UNIVERSITÄT KAISERSLAUTERN

ON ITERATIVE SOLUTION OF A LINEAR EQUATION WITH MARKOV OPERATOR

ON EXPONENTIAL MIXING FOR STATIONARY PROCESSES

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FACHBEREICH MATHEMATIK

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On iterative solution of a linear equation with Markov operator

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Abstract

The convergence of an iterative procedure for approximate solution of an equation $\nu A = \mu$ is studied where A is a Markov operator, ν and μ are probability measures. The iterative procedure is based on Bayes formula. The convergence is studied with the help of the coupling method.

1 Introduction

Let $A = A(x, dy)$ be a transition probability measure from some measurable space $(E, \mathcal{E})$ to another measurable space $(E', \mathcal{E}')$. Assume that $A(x, \cdot)$ is a probability measure for any $x \in E$ and $\int A(x, dy) I(y \in \Gamma)$ is measurable for any $\Gamma \in \mathcal{E}'$. Consider the equation

$$\nu A = \mu, \tag{1}$$

where μ is a given element of the space $V(E')$ of all probability measures on E', ν is unknown element of $V(E)$ and

$$(\nu A)(\Gamma) = \int \int \nu(dx) A(x, dy) I(y \in \Gamma).$$

It is an empirical observation that the following procedure based on Bayes formula converges to the exact solution ν_* (proved earlier for finite E and E'):

Procedure

1) Let $\nu_0 \in V(E)$.

2) Let $\mu_0 = \nu_0 A$.

3) Let

$$A_0(dx, \Gamma) = \int \nu_0(dx) A(x, dy) I(y \in \Gamma), \qquad \Gamma \in \mathcal{E}',$$

$$\hat{A}_0(dx \mid y) = \frac{A_0(dx, dy)}{\mu_0(dy)}.$$

4) Let

$$\nu_1 = \mu \hat{A}_0.$$

5) By induction, let

$$\mu_{n+1} = \nu_{n+1} A;$$

$$A_{n+1}(dx, dy) = \nu_{n+1}(dx) A(x, dy),$$

$$\hat{A}_{n+1}(dx, dy) = \frac{A_{n+1}(dx, dy)}{\mu_{n+1}(dy)};$$

$$\nu_{n+2} = \mu \hat{A}_{n+1}, \cdots$$

2 Main result

We'd like to investigate conditions under which $\nu_n \to \nu_*$ in some sense. In fact, we'd obtain the convergence in variance.

Our main (and unique) assumption is the following:

Assumption. There exists such such a measure $\lambda(dy) \in V(E')$ that

$$0 < C^{-1} = \inf_{x,y} \frac{A(x, dy)}{\mu(dy)} \le \sup_{x,y} \frac{A(x, dy)}{\mu(dy)} = C < \infty. \tag{2}$$

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Theorem 1 . *Let a solution ν_* of (1) exists. Then under the Assumption for any "initial data" $\nu_0 \in V(E)$ the convergence in variance holds:*

$$var(\nu_n - \nu_*) \to 0. \tag{3}$$

Moreover, there exist such constants $C, \lambda > 0$ that

$$var(\nu_n - \nu_*) \leq C \exp(-\delta n), \qquad n \geq 0, \tag{4}$$

uniformly in initial data ν_0.

3 Proof

We use the coupling method or the method of unique probability space, see Nummelin (1984), Veretennikov (1991). When one use this method it is preferable to consider processes instead of measures. So we consider the sequences (X_n) and (Y_n) with

$$\mathcal{L}(X_n) = \nu_n, \qquad \mathcal{L}(Y_n) = \mu_n, \qquad n \geq 0$$

(here $\mathcal{L}$ is a distribution). Random values with given distributions may be constructed, for example, with the help of Kolmogorov's theorem. We may consider all X_n and Y_n on the unique probability space, say, (Ω', F', P'). At the same time we consider the stationary Markov sequences $(X_n^* = X^*)$ and $(Y_n^* = Y^*)$ with distibutions

$$\mathcal{L}(X_n^*) = \nu_*, \qquad \mathcal{L}(Y_n^*) = \mu, \qquad n \geq 0$$

on another probability space (Ω'', F'', P'').

Now we shall try to construct such new sequence $(\hat{X}_n)$ on some extension $(\Omega, F, P) = (\Omega', F', P') \times (\Omega'', F'', P'')$ with the following properties:

(1) $\mathcal{L}\left(\hat{X}_n\right) = \mathcal{L}(X_n)$,

(2) $\hat{X}_n = X_n^*$ with the maximal possible probability.

