between the observation that the unary functions on a set obey the laws of a semigroup and the abstract notion of a semigroup itself given by axioms. This step cannot be done for clones in the same way. The infinte set of axioms which are suggested in the following are presented only for a further discussion of this problem.

Axioms:

A1 $(\square_n X * \square_m Y)((e^{m+n-1}_1,\dots,(e^{m+n-1}_{m+n-1}) = \square_n X(\square_m Y((e^{m+n-1}_m,\dots,e^{m-n-1}_m),e^{m+n-1}_{m+1},\dots,e^{m+n-1}_{m+1})$

A2 $(\xi(\square_n X))(e^n_1,\dots,e^n_n) = \square_n X(e^n_2,\dots,e^n_n,e^n_1) \in \mathbb{N}\setminus\{1\} \quad n,m \in \mathbb{N}$

A3 $(\tau(\square_n X)(e^n_1,\dots,e^n_n) = \square_n X(e^n_2,e^n_1,e^n_3,\dots,e^n_n) \quad n \in \mathbb{N}\setminus\{1\}$

A4 $(\Delta(\square_n X))(e^{n-1}_1,\dots,e^{n-1}_{n-1} = \square_n X(e^n_1,e^n_1,e^n_2,\dots,e^n_{n-1}) \quad n \in \mathbb{N}\setminus\{1\}$

A5 $\xi(\square_1 X) = \tau(\square_1 X) = \Delta(\square_1 X) = \square_1 X$

A6 $\square_m(\square_n X)((e^m_1,\dots,e^m_m) = \square_n X(e^m_1,\dots,e^m_n) \quad n \in \mathbb{N}$ for $n \leq m$

A7 $\square_m(\square_n X)((e^m_1,\dots,e^m_m) = \square_n X(e^m_1,\dots,e^m_m,e^m_m,\dots,e^m_m) \quad n,m \in \mathbb{N}$ for $n > m$

A8 $\square_n X(e^n_1,\dots,e^n_n) = \square_n X, \quad n \in \mathbb{N}$

A9 $e^2_1(\square_n X),e^2_2) = \square_n X, \quad n \in \mathbb{N}$.

Remark 3.2.3. We have used a lot of abbreviations for these axioms. e^n_i for instance is generated from the nullary operation e contained in every clone. For the construction one uses the operation $\square_n$ and a permutation which effects the exchange of the 1^{st} place with the i^{th} place. Also the various kinds of compositions like

$\square_n X(e^n_1,\ldots,e^n_1)$ are thought as an abbrevation. The description of this by the substitution *, the permutations ξ,τ and the other operations is to lengthy. Of course an axiomatization of clones should fulfill similar requirements as abstract semigroups do for semigroups of functions.

Theorem 3.2.4. *Let* $C = (C;*,\xi,\tau,\Delta,e,\square,n(n \in \mathbb{N}))$ *be an algebra fulfilling the above set of axioms.* C *is isomorphic to a clone of functions if the following conditions hold:*

a) *There is a least natural number* n *such that* $\square_n X = X$ *for every* $X \in C$

b) $e^i_i \neq e^j_j$ *for all* $i,j \in \mathbb{N}$ *with* $i \neq j$.

Proof. To construct a clone of functions we take as a carrier set $A = \{Z \mid Z \in C\}$. To every $X \in C$ with a least natural number n for $\square_n X = X$ we define a function $f_X\colon A^n \longrightarrow A$ by $f_X(Z_1,\ldots,Z_n) = X(\square_n Z_1,\ldots,\square_n Z_n)$. Furthermore we consider $F_C = \{f_X \mid X \in C\}$ as a clone of function on the set A. The map $\alpha\colon C \longrightarrow F_C$ is defined by $\alpha(X) = f_X$ and we have to show that α is a clone isomorphism. α is injective because for $f_X = f_Y$ we have

$$X = X(e^n_1,\ldots,e^n_n) = f(e^n_1,\ldots,e^n_n) = f_Y(e^n_1,\ldots,e^n_n) = Y(e^n_1,\ldots,e^n_n).$$

α is compatible with * if $f_X * f_Y = f_{X*Y}$ holds. If f_X is n–place and f_Y is m–place we have

$$\begin{aligned}
& f_X * f_Y(e^{m+n-1}_1,\ldots,(e^{m+n-1}_{m+n-1}) \\
&= f_X(f_Y(e^{m+n-1}_1,\ldots,e^{m+n-1}_m)e^{m+n-1}_{m+1},\ldots,e^{m+n-1}_{m+n-1}) && (F_C \text{ is a clone of functions}) \\
&= \square_n X(\square_m Y(e^{m+n-1}_1,\ldots,(e^{m+n-1}_m),(e^{m+n-1}_{m+1},\ldots,e^{m+n-1}_{m+n-1}) && (\text{by definition}) \\
&= \square_n X * \square_m Y(e^{m+n-1}_1,\ldots,e^{m+n-1}_{m+n-1}) && (\text{by axiom scheme A1}) \\
&= X * Y(e^{m+n-1}_1,\ldots,e^{m+n-1}_{m+n-1}) && (\text{by hypothesis}) \\
&= f_{X*Y}(e^{m+n-1}_1,\ldots,e^{m+n-1}_{m+n-1}) && (\text{by definition})
\end{aligned}$$

We proved $f_X * f_Y = f_{X*Y}$ only for some special elements of A. But it is easy to see that it holds also for all the other elements if one uses composition and (A1). The other operations like ξ, τ, etc. we get in a similar way.

Comment. It is impossible to give an axiomatization of a clone with the property that the models of the axioms are precisely the clones isomorphic to concrete clones. The reason for this is that condition a) of the theorem 3.2 is not preserved under ultraproduct constructions.

Remark 3.2.5. Every subclone D of a clone of functions on a set is again a clone of functions. This holds because the elements of D as a subset of C are functions which are closed under the operations of a clone.

Remark 3.2.6. A countable power of a clone C of finitary functions gives rise to a clone which contains infinitary functions. For this consider $C^\infty = \{(f_1,f_2,f_3,\dots) \mid f_i \in C, \quad i = 1,2,3,\dots\}$. The sequence $(e_1^1,e_1^2,e_1^3,\dots,)$ with $e_1^i(x_1,\dots,x_i) = x_1$, $i = 1,2,3,\dots$ can be considered as a function but of infinite arity.

Remark 3.2.7. Not every homorphic image of a clone of functions is again isomorphic to a clone of functions. Already A.I. Mal'cev has found the following clone $M = \{c,a,a^2,a^3,\dots\}$ where one defines $a^k * a^l = a^{k+l}$, c unit, $\xi a^l = a^l$, $\tau a^l = a^l$ and $\Delta a^l = a^{l \setminus 1}$ for k,l,n $\in$ N. This clone comes up when a clone of functions is factorized by the congruence relation κ. We have $(f,g) \in \kappa$ if and only if the arity of f is equal to the arity of g. This equivalence relation is compatible with the operations $*$, ξ, τ, Δ and hence a congruence relation (in the sense of universal algebra). Whereas M is yet isomorphic to the clone of functions of the one element set, this is not the case for

$H = \{c\}$ which is homorphic image of M, yet is not isomorphic to the clone of functions of any set.

Finally we like to pose the following problem and ask for an improvement of the solution presented above.

1) **Problem 3.2.8** Describe a variety C which contains (properly) the clones of function as algebras.

2) The variety C should be axiomatized by a few identities or schemes of identities.

3) The part of identities of C which correspond to hyperidentities should be clearly presented.

4) The subclass of algebras of C which correspond to clones of functions should be clearly presented.

§3.3 Varieties generated by clones of polynomial functions

In the following we consider the clone $P(\mathbf{A})$ of polynomial functions of an algebra $\mathbf{A}$ but the results also hold for the clone $T(\mathbf{A})$ of term functions.

Proposition 3.3.1. *Let* $f{:}A \longrightarrow B$ *be a surjective homomorphism from an algebra* $\mathbf{A}$ *onto an algebra* $\mathbf{B}$. *Then there is a surjective clone homomorphism from* $P(\mathbf{A})$ *to* $P(\mathbf{B})$ *(or respectively from* $T(\mathbf{A})$ *to* $T(\mathbf{B})$*).*

Proof. We define $\alpha{:}P(\mathbf{A}) \longrightarrow P(\mathbf{B})$ recursively by $\alpha(c^n_a) = c^n_{f(a)}$ for every constant function c^n_a, and $\alpha(e^n_{i(\mathbf{A})}) = e^n_{i(\mathbf{B})}$ for every projection e^n_i, $e^n_i(x_1,\dots,x_n) = x_i$, on A. Every polynomial function $\varphi \in P(\mathbf{A})$ has a representation by a word. Obviously we can extend the definition α to φ using this word. But α does not depend on the choice of this word and hence is well defined. We also have that α is surjective because to every word of the polynomial algebra $B[x_1,\dots,x_n]$ there is a corresponding word of $A[x_1,\dots,x_n]$. Clearly α preserves the operations of a clone.

Proposition 3.3.2. *Let* $\mathbf{A} = \prod_{i\in I} \mathbf{A}_i$ *be the direct product of a family* $\{\mathbf{A}_i \mid i \in I\}$ *of algebras of the variety* V. *Then* $P(\mathbf{A})$ *is isomorphic to a subdirect product* $\{P(\mathbf{A}_i) \mid i \in I\}$.

Proof. We consider the projections $p_i{:}A \longrightarrow A_i$ which are surjective homomorphisms from $\mathbf{A}$ onto $\mathbf{A}_i$. By proposition 3.2.1 we have that $\alpha_i{:}P(\mathbf{A}) \longrightarrow P(\mathbf{A}_i)$ are surjective clone homomorphisms. We consider $\prod_{i\in I} P(\mathbf{A}_i)$ and the function $\gamma{:}P(\mathbf{A}) \longrightarrow \prod_{i\in I} P(\mathbf{A}_i)$ defined by $p_i(\gamma(\varphi)) = \alpha_i(\varphi)$. We have that γ is a clone homomorphism because of α_i. Let $\gamma(\varphi) = \gamma(\psi)$ for some $\varphi,\psi \in P_n(\mathbf{A})$. If $\varphi(a_1,\dots,a_n) \neq \psi(a_1,\dots,a_n)$ for some $a_1,\dots,a_n \in A$ then $p_j(\varphi(a_1,\dots,a_n)) \neq p_j(\psi(a_1,\dots,a_n))$ for some $j \in I$. We have $\alpha_j(\varphi) \neq \alpha_j(\psi)$ and hence $\gamma(\varphi) \neq \gamma(\psi)$. It is clear that $\gamma(P(\mathbf{A}))$ is a subclone of $\prod_{i\in I} (\mathbf{A}_i)$.

Proposition 3.3.3. *Let* $\mathbf{B}$ *be a subalgebra of* $\mathbf{A}$. *Then* $P(\mathbf{B})$ *is isomorphic to a homomorphic image of a subclone of* $P(\mathbf{A})$.

Proof. If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ we consider the clone of polynomial functions $P = \{\varphi \mid \varphi \in P(\mathbf{A}),\ \varphi(x_1,\dots,x_n) \in B[x_1,\dots,x_n]\}$ where $B[x_1,\dots,x_n]$ denotes the polynomial algebra of $\mathbf{B}$. It is clear that $\alpha{:}P \longrightarrow P(\mathbf{B})$ defined by $\alpha(\varphi) = {}^{\varphi}/_{B}$ is a surjective clone homomorphism. On the other hand P is a subclone of $P(\mathbf{A})$.

All the above results also hold for the clones T(**A**) of term functions.

Theorem 3.3.4. *Let* V *be a variety and* W = HSP(T(V)) *the variety of clones generated by* T(V). *If* A *is an algebra of* V *then* T(**A**) *is in the variety of* W *of clones.*

An example of application is

Proposition 3.3.5. *Let* V *be variety of lattices.* V *is the variety of distributive lattices if and only if* $\square_2 x \circ \square_2 x = \square_2 x(F^2(x,y) = F(x,y))$ *hold for the variety*

W = HSP(T(V)) *of clones.*

§3.4. Projection algebras

Let **A** = (A,Ω,) be an algebra of type $(n_1,\ldots,n_\delta,\ldots,)$. Then **P** = $(A,\{e_1^i\}\ i \in I)$ with $e_1^i(x_1,\ldots,x_{ni}) = x_1$. $I = \{n_1,\ldots,n_\delta,\ldots\}$ is a derived algebra from **A**. If **A** has at least an n–ary operation with $n \geq 2$ then every projection is generated.

Definition 3.4.1. **P** = (A; Ω) *is called a projection algebra if every operation of* **P** *is a projection.*

If we have to test whether a given hyperidentity * holds for an algebra **A** we also have to test whether * holds for the projection algebras **P**.

Let V be a variety then the trivial subvariety defined by x = y is a solid subvariety. We present here some results on the phenomena that there are a lot of varieties which have only trivial solid subvarieties [Denecke, Lau, Pöschel, Schweigert]

Proposition 3.4.2. *Every congruence modular variety has only a trivial solid subvariety. (In particular, if it is not trivial then it is not solid).*

Proof. Let V be a non–trivial solid variety. Then V contains all projection algebras of the same type. The congruence lattice of a projection algebra $P = (A_j, e^i, i \in I)$ is the lattice of all equivalence relations on A. For $|A| \geq 4$ this lattice is not modular anymore.

Proposition 3.4.3. *Let* **A** *be an algebra such that* $T(\mathbf{A})$ *contains a constant function. Then* $HSP(\mathbf{A})$ *has only a trivial solid subvariety.*

Proof. For this constant term function $t(x_1,\dots,x_n) = a$ we would have $d(x) = t(x,\dots,x) = a$ and hence $d(x) = d(y)$ which is never satisfied in a non–trivial projection algebra.
Instead of testing a hyperidentity by projection algebra one can take also a syntactical point of view. Every valid hyperidentity can be derived from hyperidentities of projection algebras. Hence we like to present a hyperidentity basis for projection algebras. Misusing the language we call the variety of type τ generated by the projection algebra of type τ the variety of projection algebras of type τ (compare 3.2.5 – 3.2.7).

Notation 3.4.4. For $n \geq 1$ let $\Gamma\langle n\rangle$ consist of n–ary hyperidentities

$$F(x,\dots,x) = x$$

$$F(F(x_{11},\dots,x_{1n}),\dots,F(x_{n1},\dots,x_{nn})) = F(x_{11},\dots,x_{nn})$$

Theorem 3.4.5. *For* $n \geq 1$ $\Gamma(n)$ *is a basis for all hyperidentities of type* $\langle n\rangle$ *for the variety of projections algebras of type* $\langle 2\rangle$.

Notation 3.4.6.

$M(n,m)$:

$$F(G(x_{11},\dots,x_{1m}),\dots,G(x_{n1},\dots,x_{nm}) = G(F(x_{11},\dots,x_{n1}),\dots,F(x_{1m},\dots,x_{nm}))$$

Theorem 3.4.7. *$\Gamma(n) \cup \{M(n,n)\}$ is a basis for all hyperidentities of type $<n,n,n,\dots,>$ of the variety of projection algebras.*

Example. $F(x,G(y,z)) = G(F(x,y),F(x,z))$ is a valid hyperidentity (it holds for the variety of lattices). Consider

$$\Gamma(2) \cup \{M(2,2)\} F(x,G(y,z)) \underset{\uparrow\,\Gamma(2)}{=} F(G(x,x),G(y,z)) \underset{\uparrow\,M(2)}{=} G(F(x,y),F(y,z)).$$

Fact. If a hyperidentity holds from some variety then it can also be derived by the hyperidentities which hold for the variety of projections algebras.

Theorem 3.4.8. *Let $\Gamma = \bigcup\limits_{n\geq 1} \langle n\rangle \cup \bigcup\limits_{\substack{n\geq 1\\ n\geq 1}} \{M(n,m)\}$. Then Γ is a countably infinite basis for the hyperidentities of any type for the variety of projection algebras.*

Theorem 3.4.9. *The hyperidentity of any type for the variety of projection algebras are non–finitely based.*

For all these results we have omitted the proofs. Knoebel has shown that the variety RB of rectangular bands is generated by the projection algebras (A,e_1^2) and $(A;e_2^2)$. Therefore the variety RB of rectangular bands satisfies a hyperidentity $S = T$ if and only if $S = T$ can be derived from the hyperidentities of the variety of projection algebras of type $\langle 2 \rangle$. Obviously (RB) is the minimal (non–trivial) solid variety of type $\langle 2 \rangle$.

In §3.5 we are presenting the results of S. Wismath on hyperidentities of the variety RB of regular bands. The above hyperidentity bases are given and the theorem that the hyperidentities of any type are not finitely based is proved.

Remark 3.4.10 Let $\tau = (n_0,\ldots,n_\delta,\ldots)$ be a type with $n_0 > 1,\ldots,n_\delta > 1,\ldots$. There exists only one minimal solid variety of type τ namely the variety of projections algebra of this type. These varieties are also described in the work of Plonka [66].

§3.5 Hyperidentity bases for rectangular bands

The results of this section are due to S. Wismath [Wismath 91] if not quoted otherwise.

We have already recognized that every hyperidentity S = T of variety V has also to hold for the variety of projection algebras of the same type. Knoebel has shown that the variety RB is generated by the class of all projection algebras $(A;e_1^2)$ and $(A;e_2^2)$.

Lemma 3.5.1 *[Penner 84] The variety* RB *of rectangular bands satisfies the hyperidentity* S = T *if and only if* S = T *can be derived from the hyperidentities of the variety of projection algebras of type* $\langle 2 \rangle$.

Besides the notations of section 3.4 we use the following:

Notations 3.5.2

$H(V)$	set of all hyperidentities of any type satisfied by the variety V.
$H^m(V)$	set of all hyperidentities in $H(V)$ with at most m hypervariables.
$H_n(V)$	set of all hyperidentities in $H(V)$ with at most n variables.
$H(V)\langle\underline{n}\rangle$	set of all hyperidentities in $H(V)$ with hypervariable of type $\langle n,n,n,..,n\rangle$ (k–factors).
$H(V)\langle\underline{n}\rangle$	set of all hyperidentities in $H(V)$ with hypervariables of type $\langle n,n,n...\rangle$ (infinetely many factors).

Lemma 3.5.3. *The variety* RB *of rectangular bands fulfill* $\Gamma(n)$.

We have already seen that both hyperidentities hold for projection algebras. The second hyperidentity essentially says that variables in a type $\langle n\rangle$ hyperidentity for RB which are nabp (not accessible by projections) may be eliminated. This is perhaps more clearly seen in the equivalent set of n+1 hyperidentities

$$F(x,\dots,x) = x$$

$$F(F(x_{11},\dots,x_{1n}),x_2,\dots,x_n) = F(x_{11},x_2,\dots,x_n)$$

$$F(F(x_1,\dots,x_{n-1}),F(x_{n1},\dots,x_{nn})) = F(x_1,\dots,x_{n-1},x_{nn})$$

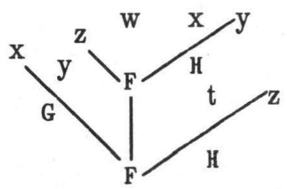
For the case $\langle n\rangle = \langle 2\rangle$ the associative hyperidentity $F(x,F(y,z)) = F(F(x,y),z)$ can be derived from $\Gamma(2)$.

Theorem 3.5.4 *For* $n \geq 1$, $\Gamma(n)$ *is a basis for all hyperidentities of type* $<n>$ *of the variety* RB *of rectangular bands.*

It will sometimes be useful to consider hyperterms concerning their formation tree.

Example. (G(x,y),F(z,u,H(x,y)),H(t,z))

Formation tree



In particular, if x is a variable of a hyperterm, we record the sequence of hypervariables and turnings in the path in the tree from the root to the variable x, in the string of x. In our example the second occurence of x is recorded as (F2,F3,H1,x). The height of a variable y is the number of not necessarily distinct hypervariables in its string. The height of a hyperterm is the maximum of the heights of its variables. The above hyperterm is of height 3.

Proof. Let $P = Q$ be any hyperidentity satisfied by RB. If P and Q both consist only of a single variable x_i, then $P = Q$ must be trivial. Thus we will assume that at least one of P or Q involves at least one occurrence of the hypervariable F. Now there is a unique hyperterm $P^*(Q^*)$ of height 1, such that $\Gamma\langle n\rangle \vdash P = P^*(Q = Q^*)$. For if P involves no occurrences of F, use the idempotent hyperidentity from $\Gamma\langle n\rangle$ to introduce one occurrence of F; if P involves more than one occurrence of F, use the two hyperidentities in $\Gamma\langle n\rangle$ to eliminate all but one occurrence of F.
Since RB $\models \Gamma\langle n\rangle$, we have RB $\models P^* = Q^*$, and $P^* = Q^*$ must model projections. But by definition all variables in $P^* = Q^*$ are abp, so $P^* = Q^*$ must be trivial. Therefore $\Gamma\langle n\rangle \vdash P = Q$, as required.

We will prove that $\Gamma\langle \underline{n}\rangle$ is a basis for hyperidentities of type $\langle n,n\rangle$ of RB by means of two lemmas. The first one deals with the special case of hyperidentities in which all variables are abp, the second one shows how any hyperidentity may be reduced to one of this special kind using $\Gamma\langle \underline{n}\rangle$.

Lemma 3.5.5. *If* $P = Q$ *is hyperidentity of type* $\langle n \rangle$ *for the variety of projection algebras and has all variables accessible by projection then*

$$\{F(x,...,x) = x, M(n,n)\} \vdash P = Q.$$

Proof. If $P = Q$ has the form $F(-) = x$, then the condition that all variables are abp ensures that all variables in $F(-)$ are x's, so that $P = Q$ is a consequence of idempotence. Hence we my now assume that $P = Q$ has the form $F(\ ) = G(\ )$. Again by the abp condition, after the first occurrences of F at the root of P, there can be no further F's in P. Moreover, if $F \neq G$, we may assume that every branch in P contains exactly one occurrence of G, since more than one G would lead to variables nabp, while if a branch has no G's we can always inflate the final variable on the branch, x say, into $G(x,...,x)$ using the idempotent hyperidentity.
Using the observations, we proceed by induction on the height of $P = Q$. Any hyperidentity of height 1 meeting these conditions must be trivial, so we begin with height 2. Then $P = Q$ would look like

$$F(H_1(\bar{x}_1),...,H_n(\bar{x}_n)) = G(K_1(\bar{y}_1),...,K_n(\bar{y}_n)),$$

where $\bar{x}_i$ and $\bar{y}_i$, $1 \leq i \leq n$, represent n–tuples of variables. If F and G are the same hypervariables, we subsitute for F the n–ary projection terms, to obtain n new hyperidentities $H_i(\bar{x}_i) = K_i(\bar{x}_i)$ of height 1, which must then be trivial. Thus $P = Q$ is trivial in this case. If $F \neq G$, the observations above show that we must have $H_i = G$ and $K_i = F$, for all $1 \leq i \leq n$, that is, $P = Q$ is actually $M(n,n)$.

Now consider $P = Q$ of height $k > 2$. If Q also has the form $F(-)$, we use the n projection terms to reduce to n hyperidentities of height $k - 1$ with the same properties. Then, $P = Q$ is a consequence of these, so by induction, $M(n,n)$ and $F(x,...,x) = x$ yield $P = Q$. So we now suppose Q has the form $G(-)$, where $G \neq F$. We give a procedure for forming a new hyperterm P^* from P. As above, every branch of P must contain exactly one occurrence of G. For each such branch, count the number of hypervariables other than G occurring on the path from F to G. Choose any such G where this number is maximal, say p. Now go back along the branch of this G to the previous hypervariable, say H. Each branch coming out of H must contain an occurrence of G, and by maximality of p these occurrences must also be at height p. So this part of P looks like H(G(–),...,G(–)), and we can use the medial identity to change it to G(H(–),...,H(–)). In this new identity, we repeat this process, first with any remaining G's at height p, then with G's at lower height. Eventually we reach a new hyperterm P* of the form G(–), such that $M(n,n) \vdash P = P^*$. Now then hyperidentity $P^* = Q$ still models projections and has all variables abp, and it has the form G( ) = G( ), so by earlier case it is a consequence of M(n,n) and idempotence. Thus M(n,n) and idempotence yield P = Q, as required.

Lemma 3.5.6. *For any hyperterm* P *there is a hyperterm* P* *with no variable which are not accessible by projection* (napb) *such that* $\Gamma\langle\underline{n}\rangle \vdash P = P^*$.

Proof. Obviously if P has no variables nabp, we may take P* to be P. We show how the hyperidentities in $\Gamma<\underline{n}>$ must be used to eliminate any variable x nabp in P. For any such variable x, there is a hypervariable F and induces $I \neq j (1 \leq i,j \leq n)$ such that the path form the root of P to x involves first F_i, then F_j.
If the two occurences of F are adjacent, then a part of P looks like

$$-F(-,F(-,R,-),-)-$$

where the second F occurs in the ith place of the first F, R is a hyperterm involving x which occurs in the jth place of the second F, and – indicates other hyperterms in P. We use the idempotent hyperidentity to inflate so that all n entries in the first F have the form F(–), then use the other hyperidentity in $\Gamma\langle n\rangle$ to reduce to

$$-F(-,\ldots,-,-,\ldots-),$$

there by eliminating the nabp variable x.

If two occurrences of F are separated by one or more other operation symbols, say $G_1,\ldots,G_k, k\geq 1$, then P has the form

$$-F(-,\ldots,-,G_1(\ldots G_k(-,\ldots F(-,R-)-)\ldots)\ldots)-\ .$$

Here we again use idempotence to inflate so that the last hypervariable before the second F has all entries of the form F(–); then use M(n,n) to replace the part $G_k(F(-),\ldots,F(-R-),\ldots F(-))$ by $F(G(-),G(-),\ldots,G(\ ))$. This moves the second occurrence of F one step closer to the first. By repeating this process we eventually reach a stage where the wo occurrencs of F are adjacent, when the method from above may be used to eliminate x. In this way all nabp variables in P may be eliminated, giving us P* as required.

Thereom 3.5.7. $\Gamma\langle\underline{n}\rangle$ *forms a basis for the hyperidentities of type* <n,n,n,...> *for the variety* RB *of rectangular bands.*

Corollary 3.5.8. Let $\Gamma = \bigcup_{n\geq 1} \Gamma\langle n\rangle \cup \bigcup_{\substack{n\geq 1\\ m\geq 1}} \{M(n,m)\}$. Then Γ is a countably infinite basis for the hyperidentities of any type for the variety RB.

Theorem 3.5.9. *The hyperidentities of any type of the variety* RB *are not finitely based.*

Proof. We will prove that for any two positive integers m and n, there is a hyperidentity H such that RB satisfies H, but H is not a consequence of $H^m(RB) \cup H_n(RB)$. Take $k = \max\{m,n\} + 1$. Define H to be the following hyperidentity, with one k–ary operation symbol F:

$$(F(x_1,x_2,\ldots,x_k),x_1,\ldots,x_1) = F(F(x_1,x_1,x_3,\ldots,x_k),x_1,\ldots,x_1) \quad (H).$$

Since H models projections, it is clear that RB $\models$ H. Now define an algebra $\underline{A} = (A;f)$ as follows. Take

$$A = \{a_1,\ldots,a_k,a_1a_2,\ldots,a_ka_{k-1}\},$$

the free rectangular band on the k generators $a_1,\ldots,a_k$. f is k–ary, given by

$$f(x_1,\ldots,x_k) = \left\{\begin{array}{ll} x_1x_2, & \text{if } \{x_1,\ldots,x_k\} = \{a_1,\ldots,a_k\} \\ x_1, & \text{otherwise}\end{array}\right\}.$$

Using f for the operation symbol in H leads to an identity which does not hold in $\underline{A}$, since the evaluation $x_1 = a_i$, $1 \leq i \leq k$, in the identity leads to $a_1a_2 = a_1$.

Therefore $\underline{\mathbf{A}}$ does not satisfy H. However, we claim that $\underline{\mathbf{A}}$ does not satisfy all the hyperidentities in $H^m(RB) \cup H_n(RB)$.

For if a hyperidentity involves at most n variables, or operation symbols all of arity $\leq$ m, since k > m,n follows that the only $\underline{\mathbf{A}}$–terms used in the hyperidentity amount to projections. So in this case $\underline{\mathbf{A}}$ satisfies the hyperidentity iff RB does.

Thus we see that H is a hyperidentity satisfied by RB, which is not a consequence of $H^m(RB) \cup H_n(RB)$. Therefore H(RB) is not finitely based.

Theorem 3.5.10. (Padmanabhan, Penner). *Let* SL *be the variety of semi–lattices. Then the following set* $H_2^{(2)}$ *of hyperidentities is a bases for all hyperidentities of type* $\langle 2\rangle$ *for* SL.

H(2): (1) $F(x,F(y,z)) = F(F(x,y),z)$

(2) $F(x,x) = x$ (3) $F(F(u,x),F(y,u)) = F(F(u,y),F(x,w))$

Proof. By the above hyperidentities we can present every hyperterm $T(x_1,\ldots,x_n)$ in a (normal) form $F\ldots F(F(x_a,x_{\pi(1)}),x_{\pi(2)},\ldots,x_{\pi(t)},x_b)$. Here we use the associativity (1) for F to put all hypervariables left hand, the comunativity (3) of the variables $x_{\pi(i)}$ inside x_a and x_b and the idempotency (2) for eliminating $x_{\pi(i)}$ if it appears twice. Inside we can order $\pi(1) < \pi(2) <\ldots< \pi(t)t \leq n-2$. Let $T(x_1,\ldots,x_n) = S(x_1,\ldots,x_m)$ be a hyperidentity of type $\langle 2\rangle$ which holds for SL. Then both sides can be presented in (normal) forms

$$F\ldots F(F(x_a,x_{\pi(1)},\ldots,x_{\pi(n-2)},x_b) = F\ldots F(F(x_c,x_{\mu(1)}),\ldots,x_{\mu(m-2)},x_d).$$

If we hypersubstitute F by the first projection we have $x_a = x_c$ (and respectively by the second $x_b = x_d$). Now we put $x_a = x_{\pi(1)}$ and $x_c = x_{\mu(1)}$. In this way we show that both sides have to be formally equal. Hence every hyperidentity of type $\langle 2\rangle$ is implied by H(2).

Problem 3.5.11. *Give the example of a variety* V *of type* $\langle 2\rangle$ *such that the hyperidentities of* V *of type* $\langle 2\rangle$ *are not finitely based and* V *is generated by an algebra* $A = (A;\circ)$ *with* A *as small as possible.*

§3.6. Normal and regular hyperidenties

Definition 3.6.1. *An identity* $t_1 = t_2$ *is regular if the set of all variables occurring p and q coincide.*

Example. $x\cdot y = y\cdot x\cdot y$

Notation. An identity is called to be "trivializing" if it is of the form $x = y$ (where x, y are different variables) or $x_k = t(x_1,\ldots x_n)$ where t is a term which is not a variable.

Definition 3.6.2. *An identity* $t_1 = t_2$ *is normal if* $t_1 = t_2$ *is not trivializing.*

It is now obvious how we have to define normal and regular hyperidentities. In this chapter we present beautiful results of Ewa Graczynska on this topic.

Notations.

Σ :	= set of identities of type τ
Mod (Σ) :	= variety of type τ defined by Σ
E(K):	= set of identities of a variety K
N(K):	= set of normal identities of K
R(K):	= set of regular identities of K
H(K):	= set of hyperidentities of K
NR(K):	= N(K) $\cap$ R(K)

One can consider N,R,H, as operators on classes of varieties. Let L(Mod(N(V)) be the lattice of all subvarieties of the variety Mod N(V). Then we have the following results:

Theorem 3.6.3. *If* V *is not a normal variety* (V $\neq$ Mod N(V)) *then the operator* N:L(V) $\longrightarrow$ L(Mod N(V)) *is an embedding of the lattice* L(V) *of the subvarieties of* V *into the lattice* L(Mod(N(V)).

Proposition 3.6.4. *Let* V *and* W *be two varieties of type* τ *with* N(V) $\subseteq$ E(W). *If* W *is not normal then* W $\subseteq$ V.

Proof. We present here a proof, proposed by N. Newrly without using the representation theorem for algebras from Mod(N(V)). Let f be a not normal identity from E(W). If f is a trivial identity of the form x = y, where x and y are different variables, then obviously W $\subseteq$ V. Assume, that f is of the form $x_k = p(x_1,\dots,x_n)$, where $1 \leq k \leq n$ and p is a proper term (i.e. not a variable). Consider the identity g of the form $x = p(x,\ldots,x)$, for a variable x. Obviously g is a consequence of f. If $E(V) = N(V)$, then obviously $W \subseteq V$. Otherwise, let e be a not normal identity of V. Assume, that e us if the form $x_2 = q(x_1,\ldots,x_m)$, with $1 \subseteq 2 \subseteq m$. If q is a variable, i.e. $x = y$ is an identity of V, then $V = W$, because $x = p(x,\ldots,x) = P(y,\ldots,y) = y$ is a proof of $x = y$ from $N(V) \cup \{g\} \subseteq E(W)$. If q is a proper term, take $\tau = \max(k,m)$. Then $x_1 = p(x_1,\ldots,x_{k-1}\ x_1,x_{k+1},\ldots,x_\tau) = P(x_1,\ldots,x_{k-1},\ q(x_1,\ldots,x_\tau) = q(x_1,\ldots,x_\tau)$ is a proof of $x_\tau = q(x_1,\ldots,x_\tau)$ from the set $N(V) \cup \{f\}$. This gives that e is an identity of E(W) and we conclude, that $E(V) = \mathcal{E}(N(V) \cup \{e\} \subseteq E(W)$, i.e. $W \subseteq V$.

Theorem 3.6.5. *If* V *is not a normal variety then the lattice* L(ModN(V)) *is isomorphic to the direct product of the lattice* **L**(V) *and a two – element lattice.*

Proof. Denote by $\mathbf{2} = (\{0,1\},\leq)$, the two–element lattice with $0 < 1$. Consider the following mapping $h{:}\mathbf{L}(V)\mathrm{x}\mathbf{2} \longrightarrow \mathbf{L}(Mod(N(V))$ given by the rule $h(K,O)) = K, h(K,1)) = Mod\ (N(K))$, for $K \in L(V)$. Then for $K_1,K_2 \in L(V)$ we obtain:

$$h((K_1,O) \cap (K_2,O)) = h(K_1 \cap K_2,O) = K_1 \cap K_2 = h(K_1,O)) \cap h((K_2,O)),$$

$$h((K_1,1) \cap (K_2,1)) = h(K_1 \cap K_2,1) = Mod(N(K_1 \cap K_2)) = h((K_1,1)) \cap ((K_2,1))$$

by Theorem 2. Let us notice, that for $K_1,K_2 \in L(V)$ we obtain the following inequalities in L(Mod(N(V)):

$$K_1 \cap K_2 \leq K_1 \cap Mod(N(K_2)) \leq Mod(N(K_1)) \cap Mod(N(K_2)) = Mod(N(K_1 \cap K_2)),$$

but $K_1 \cap K_2$ is not normal and $Mod(N(K_1 \cap K_2))$ covers $K_1 \cap K_2$ in the lattice **L**(Mod(N(V)), thus we get the equality: $K_1 \cap K_2 = K_1 \cap Mod(N(K_2))$, because the variety $K_1 \cap Mod(N(K_2))$ is not normal.

So we have:

$$h((K_1,O)\cap(K_2,1)) = h(K_1\cap K_2,O)) = K_1\cap K_2$$
$$= K_1\cap Mod(N(K_2)) = h((K_1,0))\cap h((K_2,1)).$$

Similarly we obtain that h is a join–homomorphism.

To complete the proof we should show that h is onto L(Mod(N(V). Let K' $\in$ L(Mod(N(V)). By Proposition 3.6.4 we conclude, that there are only two possibilities:

(i) K'$\in$ L(V) or (ii) E(K') = N(K').

In the case (i) we obtain that K' = H((K',O)), where K' $\in$ L(V).

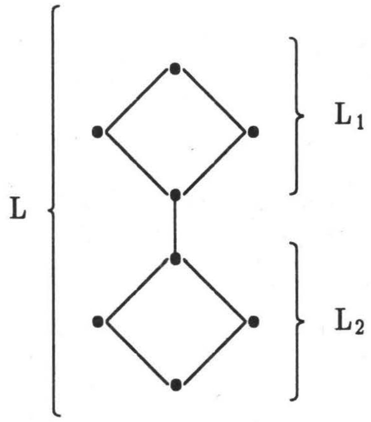
In the second case, we consider the variety K = K' $\cap$ V. Then K $\in$ L(V) and by Theorem 1,

$$h((K,1)) = Mod(N(K)) = Mod(N(K' \cap V))$$
$$= Mod(N(K')) \cap Mod(N(V)) = Mod(N(K')) = Mod(E(K')) = K',$$

thus we conclude that h is an isomorphism.

Recall that a lattice **L** is the double of its sublattice $\mathbf{L}_1$ if $L_2 = L\backslash L_1$ is also a sublattice of L and there exists an isomorphism $f:\mathbf{L}_1 \longrightarrow \mathbf{L}_2$ such that $x \leq f(x)$ in the lattice **L**.

Remark. Theorem 3.8.5. says that the lattice **L**(Mod(N(V)) is a double of the lattice **L**(V), for a given not normal variety V. But the double of a lattice $\mathbf{L}_1$ need not to be isomorphic to the direct product of $\mathbf{L}_1$ and a two–element lattice, as it can be shown by the following diagram:



The lattice **L** on the diagramm is the double of the lattice L_1 but is not isomorphic to the direct product of $\mathbf{L}_1$ and 2.

Recall, that a variety V is called solid if E(V) = H(V), i.e. all identities of V are satisfied in V as hyperidentities.

Example. Consider varieties of type (1) with one unary operation f(x). Take the variety V defined by the identity x = f(x). This variety is solid and not normal. Moreover, the variety Mod(N(V)) is also solid. Let us note the following:

Theorem 3.6.6. *Assume that* V *is a variety of unary type* τ *(i.e. such that* $\tau(T) = 1$ *and* V *is solid. Then* Mod (R(V)) *is solid.*

Proof. Assume that V is solid variety of unary type τ. Let p(x) and q(y) be two polynomial symbols of type τ. If p(x) = q(x) is a hyperidentity of V, then p(x) = q(x) is a hyperidentity of Mod(R(V), because any hypersubstitution of a regular identity of unary type is regular. By the same argument, if p(x) = q(y) is a hyperidentity of V, then p(x) = q(x) is a hyperidentity of Mod(R(V)). Thus R(V) = H(R(V)), i.e. Mod (R(V)) is solid.

Example. Consider the trivial variety T (i.e. defined by the identity $x = y$) of unary type with two unary operations f and g. Then T is solid, but Mod(N(T)) is not solid, because $f(x) = g(x)$ is its hypersubstitution and is not normal, i.e. Mod(N(T)) is not solid.

Generally, the normal (or regular) part of a solid variety need not to be solid. For example take trivial variety T in type (2). Then the identity

$$f(f(x,y),z) = f(f(x,z),y)$$

is a normal hyperidentity of T, but not a hyperidentity of Mod(n(T)), therefore Mod(n(T)) is not solid. The same example shows also that Mod (R(T)) may not be solid.

We do not know if a similar theorem as theorem 3.6.6 can be proved for normal varieties. However there are some similarities between normal parts of varieties and solid varieties. For example the fact that the variety Mod(N(V)) is a cover of not normal variety V, in the lattice **L**(Mod(N(V)) can be expressed for solid varieties in the following way.

Theorem 3.6.7. *Let* V *be a solid variety, which is not normal. Let* e *be a hyperidentity of* V *from* $H(V)\setminus N(V)$. *Then:* $E(N(H(V)) \cup \{e\}) = H(V)$.

Proof. We present here a syntactic method, which is useful for the next theorem on the word problem.

The inclusion $\subseteq$ follows from the fact that consequences of hyperidentites of V are hyperidentities of V and e has the form $x_k = r(x_1,\ldots,x_n)$, $1 \leq k \leq N$. If τ is a variable (different of x_k), then the inclusion obviously holds, If $r(x_1,\ldots,x_n)$ is a proper term

(i.e. r is not a variable) then take a hyperidentity $x_j = q(x_1,...,x_m)$ from the set $(H(V)\setminus N(H(V))$. We can assume, that $n = m$ (otherwise we treat r and q as terms on $l = \max(n,m)$ variables). If $k = j$ and $r(x_1,...,x_n) = q(x_1,...,x_n)$ is a normal hyperidentity of V then $N(H(V)) \cup \{e\} \vdash x_k = q$. If $k \neq j$ then let $r^*(x_1,...,x_n)$ denotes the term $r(x_1,...,x_{k-1},x_j,x_{k+1},...,x_n)$, obtained from r by substituting x_k by,x_j. Then $x_k = r \vdash x_j = r^*$ and $x_j = r^*$ is a hperidentity of V.
If q is not a variable, then $r^* = q$ is a normal hyperidentity of V and thus e, $x_j = r^*$. $\rho^* = q$, $x_j = q$ is a proof of $x_j = q$ from the set $N(H(V)) \cup \{e\}$. Otherwise, i.e. if q is a variable y, different of x_j, then $r(z,...,z) = z$, $r(z,...,z) = r(x_1,...,x_n)$ are hyperidentities of V, for any variable z different of x_k. Thus, $r(x_1,...,x_n) = r(z,...,),r(z,...,z) = z$, $x_k = z$, $x_j = y$ is a proof of $x_j = q$ from the required set.

Remark. The theorem 4 can be reformulated for varieties V in the following way. Let V be a variety and e be a hyperidentity of V. Assume that e is not normal. Then $H(V) = E(N(H(V)) \cup\{e\})$, i.e. each hyperidentity of V can be deduced from all normal hyperidentities of V and any fixed not normal hyperidentity of V.
We deal with axiomatic theories, i.e. varieties V of universal algebras, of a given type defined by a set Σ of axioms. We say that the word problem for the variety $Mod(\Sigma)$ is solvable, if there is an effective procedure to decide for any identiy $p = q$ of a given type, if it is a consequence of Σ.
Similarly for solid varities, defined by a set of hyperidentities.

Because consequences of normal identities (hyperidentities) are normal, thus if the variety Mod(N(V) is not normal, then there exists an identity (hyperidentity) from Σ which is not normal. Therefore, immediately from the proof of theorem 2 of [Graczynska 84] we get:

Theorem 3.6.8. *The word problem for a variety* V *is solvable iff the word problem for the variety* Mod(N(V)) *is solvable.*

Proof. Necessity is obvious. To prove the sufficiency, let e be a not normal identity from E(V), then either e is of the form $x = y$ or an identity of the form $x = r(x,\dots,x)$ is its consequence, where τ is a proper term. Therefore V is trivial and $N(V) = N(\tau)$ or $p(x_1,\dots,x_n) = q(x_1,\dots,x_n)$ belongs to E(V) if an only if $p^*(x_1,\dots,x_n) = q^*(x_1,\dots,x_n)$ is an identity of N(V), where $p^*(x_1,\dots,x_n)$ denotes the term obtained from $p(x_1,\dots,x_n)$ by substituting x_j by $r(x_j,\dots,x_j)$ for all $1 \leq j \leq n$ and similarly for q.

Similarly, we conclude:

Theorem 3.6.9. *The word problem for a solid variety* V *defined by a set* Σ *of hyperidentities is solvable iff the word problem for the variety* Mod(N(H(V)) *is solvable.*

Remark. Generally, operators H and N (H and R) do not commute. For example take the trivial variety T in type (2) and the identity $f(f(x,y),z) = f(f(x,z),y))$. Then this identity is normal (regular) hyperidentity of T, but it is not a hyperidentity of Mod(N(T)) (Mod(T)), i.e. N(H(T)) in not included in H(MOD(N(T)) and R(H(T)) is not included in H(Mod(R(T))).

Generally, only the opposite inclusion holds, namely:
$H(Mod(N(K))) \subseteq N(H(K))$ and $H(Mod(R(K)) \subseteq R(H(K))$, for any variety K, because $H(Mod(N(K)) \subseteq N(K)$ and $H(Mod(N(K)) \subseteq H(K)$ by the definition of H(V), for a variety V.