(2) is really the idea of the coupling method and in all known to the author cases it is based on the following simple "folklore" fact:

Lemma 1 . *Let ξ and η be two random vectors on some probability space. Then there exist such extension of the probability space and such random*

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value ς on this extension that

$$\mathcal{L}(\varsigma) = \mathcal{L}(\xi)$$

and

$$P(\varsigma = \eta) = \int \min\left(1, d\mathcal{L}\left(\xi\right)/d\mathcal{L}\left(\eta\right)\right) d\mathcal{L}\left(\eta\right).$$

Under the Assumption the right-hand side μ in (1) has a bounded derivative f with respect to λ: $f = d\mu/d\lambda$. Moreover,

$$C^{-1} \leq f \leq C. \tag{5}$$

Note that for any μ_n this is also true: $d\mu_n \ll d\lambda$, and

$$C^{-1} \leq f_n := d\mu_n/d\lambda \leq C. \tag{6}$$

Thus, it follows from the Procedure (see (5)) that distributions of X_n^* and X_n for $n \geq 1$ are equivalent and, moreover,

$$0 < C^{-2} \leq d\mathcal{L}\left(X_n\right)/d\mathcal{L}\left(X_n^*\right) \leq C^2. \tag{7}$$

So we can apply our lemma to construct a new process $\hat{X}_n$ we spoken earlier:

A) if $\hat{X}_n = X_n^*$ then $\hat{X}_{n+1} = \hat{X}_{n+2} = \cdots = X^*$.

B) if $\hat{X}_n \neq X_n^*$ then we go on this construction and apply our lemma at the next step $n+1$.

By virtue of (7) we have

$$P(\hat{X}_n = X_n^*) \geq C^{-1} > 0 \tag{8}$$

on each step of our construction. Denote $L := \inf(n \geq 0 : \hat{X}_n = X_n^*)$. Then (8) implies the inequality

$$P(L > n) \leq (1 - C^{-1})^n. \tag{9}$$

Thus,

$$var\left(\mathcal{L}\left(X_n\right) - \mathcal{L}\left(X^*\right)\right) \leq 2P\left(L > n\right) \leq 2(1 - C^{-1})^n. \tag{10}$$

The first obvious inequality in (10) is known as the main inequality of the coupling method. Theorem is proved.

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References

Nummelin, E. (1984) General irreducible Markov chains and non–negative operators. Cambridge et al.: Cambridge Univ. Press.

Veretennikov, A.Yu. (1991) On the estimates of the mixing rate for Markov processes. Lietuvos Matem. Rink. Vol. 31(1). P. 40-49 (in Russian)

On exponential mixing for stationary processes

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Abstract

The rate of mixing for stationary sequences is studied for some
"ratio-mixing" coefficient under conditions of weak dependance of the
"previous path".

1 Introduction

In Veretennikov (1988), (1991), (1992), Gulinskii and Veretennikov (1992)
the rate of mixing for Markov and stationary processes is studied. In this
paper we go on the investigation of this problem for stationary processes.
Now we study more strong mixing coefficient in compare with Veretennikov
(1992).

Earlier and in recent years results on the rate of mixing for various
processes were obtained mainly for Gaussian stationary processes, see Kol-
mogorov and Rozanov (1960), Ibragimov (1961, 1962), and for Markov pro-
cesses, see Doob (1953), Davydov (1973), Veretennikov (1988, 1991). Vari-
ous examples and properties of mixing for stationary processes were estab-
lished by Bradley (1980), Berbee and Bradley (1984) et al. In this paper
we extend the approach from Veretennikov (1988, 1991) to nonmarkov and
nongaussian case.

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2 Main results

We consider the stationary process $(X_n, -\infty < n < \infty)$ with values in the torus $[0,1]$ and regular transition probability densities

$$p(x^n_{-\infty}; x_{n+1}) = P(X_{n+1} \in dx_{n+1} \mid X^n_{-\infty} = x^n_{-\infty})/dx_{n+1}$$

which do not depend on n. Here again $x^n_{-\infty} = (x_n, x_{n-1}, ...)$, $x_i \in [0,1]$. We study here the following mixing coefficient:

$$\psi(s) = ess_{(P)} \sup_{(B \in F_{\geq s})} |\frac{B(B|F_0)}{P(B)} - 1|,$$

where $F_{\geq s} = \sigma(X_i, i \geq s)$. Let us state assumptions on the density p:

Assumption 1 . *For any* $X^0_{-\infty}, Z^0_{-\infty}, y$

$$\frac{p(X^0_{-\infty}, y)}{p(Z^0_{-\infty}, y)} \leq C < \infty.$$

Assumption 2 . *There exists such* $q \in (0, 1/2)$ *that*

$$|\frac{p(x^0_{-\infty}; y)}{p(z^0_{\infty}; y)} - 1| \leq q^{i+1}$$

if

$$x_0 = z_0, \cdots, x_{-i+1} = z_{-i+1} \cdots$$

Theorem 1 . *If assumptions 1 + 2 are satisfied with some* $q \in (0, 1/2)$ *then there exist such* $C, \lambda > 0$ *that*

$$\varphi(s) \leq C \exp(-\lambda s).$$

Let

$$(0 <) 1 - c = inf \int min \left(p\left(x^0_{-\infty}, y\right), p\left(z^0_{-\infty}, y\right)\right) dy$$

Let us state another assumptions for the following theorem.

Assumption 3 . *There exists such* $q < c^{1/2} - c$ *that*

$$|\frac{p(x^0_{-\infty}; y)}{p(z^0_{\infty}; y)} - 1| \leq q^i$$

if

$$x_0 = z_0, \cdots, x_{-i+1} = z_{-i+1} \cdots$$

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Theorem 2 . *If assumptions 1 + 3 holds then there exist such $C, \lambda > 0$ that*

$$\psi(s) \leq C \exp(-\lambda s).$$

Comment. Assumption 3 is natural in the following sense. It means that the dependence of the density from the "far" coordinates is exponentially weak. Assumption 2 with a small q is even stronger: it means, moreover, that dependence from *any* coordinates of the previous path is weak.

3 Proof of theorem 2

I. Let (X_n) and (Y_n) be two independent copies of the same stationary process. Denote by (Ω, F, P) the direct product of correspondent probability spaces. Fix some $x^o_{-\infty} = (x_0, x_{-1}, ...), y^o_{-\infty} = (y_0, y_{-1}, ...)$. To prove the statement of the theorem it is sufficient to show that for any $\Gamma \in \mathcal{B}^1 \times \mathcal{B}^2 \times ...$

$$\left| \frac{P\left(X_n^\infty \in \Gamma \mid X^0_{-\infty} = x^0_{-\infty}\right)}{P\left(Y_n^\infty \in \Gamma \mid Y^0_{-\infty} = y^0_{-\infty}\right)} - 1 \right| \leq C \exp(-\lambda n) \qquad (1)$$

with some $C, \lambda > 0$ which do not depend on $x^o_{-\infty}, y^o_{-\infty}, \Gamma$. Let $\Gamma = \Gamma_n \times \Gamma_{n+1} \times ..., \Gamma_i \in \mathcal{B}^1$. Firstly let us show that for any $\Gamma_1 \in \mathcal{B}^1$

$$\left| P(X_n \in \Gamma \mid X^0_{-\infty} = x^0_{-\infty}) - P(Y_n \in \Gamma \mid Y^0_{-\infty} = y^0_{-\infty}) \right| \leq C \exp(-\lambda n). \quad (2)$$

II. Now assume $\Gamma = \Gamma_n \in \mathcal{B}^1$. We use the coupling method although there is no Markov process here. Nevertheless, we may try to construct such a new process $\left(\tilde{Y}_n\right)$ on the extension of the probability space (Ω, F, P) that $\left(\tilde{Y}_n\right)$ is equal to (Y_n) in distribution and $\left(\tilde{Y}_n\right)$ is equal to Y_n pointwise with the maximal possible probability. (This is actually the main idea of the coupling method). It follows from assumptions that for any $X^0_{-\infty}$ and $Y^0_{-\infty}$

$$\int_0^1 min\left(p\left(X^0_{-\infty}; y\right), p\left(Y^0_{-\infty}; y\right)\right) dy \geq 1 - c > 0.$$

Hence, as in the coupling method we may construct such a new random value $\tilde{Y}_1$ that

$$\mathcal{L}\left(\tilde{Y}_1, Y^0_{-\infty}\right) = \mathcal{L}\left(Y_1, Y^0_{-\infty}\right)$$