Remark. Some properties of the operator Hon varieties were considered in [Grazynska, Schweigert] in connection with Problem 4 of W. Taylor [Taylor 79].

Theorem 3.6.9. *For a variety* V, *the following conditions are equivalent:*

(i) *The word problem for* V *is solvable*

(ii) *The word problem for* Mod(N(V)) *is solvable,*

(iii) *The word problem for* Mod(R(V)) *is solvable,*

(iv) *The word problem for* Mod(N(R(V) *is solvable.*

§3.7. On the Unification of hyperterms

This chapter deals with the unification problems (For the notation see §3.8) The results are due to Ewa Gracynska. Before we proceed with our topic we like to mention the following.

Fact. Let V be a variety of type τ. Then the following are equivalent.
(i) H(N(V)) = N(H(V)), (ii) Mod(N(V)) is solid.

Proof. Let H(N(V)) = N(H(V)). Take a normal identity p = q from N(V). Consider a substitution $\sigma(p) = \sigma$ (q). Then $\sigma(p) = \sigma(q)$ belongs to H(N(V) = N(H(V)), i.e. it is a normal hyperidentity of V. Therefore p = q is a hyperidentity of Mod(N(V)) and we conclude, that Mod(N(V)) is solid.
Assume now, that Mod(N(V) is a solid variety, i.e. H(N(V)) = N(V). But H(V) $\subseteq$ E(V), therefore N(H(V)) $\subseteq$ N(V) $\subseteq$ H(N(V)). To show the converse, assume that p = q be a normal identity of V. By the assumption, we get that any hypersub-

stitution of $p = q$ is an identity of Mod(N(V). Thus $p = q$ is a normal hyperidentity of V. Similarly for H and R.

Notations.

$P(\tau)$	denotes the free algebra of type τ, the algebra of all terms of type τ.
$P(\tau)/=>$	denotes the quotient algebra of $P(\tau)$ by the equivalence relation (= equation) on terms of type τ.

Definition 3.7.1. *(see [Siekmann]). Let* s *and* t *be two terms of a given type* τ. *An identity* $s = t$ *is unifiable in a given algebra* **A** *of type* τ *if and only if there exists a homorphism* γ: $\mathbf{P}(\tau) \longrightarrow \mathbf{A}$ *such that* $\gamma(s) = \gamma(t)$.

Example. In a one–element algebra, any identity $s = t$ is unifiable.

Example. Take any algebra $\mathbf{A} = (A,(f_t : t \in T)$ and its one–element extension A^*, which is defined as follows: $A^* = A \cup \{1\}$, where 1 is a new element not belonging to A and $\mathbf{A}^* = (A^*(f_t^* : t \in T))$ with

$$f_t^*(a_1,\ldots,a_n) = 1, \text{ if } 1 \in \{a_1,\ldots,a_n\} \quad \text{and}$$
$$f_t^*(a_1,\ldots,a_n) = f_t(a_1,\ldots,a_n), \quad \text{otherwise.}$$

Then any identity $s = t$ is unifiable in $\mathbf{A}^*$. Namely γ can be defined as $\gamma(x) = 1$, for any variable x. Then $\gamma(s) = \gamma(t) = 1$ in A^*.

Definition 3.7.2. ***The first unification problem*** *(see [Siekmann]). Take a variety* V *of type* τ *and any two terms* s *and* t *of type* τ. *The problem is to decide whether or not* $s = t$ *is unifiable in the algebra* $\mathbf{P}(\tau)/=$ *(i.e. if there is a substitution* α, *such that* $\alpha(s) = \alpha(t)$ *is an identity of* E(V)).

Theorem 3.7.3. *Given two terms* s *and* t *and a variety* V *of type* τ. *Then*

(1.1) s *and* t *are unifiable in* V *iff they are unifiable in* Mod(R(V)),

(1.2) *the first unification problem is solvable in* V *iff it is solvable in* Mod(R(V)).

Proof. Let α be a substitution, such that $\alpha(s) = \alpha(t)$ is an identity of V. If x is a variable from $\mathrm{Var}(\alpha(s))\setminus\mathrm{Var}(\alpha(t))$ (or $\mathrm{Var}(\alpha(t))\setminus\mathrm{Var}(\alpha(s))$), then let $\beta(x) = y$ for any fixed variable y of $\mathrm{Var}(\alpha(t))$ $\mathrm{Var}(\alpha(s))$). Then for $\gamma = \beta \circ \alpha$ we obtain that $\gamma(s) = \gamma(t)$ is an identity of R(V), i.e. s and t are unifiable in the variety Mod(R(V)). Obviously this procedure gives a method, how to rewrite a decision algorithm for the first unification problem for E(V) to obtain an algorithm for R(V). The opposite implication is obvious.

The same fact holds for normal identities of a given variety V. Not normal identities are known under the name absorption law (see [*Ježek*, McNulty]).

Theorem 3.7.4. *Given two terms* s *and* t *of type* τ. *Then:*

(2.1) s *and* t *are unifiable in* V *iff they are unfiable in the variety* Mod(N(V));

(2.2) *the first unification problem is solvable in* V *iff it is solvable in* Mod(N(V)).

Proof. The sufficiency in (2.1) and (2.2) is obvious. We show the necessity. If E(V) = N(V) then the theorem is obvious. Let $x = p(x_1,\ldots,x_n)$ be an absorption law satisfied in V. If α is a substitution such that $\alpha(s) = \alpha(t)$ is an absorption law in V, i.e. $\alpha(s)$ is a variable y and $\alpha(t)$ is not a variable or $\alpha(t)$ is a different variable z), then take the substitution β with $\beta(w) = p(w)$ for any variable w, where p(w) is the term $p(w,\ldots,w)$. Let $\gamma = \beta \circ \alpha$. Then $\gamma(s) = \gamma(t)$ is a normal identity of V. This gives a procedure how to rewrite a decision procedure for the first unification problem for E(V) to N(V).

Next theorem shows the role of operators N and R in the problem of description of some special theories (called permutative by Siekmann or term finite by Ježek and McNulty.

Definition 3.7.5. *Let* E(V) *be an equational theory of a variety* V *of type* τ. V *is called term finite iff for any term p of type* τ *the class* $[\mathrm{p}]/=\mathrm{v}$ *is finite (i.e. there is only a finite number of terms* s *such that* s = t *is an identity of* V*).*
If the equational theory E(V) *of a variety* V *is term finite, then we say that* V *is term finite.*

From now on we assume that type ⌙ is not emtpy, i.e. that T ≠ ∅.

Proposition 3.7.6. *[Siekmann] Term finite theories are regular (i.e. if a variety* V *is term finite, then* E(V) = R(V))

Proposition 3.7.7. *[Ježek, McNulty] Term finite theories are normal (i.e. if* V *is term finite, then* E(V) = N(V)).

Example. The variety S of semigroups is term finite.

Example. Any variety with an idempotent law is not term finite.

Theorem 3.7.8. *Let* V *be a term finite variety of type* τ. *Then:*

(3.1) *V is not of the form* Mod(R(W) *for any nonregular variety* W *of type* τ,

(3.2) *V is not of the form* Mod(N(V) *for any not normal variety* W *of type* τ.

Proof. Assume that W is a non–regular variety of type τ and $V = Mod(R(V))$. By Lemma of [Plonka 69] one of the following conditions holds (where x and y denotes different variables):

(i) an identity $x = y$ belongs to $E(V)$;

(ii) type τ is unary and an identity of the form $p(x) = p(y)$ belongs to $E(V)$ where p is a proper term;

(iii) an identity of the form $x = p(x,y)$ belongs to $E(V)$, where p is a binary term and $x,y \in p(x,y)$;

(iv) an identity of the form $p(x,x) = p(x,y)$ belongs to $E(V)$, for some binary term p, such that $x,y \in p(x,y)$.

In case (i) takes a functional symbol $F(x_1,\ldots,x_n)$ of type τ. Then $x = F(x,\ldots,x)$ $= F(Fx,\ldots,x),x\ldots,x)) = \ldots$ constitute an infinite sequence of identities from $E(V)$.

In case (ii) we obtain infinite sequence $p(x) = p(p(x)) = p(p(p(x))) = \ldots$ of identities satisfied in V.

In case (iv) consider $p(x,x) = p(x,p(x,x)) = p(x,p(x,p(x,x))) = \ldots$.

Case (iii) is similar. This proves (3.1), by contradiction.

To prove (3.2) assume that W is a variety of type τ with an absorption law $x = p(x,\ldots,x)$. which must exists, because there are functional symbols of type τ. If $V = Mod(N(W))$, then

$$p(x,\ldots,x) = p(p(x,\ldots,x),x,\ldots,x)) = p(p(p(x,\ldots,x),x,\ldots,x),x,\ldots,x) = \ldots$$

constitute an infinte sequence of identities of $E(V)$, i.e. V is not a term finite variety, a contradiction.

Remark. Note that if type τ is empty, then the trivial variety T of type τ is not term finite, but varieties $Mod(R(T))$ and $Mod(N(T))$ are term finite.

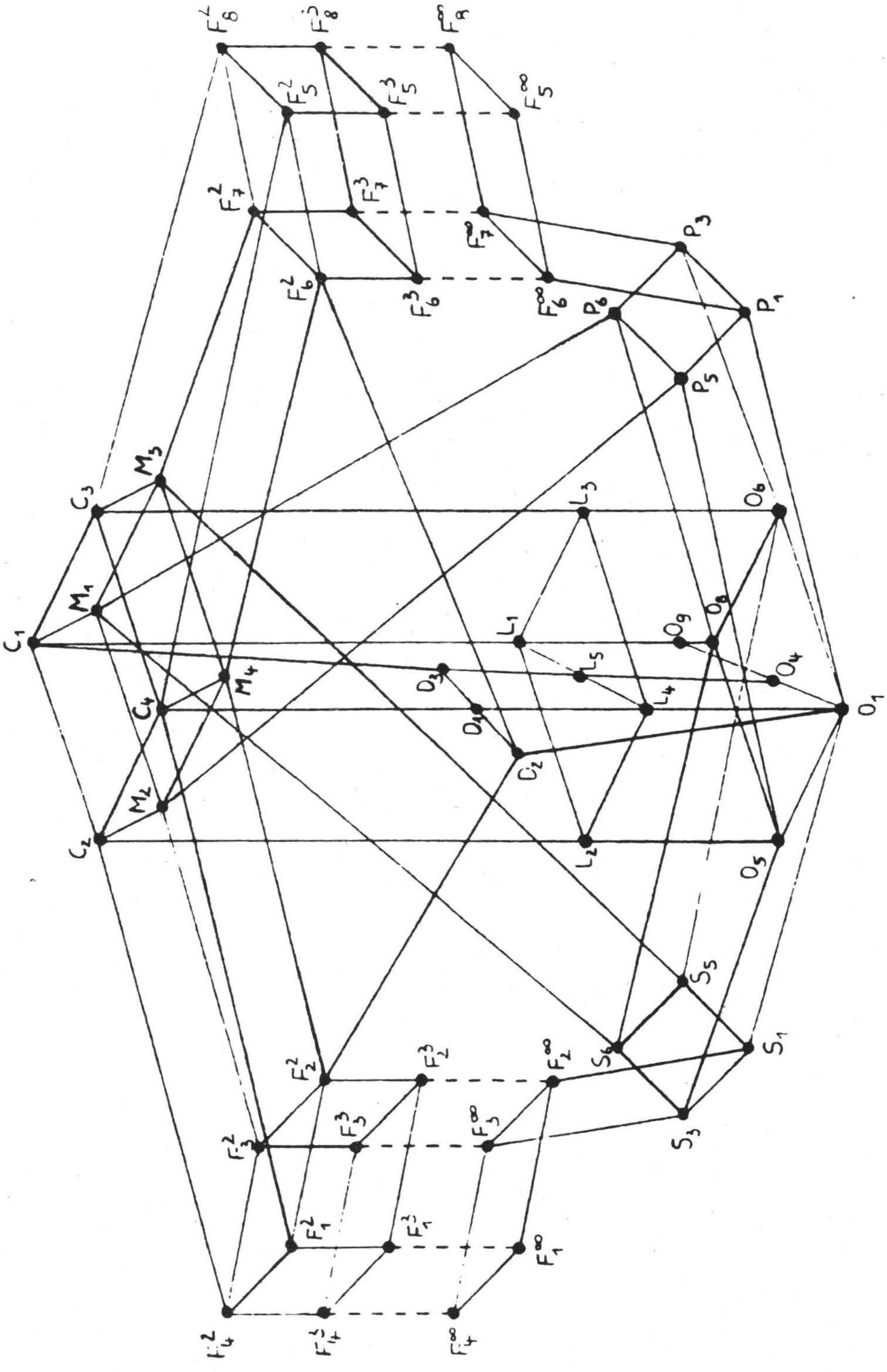
Corollary 3.7.10. The variety S of semigroups is not of the form Mod(R(W)) or Mod(N(W)) for any non–regular (or not normal) variety W of type (2).

Remark. The conditions (3.1) and (3.2) are not sufficient to describe term finite theories, which can be visualized by an easy example of the variety of semigroups defined by the identity $x^2 = x^4$.

In fact many of such examples can be produced, by taking any equation $p = q$ with the property, that p is a proper subterm of q.
The problem above is directly connected with practical questions appearing in computer programming (for example the procedure presented above is similar to so called occur check in Prolog programming ([Closcin],p. 224).

§ 3.8 Boolean Clones

Two algebras **A**, **B** of the same type can be defined by different sets Id **A**, Id **B** of identities if and only if A and **B** generate different varieties. In this case we say that **A** and **B** can be separated by identities.
In the following we show that non–isomorphic clones on the two–element can be separated by hyperidentities. Indeed every clone on $\{0,1\}$ is subdirectly irreducible (see 4.4) and the non–isomorphic clones on $\{0,1\}$ generate different varieties.
It is obvious that there are non–isomorphic clones on $\underset{\sim}{n} = \{0,1,..,n\}$ $n > 2$ which generate the same variety and hence cannot be separated anymore by hyperidenties. (see 4.5)



C_1 C_2 C_3 C_4
M_1 M_2 M_3 M_4
L_1 L_2 L_3 L_4 L_5
D_3 D_4
O_1 O_4 O_5 O_6 O_8 O_9
P_1 P_3 P_5 P_6
S_1 S_3 S_5 S_6
F_1^2 F_1^3 F_1^∞
F_2^2 F_2^3 F_2^∞
F_3^2 F_3^3 F_3^∞
F_4^2 F_4^3 F_4^∞
F_5^2 F_5^3 F_5^∞
F_6^2 F_6^3 F_6^∞
F_7^2 F_7^3 F_7^∞
F_8^2 F_8^3 F_8^∞

Let A be a finite nonempty set and let $O_A^{(n)}$ be the set of all n–ary functions $f:A^n \longrightarrow A$. We put $O_A = \bigcup_{n=1}^{\infty} O_A^{(n)}$ and consider the algebra $\mathbf{O}_A = (O_A;{}^*,\xi,\tau,\Delta,e_1^2)$. The lattice of all subclones of the clone $\mathbf{O}_A$ with $A = \{0,1\}$ was investigated by E.L. Post (see picture at the end of this paragraph). This lattice is atomic, dually atomic, countably infinite and every clone has a finite basis of generators. Clones which are symmetric in this picture are isomorphic.

There were several attempts to simplify Post's proof from 1920. A very detailed proof of Post's theorem was worked out by Jablonskij, Gawrilov and Kudrjawzev in 1966 [Jablonskij, Gawrilov, Kudrjawzev].

Other approaches using Mal'cev type theorems can be found in several papers especially by [McKenzie, McNulty, Taylor 87].

An elementary and short approach to the results of Post was presented by Lau in [Lau] which uses no theorems of universal algebra.

We present a description in detail for every Boolean clone in the following table. These results are due to [Denecke, Mal'cev, Reschke].

clone	description	generating system (as example)
C_1	set of all Boolean functions	$\{\wedge, N\}$, $\wedge$ conjunction $x\wedge y := xy$, N negation
C_3	$\{0\}$–preserving functions, i.e. $f(0,\ldots,0) = 0$	$\{\wedge,+\}$ + addition mod 2
C_4	$\{0\}$–, and $\{1\}$–preserving functions, i.e. $f(0,\ldots,0)=0$ and $f(1,\ldots,1) = 1$	$\{\vee, g_1\}$ $\vee$ disjunction, $g_1(x,y,z) = x\wedge(y+z+1)$

M_1	monotone functions	$\{\wedge, \vee, c_0^1, c_1^1\}$, c_0^1, c_1^1 unary constant functions
M_3	monotone, $\{0\}$–preserving functions	$\{\wedge, \vee, c_0^1\}$
M_4	monotone, $\{0\}$–, and $\{1\}$–preserving functions	$\{\wedge, \vee\}$
D_3	selfdual functions, i.e. $f(x_1,\dots,x_n)=Nf(Nx_1,\dots,Nx_n)$	$\{u_2, x+y+z, N\}$ $u_2(x,y,z) = xy \vee xz \vee yz$
D_1	selfdual, $\{0\}$–, and $\{1\}$–preserving functions	$\{u_2, x+y+z\}$
D_2	selfdual, monotone functions	$\{u_2\}$
L_1	linear Boolean functions, i.e. $f(x_1,\dots,x_n)=c_0+a_1x_1+\dots+a_nx_n$, $a_i \in \{0,1\}$	$\{+, N, c_0^1, c_1^1\}$
L_3	linear, $\{0\}$–preserving Boolean functions	$\{+, c_0^1\}$
L_4	linear, $\{0\}$–, and $\{1\}$–preserving Boolean functions	$\{x+y+z\}$
L_5	linear, self–dual functions	$\{x+y+z, N\}$
P_6		$\{\wedge, c_0^1, c_1^1\}$
P_3		$\{\wedge, c_0^1,\}$
P_1		$\{\wedge\}$
P_5		$\{\wedge, c_1^1\}$

O_9		$\{N,c_0^1\}$
O_8	clones consisting only of	$\{id, c_0^1, c_1^1\}$
	essentially unary functions	id = identity
O_4	functions	$\{N\}$
O_6		$\{id, c_0^1\}$
O_1		$\{id\}$
F_8^m	1–separating function of degree	$\{u_m, g_4\}$
	$m \geq 2$, i.e. each m–elementic sub–	$g_4(x,y,) = x \wedge Ny$
	set of $f^{-1}(1)$ has a common i–th	$u_m(x_1,\ldots,x_{m+1}) =$
	component of the value 1	$\bigvee_{i=1}^{m+1} x_1\ldots x_{i-1}x_{i-1}\ldots x_{m+1}$
F_5^m	1–separating of degree $m \geq 2$,	$\{u_m, g_3\}$,
	$\{0\}$–preserving functions	$g_3(x,y,z) = x(y \vee Nz)$
F_7^m	1–separating of degree $m \geq 2$,	$\{u_m, c_0^1,\}$
	monotone functions	
F_6^m	1–separating of degree $m \geq 2$, mo–	$\{g_2, u_2\}$
	notone, $\{0\}$–preserving functions	$g_2(x,y,z) = x(y \vee z)$
F_8^∞	1–separating functions, i.e.	$\{g_4\}$
	each subset of $f^{-1}(1)$ has a	
	common component of value 1	
F_5^∞	1–separating, $\{0\}$–preserving	$\{g_3\}$
	functions	
F_7^∞	1–separating, monotone functions	$\{g_2, c_0^1\}$
F_6^∞	1–separating, monotone, $\{0\}$–	$\{g_2\}$
	preserving functions	

Problem. *Let* C_1 and C_2 *be nonisomorphic Boolean clones defined on the same set* $A = \{0,1\}$. *Are the sets* IdC_1 *and* IdC_2 *of its identities different?*

Case 1. We consider pairs of Boolean clones C_1, C_2 with $C_1 \subsetneqq C_2$. If $C_1 \subsetneqq C_2$ then there holds $IdC_1 \supseteq IdC_2$. In this case problem 3.5.1 is the question whether $IdC_1 \supsetneqq IdC_2$ or not. The answer is positive if there exists a separating hyperidentity – with other words: a hyperidentity of C_1 which is not a hyperidentity of C_2. If $C_2 = O_A$ and C_1 is a dual atom in the lattice of all subclones of O_A then a positive answer of problem 3.5.1 is given in [Denecke, Reichel 88]. The hyperidentity $\varepsilon : F(F(x,y),F(x,y)) = F(F(x,x),F(y,y))$ holds in all dual atoms of the Post lattice but not in O_A, because all binary Boolean functions beside the Sheffer functions fullfil ε. Therefore every subclone of O_A can be separated from O_A by a hyperidentity, namely ε. Now we want to find the separating hyperidentities also for the pairs (C_1,C_2) with $C_2 \neq O_A$. Therefore we have the following definition:

Definition 3.8.1. *Let* (C_1,C_2) *and* (C'_1,C'_2) *be two pairs of clones* $(C_1,C_2,C'_1,C'_2 \subseteq O_A)$ *with* $C_1 \subsetneqq C_2$. *Then*

$$(C_1,C_2) \lesssim (C'_1,C'_2) :\Leftrightarrow C'_1 \subseteq C_1 \text{ or } C''_1 \subseteq C_1 \text{ with } C'_1 \cong C''_1$$
$$\text{and } C_2 \subseteq C'_2 \text{ or } C_2 \subseteq C''_2 \text{ with } C'_2 \cong C''_2 .$$

Then the following lemma holds:

Lemma 3.8.2. *If* $(C_1,C_2) \lesssim (C'_1,C'_2)$ *with the assumptions in definition 3.5.2 and there exists a hyperidentity* ε *which holds in* C_1 *and not in* C_2 *then it follows that* ε *holds in* C'_1 *but not in* C'_2.

This implies that the separation of clones C'_1, C'_2 with $C'_1 \subsetneq C'_2$ by hyperidentities can be reduced to find separating hyperidentities only for the pairs of clones which are minimal with respect to the relation $\lesssim$. If (C_1, C_2) is such a minimal pair then C_1 is a maximal subclone of C_2. A positive result was achieved for this case in [Denecke 88]. From that it follows

Theorem 3.8.3. *Any two Boolean clones C_1, C_2 with $C_1 \subsetneq C_2$ can be separated by hyperidentities, i.e. $IdC_1 \supsetneq IdC_2$.*

Case 2. We consider pairs of Boolean clones which are incomparable with respect to $\subseteq$.

We define a relation $\lesssim$ between pairs of incomparable clones in the following manner:

Definition 3.8.4. *Let (C_1, C_2) and (C'_1, C'_2) be two pairs of clones*

$$(C_1, C_2, C'_1, C'_2 \subseteq 0_A) \text{ with } C_1 \nsubseteq c_2 \text{ and } C_2 \nsubseteq C_1. \text{ Then}$$

$$\begin{aligned}(C_1, C_2) \lesssim (C'_1, C'_2) :\Leftrightarrow\ & C_1 \subseteq C'_1 \text{ or } C_1 \subseteq C''_1 \text{ with } C'_1 \cong C''_1 \\ & \text{and } C'_2 \subseteq C_2 \text{ or } C''_2 \subseteq C_2 \text{ with } C'_2 \cong C''_2 .\end{aligned}$$

Then the following lemma holds:

Lemma 3.8.5. *If $(C_1, C_2) \lesssim (C'_1, C'_2)$ with the assumptions in definition 3.5.5 and $C'_1 \nsubseteq C'_2$ and $C'_2 \nsubseteq C'_1$ and there exists a hyperidentity ε which holds in C'_1 and not in C'_2 then it follows that ε holds in C_1 but not in C_2.*

This implies that the separation of clones C_1, C_2 with $C_1 \nsubseteq C_2$ and $C_2 \nsubseteq C_1$ by hyperidentities can be reduced to find separating hyperidentities only for the pairs of clones which are maximal with respect to the relation $\lesssim$.

Lemma 3.8.6. *(C_1,C_2) $(C_1 \nsubseteq C_2,\ C_2 \nsubseteq C_1)$ is maximal with respect to* $\lesssim$ if and only *if for all clones K_1, K_2 with $K_1 \supseteq C_1$ and $K_2 \subseteq C_2$ we have $K_1 \subseteq K_2$ (or $K_1 \subseteq K_2'$ with $K_2' \cong K_2$) or $K_2 \subseteq K_1$ (or $K_2 \subseteq K_1'$ with $K_1' \cong K_1$).*

In [Denecke, Mal'cev, Reschke] (lemma 4.5 and 4.6) is given one method to find all such maximal pairs of incomparable clones. Also in this paper one can find a separating hyperidentity for every (with respect to $\lesssim$) maximal pairs of incomparable pairs of Boolean clones. That leads to

Theorem 3.8.7. *Let (C_1,C_2) be a pair of Boolean clones with $C_1 \nsubseteq C_2'$ and $C_2 \nsubseteq C_1'$ for all $C_2' \cong C_2$ and all $C_1' \cong C_1$. Then C_1 and C_2 can be separated from each other by hyperidentities, i.e. there is a hyperidentity ε which holds in C_1 and not in C_2 and a hyperidentity ε' which holds in C_2 and not in C_1. This means that for the sets of identities of C_1 and C_2 we have $IDC_1 \nsubseteq IdC_2$ and also $IdC_2 \nsubseteq IdC_1$.*

The following table will give a review about all separating hyperidentities of all Boolean clones.

clone $\mathbf{X}$	separating hyperidentities which hold in maximal clones of $\mathbf{X}$ and not in $\mathbf{X}$ itself	hyperidentities which hold in $\mathbf{X}$ and not in clones which are incomparable (w.r.to $\subseteq$) with $\mathbf{X}$
$\mathbf{C}_1$	ε_0	$\mathbf{C}_1$ is comparable with all other Boolean clones
$\mathbf{C}_3$	CHI($\mathbf{C}_4$), CHI($\mathbf{M}_3$), CHI($\mathbf{L}_3$)	ε_2 for $\mathbf{O}_4,\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9,\mathbf{D}_3$
	CHI($\mathbf{F}_8^2$)	ε_6 for $\mathbf{O}_8,\mathbf{P}_6,\mathbf{M}_1$
$\mathbf{C}_4$	CHI($\mathbf{M}_4$), CHI($\mathbf{D}_1$), CHI($\mathbf{F}_5^2$)	ε_2 for $\mathbf{D}_3,\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9,\mathbf{D}_4$
		ε_5 for $\mathbf{O}_6,\mathbf{M}_1,\mathbf{M}_3,\mathbf{L}_3,\mathbf{P}_3,\mathbf{P}_5$, $\mathbf{F}_8^m,\mathbf{F}_7^m,\mathbf{F}_8^\infty,\mathbf{F}_7^\infty$ $(m \geq 2)$
		ε_6 for $\mathbf{P}_6,\mathbf{O}_8$
$\mathbf{M}_1$	CHI($\mathbf{M}_3$), CHI($\mathbf{P}_6$)	ε_3 for $\mathbf{L}_4,\mathbf{O}_4,\mathbf{C}_3,\mathbf{C}_4,\mathbf{D}_3,\mathbf{D}_1,\mathbf{L}_1,\mathbf{L}_3,\mathbf{L}_5,\mathbf{O}_9$
		ε_{10} for $\mathbf{F}_5^\infty,\mathbf{F}_8^m,\mathbf{F}_5^m,\mathbf{F}_8^\infty$ $(m \geq 2)$
$\mathbf{M}_3$	CHI($\mathbf{M}_4$), CHI($\mathbf{P}_5$), CHI($\mathbf{F}_7^2$)	ε_3 for $\mathbf{C}_4,\mathbf{D}_3,\mathbf{D}_1,\mathbf{L}_1,\mathbf{L}_3,\mathbf{L}_4,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4$
		ε_6 for $\mathbf{P}_6,\mathbf{O}_8$
		ε_{10} for $\mathbf{F}_8^m,\mathbf{F}_5^m,\mathbf{F}_8^\infty,\mathbf{F}_5^\infty$ $(m \geq 2)$
$\mathbf{M}_4$	CHI($\mathbf{F}_6^2$)	ε_3 for $\mathbf{D}_3,\mathbf{D}_1,\mathbf{L}_1,\mathbf{L}_3,\mathbf{L}_4,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4$
		ε_5 for $\mathbf{P}_3,\mathbf{P}_5,\mathbf{O}_6,\mathbf{F}_8^m,\mathbf{F}_7^m,\mathbf{F}_8^\infty,\mathbf{F}_7^\infty$
		ε_6 for $\mathbf{P}_6,\mathbf{O}_8$
		ε_{10} for $\mathbf{F}_5^m$, $\mathbf{F}_5^\infty$ $(m \geq 2)$
$\mathbf{D}_3$	CHI($\mathbf{D}_1$), CHI($\mathbf{L}_5$)	ε_1 for all other clones besides $\mathbf{D}_2,\mathbf{L}_4,\mathbf{O}_4$
$\mathbf{D}_1$	CHI($\mathbf{D}_2$), CHI($\mathbf{L}_4$)	ε_2 for $\mathbf{L}_5,\mathbf{O}_4$
		ε_1 for all other clones besides $\mathbf{C}_3,\mathbf{C}_4$
$\mathbf{D}_2$	is an atom in the lattice off all Boolean clones	ε_1 for $\mathbf{L}_1,\mathbf{L}_3,\mathbf{P}_6,\mathbf{P}_3,\mathbf{P}_1,\mathbf{P}_5,\mathbf{O}_8,\mathbf{O}_6$,
		$\mathbf{F}_8^m,\mathbf{F}_5^m,\mathbf{F}_7^m,\mathbf{F}_6^m,\mathbf{F}_8^\infty,\mathbf{F}_5^\infty,\mathbf{F}_7^\infty,\mathbf{F}_6^\infty$ $(m \geq 3)$
		ε_3 for $\mathbf{L}_4,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4$

$\mathbf{L_2}$	CHI($\mathbf{L_3}$), CHI($\mathbf{L_5}$), CHI($\mathbf{O_9}$)	ε_4 for $\mathbf{D_2,P_1,C_3,C_4,M_1,M_3,M_4,D_3,D_1,}$
		$\mathbf{P_6,P_3,P_5,F^m_8,F^m_5,F^m_7,F^m_6,F^\infty_8,F^\infty_5,F^\infty_7,F^\infty_6}$ $(m\geq 2)$
$\mathbf{L_3}$	CHI($\mathbf{L_4}$), CHI($\mathbf{O_6}$)	ε_2 for $\mathbf{D_3}$, $\mathbf{L_5}$, $\mathbf{O_9}$, $\mathbf{O_4}$
		ε_4 for $\mathbf{C_4}$, $\mathbf{M_3}$, $\mathbf{M_4}$, $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{P_3}$, $\mathbf{P_1}$,
		$\mathbf{P_5,F^m_8,F^m_5,F^m_7,F^m_6,F^\infty_8,F^\infty_5,F^\infty_7,F^\infty_6}$ $(m\geq 2)$
		ε_6 for $\mathbf{M_1}$, $\mathbf{P_6}$, $\mathbf{O_8}$
$\mathbf{L_4}$	is an atom in the lattice	ε_1 for $\mathbf{O_8,O_6}$
	of all Boolean clones	ε_2 for $\mathbf{O_9,O_4}$
		ε_4 for $\mathbf{M_1,M_3,M_4,D_2,P_6,P_3,P_1,P_5,}$
		$\mathbf{F^m_8,F^m_5,F^m_7,F^m_6,F^\infty_8,F^\infty_5,F^\infty_7,F^\infty_6}$ $(m\geq 2)$
$\mathbf{L_5}$	CHI($\mathbf{L_4}$), CHI($\mathbf{O_4}$)	ε_1 for $\mathbf{L_3,O_9,O_8,O_6}$
		ε_4 for all other clones besides $\mathbf{D_3,L_1}$
$\mathbf{P_6}$	CHI($\mathbf{P_3}$), CHI($\mathbf{P_5}$), CHI($\mathbf{O_8}$)	ε_3 for $\mathbf{C_3,C_4,D_3,D_1,L_1,L_3,L_4,L_5,O_9,O_4}$
		ε_9 for $\mathbf{D_2,F^\infty_6,M_3,M_4,F^m_8,F^m_5,F^m_7,F^m_6,F^\infty_8,F^\infty_5,F^\infty_7,}$ $(m\geq 2)$
$\mathbf{P_3}$	CHI($\mathbf{P_1}$), CHI($\mathbf{O_6}$)	ε_3 for $\mathbf{C_4,D_3,D_1,L_1,L_3,L_4,L_5,O_9,O_4}$
		ε_6 for $\mathbf{O_8}$
		ε_9 for $\mathbf{M_4,D_2,F^m_5,F^m_6,F^\infty_5,F^\infty_6}$ $(m\geq 2)$
		ε_{13} for $\mathbf{P_5}$
$\mathbf{P_1}$	is an atom in the lattice	ε_3 for $\mathbf{D_3,D_1,L_1,L_4,L_5,O_9,O_4}$
	of all Boolean clones	ε_5 for $\mathbf{L_3}$, $\mathbf{O_6}$
		ε_6 for $\mathbf{O_8}$
		ε_9 for $\mathbf{D_2}$
$\mathbf{P_5}$	CHI($\mathbf{P_1}$), CHI($\mathbf{O_6}$)	ε_3 for $\mathbf{C_4,D_3,D_1,L_1,L_3,L_4,L_5,O_9,O_4}$
		ε_6 for $\mathbf{O_8}$
		ε_9 for $\mathbf{M_4,D_2,F^m_8,F^m_5,F^m_7,F^m_6,F^\infty_8,F^\infty_5,F^\infty_7,F^\infty_6}$ $(m\geq 2)$
		ε_{11} for $\mathbf{P_3}$

$\mathbf{O}_9$	CHI($\mathbf{O}_8$), CHI($\mathbf{O}_4$)	ε_{12} for $\mathbf{L}_4,\mathbf{L}_3,\mathbf{L}_5$
		ε_4 for all other clones besides $\mathbf{L}_1,\mathbf{O}_6$
$\mathbf{O}_8$	CHI($\mathbf{O}_6$)	ε_3 for $\mathbf{L}_3,\mathbf{L}_4,\mathbf{L}_5,\mathbf{O}_4$
		ε_4 for all clones besides $\mathbf{M}_1,\mathbf{P}_6,\mathbf{O}_9$
$\mathbf{O}_4$	is an atom in the lattice	ε_1 for $\mathbf{L}_3,\mathbf{O}_8,\mathbf{O}_6$
	of all Boolean clones	ε_{12} for $\mathbf{L}_4$
		ε_4 for all other clones besides $\mathbf{D}_3,\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9$
$\mathbf{O}_6$	is an atom in the lattice	ε_3 for $\mathbf{L}_4,\mathbf{L}_5,\mathbf{O}_4$
	of all Boolean clones	ε_4 for $\mathbf{C}_4,\mathbf{M}_4,\mathbf{D}_3,\mathbf{D}_1,\mathbf{D}_2,\mathbf{P}_1,\mathbf{F}_5^m,\mathbf{F}_6^m,\mathbf{F}_5^\infty,\mathbf{F}_6^\infty$ $(m\geq 2)$
$\mathbf{F}_8^m$	CHI($\mathbf{F}_8^{m+1}$)	ε_2 for $\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4,\mathbf{D}_3$
$(m\geq 3)$	CHI($\mathbf{F}_5^m$), CHI($\mathbf{F}_7^m$)	ε_6 for $\mathbf{M}_1,\mathbf{P}_6,\mathbf{O}_8$
		ε_7 for $\mathbf{C}_4,\mathbf{M}_3,\mathbf{M}_4,\mathbf{D}_1,\mathbf{L}_3,\mathbf{L}_4$
		ε_8 for $\mathbf{D}_2$, $\mathbf{F}_6^n,\mathbf{F}_5^n,\mathbf{F}_7^n$ $(n<m)$
		ε_{13} for $\mathbf{P}_5$
$\mathbf{F}_5^m$	CHI($\mathbf{F}_5^{m+1}$)	ε_2 for $\mathbf{D}_3,\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4$
$(m\geq 3)$	CHI($\mathbf{F}_6^m$)	ε_5 for $\mathbf{M}_3,\mathbf{P}_3,\mathbf{O}_6,\mathbf{F}_8^\infty,\mathbf{F}_7^\infty,\mathbf{F}_8^k,\mathbf{F}_7^m$ $(k>m)$
		ε_6 for $\mathbf{M}_1,\mathbf{P}_6,\mathbf{O}_8$
		ε_7 for $\mathbf{M}_4,\mathbf{D}_1,\mathbf{L}_3,\mathbf{L}_4$
		ε_8 for $\mathbf{D}_2,\mathbf{F}_6^n$ $(n<m)$,
		ε_{13} for $\mathbf{P}_5$
$\mathbf{F}_7^m$	CHI($\mathbf{F}_7^{m+1}$)	ε_2 for $\mathbf{D}_3,\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4$
$(m\geq 3)$	CHI($\mathbf{F}_6^m$)	ε_3 for $\mathbf{C}_4,\mathbf{D}_1,\mathbf{L}_3,\mathbf{L}_4$
		ε_6 for $\mathbf{P}_6,\mathbf{O}_8$
		ε_7 for $\mathbf{M}_4$
		ε_8 for $\mathbf{D}_2,\mathbf{F}_8^k$ $(k>m)$, $\mathbf{F}_6^n$ $(n<m)$

		ε_{10} for $\mathbf{F}^m_5, \mathbf{F}^\infty_8, \mathbf{F}^\infty_5$
		ε_{13} for $\mathbf{P}_5$
$\mathbf{F}^m_6$	CHI($\mathbf{F}^{m+1}_6$)	ε_2 for $\mathbf{D}_3, \mathbf{L}_1, \mathbf{L}_5, \mathbf{O}_9, \mathbf{O}_4$
($m \geq 3$)		ε_3 for $\mathbf{D}_1, \mathbf{L}_3, \mathbf{L}_4$
		ε_5 for $\mathbf{P}_3, \mathbf{O}_6, \mathbf{F}^k_7$ ($k>m$), $\mathbf{F}^\infty_7$
		ε_6 for $\mathbf{P}_6, \mathbf{O}_8$
		ε_8 for $\mathbf{D}_2, \mathbf{F}^k_8$ ($k>m$)
		ε_{10} for $\mathbf{F}^k_5$ ($k>m$), $\mathbf{F}^\infty_8, \mathbf{F}^\infty_5$
		ε_3 for $\mathbf{P}_5$
$\mathbf{F}^2_8$	CHI($\mathbf{F}^3_8$)	ε_2 for $\mathbf{D}_3, \mathbf{L}_1, \mathbf{L}_5, \mathbf{O}_9, \mathbf{O}_4$
	CHI($\mathbf{F}^2_5$)	ε_6 for $\mathbf{M}_1, \mathbf{P}_6, \mathbf{O}_8$
	CHI($\mathbf{F}^2_7$)	ε_7 for $\mathbf{M}_4, \mathbf{L}_4, \mathbf{C}_4, \mathbf{M}_3, \mathbf{D}_1, \mathbf{L}_3$
		ε_{13} for $\mathbf{P}_5$
$\mathbf{F}^2_5$	CHI($\mathbf{F}^3_5$)	ε_2 for $\mathbf{D}_3, \mathbf{L}_1, \mathbf{L}_5, \mathbf{O}_9, \mathbf{O}_4$
	CHI($\mathbf{F}^2_6$)	ε_5 for $\mathbf{M}_3, \mathbf{L}_3, \mathbf{P}_3, \mathbf{P}_5, \mathbf{O}_6, \mathbf{F}^m_8$ ($m \geq 3$), $\mathbf{F}^m_7$ ($m \geq 2$), $\mathbf{F}^\infty_8, \mathbf{F}^\infty_7$
		ε_6 for $\mathbf{M}_1, \mathbf{P}_6, \mathbf{O}_8$
		ε_7 for $\mathbf{M}_4, \mathbf{D}_1, \mathbf{L}_4$
$\mathbf{F}^2_7$	CHI($\mathbf{F}^3_7$)	ε_2 for $\mathbf{D}_3, \mathbf{L}_1, \mathbf{L}_5, \mathbf{O}_9, \mathbf{O}_4$
	CHI($\mathbf{F}^2_6$)	ε_3 for $\mathbf{C}_4, \mathbf{D}_1, \mathbf{L}_3, \mathbf{L}_4$
		ε_6 for $\mathbf{P}_6, \mathbf{O}_8$
		ε_7 for $\mathbf{M}_4$
		ε_{10} for $\mathbf{F}^m_8$ ($m \geq 3$), $\mathbf{F}^m_5$ ($m \geq 2$), $\mathbf{F}^\infty_8$, $\mathbf{F}^\infty_5$
		ε_{13} for $\mathbf{P}_5$
$\mathbf{F}^2_6$	CHI($\mathbf{F}^3_6$)	ε_2 for $\mathbf{D}_3, \mathbf{L}_1, \mathbf{L}_5, \mathbf{O}_9, \mathbf{O}_4$
	CHI($\mathbf{D}_2$)	ε_3 for $\mathbf{D}_1, \mathbf{L}_4$
		ε_5 for $\mathbf{L}_3, \mathbf{P}_3, \mathbf{P}_5, \mathbf{O}_6, \mathbf{F}^m_8, \mathbf{F}^m_7$ ($m \geq 3$), $\mathbf{F}^\infty_8, \mathbf{F}^\infty_7$

		ε_6 for $\mathbf{P}_6$, $\mathbf{O}_8$
		ε_{10} for $\mathbf{F}_5^m$ ($m \geq 2$), $\mathbf{F}_5^\infty$
$\mathbf{F}_8^\infty$	CHI($\mathbf{F}_5^\infty$)	ε_2 for $\mathbf{D}_3,\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4$
	CHI($\mathbf{F}_7^\infty$)	ε_6 for $\mathbf{M}_1,\mathbf{P}_6,\mathbf{O}_8$
		ε_7 for $\mathbf{C}_4,\mathbf{M}_3,\mathbf{M}_4,\mathbf{D}_1,\mathbf{L}_3,\mathbf{L}_4$
		ε_8 for $\mathbf{D}_2$, $\mathbf{F}_5^m,\mathbf{F}_7^m,\mathbf{F}_6^m$ ($m \geq 2$)
		ε_{13} for $\mathbf{P}_5$
$\mathbf{F}_5^\infty$	CHI($\mathbf{F}_6^\infty$)	ε_2 for $\mathbf{D}_3,\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4$
		ε_5 for $\mathbf{P}_3,\mathbf{O}_9,\mathbf{F}_7^\infty$
		ε_6 for $\mathbf{M}_1,\mathbf{P}_6,\mathbf{O}_8$
		ε_7 for $\mathbf{M}_3,\mathbf{M}_4,\mathbf{D}_1,\mathbf{L}_3,\mathbf{L}_4$
		ε_8 for $\mathbf{D}_2,\mathbf{F}_7^m,\mathbf{F}_6^m$ ($m \geq 2$)
		ε_{13} for $\mathbf{P}_5$
$\mathbf{F}_7^\infty$	CHI($\mathbf{F}_6^\infty$)	ε_2 for $\mathbf{D}_3,\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4$
	CHI($\mathbf{P}_3$)	ε_3 for $\mathbf{C}_4,\mathbf{D}_1,\mathbf{L}_3,\mathbf{L}_4$
		ε_6 for $\mathbf{P}_6,\mathbf{O}_8$
		ε_7 for $\mathbf{M}_4$
		ε_8 for $\mathbf{D}_2$, $\mathbf{F}_6^m$ ($m \geq 2$)
		ε_{10} for $\mathbf{F}_5^m$ ($m \geq 2$), $\mathbf{F}_5^\infty$
		ε_{13} for $\mathbf{P}_5$
$\mathbf{F}_6^\infty$	CHI($\mathbf{P}_1$)	ε_2 for $\mathbf{D}_3,\mathbf{L}_1,\mathbf{L}_5,\mathbf{O}_9,\mathbf{O}_4$
		ε_3 for $\mathbf{D}_1,\mathbf{L}_3,\mathbf{L}_4$
		ε_5 for $\mathbf{P}_3,\mathbf{O}_6$
		ε_6 for $\mathbf{P}_6,\mathbf{O}_8$
		ε_8 for $\mathbf{D}_2$
		ε_{13} for $\mathbf{P}_5$

(– table 2 –)

The clone $\mathbf{O}_1$ is separated from all other clones by the hyperidentity CHI($\mathbf{O}_1$).

List of hyperidentities:

ε_0: $F(F(x,y),F(x,y)) = F(F(x,x), F(y,y))$

ε_1: $F(x,x) = G(F(G(x,y),G(x,y)), F(G(y,x),G(y,x)))$

ε_2: $F(x,x) = F(F(x,x), F(x,x))$

ε_3: $F(x,x,y) = F(F(x,x,y), F(x,x,x), F(y,y,y))$

ε_4: $F(x,y,y) = F(x,y,F(z,z,F(z,z,y)))$

ε_5: $F(x) = x$

ε_6: $F(G(x)) = G(F(x))$

ε_7: $G(F^*,F^*,F^*) = G(F^+,F^+,F^+)$ with

$F^+ = F(F(x,x,y), F(y,x,x), F(x,y,x))$

$F^* = F(F(x,x,G^+), F(G^+,x,x), F(x,G^+,x))$

$G^+ = G(G(y,y,x), G(x,y,y), G(y,x,y))$

ε_8: $G(T_1,T_1) = G(T_2,T_2)$ with

$T_1 := F(F(x,G^+,y,\dots,y), F(y,x,G^+,y,\dots,y),\dots\dots,F(y,\dots,y,x,G^+), F(G^+,y,\dots,y,x))$,

$T_2 := F(F(x,y,\dots,y), F(y,x,y,\dots,y),\dots,F(y,\dots,y,x,xy), F(xy,y,\dots,y,x))$,

$G^+ := G(G(y,x),G(x,y))$

where G is binary and T_1,T_2 are 4–ary in the case that ε_8 is a separating hyperidentity for $\mathbf{F}_8^3$ and $\mathbf{D}_2$, i.e. ε_8 holds in $\mathbf{F}_8^3$ and not in $\mathbf{D}_2$, and T_1,T_2 are (m+1)ary otherwise.

ε_9: $F(F(x,y,x),F(x,x,y),F(y,x,x)) = F(F(x,F(y,y,y),F(y,y,y)),F(y,x,F(y,y,y)),F(x,x,x))$

ε_{10}: $F(F(x,y,x), F(y,y,y), F(y,y,y)) = F(F(x,x,x), F(x,y,y), F(y,y,y))$

ε_{11}: $F(x,F(G(x,x),x)) = F(x,x)$

ε_{12}: $F(x,y,y) = F(F(F(x,x,x),F(x,x,x),F(x,x,y)),$
$F(F(x,x,x),F(x,y,x),F(x,y,x)),F(F(x,x,x),F(x,y,y),F(x,y,y)))$

ε_{13}: $H(H(H(x,y),x),G(y)) = H(H(H(x,x),x), G(y))$

CHI($\mathbf{C}_3$): $G(F(x,y), F(x,x)) = F(G(G(x,x), G(x,x)), G(G(x,x), G(x,x)))$

CHI($\mathbf{C}_4$): $G(x,y,z) = F(G(x,y,z), G(x,y,z), G(x,y,z))$

CHI($\mathbf{A}_1$): $F(x,y) = F(F(x,y), F(x,y))$

CHI($\mathbf{A}_3$): $F(G(x,y), G(x,y)) = G(F(x,F(x,x)), F(F(y,y),y))$

CHI($\mathbf{A}_4$): $F(x,y,z) = F(F(x, G(y,y,y), z),y,z)$

CHI($\mathbf{D}_3$): $F(F(x,x,x),F(x,x,x),F(x,x,x))$

$= F(F(x,x,y),F(F(x,x,y),x,x),F(x,F(z,z,x),x))$

CHI($\mathbf{D}_1$): $F(x,x,x) = F(F(x,x,y),F(F(x,x,y),x,x),F(x,F(z,z,x),x))$

CHI($\mathbf{D}_2$): $F(F(x,z,x),y),F(z,x,F(x,x,y)),F(F(y,x,x),z,x)) = F(x,x,x)$

CHI($\mathbf{L}_1$): $F(x,x) = F(F(F(x,y),y), F(y,F(y,x)))$

CHI($\mathbf{L}_3$): $F(G(x,x), G(x,x))$

$= G(F(G(x,F(y,y)),G(x,F(x,y))),F(G(F(y,x),x),G(F(y,y),x)))$

CHI($\mathbf{L}_4$): $F(x,x,G(G(x,y,y), G(y,x,y), G(y,y,x)))$

$= G(x,G(y,G(z,x,y),G(y,z,y),G(z,G(z,z,y),G(y,z,x)))$

CHI($\mathbf{L}_5$): $F(x,x,x) = F(x,F(y,F(z,x,y), F(y,z,z)), F(z,F(z,z,y), F(y,z,x)))$

CHI($\mathbf{P}_1$): $F(F(x,y,x), F(x,x,y), F(y,x,x))$

$= F(F(x,G(y,y,y), G(y,y,y)), F(x,x,G(y,y,y)), F(x,x,x))$

CHI($\mathbf{P}_3$): $G(F(x,y,x),F(x,x,x),F(x,y,x))= F(G(x,x,x),G(G(y,x,x),x,x),G(x,x,x)),$

CHI($\mathbf{P}_5$): $F(F(x,x),x) = F(x,F(G(x,x),x))$

CHI($\mathbf{P}_6$): $F(F(x,G(x,y)), F(G(y,x),x)) = F(x,F(G(y,y),x))$

CHI($\mathbf{O}_9$): $F(F(x,x), F(y,x)) = F(F(x,y), F(y,x))$

CHI($\mathbf{O}_8$): $F(x,x) = F(F(x,y), F(y,x))$

CHI($\mathbf{O}_6$): $G(F(x,x),F(x,x)) = F(G(x,x), G(G(x,y),x))$

CHI($\mathbf{O}_4$): $F(G(x,x,x),G(y,y,y),G(z,z,z))$

$= G(F(x,y,z),F(x,y,z),F(x,y,F(x,y,F(z,z,z))))$

CHI($\mathbf{O}_1$): $F(x,x,x) = G(F(x,F(y,x,z), F(y,y,x)),$

$= F(F(x,y,z),x,F(y,z,x)), F(F(x,y,z), F(y,x,z),x))$

CHI($\mathbf{F}_5^2$): $F(F(x,x,G^+),F(G^+,x,x),F(x,G^+,x)) = F(F(x,x,y),F(y,x,x),F(x,y,x)$ with

$G^+ := G(G(y,y,x),\ F(x,y,y),\ G(y,x,y))$

CHI($\mathbf{F}_6^2$): $G(F^{\oplus},F^{\oplus},F^{\oplus}) = F(F_1^+,F_2^+,F_3^+)$ with

$F^{\oplus} := F(F(x,x,G^+),\ F(G^+,x,x),\ F(x,G^+,x))$

$G^+ := G(G(y,y,x),\ G(x,y,y),\ G(y,x,y))$

$F_1^+ := F(F(x,x,x),\ F(x,y,x),\ F(x,x,x))$

$F_2^+ := F(F(x,x,x),\ F(x,x,x),\ F(x,x,y))$

$F_3^+ := F(F(y,x,x),\ F(x,x,x),\ F(x,x,x))$

CHI($\mathbf{F}_7^2$): $G(F^{\oplus},F^{\oplus},F^{\oplus}) = G(F',F',F')$ with $F' := F(F_1^+,F_2^+,F_3^+)$

CHI($\mathbf{F}_8^2$): ε_7

CHI($\mathbf{F}_6^m$): $G(h_1) = G'(h_2)$ with

$h_1 := F(F(x,y,x,y,\ldots,y),F(y,x,x,y,\ldots,y),F(y,y,x,y,\ldots,y),\ldots,F(y,\ldots,y,x))$,

$h_2 := F(F(x,y,\ldots,y,x),\ F(y,x,y,\ldots,y,x),F(y,y,x,y,\ldots,y),\ldots,F(y,\ldots,y,x))$

where F is an (m+1)–ary operation symbol and G,G' are unary operation symbols

CHI($\mathbf{F}_5^m$): $H(T_1) = H'(T_2)$ with T_1,T_2 and G^+ from ε_8,

CHI($\mathbf{F}_7^m$): $h_1 = h_2$

CHI($\mathbf{F}_8^m$): ε_8 where T_1 and T_2 are (m+1)–ary

CHI($\mathbf{F}_i^{\infty}$): $i = 5,6,7,8$ $\{\mathrm{CHI}(\mathbf{F}_i^m) : m \geq 3\}$

Part IV Clone congruences

§4.1. The lattice of hypervarieties

Proposition 4.1.1. *The set of all hypervarieties of a given type τ forms a lattice $(\mathscr{L}(\tau), \subseteq)$. If V_1, V_2 are two hypervarieties of type τ defined by the closed sets of hyperidentities E_1, E_2, respectively, then $V_1 \wedge V_2$ is the hypervariety defined by the set $[E_1 \cup E_2]$, where $[E]$ denotes the closure of a set E under the rules of inference (1)–(6) and $V_1 \vee V_2$ is the hypervariety defined by $E_1 \cap E_2$. $V_1 \wedge V_2 = \text{g.l.b.}\ (V_1, V_2)$ and $V_1 \vee V_2 = \text{l.u.b.}\ (V_1, V_2)$ in $(\mathscr{L}(\tau), \subseteq)$.*

Proof. Notice that the set $[E_1 \cup E_2]$ is closed under rule (6). Thus $[E_1 \cup E_2]$ is the smallest set, closed under (1)–(6) and containing the set E_1 and E_2. Thus, the g.l.b. (V_1, V_2) exists in $(\mathscr{L}(\tau), \subseteq)$ and equals $V_1 \wedge V_2$, by the completeness theorem. Obviously the set $E_1 \cap E_2$ is closed under (1)–(6). Thus $E_1 \cap E_2$ is the greatest set, closed under (1)–(6) and contained in E_1 and E_2. Thus, by the completeness theorem, the l.u.b. (V_1, V_2) exists in $(\mathscr{L}(\tau), \subseteq)$ and equals $V_1 \vee V_2$.

Theorem 4.1.2. *The lattice $(\mathscr{L}(\tau), \wedge, \vee)$ of all hypervarieties of type τ is isomorphic to a sublattice of the lattice $(L(\tau), \wedge, \vee)$ of all varieties of type τ.*

Proof. We consider the map $k: \mathscr{L}(\tau) \longrightarrow L(\tau)$ which is defined for a hypervariety C of type τ in the following way. If $C = \{V_i : i \in I$ then $k(C) = \bigcup(V_i : i \in I)$, i.e. k(C) is the class of all algebras contained in the varieties of the hypervariety C. Because C is a hypervariety, k(C) is closed under H, S, P and hence is a variety. From $C_1 \subseteq C_2$ it is easy to see that $k(C_1) \subseteq k(C_2)$, i.e. k is monotone. Now let $k(C_1) = k(C_2)$ for hypervarieties C_1, C_2 and let E_1 be the set of hyperidentities of type τ holding in C_1 and E_2, respectively for C_2. Let (T_1,T_2) be a hyperidentity of E_1. As $k(C_1) = k(C_2)$, all algebras of $k(C_2)$ satisfy the hyperidentity (T_1,T_2), i.e. $E_1 \subseteq E_2$. Similarly $E_2 \subseteq E_1$. We conclude that $E_1 = E_2$, and $C_1 = C_2$. Let $\mathbf{A} \in k(C_1) \wedge k(C_2)$ and let E_i be the set of all hyperidentities holding for C_i, $i = 1,2$. Then $H_\tau(\mathbf{A}) \supseteq [E_1 \cup E_2]$. Furthermore, $[E_1 \cup E_2]$ is the set of hyperidentities defining $C_1 \wedge C_2$, by Proposition 4.1.1. Hence $\mathbf{A} \in k(C_1 \wedge C_2)$. Since k is monotone, we conclude that $k(C_1) \wedge k(C_2) = k(C_1 \wedge C_2)$. Now take $\mathbf{A} \in (C_1 \vee C_2)$. Thus the algebra $\mathbf{A}$ satisfies the hyperidentities of $C_1 \vee C_2$, i.e. $H_\tau(\mathbf{A}) \supseteq E_1 \cap E_2$, by Proposition 4.1.1. Because $E_1 \cap E_2$ is closed under rules (1)–(6), we conclude also by Proposition 4.1.1 that $\mathbf{A} \in k(C_1) \vee k(C_2)$. Since k is monotone, we have $k(C_1 \vee C_2) = k(C_1) \vee k(C_2)$, i.e. k is a lattice homomorphism.

Remark 4.1.3. The lattice $(\mathscr{L}(\tau), \wedge, \vee)$ is a complete lattice.

Proof. Similarly, as in the proof of Proposition 4.1.1 it is easy to see that for a family $(V_i : i \in L)$ of hypervarieties of type τ, defined by the sets E_i, $i \in I$ of hyperidentities, respectively, the hypervarieties: $\bigwedge(V_i : i \in I)$ and $\bigvee(V_i : i \in I)$ are defined by the sets of hyperidentities: $\bigcup(E_i : i \in I)$ and $\bigcap(E_i : i \in I)$, respectively.
Let V be a variety of type τ. Then $h_\tau(V)$ denotes the hypervariety of type τ and h(V) the hypervariety which is generated by V, i.e. defined by all the hyperidentities of V. Obviously, we have: $h(V) \subseteq h_\tau(V)$.

Proposition 4.1.4. *The map* $h_\tau : \mathscr{L}(\tau) \longrightarrow L(\tau)$ *defined by* $V \longrightarrow h_\tau(V)$ *is a surjective complete join–homomorphism.*

Proof. Let C be a hypervariety of type τ, $C = \{V_i : i \in I\}$ where V_i are varieties of type τ for $i \in I$. Take $V = \bigvee(V_i : i \in I)$, the join of the family $\{V_i : i \in I\}$ in the lattice $L(\tau)$. Then C is generated by V, i.e. $h_\tau(V) = C$. Hence h_τ is surjective. Obviously h_τ is a monotone map.
To show that $h_\tau(\bigvee(V_i : i \in I)) = \bigvee(h_\tau(V_i) : i \in I)$, notice that the hypervariety C, generated by the join of the family $(V_i : i \in I)$ of varieties of type τ, is defined by the set $\bigcap(E_\tau(V_i) : i \in I)$ of hyperidentities. But this is exactly the join of hypervarieties $(h_\tau(V_i) : i \in I)$.

Remark 4.1.5. According to the results of [Bergman] the map h is not one–to–one, in case of semigroups and groups considered as varieties of the same type.

Proposition 4.1.6. V *is a solid variety if and only if there exists a hypervariety* C *of the same type, such that* $k(C) = V$.

Proof. Let V be a solid variety, i.e. $E_\tau(V) = H_\tau(Id(V))$. Take the set $\Sigma = E_\tau(V)$ of all hyperidentities of V and the hypervariety $C = \{V_i : i \in I\}$ of the same type as V, defined by Σ, i.e. $E_\tau(V_i) \supseteq \Sigma$, for all $i \in I$. Thus $V \in C$ and $k(C) = \bigcup(V_i : i \in I)$ and $E_\tau(k(C)) = \bigcap(E_\tau(V_i) : i \in I) = E_\tau(V)$, because $V \in C$ and $E_\tau(V_i) \supseteq \Sigma$ for all $i \in I$. V is solid; thus $E_\tau(V) = H_\tau(Id(V))$. Also, $Id(V_i) \supseteq I_V(E_\tau(V_i)) \supseteq I_V(E_\tau(V)) = Id(V)$ for $i \in I$. Thus $V_i \subseteq V$, for all $i \in I$, i.e. $Id(k(C)) = \bigcap(Id(V_i) : i \in I) = Id(V)$, and thus $k(C) = V$.

Now let $V = k(C)$ for some hypervariety $C = \{V_i : i \in I\}$, such that $E_\tau(V_i) \supseteq \Sigma$, for some set Σ of hyperidentities, which is closed under rules (1)–(6) by the completeness theorem. Thus a solid variety W, defined by $I_V(\Sigma)$ belongs to C, i.e. $Id(W) = I_V(\Sigma)$. Thus $V = k(C) = \bigcup(V_i : i \in I)$ and $Id(V) = \bigcap(Id(V_i) : i \in I) = I_V(\Sigma)$, because $Id(V_i) \supseteq I_V(E_\tau(V_i)) \supseteq I_V(\Sigma)$ for all $i \in I$ and $W \in \{V_i : i \in I\}$ and also $E_\tau(V) = \bigcap(E_\tau(V_i) : i \in I) = \Sigma$ i.e. $Id(V) = I_V(E_\tau(V)$, and thus V is a solid variety.

Corollary 4.1.7. Let $S(\tau)$ be the set of all solid varieties of type τ. Then $S(\tau)$ forms a (complete) sublattice of $L(\tau)$.

The **proof** follows from remark 4.1.3, theorem 4.1.2 and proposition 4.1.6.

§4.2 Solid kernels and solid envelopes

We have already considered solid envelopes in the first paragraph. Now we like to study this concept from the point of view of monotone operators.

Notation 4.2.1. Let Σ be the identities of the variety V of type τ and $H_\tau(\Sigma)$ the set of all transformations to hyperidentities. We define the solid kernel k(V) of the variety V as the subvariety of V which is given by $Id(H_\gamma(\Sigma))$.

Example. Let D be the variety of distributive lattices. Then the solid kernel k(V) is the trivial variety because we have $x = y$ from $F(x,y) = F(y,x)$ by $x \wedge y = y \wedge x$.

If V is solid we have $k(V) = V$. Let $U \subseteq V$ be varieties for U,V of some type τ and let denote $\Sigma(U)$, $\Sigma(V)$ denote the identities of U,V respectively. From $U \subseteq V$ we have $\Sigma(V) \subseteq \Sigma(U)$, $H_\gamma(\Sigma(V)) \subseteq H_\gamma(\Sigma(V))$ and $IdH_\gamma(\Sigma(V)) \subseteq Id\, H_\gamma(\Sigma(V))$. Hence $k(U) \subseteq k(V)$.

Theorem 4.2.2. [Gracyńska 89] *Let* $L(\tau)$ *be the lattice of varieties of type* τ *and* $S(\tau)$ *the lattice of solid varieties of type* τ. *Then* k: $L(\tau) \longrightarrow S(\tau)$ *is a meet–homomorphism.*

Proof. We have to show $k(V_1 \cap V_2) = k(V_1) \cap k(V_2)$ for $V_1,V_2 \in L(\tau)$. Obviously k is a monotone map and hence we have $k(V_1 \cap V_2) \subseteq k(V_1) \cap k(V_2)$. For the other direction let ϵ be a hyperidentity which holds for $k(V_1) \cap k(V_2)$. The hyperidentity of ϵ is a hyperconsequence of $H(V_1)$ as well as $H(V_2)$ and hence also of $H(V_1 \cap V_2)$.

Remark. k(V) is the greatest solid variety contained in V. The theorem holds also for complete meets.

We have already seen that for every variety V there exists a least solid variety s(V) which contains V. We call s(V) the solid envelope of V.

Remark. Let V be a variety which is generated by a set K of subdirectly irreducible algebras A_i, $i \in I$. Then s(V) is generated by D(K) the set of the derived algebras of V.

We use the fact that s(V) = HSPD(V).

Example. Let D be the variety of distributive lattices. Let $D_2 = (\{0,1\};\wedge,\vee)$ be the two–element simple lattice. Then we have the following simple derived algebras:

$E_1 = (\{0,1\},e_1^2,e_2^2)$, $S_1 = (\{0,1\};e_1^2,\wedge)$, $S_2 = (\{0,1\};\wedge,e_1^2)$ and

$S_3 = (\{0,1\};e_1^2,\vee)$, $S_4 = (\{0,1\};\vee,e_1^2)$, $S_5 = (\{0,1\};\wedge,\wedge)$

$S_6 = (\{0,1\};e_1^2,\vee)$

Nevertheless, it will be usually difficult to find the subdirectly irreducible algebras of s(V).

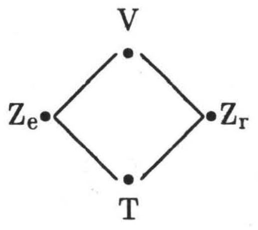
Let u $\subseteq$ V for varieties u,v of some type τ. Then $H_\gamma(V) \subseteq H_\gamma(U)$ for the sets of hyperidentities of type τ which hold for V or U respectively. Hence s(U) $\subseteq$ (V).

Theorem 4.2.3. [Graczyńka 89] *Let* $L(\tau)$ *be the lattice of varieties of type* τ *and* $S(\tau)$ *the lattice of solid varieties of type* τ. *Then* s: $L(\tau) \longrightarrow S(\tau)$ *is a join–homomorphism.*

Proof. We have to show that $s(V_1 \vee V_2) = s(V_1) \vee s((V_2)$ for every $V_1,V_2 \in L(\tau)$. Obviously s is monotone and hence $s(V_1 \vee V_2) \leq s(V_1) \vee (V_2)$. Let Σ,Σ_1,Σ_2 be the sets of hyperidentities of $V_1 \vee V_2$, V_1 and V_2 respectively. We have $\Sigma \supseteq \Sigma_1 \cap \Sigma_2$ because of the rule (6) and conclude that $s(V_1) \vee s(V_2) \supseteq s(V_1 \vee V_2)$.

Remark. The theorem holds for complete joins.

Example [Gracyńska]. Consider the following varieties of semigroups.



T the trivial variety of semigroups
Z_e the variety defined by $xy = x$
Z_r the variety defined by $xy = y$
$V = Z_e \vee Z_r$

V is the solid variety defined by $xyz = xz$ and $x^2 = x$. We have $s(Z_e \cap Z_r) = s(T) = T$ but $s(Z_e) = s(Z_r) = V$ and hence $s(Z_e \cap Z_r) \neq s(Z_e) \cap s(Z_r)$. Furthermore we have $k(Z_e \vee Z_r) = K(v) =$ but $k(Z_e) \vee k(Z_r) = T \vee T = T$.

Problem 4.2.4 *Let* V *be a given solid variety of type* τ

a) *Describe all varieties* W *of type* τ *such that* $k(W) = V$.

b) *Describe all varieties of* W *of type* τ *such that* $s(W) = V$.

The extreme cases where V is trivial or the variety of all algebras of type τ deserve special interest.

Problem 4.2.5. *Let* V *be a given variety of type* τ. *The variety* W *of type* τ *is called a flexible complement of* V *if* $k(V) = k(W)$ *and* $s(V) \cap W = V$. *Determine all maximal flexible complements. (As an example take* D *the variety of distributive lattices)*

§4.3. Clone congruences

The results of this section are due to Schweigert (compare [Schweigert 87a,89])

Definition 4.3.1. *Let* $\mathbf{H} = (H;*;\xi,\tau,\Delta,e)$ *be a clone of functions on a set A. An equivalence relation* ζ *is called a clone congruence of* $\mathbf{H}$ *if* ζ *is compatible with the clone operations* $*,\xi,\tau,\Delta$.

Example. Consider a clone **H** of function on a set A. Then $\kappa = \{(f,g) \mid \text{ar } f = \text{ar } g, f,g \in H\}$ is a clone congruence ($\text{ar } f = m$ denotes the arity of the function

$f{:}A^m \longrightarrow A$). Obviously κ is an equivalence relation. Let $(f,g) \in \kappa$, $h \in H$ with $\mathrm{ar}\, f = \mathrm{ar}\, g = m$ and $\mathrm{ar}\, h = n$. Then $(f{*}h, g{*}h) \in \kappa$ because $\mathrm{ar}\, f{*}h = n+m-1 = \mathrm{ar}\, g{*}h$. Similarly we have $(h{*}f, h{*}g) \in \kappa$. Now let $(d,b) \in \kappa$. For $(f,g) \in \kappa$ we have $(f{*}d, g{*}d) \in \kappa$ and we have proved the compatibility of κ with $*$.

Notation 4.3.2. The clone congruence κ of **H** is called the arity congruence of **H**. On every clone of functions there are at least three clone congruences κ_0, κ, and κ_1 where κ_0 is the identity and κ_1 the all relation.

Remark 4.3.3. If ζ is a clone congruence of **H** with $\zeta \neq \kappa_1$ then $\kappa_0 \subseteq \zeta \subseteq \kappa$.

Proof. We assume that $\zeta \nsubseteq \kappa$. Then there are functions $f,g \in H$ $\mathrm{ar}\, f = n > m = \mathrm{ar}\, g$ with $(f,g) \in \zeta$. Let $\bar{f}(x_1,\dots x_n) = f(x_1,\dots,x_1,x_2)$ $\bar{g}(x_1,\dots,x_m) = g(x_1,\dots,x_1)$. We have $(\bar{f},\bar{g}) \in \zeta$ and $(e^n_n{*}\bar{f}, e^n_n{*}\bar{g}) \in \zeta$ where $e^n_n(x_1,\dots,x_n) = x_n$. We conclude that $(e^{n+1}_{n+1}, e^n_n) \in \zeta$. But from this it follows immediately that any two given functions are in the clone congruence ζ and $\zeta = \kappa_1$.

Fact. κ is a maximal clone congruence.

Notation. Every clone congruence $\zeta \subseteq \kappa$ is called a proper clone congruence.

Notations. Let $\mathbf{F}(X) = (F(X),\Omega)$ denote the free algebra of the variety V. Con $\mathbf{F}(X)$ is the lattice of the fully invariant congruences of $\mathbf{F}(X)$. By the terms of $\mathbf{F}(X)$ we define term functions on the set $\mathbf{F}(X)$. $\mathbf{T}(X)$ denotes the clone of all term functions on $\mathbf{F}(X)$. Con$\mathbf{T}(X)$ is the lattice of all proper clone congruences of $\mathbf{T}(X)$.

Theorem 4.3.4. *Every proper clone congruence of the clone* $\mathbf{T}(X)$ *of term functions corresponds to a fully invariant congruence of the free algebra* $\mathbf{F}(X)$. *There is a lattice isomorphism* $h:\mathrm{Con}\mathbf{T}(X) \longrightarrow \mathrm{Con}\mathbf{F}(X)$.

Proof. We define a map $h: \mathrm{Con}\mathbf{T}(X) \longrightarrow \mathrm{Con}\mathbf{F}(X)$ in the following way. $(t(x_1,\ldots,x_k),u(x_1,\ldots,x_k)) \in \theta_h$ if and only if $(t,u) \in \theta$ for $t(x_1,\ldots,x_k) \in \mathbf{F}(X)$ $u(x_1,\ldots,x_k) \in \mathbf{F}(X)$ and the corresponding term functions $t,u \in \mathbf{T}(X)$. The equivalence relation θ_n is compatible with any operation $w \in \Omega$ of the free algebra $\mathbf{F}(X)$ because we have $(w(t_1,\ldots,t_n),w(u_1,\ldots,u_n)) \in \theta$ for $(t_i,u_i) \in \theta$ $i = 1,\ldots n$. If κ is an endomorphism κ: $\mathbf{F}(X) \longrightarrow \mathbf{F}(X)$ with $\kappa(x_i) = s_i$, $i = 1,\ldots,n$ then by the substitution property of a clone we have $(t(s_1,\ldots,s_n),u(s_1,\ldots,s_u)) \in \theta$ and hence $(\kappa(t), \kappa(u)) \in \theta_h$. Therefore θ_h is fully invariant. On the other hand if θ_h is a fully invariant of $\mathbf{F}(X)$ and $(t(x_1,\ldots,x_k),s(x_1,\ldots,x_m)) \in \theta_h$ then by adding fictitions variables we get pairs (t,s) of term functions on the set A with ar t = ar s. The set θ of these pairs is an equivalence relation on $\mathbf{T}(X)$ which is compatible with the substitution because θ_h is a congruence of $\mathbf{F}(X)$. But θ is also compatible with a permutation Π of variables as $\kappa(x_i) = x_{\Pi(i)}$ extends to an e endomorphism and θ_h is fully invariant. The same argument holds for the identification of variables. Hence θ is a proper clone congruence of $\mathbf{T}(X)$. $h: \mathrm{Con}T(X) \longrightarrow \mathrm{Con}F(X)$ with $h(\theta) := \theta_h$ is a lattice isomorphism.

Corollary 4.3.5. There is a polarity (dual–isomorphism) from the lattice $\mathrm{Con}\mathbf{T}(X)$ of the proper clone congruence to the lattice $L(V)$ of all subvarieties of the variety V.

Notations. $\Omega(X)$ *denotes the set of all fundamental operations* f_δ *of* $\mathbf{F}(X) = (F(X),\Omega)$. *By definition of* $\mathbf{T}(X)$ *they are contained in* $\mathbf{T}(X)$. *A term substitution* β *is a map* β*:* $\Omega(X) \longrightarrow T(X)$ *such that ar* f_δ = *ar* $\beta(f_\delta)$. *We write*

$$\beta(f_\delta)(x_1,\dots,x_n) = \beta(f_\delta(x_1,\dots,x_n)).$$

As an example consider $(\mathbb{Z};+)$ with $\beta(x+y) = ax+bx$ for some fixed $a,b \in \mathbb{Z}$.

Proposition 4.3.6. *The variety* V *is solid if and only if every term substitution* B: $\Omega(X) \longrightarrow T(X)$ *can be extended to a clone endomorphism* $\bar{\beta}$*:* $T(X) \longrightarrow T(X)$.

Proof. If B is solid then we define $\bar{\beta}(t)$ for $t(x_1,\dots,x_n) \in F(X)$ by the term where every operation symbol is substituted by a term according to the map β. If this map $\bar{\beta}$ is well defined then it will obviously be a clone endomorphism. Therefore let us consider the equation $t_1 = t_2$ in $T(X)$ or $t_1(x_1,\dots,x_n) = t_2(x_1,\dots,x_n)$ respectively. As V is solid any such term substitution provides a valid equation for V. Therefore we have $\bar{\beta}(t_1) = \bar{\beta}(t_2)$.

We repeat the

Definition. *A congruence* θ *of* $\mathbf{A} = (A,\Omega)$ *is called totally invariant if* $(a,b) \in \theta$ *implies* $(h(a),h(b)) \in \theta$ *for every type preserving weak endomorphism* h *of* $\mathbf{A}$ *and every* $a,b \in A$.

Remark. A totally invariant congruence is also fully invariant.

Theorem 4.3.7. *Every fully invariant proper clone congruence of the clone* $\mathbf{T}(X)$ *of term functions corresponds to a totally invariant congruence of the free algebra* $\mathbf{F}(X)$. *There is a lattice homomorphism* s:$\mathrm{Con}_f\mathbf{T}(X) \longrightarrow \mathrm{Con}_t\mathbf{F}(X)$.

Proof. Let θ be a fully invariant proper clone congruence of $T(X)$. We define $(t(x_1,\dots,x_k),u(x_1,\dots,x_k)) \in \theta_s$ if and only if $(t,u) \in \theta$ for the corresponding term functions $t,u \in \mathbf{T}(X)$. We have already shown in 4.3.4 that θ_s is a fully invariant congruence of $\mathbf{F}(X)$. Now let $h{:}F(X) \longrightarrow F(X)$ be a type preserving weak endomorphism. Let $\bar{h}$: $T(X) \longrightarrow T(X)$ be defined by $\bar{h}(t) = s$ iff $h(t(x_1,\dots,x_k)) = s(x_1,\dots,x_k)$. $\bar{h}$ is compatible with the substitution, i.e. $\bar{h}(u^*v) = \bar{h}(u)^*\bar{h}(v)$ because

$$h(u(v(x_1,\dots,x_n),x_2,\dots,x_{m+n-1})) = h(u(h(v(x_1,\dots,x_n),x_2,\dots,_{m+n-1}))).$$

Obviously $\bar{h}$ is comaptible with the other operatons ξ,τ,Δ of a clone. h is a clone endomorphism and we have $(\bar{h}(t),\bar{h}(u)) \in \theta$. Hence θ_s is totally invariant.

On the other hand if θ_s is a totally invariant congruence of $\mathbf{F}(X)$ then from 4.3.4 it follows that θ is a proper clone congruence. Let $\bar{f}{:}\mathbf{T}(X) \longrightarrow \mathbf{T}(X)$ be a clone endomorphism. Let $f{:}\mathbf{F}(X) \longrightarrow \mathbf{F}(X)$ be defined by $f(u(x_1,\dots,x_k)) = v(x_1,\dots,x_k)$ iff $\bar{f}(u) = v$. f is type–preserving. Let $w \in \Omega$ for $\mathbf{F}(X) = (F(X),\Omega)$. Then

$$f(w(u_1(x_1,\dots,x_{k1}),\dots,u_n(x_1,\dots,x_{kn})) = f(w)(f(u_1(x_1,\dots,x_{k1}),\dots,f(u_n(x_1,\dots,x_{kn}))$$

by the substitution property of f. f is a weak endomorphism. Hence $\bar{f}(u),\bar{f}(v)) \in \theta$ because $(f(u(x_1,\dots,x_k),f(v(x_1,\dots,x_k)) \in \theta_s$. θ is a fully invariant proper clone congruence.

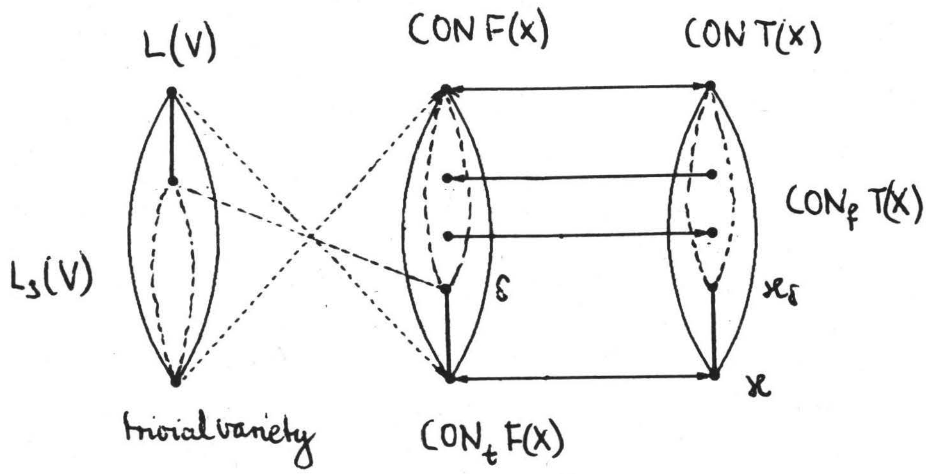
Proposition 4.3.8. *Every totally invariant congruence of the free algebra* $\mathbf{F}(X)$ *corresponds to a solid subvariety of* V.

Proof. Let θ_s be a totally invariant congruence of $\mathbf{F}(X)$. Then θ_s is fully invariant and corresponds to a subvariety U of V. Consider a term substitution $\beta{:}\Omega(X) \longrightarrow T_u(X)$ where $T_u(X)$ is the clone of all term functions is the variety U. Consider $\bar{\beta}$: $T_u(X) \longrightarrow T_u(X)$ as in 4.3.6. Let $t_1\theta t_2$. Hence $(t_1(x_1,\dots,x_k),t_2(x_1,\dots,x_k)) \in \theta_s$ and $\bar{\beta}(t_1) = \bar{\beta}(t_2)$. $\bar{\beta}$ is a well defined clone endomorphism of $T_u(X)$ and U is solid.

Corollary 4.3.9. There is a polarity (dual isomorphism) from the lattice $Con_t\mathbf{F}(X)$ of all totally invariant congruence of $\mathbf{F}(X)$ to the lattice $\mathbf{L}_s(V)$ of the solid subvarieties of V.

Remark. The kernel k(V) is the greatest solid subvariety of V and the trivial variety is the least solid subvariety of V.
The meet of totally invariant congruences is again a totally invariant congruence. The all congruence is totally invariant. Hence there exist a least totally invariant congruence δ which is the identity relation of Con$\mathbf{F}$(X)only in case that V is solid.

Illustration 4.3.10



Con**T**(X):	= lattice of the proper clone congruences of the clone T(X)
Con**F**(X):	= lattice of the fully invariant congrunces of the free algebra F(X)
L(V):	= lattice of all subvarities of the variety V
$Con_f\mathbf{T}(X)$:	= lattice of the fully invariant proper clone congruences
$Con_t\mathbf{F}(X)$:	= lattice of the totally invariant congruences of the free algebra F(X)
κ:	= arity congruence
κ_δ:	= least fully invariant proper clone congruence
δ:	= least totally invariant congruence of F(X)

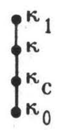
§4.4. Subdirectly irreducible clones

Notations. Let $\mathbf{P}_A = (P_A;*,\xi,\tau,\Delta,e)$ be a clone of all functions on the set A. A function $f \in \mathbf{P}_A$ is called w–function if $x_1 = w$ or ... $x_n = w$ implies $f((x_1,...,x_n) = w$. Obviously the set of all w–functions of P_A forms a subclone H_w.
We define the relation κ_w on P_A by taking $(f,g) \in \theta_w$ iff either $f = g$ or f,g are functions of different arity which take the constant value w on A. It is easy to check that θ_w is a clone congruence of H_w. With this notation we present

Theorem 4.4.1. [A. I. Malcev 73] *Let* $\mathbf{H}$ *be a subclone of* $\mathbf{P}_A$ *such that* $\mathbf{H}$ *contains properly the clone* H_w. *Then the only clone congruences of* $\mathbf{H}$ *are* κ_0,κ,κ_1.

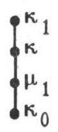
Corollary 4.4.2. Every primal algebra $\mathbf{A} = (A,\Omega)$ has a subdirectly irreducible clone $T(\mathbf{A})$ of term functions.
We restrict in the following our consideration to Boolean clones (clones of function on set $A = \{0,1\}$) and define $(f,g) \in \kappa_c$ if and only if ar f = ar g and there is an element $c \in \{0,1\}$ with $f(x_1,...,x_n) = g(x_1,...,x_n)+c$. $(f,g) \in \mu$ if and only if $f = g$ or there is an element $n \in \mathbb{N}$ sucht that $f,g \in \{c_0^n,c_1^n\}$ with the constant functions $c_0^n(x_1,...,x_n) = 0$ and $c_1^n(x_1,...,x_n) = 1$.

Theorem 4.4.3. [Gorlov] *The congruence lattices of all Boolean clones are of the following form*

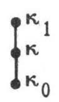


L_1, L_5, O_4

O_9



O_8



all other Boolean clones

Corollary 4.4.4. Every Boolean clone is subdirectly irreducible.

Now we change the direction of this topic and consider algebras **A** with a subdirectly irreducible clone T(**A**).

Definition 4.4.5. *The algebra* $\mathbf{A} = (A,\Omega)$ *is called 2–subdirectly irreducible if* **A** *is a subdirectly irreducible algebra and* T(**A**) *is a subdirectly irreducible clone.*

Example. Every algebra $\mathbf{A} = (\{0,1\},\Omega)$ is 2–subdirectly irreducible.

Problem 4.4.6. *Is every solid variety generated by its 2–subdirectly irreducible algebras?*

§4.5 Clone–products of algebras

Definition 4.5.1. *The algebra* $\mathbf{A} = \mathbf{A}_1 * \mathbf{A}_2$ *is called a direct clone–product of the algebras* $\mathbf{A}_1,\mathbf{A}_2$ *provided that there exist clone congruences* ζ_1,ζ_2 *of* T(**A**) *such that*

(i) *for every* $f \in T(A)$ *there is* $\bar{f} \in T(A)$ *with* $\bar{f} = \begin{cases} f/A_i^n \\ e/A^n \ A_i^n \end{cases}$ *such that*

$(f,\bar{f}) \in \zeta_i$, $i = 1,2$ $e/A^n \backslash A_i^n$

(ii) $\zeta_1 \wedge \zeta_2 = \omega$ *where* ω *is the identity relation*

(iii) $\zeta_1 \vee \zeta_2 = \zeta_2 \circ \zeta_1 = \kappa$ *where* κ *is the arity relation.*

e *is the first projection* $e(x_1,\ldots,x_n) = x_1$

Definition 4.5.2. *If* **A** *is isomorphic to a subalgebra for a direct clone–product of* $\mathbf{A}_1$, $\mathbf{A}_2$ *then* **A** *is called a subdirect clone–product of* A_1,A_2.

Theorem 4.5.3. *Let* $\mathbf{A}_1 = (A_1,\Omega_1)$ *and* $\mathbf{A}_2 = (A_2,\Omega_2)$ *be algebras of not necessarily the same type. Let* $A_1 \cap A_2 = 0$ *and* $A = (A_1 \cup A_2)$. *Let* $T(\mathbf{A})$ *be the clone generated by the functions* f *such that* $f/A_i^n \in T(A_i)$, $i = 1,2$ *and* $f(a_1,...,a_n) = a_1$ *for* $(a_1,...,a_n) \notin A_1^n$ *and* $(a_1,...,a_n) \notin A_2^n$. *Let* Ω *be a set of generators of* $T(\mathbf{A})$. *Then* $\mathbf{A} = (A,\Omega)$ *is a direct clone–product of* $\mathbf{A}_1,\mathbf{A}_2$.

Proof. We define $f\ \theta_i g$ if and only if for the arity we have ar f = ar g and $f/A_i^n = g/A_i^n$, $i = 1,2$ and $f(x_1,...,x_n) = g(x_1,...,x_n)$ for $(x_1,...,x^n) \notin A_2^n$.
By definition θ_i is an equivalence relation contained in κ. Also by definition the condition (i) is fulfilled. Obviously θ_i is a clone congruence relation (i.e. compatible with the clone operations).
From $f\ \theta_1 \wedge \theta_2\ g$ it follows $f(x_1,...,x_n) = g(x_1,...,x_n)$ for every $(x_1,...,x_n) \in A^n$. Hence we have $\theta_1 \wedge \theta_2 = \omega$.
Let $f,g \in T(A)$ and $(f,g) \in \kappa$. Then we consider $h{:}A_n \longrightarrow A$ such that $h/A_1^n = f/A_1^n$, $h/A_2^n = g/A_2^n$ and $h(a_1,...,a_n) = a_1$ else. Obviously we have $f\ \theta_1\ h$ and $h\ \theta_2\ g$. Hence $f\ \theta_1 \circ \theta_2\ h$ and also $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1 = \kappa$.

Lemma 4.5.4. *Let* $\mathbf{A} = \mathbf{A}_1 * \mathbf{A}_2$ *be a subdirect clone product of* $\mathbf{A}_1$ *and* $\mathbf{A}_2$. *Then* $T(\mathbf{A})$ *is a subdirect product of* $T(\mathbf{A}_1)$ *and* $T(\mathbf{A}_2)$.

(This follows from 4.5.1. (ii)).

§4.6. Clone–unions of algebras

The construction in 4.5 has a lot of beautiful properties which will fail for the method of clone–union.

Definition 4.6.1. *Let* $\mathbf{A}_i = (A,\Omega_i)$ *be algebras and let* $\Omega = \Omega_1 \cup \Omega_2$ *and* $\tau = \tau_1 \cup \tau_2$. *The algebra* $\mathbf{A}$ *is a clone–union of the algebras* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *if the following holds*

1) $T(A_1) \cap T(A_2) = P$

2) $\mathbf{A}$ is weakly isomorphic to (A,Ω) of type τ

(P is the clone of projections on A).

Example 4.6.2. The distributive lattice $D = (\{0,1\}\wedge,\vee)$ is a clone–union of the semilattices $D_1 = (\{0,1\};\wedge)$ and $D_2 = (\{0,1\};\vee)$

Example 4.6.3. The cyclic group $C_3 = (A;+)$ of order 3 where $C = \{0,1,2\}$ is the clone–union of the quasi–groups $C_3^{2x+y} = (A;+_1)$ and $C_3^{2x+2y} = (A;+_2)$. The fundamental term function $x+_1y$ of C_3^{2x+y} is defined by $x+_1y = 2x+y$ in terms of C_3 and $x+_2y = 2x+y$ respectively.

This construction may have a lot of disadvantages but can also be consdidered as a tool to decompose algebras. It is far from being unique in any sense. Nevertheless, let us state the following

Problem 4.6.4. *Can every finite abelian group be presented as a direct product of clone–unions of simple quasi–groups?*

Part V Hybrid logic

§ 5.1 Hyperquasi–identities

The approach to hyperidentities in § 1.1 and §1.2 is extended to quasi–identities and sentences in the following.

We develope logics containing the hypersubstitution as an additional rule and prove completeness. Compared to §1.2 we have chosen a different way to these results. Proofs in logics with hypersubstitutions are transformed in proofs logics without hypersubstitutions and vice versa. This method clearly points out that a logic with hypersubstitution has more expressive power and the proofs are usually shorter. These logics with hypersubstitutions proceeds beyond first order (but they are only a fragment of the second order logic). One can strengthen the expressive power of these logics furthermore if operation symbols and hypervariables are admitted simultaneously in the language.
All sentences, quasi–identities and identities are written without quantifiers but are considered as universally closed. We consider varieties of algebras of given type. A type of algebras τ is a sequence $(n_0,n_1,\dots,n_\tau,\dots)$ of positive integers, $\tau < 0(\tau)$, where $0(\tau)$ is an ordinal, called the order of τ. For every $\tau < 0(\tau)$ we have a symbol f_τ for an n_τ–ary operation. Moreover, for every τ the symbol F_τ is associated. F_τ is called an n_τ–ary hypervariable.

Definition 5.1.1. *Let τ be a given type. The n-ary hyperterms of type τ are recursively defined by:*

(1) *the variables* $x_1,\ldots,x_n$ *are n-ary hyperterms,*

(2) *if* $T_1,\ldots,T_m$ *are n-ary hyperterms and* F_τ *is a m-ary hypervariable of type τ, then* $F_\tau(T_1,\ldots,T_m)$ *is n-ary hyperterm of type τ.*

$H^n(\tau)$ is the smallest set containing (1) which is closed under finite application of (2). $H(\tau) = \cup(H^n(\tau) : n \in \mathbb{N})$ is called the set of all hyperterms of type τ (where $\mathbb{N}$ is the set of all positive integers). A hyperidentity of type τ is a pair of hyperterms (T_1,T_2), which also denoted by $T_1 = T_2$. The free algebra in countably many variables of a variety V of type τ is denoted by T(V) and its elements t are called terms. A quasi—identity is an implication of the form

$$(t_0 = s_0) \wedge (t_1 = s_1) \wedge\ldots\wedge (t_{n-1}= s_{n-1})) \longrightarrow (t_n= s_n).$$

Definition 5.1.2. *A hyperquasi-identity is an implication of the form*

$$(T_0 = S_0) \wedge (T_1 = S_1) \wedge\ldots\wedge (T_{n-1} = S_{n-1})) \longrightarrow (T_n = S_n).$$

where $T_i = S_i$ *are hyperidentities,* $i = !,\ldots,n$.

Definition 5.1.3. *A mapping* $\sigma : \{F_i \mid i \in I\} \longrightarrow T(V)$ *which assigns to every* n_i*-ary hpervariable an* n_i*-ary term is called hypersubstitution. Such a mapping σ can be extended to a mapping $\bar{\sigma}$ from the set of hyperterms into* T(V) *by defining recursively* $\bar{\sigma}(x) = x$ *for every variable* x *in* T(V) *and*

$$\bar{\sigma}(F_i(T_1,\ldots,T_n)) = \sigma(F_i)(\bar{\sigma}(T_1),\ldots,\bar{\sigma}(T_n)).$$

In the following both maps σ, $\bar{\sigma}$ are denoted by σ only and we call $\sigma(T) = \sigma(S)$ a transformation of the hyperidentity $T = S$ into an identity. Similarly we have transformation σ of hyperquasi–identities into quasi–identities. Z is the set of all these transformations. For a hyperquasi–identity e the set

$$Z(e) = \left\{\sigma(e) \mid \sigma \in Z\right\}$$

denotes all transformations of E.

Example. Consider the following quasivariety V of type (2)

(K1) $x \circ (y \circ z) = (x \circ y) \circ z$

(K2) $x \circ x = x$

(K3) $(u \circ x) \circ (y \circ w) = (u \circ y) \circ (x \circ w)$

(K4) $(x \circ y = y \circ x) \longrightarrow x = y$.

This quasivariety is not trivial as it contains for instance the algebra ({ 0,1)};o) with $x \circ y = y$. The terms in two variables x,y of T(V) can be listed up:

$t_1(x,y) = t_2(x,y) = y$, $t_3(x,y) = x \circ y$, $t_4(x,y) = y \circ x$, $t_5(x,y) = x \circ y \circ x$, $t_6(x,y) = y \circ x \circ y$. We consider the hyperquasi–identy

$$(F(x,y) = F(y,x)) \longrightarrow (x = y).$$

If we replace the hypervariable F by the term t_5 this transformation produces

$$(x \circ y \circ x = y \circ x \circ y) \longrightarrow (x = y).$$

Definition 5.1.4. *A quasivariety* V *of type* τ *satisfies a hyperquasi–identity* e *if the set* Z(e) *of quasi–identities holds for* V.

Example. $(F(x,y) = F(y,x) \longrightarrow x = y)$ holds for V. We would have to consider all terms listed up but confine us to t_5. Now $x \circ y \circ x = y \circ x \circ y$ implies $(x \circ y) \circ (y \circ x) = (x \circ y) \circ (x \circ y)$ by (K2) and (K1) and furhtermore $x = y$ by (K4).

Definition 5.1.5. *A mapping*

$$h : \{f_i \mid i \in I\} \longrightarrow \{F_i \mid i \in I\}$$

which assigns to every n_i–ary *operation symbol* f_i *an* n_i–ary *hypervariable* F_i *is called a transformation of terms if the variables* x,y,z,... *are left unchanged. Of course we extend to the set* T(V) *of all terms recursively.*
The set of all these transformations is denoted by Z^{-1}.

Example. The quasi–identity $(x \circ y \circ x = y \circ x \circ y) \longrightarrow (x = y)$ is transformed to the hyperquasi–identity.

$$(F(F(x,y),x) = F(F(y,x),y)) \longrightarrow (x = y).$$

This hyperquasi–identity holds for the above quasivariety V because it can be derived from h(K1), h(K2), h(K3) and h(K4).

Definition 5.1.6. *A quasivariety* V *is called solid if every quasi–identity of* V *can be transformed to a hyperquasi–identity which hold for* V.

Notation. Let Σ be the set of identities which hold for V. If V is solid the $Z^{-1}(\Sigma) \subseteq E$ where E is the set of all hyperquasi–identities which hold for V.

Examples. 1) The quasivariety V of type 2 with the axioms K(1) – K(4) is solid. 2) Every hyperquasivariety of a given type (i.e. a quasivariety defined by hyperquasi–identities).

§ 5.2. Preservation properties

We quote the following results.

Theorem, 5.2.1. [Malcev 71] *A class* K *of algebras of a type* τ *is a quasivariety if and only if* K *is closed under the formation* S *of subalgebras,* I *isomorphic images and* P_R *reduced products.*

$$SK \subseteq K,\ P_R K \subseteq K.$$

(Here we put $P_R := IP_R$*).*

Theorem 5.2.2 *Let* K *be a class of algebras of type* τ*.* S P_R K *is the class of all models of the set of quasi–identities true in* K.

Notation. Let K be a class of algebras of a given type $\tau = (n_0,n_1,\ldots n_\gamma,\ldots)$. The algebra *B* is called a derived algebra of $A = (A;\ f_0,f_1,\ldots,f_\gamma,\ldots)$ if there exist term operations $t_0,t_1,\ldots,t_\gamma,\ldots$ of type τ such that $B = (A;\ t_0,t_1,\ldots,t_\gamma,\ldots)$. For a class K of algebras of type τ we denote by D(K) the class of all derived algebras of type τ of K.

Theorem 5.2.3. *A class* K *of algebras of a type* τ *is a solid quasivariety if and only if we have*

$$SK \subseteq K,\ P_R K \subseteq K,\ DK \subseteq K.$$

Proof by the following lemma.

Lemma 5.2.4. *A quasivariety* V *of type* τ *is solid if and only if it is closed under the condition:*

Let A *be an algebra of* V, *of type* $\tau = (n_1 n_2,\ldots,n_\gamma,\ldots:\ \tau < 0(\tau))$.

(*) *If* t_γ *is the realization of an* n_γ–ary *term operation of type* τ *in* A *then* $\bar{A} = (A,(t_1,t_2,\ldots,t_\gamma,\ldots: \tau < 0(\tau)))$ *is an algebra of* V.

Proof. Let V be a solid quasivariety. Consider the algebra

$$\bar{A} = (A,(t_1,t_2,\ldots,t_\tau,\ldots: \tau < 0(\tau))).$$

The quasi–identities of V are transformed into hyperquasi–identities of V and hence hold for the term functions t_τ. Especially they hold for $\bar{A}$. Hence $\bar{A} \in V$. Let the condition (*) hold for V. Then the quasi–identities of V hold for all term functions of the suitable arity and hence are transformed into hyperquasi–identities, i.e. V is a solid variety.

Theorem 5.2.5. K *of type* τ *is a solid quasivariety if and only if*

$$K = S\, P_R\, DK.$$

Proof. We have to show $DP_R K \subseteq P_R DK$. For $B \in DP_R(K)$ we have $B_0 = (A,t_0,t_1,\ldots,t_\gamma,\ldots)$ with $A = (A,f_0,f_1,\ldots f_\gamma,\ldots)$ and $A = \Pi A_i$, $A = (A;f_0,f_1,\ldots,f_\gamma\ldots)$. Consider $X_i := (A_i,t_0,t_1,\ldots,t_\gamma\ldots)$ then we have $B = \Pi B_i$ and hence $B \in P_R D(K)$.

§ 5.3. Solid classes of models

We are considering a class of relational structures of given type τ. The type of a structure is a sequence $(n_0,n_1,\ldots,n_\gamma,\ldots)$ of positive integers, $\tau < 0(\tau)$, where $0(\tau)$ is an ordinal. For every $\tau < 0(\tau)$ we have a predicate symbol r_γ for an n_γ–ary relation. Moreover for every τ the symbol R_γ is associated. R_γ is called a hyper predicate variable.

Definition 5.3.1. *An atomic hyperformula is an expression of the form* $P(T_1,...,T_n)$ *where* P *is an n–place hyper predicate variable and* $T_1,...,T_n$ *are hyperterms.*

Definition 5.3.2. *The hyperformulas are built up from the atomic formulas by use of the connective symbols and the quantifier symbol* $(R \wedge Q)$, $\forall x_i R$.

Definition 5.3.3. *A hypersentence is a hyperformula where every variable and every hyperpredicate variable is bound.*

Example. $\forall P\ \forall x \forall y\ (P(x,y)) \longrightarrow P(y,x))$. We only write: $P(x,y) \longrightarrow P(x,y)$ dropping all quantifiers.

Notations 5.3.4. Given a class K of models of type τ and a hypersentence $C(R_1,...,R_n)$ of type τ. Let σ be a map of all hyperpredicate variables into the set of quantifier–free formulas. σ transforms $C(R_1,...,R_n)$ into a first–order–formula $\sigma(C(R_1,...,R_n))$. Let Z be the set of all these transformations. The hypersentence $C(R_1,...,R_n)$ holds in the class **K** if for all $\sigma \in Z$ $\sigma(C(R_1,...,R_n))$ is valid formula of first order for **K**. We write

$$\models_{\text{hyp}} C(R_1,...,R_n).$$

Similarly we define Σ of hypersentences and a hypersentence U.

Notation 5.3.5. Let $c(r_1,...,r_n)$ be a quantifierfree formula of first–order and let m be the maximum of the arities of the predicate symbols $r_1,...,r_n$. Then we define the derived relation r by

$$(x_1,...,x_m) \in r \Rightarrow (x_1,...,x_n) \in c(r_1,...,r_n)$$

or in the usual notation:

$$r(x_1,\ldots,x_m) \Leftrightarrow c(r_1,\ldots,r_n)(x_1,\ldots,x_m).$$

Let $A = (A,\rho)$ be a relational system of a class K. A derived relational system $A = (A,\bar{\rho})$ is a system where every relation in ρ is substituted by a derived relation of the same arity.

D(K) denotes the class of all derived relational systems of K.

Definition 5.3.6. *A class of models of type τ is called solid if every sentence valid in* K *hold as hypersentence in* K *substituting the predicate symobls by hyperpredicate symbols of the same arity.*

Notation. We denote these transformations by h, the set of these transformations by Z^{-1} and we also write $Z^{-1}[\Sigma]$ of a set Σ of sentences.

Example of a solid model of type (2,2): (A;p,q) axioms:

(*) p(x,y), (**) $q(x,y) \longrightarrow q(y,x)$. We show that (***) $P(x,y) \longrightarrow P(y,x)$ is a hypersentence.

Proof. If w(x,y): = p(x,y) then (***) holds by (*). All sentences can be built up by the connectives $\wedge$, $\rightarrow$.

a) We assume $w(x,y) \equiv (k(x,y) \wedge l(x,y))$ and (***) holds for k(x,y). We have

$$k(x,y) \longrightarrow k(y,x)$$
$$l(x,y) \longrightarrow l(y,x)$$

and hence

$$k(x,y) \wedge l(x,y) \longrightarrow k(x,y)$$
$$k(x,y) \wedge l(x,y) \longrightarrow l(y,x)$$

and finally

$$k(x,y) \wedge l(x,y) \longrightarrow k(y,x) \wedge l(y,x)$$
$$w(x,y) \longrightarrow w(y,x).$$

b) We assume $w\,(x,y) \equiv (k(x,y) \longrightarrow l(x,y))$. By (***) we have

$$\begin{array}{ccc} k(x,y) & \longleftrightarrow & k(y,x) \\ \downarrow & & \\ l(x,y) & \longleftrightarrow & l(y,x). \end{array}$$

It follows $k(y,x) \longrightarrow l(y,x)$ and hence $w(x,y) \longrightarrow w(y,x)$

q.e.d.

§ 5.4. Completeness for Hypersentences

We follow the notation of [Enderton] and present the following axiom schemes for a logic of hypersentences.

1. Tautologies
2. Substitution of variables: $\forall x \alpha \longrightarrow \alpha^{x}_{T}$
3. $\forall P \alpha \longrightarrow \alpha^{P}\ C(R_1,\ldots,R_n)$.

4a) $\forall x(\alpha \rightarrow \beta) \longrightarrow (\forall x \alpha \rightarrow \forall x \beta)$

b) $\forall P(\alpha \rightarrow \beta) \longrightarrow (\forall P \alpha \rightarrow \forall x \beta)$

5a) $\alpha \longrightarrow \forall x \alpha$ where x does not occur free in $\alpha\alpha$

$\alpha \longrightarrow \forall P \alpha$ where P does not occur free in α.

Rule of inference: Modus ponens $\dfrac{\alpha, \alpha \rightarrow \beta}{\beta}$.

Definition 5.4.1. *Let* E *be a set of hypersentences. The hypersentence can be derived from* Σ *if there is finite sequence* $(\alpha_0,\dots,\alpha_n)$ *of hyperformulas such that* $\alpha_n \equiv e$ *and for each* $0 \leq i \leq n$ *either*

a) $\alpha_i \in \Sigma \cup \Lambda$ *denotes the axiom schemes, or*

b) *for some* j *and* k *less than* i α_j *is obtained by the modus ponens from* α_j *and* α_k.

We write $\Sigma \vdash_{\text{hyp}} e$.

Lemma 5.4.1. *Let* E *be a set of hypersentences and* e be *a hypersentence*

$$E \vdash_{\text{hyp}} e \textit{ iff } \bigcup_{\sigma \in Z} \sigma'(E) \vdash \sigma(e) \textit{ for every } \sigma \in Z.$$

The proofs for lemma 5.4.1. and theorem 5.4.2. will be given in section 5.9 in a more general setting.

Remark. For quasi–identities we consider the axioms and rules given in Selman [10]. The above lemma 4.1 and the theorem 4.2 hold also for quasi–identities after changing the notations. This also is the case for the following results of this section 4.

Notation. Let us denote the substitution of every n_γ–ary predicate symbol by a n_γ–ary hyperpredicate symbol R_γ by the bijective map h. If Σ is a set of sentences then $H(\Sigma)$ denotes the corresponding set of hypersentences. We formalize the

Definition 5.4.3. *Let* E *be the set of hypersentences and* Σ *the set of sentences which hold for the class* K *of models* K *is a solid class of models if* $H(\Sigma) \subseteq E$*: Obviously we have* $H(\Sigma) = E$.

Theorem 5.4.4. K *is solid and only if* $\bigcup_{\sigma \epsilon Z} \sigma(E) \subseteq \Sigma$.

Proof. From $\bigcup_{\sigma \epsilon Z} \sigma(E) = \Sigma$ we conclude that $h(\Sigma) \subseteq E$ and hence K is solid. On the other hand assume that we have $h(\Sigma) \subseteq E$. We conclude that $E \vdash_{hyp} h(\epsilon)$ for every sentence ϵ of Σ. By Lemma 4.1 we have $\bigcup_{\sigma' \epsilon Z} \sigma'(E) \vdash \sigma(h(\epsilon))$ for every $\sigma \epsilon Z$. We choose σ such that $\sigma(h(\epsilon)) = \epsilon$ and have $\bigcup_{\sigma' \epsilon Z} \sigma'(E) \vdash \epsilon$ for every sentence ϵ of Σ. As $\bigcup_{\sigma' \epsilon Z} \sigma'(E)$ is closed under the axioms schemes and modus ponens of the predicate calculus we have $\bigcup_{\sigma \epsilon Z} \sigma(E) \supseteq \Sigma$.

§5.5 Hybrid terms

Terms are built up with variables and operation symbols, hyperterms with variables and hypervariables. If one admits operation symbols and hypervariables simultaneously in a language then one gets hybrid terms. Therefore the concept of hybrid terms is a generalization of hyperterms. In our approach we restrict the hypervariables respectively hyperpredicate variables to a fixed type. By this restriction a lot of problems become solvable, a fact which can also be concluded from Henkin's work on completeness. Furthermore we restrict the operator variables which are called hypervariables to terms. These hybrid logics have not the expressive power of a general second order logic. Nevertheless proofs may be shorter and axioms systems may become finite in a hybrid logic.

We consider varieties of a given type. A type τ of algebras is a sequence $(n_0, n_1, \ldots, n_\gamma, \ldots)$ of positive integers, $\gamma < 0(\tau)$, where $0(\tau)$ is an ordinal, called the order of τ. For every $\gamma < 0(\tau)$ we have a symbol f_γ for an n_γ–ary operation. Moreover for every γ with $n_\gamma \neq 0$ the symbol F_γ is associated. F_γ is called an n_γ–ary hypervariable.

Definition 5.5.11. *Let τ be a given type. The n–ary hybrid terms of type τ are recursively defined by:*

(1) *the variables* $x_1,\ldots,x_n$ *are n–ary hybrid terms*

(2) *if* $T_1,\ldots,T_m$ *are n–ary hybrid terms and* f *is a m–ary operation symbol then* $f(T_1,\ldots,T_m)$ *is an n–ary hybrid term*

(3) *if* $T_1,\ldots,T_m$ *are n–ary hybrid terms and* F *is a m–ary hypervariable then* $F(T_1,\ldots,T_m)$ *is an n–ary hybrid term.*

$B^n(\tau)$ *is the smallest set containing (1) which is closed under finite application of (2) and (3).* $B(\tau) = \bigcup\{B^n(\tau) \mid n \in \mathbb{N}\}$ *is called the set of hybrid terms of type τ. A hybrid identity of type τ is a pair of hybrid term* (T_1,T_2) *which is also denoted by* $T_1 = T_2$. *The free algebra in countably many variables of a variety* V *of type τ is denoted by* T(V) *and its elements are called terms. If* V *is generated by the algebra* **A** *we write* T(**A**) *instead of* T(V).

Definition 5.5.2. *Let* (T_1,T_2) *be a hybrid identity of type τ and* V *a variety of type τ. If every n_γ–ary hypervariable occurring* (T_1,T_2) *is replaced by an n_γ–ary term $t_\gamma \in$* T(V) *leaving the variables and operation symbols unchanged in* (T_1,T_2) *then the resulting identity* (t_1,t_2) *is called a transformation of the hyperidentity* (T_1,T_2).

Example. Let $F(x\wedge y,z) = F(x,z)\wedge F(y,z)$ be a hybrid identity with a binary hypervariable F and a binary operation symbol. Let V be the variety of distributive lattices of type (2,2). If we replace F(x,y) by the term $x\vee y$ we get the transformation $(x\wedge y)\vee z = (x\vee z)\wedge(y\vee z)$. To get all four possible transformations F has to be replaced by the four terms x, y, $x\wedge y$, $x\vee y$.

Example. $F(x,F(y,z)) = F(F(x,y),z)$ is a hybrid identity which does not contain any operations symbol. This hybrid identities are called hyperidentities. If E is a set of hybrid identities of type τ then the set of all transformations of E for a variety V of type τ is denoted by $I_V(T_1,T_2)$.

Definition 5.5.3. *A variety* V *of type* τ *satisfies the hybrid identity* (T_1,T_2) *of type* τ *if the set* $I_V((T_1,T_2))$ *of all transformations of* T_1,T_2 *is contained in the set of identities which hold in* V.

Example. The hybrid identity $F(x \wedge y,z) = F(x,z) \wedge F(y,z)$ holds for the variety of distributive lattices.

Definition 5.5.4. *Let* (t_1,t_2) *be an identity which holds for a variety* V. *If one substitutes some* n_γ*-ary operation symbols* f_γ *by* n_γ*-ary hypervariables* F_γ *having the variables unchanged then the resulting hybrid identity* (T_1,T_2) *is called a transformation of* (t_1,t_2).

Example. Consider the identity $(x \wedge y) \vee z) = (x \vee z) \wedge (y \vee z)$ for the variety of distributive lattices V. If we substitute the operations symbol $\vee$ by the binary hypervariable F we get the hybrid identity $F(x \wedge y,z) = F(x,z) \wedge F(y,z)$. Of course one can get transformations like $F(x,y) = F(y,x)$ from $x \vee y = y \vee x$ which do not hold as hybrid identites for V.

A transformation of (t_1,t_2) which contains a maximal number of different hypervariables is called general. For instance, $F(G(x,y),z) = G(F(x,z),F(y,z))$ is a general transformation from the law of distributivity.

We use a slight generalization of the concept of hypersubstitution [6] p.308.

Rule of hybrid substitution.

(6) The hybrid identity (T_1,T_2) implies the hybrid identity (T^*_1,T^*_2) if (T^*_1,T^*_2) is the result of any simultaneous substitution of hypervariables in T_1 and T_2 by a hybrid term of the same arity.

Together with the rules (1)–(5) which are reformulations from the classical equational logic the derivation of hybrid identities is defined.

(1) $T_1 = T_1$ for every hybrid term $T_1 \in B(\tau)$

(2) $T_1 = T_2$ implies $T_2 = T_1$ for hybrid terms $T_1,T_2 \in B(\tau)$

(3) $T_1 = T_2$, $T_2 = T_3$ implies $T_1 = T_3$ for hybrid terms $T_1,T_2,T_3 \in B(\tau)$

(4) $T_i = S_i$ for $i = 1,\dots,m_\gamma$ implies $F_\gamma(T_1,\dots,T_{m_\gamma}) = F_\gamma(S_1,\dots,S_{m_\gamma})$ for hybrid terms T_i, $S_i \in B(\tau)$ and m_γ–ary hypervariables F_γ.

(5) $T(x_1,\dots,x_n) = S(x_1,\dots,x_n)$ implies $T(R_1,\dots,R_n) = S(R_1,\dots,R_n)$ for $T,S,R_1,\dots,R_n \in B(\tau)$.

Given a variety V of type τ, $E_\tau(V)$ denotes the set of a hybrid identities of type τ which are satisfied in V.

The following is a slight modification of G. Birkhoff's theorem [2]. (Compare also [6]).

Completeness theorem. A set Σ of hybrid identities can be presented in the form $E_\tau(K)$ for some variety k of type τ if and only if Σ is closed under the rules (1)–(6).

§5.6. Basis of hybrididentities

The hybrid equational logic has more expressive power than an equational logic. Hence one can expect that some varieties can be described by a shorter system of axioms. Let D be a set of hybrid identities of some variety V. We call D a hybrid basis of identities of V if every identity of V is implied by D. D is a basis of hybrid identities of V if every hybrid identity V is implied. (D is called a basis of hyperidentities if every hyperidentity is implied.)

Proposition 5.6.1. *Let*
$D = \{x\cdot(y\cdot z) = (x\cdot y)\cdot z,\ xyzw = xzyw,\ yx^2y = xy^2x,\ y\cdot G(x)\cdot x^2y = x\cdot y\cdot G(x)\cdot y\cdot x\}$
be a set of hybrid identities involving an associative binary operation symbol and a unary hypervariable G. *Then* D *cannot be presented by a finite basis of identities but by a finite hybrid basis of identities.*

Proof. We replace the hyperterms G(x) by x^k, $k \in \mathbb{N}$, and get an infinite set E of identities. By [Perkins] this infinite set E of identities has no finite basis of identities.

Problem 5.6.2. *Determine an algebra of minimal cardinality without a finite hybrid basis of identities.*

Remark 5.6.3. Let V be a variety which has no finite bases of hyperidentities. Then there exists no finite bases of hybrid identities. Because of the rules (1)–(6) of hybrid logic it is impossible to derive hyperidentities from other hybrid identities than hyperidentities.

Solid varieties [6] are varieties where every identity can be transformed into a hyperidentity. The following results give a new description.

Lemma 5.6.4. *A variety* V *of type* τ *is solid if and only if every transformation of any of the identities of* V *holds as a hybrid identity of* V.

Proof. Let (t_1,t_2) be an identity of V and (T_1,T_2) be a transformation into a hybrid identity. Let (T^*_1,T^*_2) be a transformation into a hyperidentity with a maximal number of different hypervariables. We replace the appropriate hyperterms in (T^*_1,T^*_2) to get (T_1,T_2). As V is solid (T^*_1,T^*_2) holds for V and hence also (T_1,T_2).

Corollary 5.6.5. Let Σ be a basis for the identities of V of type τ. V is solid if and only if every transformation of Σ holds as a hybrid identity of V.

§5.7. Hybrid Terms of Distributive Lattices

Notation 5.7.1. We consider the following set B of hybrid identities of type (2,2) using the binary operation symbols $\wedge,\vee$ and the binary hypervariables F,G.

(H1) $F(x,F(y,z)) = F(F(x,y),z)$

(H2) $F(x,x) = x$

(H3) $F(F(u,x),F(y,w)) = F(F(u,y)F(x,w))$

(H4) $F(G(x,y),z) = G(F(x,z),F(y,z))$

(H5) $F(x,G(y,z)) = G(F(x,y),F(x,y))$

(E1) $x\wedge y = y\wedge x,\ \ x\vee y = y\wedge x$

(E2) $x\wedge(y\vee x) = x,\ x\vee(y\wedge x) = x$

Remark 5.7.2. An algebra L of type (2,2) is a distributive lattice if the hybrid identities H1, H2, H4, H5, E1, E2 hold.

Proof. From H1 follows the associativity, from H2 the idempotency and from H4 and H5 the distributivity of the lattice operations $\wedge$ and $\vee$ (putting $x := u$, $y := z$, $z := y = w$ and hypersubstituting F and G by $\wedge$ and $\vee$ respectively).

Remark 5.7.3. The hyperidentity (H4) respectively (H5) implies

(M1) $\quad F(x\wedge y,z) = F(x,z)\wedge F(y,z)$

(if we hypersubstitute G by $\wedge$).

(M2) $\quad F(x\vee y,z) = F(x,z)\vee F(y,z)$

(M3) $\quad F(x,y\wedge z) = F(x,y)\wedge F(x,z)$

(M4) $\quad F(x,y\vee z) = F(x,y)\vee F(x,z)$

Proposition 5.7.4. *Every hybrid term* T *can be presented as a* *d̲isjunction of c̲onjunctions of h̲yperterms i.e. in dch–form.*

Proof. If T is a hyperterm then 3.5 holds. If $T = T_1 \vee T_2$ and T_1,T_2 are in dch–form then T is in dch–form. If $T = T_1 \wedge T_2$ then by the distributive law T can be presented in dch–form. If $T = F(T_1,T_2)$ we apply (M1)–(M4) to get a dch–form.

Example. Consider $T = G(F(x\wedge y,z),x)$. T can be transformed in the dch–form in the following way

$$G(F(x\wedge y,z),x) \underset{(M1)}{\longrightarrow} G(F(x,z)\wedge F(y,z),x) \underset{(M1)}{\longrightarrow} G(F(x,z),x)\wedge G(F(y,z),x).$$

Notation. A hyperterm T is called a F–hyperterm respectively G–hyperterm if T contains only hypervariables F or respectively G.

Proposition 5.7.5. *Every hyperterm* T *can be represented as a* F*–hyperterm substituted by* G*–hyperterms.*

Proof. One applies (H4) and (H5)

Example.

$$G(F(x,y),F(u,v)) \underset{(H4)}{\longrightarrow} F(G(x,F(u,u),G(y,F(u,u)) \underset{(H5)}{\longrightarrow} F(F(G(x,u),G(x,u),F(G(y,u),$$
$$G(y,u))) \underset{(H1)}{\longrightarrow} F(F(F(G(x,u),G(x,u)),G(y,u)),G(y,u))$$

Remark 5.7.6. As F,G are associative one may write by abusing the notations

$$F(x_1,\dots,x_n) := F(F\dots(F(x_1,x_2),x_3,\dots,x_n)$$

§5.8. Unification of Hybrid Terms of 2–groups

In automatic theorem proving the unification of formulas plays an important role. A unifier of two formulas is a substitution such that the two formulas under this substitution become equal.

The problem of unification has already been studied for second and higher order logics. By a result of [Goldfarb] it is shown that unification is undecidable for the second order logic. Hybrid logic is a fragment of second order logic but it is an open question whether unification is decidable. It is obvious that specific examples in hybrid logic can be handled by a transformation to first order logic. If V is a variety where every n generated free algebra is finite then the unification of hybrid identities is decidable if and only if the unification of identities is decidable.

There are only a few varieties where the unification problem is explicitly solved. We use an idea of Löwenheim to study the unification in the variety of 2–groups (groups of exponent 2).

(H1) $F(x,F(y,z)) = F(F(x,y),z)$

(H2) $F^3(x,y) = F(x,y)$

(H3) $F(F(u,x),F(y,v)) = F(F(u,y),F(x,v))$

(M1) $F(x+y, u+v) = F(x,u) + F(y,v)$

(M2) $F(0,0) = 0$

(A1) $x+x = 0$

(A2) $x+y = y+x$

Here we define

$$F^3(x,y) := F(F(Fx,y),y),y).$$

In a more general form we consider a hybrid term $T(x_1,\dots,x_n)$ and use

($\mathbf{M}_1$) $$T(x_1+y_1,\dots,x_n+y_n) = T(x_1,\dots,x_n) + T(y_1,\dots,y_n).$$

A term for 2–groups can be written in the general form $a_1x_1+\dots+a_nx_n$, $a_i \in \{0,1\}$, $i = 1,\dots,n$. Obviously $a_1(x_1+y_1)+\dots+a_n(x_n+y_n) = a_1x_1+\dots+a_nx_n + a_1y_1+\dots+a_ny_n$.

Similarly we have

($\mathbf{M}_2$) $$T(0,\dots,0) = 0.$$

We denote $T_i^2(x_1,\dots,x_n) = T(x_1,\dots,x_{i-1}, T(x_1,\dots,x_n),x_{i+1},\dots,x_n)$ and recursively $T_i^3(x_1,\dots,x_n) = T(x_1,\dots,x_{i-1}, T_i^2(x_1,\dots,x_n),x_{i+1},\dots,x_n)$. We have for $i \in \{1,\dots,n\}$

($\mathbf{H}_2$) $T_i^3(x_1,\dots,x_n) = T(x_1,\dots,x_n)$

$$a_1x_1+\dots+a_i(a_1x_1+\dots+a_i(a_1x_1+\dots+a_nx_n)+\dots+a_nx_n)+\dots+a_nx_n$$
$$= a_1x_1+\dots+(a_ia_1x_1+\dots+a_ia_1x_1 + a_ix_i + a_ia_nx_n+\dots+a_ia_nx_n)+\dots+a_nx_n$$
$$= a_1x_1+\dots+a_ix_i+\dots+a_nx_n$$

In the following a hybrid substitution is a finite set $\{v_1|T_1,...,v_n|T_n\}$ of pairs where $v_1,...,v_n$ are variables and hypervariables and $T_1,...,T_n$ are hybrid terms such that if v_i is an n_γ–ary hypervariable then T_i is an n_γ–ary hybrid term.

The result θT of applying a hybrid substitution $\theta = \{v_1|T_1,...,v_n|T_n\}$ to a hybrid term can be defined recursively in an obvious way. We hypersubstitute the hypervariables and then substitute the variables.

Example. Consider the hybrid equation $F(x,y)+z = F(y,x)$ which does not hold as a hybrid identity for 2–groups. Consider $\theta = \{F|+, z|0\}$. The result of $\theta(F(x,y)+z)$ is $x+y$ and of $\theta F(y,x)$ is $y+x$.

Definition 5.8.1. *A hybrid substitution θ is unifier to a pair (T_1,T_2) of hybrid terms if $\theta T_1 \doteq \theta T_2$.*

Definition 5.8.2. *A unifier θ for a pair of hybrid terms is a most general unifier if and only if for each unifier σ for the set there is a hybrid substitution λ such $\sigma = \lambda \circ \theta$.*

To find a non–trivial unifier we use an approach similar to Löwenheim (compare [11]). Instead of considering the unification problem for the hybrid equation $T = S$ we study the hybrid equation $T+S = 0$. These equations are equivalent because $T = T+(S+S)$ and $(T+S)+S = 0+S = S$. Hence we search for unifier of the hybrid equation $T(x_1,...,x_n) = 0$.

Lemma 5.8.3. *Let* $T(x_1,\dots,x_n) = 0$ *be a hybrid equation. Then there exist a non–trivial unifier*

$$\theta = \{x_1 | x_1 + T_1^2(x_1,\dots,x_n),\dots,x_n | x_n + T_n^2(x_1,\dots,x_n)\}$$

Proof. We show that θ is a unifier applying (**M1**) and (**H2**). Let n be odd.

$$T(x_1 + T_1^2(x_1,\dots,x_n),\dots,x_n + T_n^2(x_1,\dots,x_n)\}$$

$$= T(x_1 + (\underbrace{x_1+\dots+x_1}_{(n-1)\ \text{times}})+ T_1^2(x_1,\dots,x_n),\dots,x_n + (\underbrace{x_n+\dots+x_n}_{(n-1)\ \text{times}}) + T_n^2(x_1,\dots,x_n))$$

$$= T(x_1,\dots,x_n) + T(T_1^2(x_1,\dots,x_n),x_2,\dots,x_n)+\dots+T(x_1,\dots,x_{n-1}, T_n^2(x_1,\dots,x_n))$$

$$= T(x_1,\dots,x_n) + \underbrace{T(x_1,\dots,x_n)+\dots+T(x_1,\dots,x_n)}_{n\ \text{times}} = 0$$

Let n be even.

$$T(x_1 + T_1^2(x_1,\dots,x_n),\dots,x_n + T_n^2(x_1,\dots,x_n))$$

$$= T(x_1+(\underbrace{x_1+\dots+x_1}_{(n-2)\ \text{times}}) + T_1^2(x_1,\dots,x_n),\dots\dots,x_n + (\underbrace{x_n+\dots+x_n}_{(n-2)\ \text{times}}) + T_n^2(x_1,\dots,x_n))$$

$$= T(T_1^2(x_1,\dots,x_n),x_2),\dots,x_n)+\dots+T(x_1,\dots,x_{n-1}, T_n^2(x_1,\dots,x_n))$$

$$= \underbrace{T(x_1,\dots,x_n)+\dots+T(x_1,\dots,x_n)}_{n\ \text{times}} = 0$$

Let $\sigma = \{F | \sigma F, x_1 | y_1,\dots,x | y_n\}$ be a unifier and let $\sigma T(x_1,\dots, x_n)$ be the result by hypersubstituting F without changing the variables.

Lemma 5.8.4. $\theta(\sigma) = \{x_i | x_i+\sigma T_i^2(x_1,\dots,x_n) + \sigma T(0,\dots,0,y_i,0\dots0), i = 1,\dots,n\}$ is a unifier for $\sigma\, T(x_1,\dots,x_n)$.

Proof. We have only to consider

$$\sigma T(\sigma T(y_1,0,\dots,0),\dots,\sigma T(0,\dots,0,y_n))$$

because of lemma 5.8.3. Interpretating $\sigma T(x_1,\dots,x_n)$ by terms $a_1x_1+\dots+a_nx_n$ we have the result $a_1a_1y_1+\dots+a_na_ny_n = a_1y_1+\dots+a_ny_n = 0$.

Theorem 5.8.5. *Every unifier σ can be presented by $\sigma = \sigma \odot \theta(\sigma)$.*

Proof. We have to consider

$$y_i+\sigma T^2(y_1,\dots,y_n)+\sigma T(0,\dots,0,y_i,0\dots0)$$
$$= y_i+\sigma T_i(y_1,\dots,y_{i-1},0,y_{i+1},\dots y_n)+\sigma T(0,\dots,0,y_i,0\dots0) = y_i$$

§5.9. Hybrid Sentences

We use hypervariables and hyperpredicate variables of a fixed type to define hybrid sentences in the usual recursive way.

Definition 5.9.1. *An atomic hybrid formula is an expression of the form*

$$P(T_1,\dots,T_n)$$

where P *is an n−place hyperpredicate variable and* $T_1,\dots,T_n$ *are hybrid terms.*

The hybrid formulas are built up from the atomic formulas by the use of connective symbols and the quantifier symbol: $(R \wedge Q)$, $(R \longrightarrow Q)$, $\forall x_i\, R$.

A hybrid sentence is a hybrid formula where every variable, every hypervariable and every hyperpredicate variable are bound.

We follow the notation of [Enderton] and present the following axiom scheme for a hybrid logic.

1. *Tautologies*

2. *Substitution of variables* $\forall x\alpha \longrightarrow \alpha^{x}_{T}$

3. $\forall P\alpha \longrightarrow \alpha^{P}_{c}(R_1,\dots,R_n)$

4.a) $\forall x(\alpha \to \beta) \longrightarrow (\forall x\alpha \to \forall x\beta)$

b) $\forall P(\alpha \to \beta) \longrightarrow (\forall P\alpha \to \forall P\beta)$

5.a) $\alpha \longrightarrow \forall x\alpha$ *where* x *does not occur free in* α

b) $\alpha \longrightarrow \forall P\alpha$ *where* P *does not occur free in* α.

Rule of inference: Modus ponens $\dfrac{\alpha, \alpha \to b}{\beta}$

for a hybrid term T, *for hyperpredicate variable* P *or respectively a hypervariable, for hybrid formula* $C(R_1,\dots,R_n)$ *and hybrid formulas* α,β.

Definition 5.9.2. *Let* E *be a set of hybrid sentences. The hybrid sentence e can be derived from* Σ *if there is a finite sequence* $(\alpha_0,\dots,\alpha_n)$ *of hybrid formulas such that* $\alpha_n \equiv e$ *and for each* $0 \leq i \leq n$ *either*

a) $\alpha_i \in \Sigma \cup \Lambda$ *where* Λ *denotes the axiom schemes, or*

b) *for some* j *and* k *less than* i α_i *is obtained by the modus ponens from* α_j *and* α_k. We write $\Sigma \vdash_h e$.

Notations. Let $\bar{\sigma} : \{F_i \mid i \in I\} \longrightarrow T(L)$ assign to every n_i–ary hypervariable F_i an n_i–ary term t of the language L. Such a map $\bar{\sigma}$can be extended to a map $\bar{\bar{\sigma}}$ from the set of hybrid terms into T(V). We define furthermore

$$\bar{\bar{\bar{\sigma}}} : \{P_j \mid j \in J\} \longrightarrow \{p_j(r_{j_1},\dots,r_{j_n}) \mid j \in J\}$$

which assigns to every hyperpredicate variable an atomic formula of the same arity. Altogether we get a transformation σ which assigns to every hybric formula a formula of first order. Z denotes the set of all these transformations σ.

Lemma 5.9.3. *Let E be a set of hybrid sentences and e a hybrid sentence*

$$E \vdash_h e \quad \textit{iff} \quad \bigcup_{\sigma^{\curlyvee} \in Z} \sigma^{\curlyvee}(E) \vdash \sigma(e) \quad \textit{for every } \sigma \in Z.$$

Proof. Let $E \vdash_h e$ and let $(e_1,\ldots,e_n)$, $e_n \equiv e$, be a sequence of hybrid sentences where e_i either follows from e_j, e_k, $j,k \leq i$, by modus ponens or is from the axiom scheme or from E. We choose a $\sigma \in Z$ and transform ervery hybrid sentence e_i to a sentence $\sigma(e_i)$. The sequence $(\sigma(e_1),\ldots,\sigma(e_n))$ need not to be derivation in the predicate calculus because axiom scheme 3, 4b, 5b get meaningless after applying σ. Let us consider a step according axiom scheme 3 from the hybrid sentence $e_h(T_1,\ldots,T_n)$ to $e_k(T_1,\ldots,T_n)$. Then e_k arises from e_h by substituting hyper predicate variables P_γ by atomic hybrid formulas T_γ. For every e_i, $1 \leq i \leq k$ we consider a transformation $\sigma^{\curlyvee}$ such that we have a sequence $(\sigma^{\curlyvee}(e_1),\ldots,\sigma^{\curlyvee}(e_i))$ with $\sigma^{\curlyvee}(e_i) = \sigma(e_i)$. We include this sequence before $\sigma(e_i)$ and get i additional members. Proceeding in such a way we finally end with a possibly much longer sentence within the predicate calculus and have $\bigcup_{\sigma^{\curlyvee} \in Z} \sigma^{\curlyvee}(E) \vdash \sigma(e)$ for $\sigma \in Z$. On the other hand if we have $\bigcup_{\sigma^{\curlyvee} \in Z} \sigma^{\curlyvee}(E) \vdash \sigma(e)$ for every $\sigma \in Z$ we consider a transformation $h \in Z^{-1}$ such that $\sigma(e)$ is transformed to the hybrid sentence e and $\bigcup_{\sigma^{\curlyvee} \in Z} \sigma^{\curlyvee}(E)$ is transformed to a set $\bar{E}$. By axiom scheme 3 it is obvious that $E \vdash_h \bar{E}$ and hence we have $E \vdash_h e$.

Theorem 5.9.4. $E \models_h e$ *iff* $E \vdash_h e$.

Proof. Let $E \models_h e$ and let K be the class of models which fulfill every hybrid sentence of E. Then K fulfills e by hypothesis and furthermore we conclude $\bigcup_{\sigma^{\curlyvee} \in Z} \sigma^{\curlyvee}(E) \models \sigma(e)$ for every transformation σ. By the completeness of the predicate calculus we have $\bigcup_{\sigma^{\curlyvee} \in Z} \sigma^{\curlyvee}(E) \vdash \sigma(e)$for every transformation σ abd by kenna 5.3 $E \vdash_h e$.

For the reverse direction we use again lemma 5.3 and the correctness of the predicate calculus to get $\bigcup\limits_{\sigma^{\curlyvee}\in Z} \sigma^{\curlyvee}(E) \models \sigma(e)$. There is a transformation h such that $h(\sigma(e)) = e$. With the transformation we get a set $\bar{E}$ from $\bigcup\limits_{\sigma^{\curlyvee}\in Z} \sigma^{\curlyvee}(E)$. It is obvious that $E \models_h \bar{E}$ and hence $E \models_h e$.

Remark 5.9.5. One may use lemma 5.3 to show that the models of a set E of hybrid sentences are closed under ultraproducts. It is clear that Craig's interpolation theorem holds for hypersentences.

Additional remark. One should feel free to interprete the hypervariables by special sets of term functions. For instance in the case of Boolean algebras a binary hypervariable may stand only for monotone term functions generated by the operations join and meet. This will yield a different hybrid logic which may have its own merits. Therefore a manifold of hybrid logics concerning types and restrictions for interpretation are possible and may be of good use in applications (for instance in knowledge representation).

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