(here $\mathcal{L}$ is a law) and

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$$P\left(\tilde{Y}_1 = X_1 | X^0_{-\infty}, Y^0_{-\infty}\right)$$

$$= \int_0^1 \min\left(p\left(X^0_{-\infty}; y\right), p\left(Y^0_{-\infty}; y\right)\right) dy$$

$$\geq 1 - c > 0.$$

(see Nummelin (1984)). Here P is already a new extension of "old" probability measure, but we do not change the notation P. The value $\tilde{Y}_1$ is thus constructed. Now, $\tilde{Y}_2, \ldots, \tilde{Y}_n, \ldots$ are built in the same way by induction with the properties

$$\mathcal{L}\left(\tilde{Y}_{n+1}, Y^n_{-\infty}\right) = \mathcal{L}\left(Y_{n+1}, Y^n_{-\infty}\right)$$

and

$$P\left(\tilde{Y}_{n+1} = X_{n+1} | X^n_{-\infty}, Y^n_{-\infty}\right)$$

$$= \int_0^1 \min\left((X^n_{-\infty}; y), p\left(Y^n_{-\infty}; y\right)\right) dy \geq 1 - c > 0.$$

IV. It follows that $P\left(\tilde{Y}_m = X_m\right) \geq c > 0$. Moreover, Lipshitz condition on p implies that if $\tilde{Y}_{n-1} = X_{n-1}$ then

$$P\left(\tilde{Y}_n = X_n \mid \tilde{Y}_{n-1} = X_{n-1}\right)$$

$$= E\left(\int_0^1 \min\left(p(X^n_{-\infty}; y), p\left(\tilde{Y}^n_{-\infty}; y\right)\right) dy \mid \tilde{Y}_{n-1} = X_{n-1}\right) \geq 1 - q/(1-q).$$

If $\tilde{Y}^{n-1}_{n-k} = X^{n-1}_{n-k}$ then the estimate is better:

$$P\left(\tilde{Y}_n = X_n \mid \tilde{Y}^{n-1}_{n-k} = X^{n-1}_{n-k}\right) \tag{3}$$

$$= E\left(\int_0^1 \min\left(p\left(X^n_{-\infty}; y\right), p\left(\tilde{Y}^n_{-\infty}; y\right)\right) dy \mid \tilde{Y}^{n-1}_{n-k} = X^{n-1}_{n-k}\right)$$

$$\geq 1 - q^k/(1-q).$$

V. Consider the sequence $(\alpha_1, ..., \alpha_n)$: $\alpha_k = I\left(X_k = \tilde{Y}_k\right), 1 \leq k \leq n$. By virtue of assumptions for any sequence $(a_1, ..., a_{n-1})$ the probability

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$$P(\alpha_1 = a_1, ..., \alpha_{n-1} = a_{n-1})P(\alpha_n = 0 \mid \alpha_1 = a_1, ..., \alpha_{n-1} = a_{n-1}) =$$

$$= P(\alpha_1 = a_1, ..., \alpha_{n-1} = a_{n-1}, \alpha_n = 0)$$

may be estimated as follows. Let the sequence $(a_1, ..., a_{n-1})$ has the form

$$a_1 = ... = a_{k_1} = 1, a_{k_1+1} = ... = a_{k_1+m_1} = 0,$$

$$a_{k_1+m_1+1} = ... = a_{k_1+m_1+k_2} = 1,$$

$$a_{k_1+m_1+k_2+1} = ... = a_{k_1+m_1+k_2+m_2+1} = 0, \text{ etc.}$$

Let r denote the number of series of zeros between 1 and n. Then by virtue of assumptions we have

$$P(\alpha_{k_1+1} = 0 \mid \alpha_1 = ... = \alpha_{k_1} = 1) \le q^{k_1},$$

$$P(\alpha_{k_1+m_1+k_2+1} = 0 \mid \alpha_{k_1+m_1+1} = \cdots = \alpha_{k_1+m_1+k_2} = 1) \le q^{k_2}, \cdots,$$

and, moreover,

$$P(\alpha_{k_1+2} = ... = \alpha_{k_1+m_1} = 0) \le c^{m_1-1},$$

$$P(\alpha_{k_1+m_1+k_2+2} = \cdots = \alpha_{k_1+m_1+k_2+m_2} = 0) \le c^{m_2-1}, \cdots$$

Denote $k = \sum_{i=1}^{n-1} I(\alpha_i = 1)$. Then

$$P(\alpha_1 = a_1, ..., \alpha_{n-1} = a_{n-1}, \alpha_n = 0) \le q^k c^{(n-k-r)}.$$

So

$$P(\alpha_n = 0) \le \sum_{k=0}^{n-1} C_{n-1}^k q^k c^{(n-k-r)} =$$

$$= c^{-r}(q+c)^{n-1}.$$

Since $r \le n/2$ then

$$P(\alpha_n = 0) \le (q+c)^{-1}(c^{-1/2}(q+c))^n$$

and assumption $q + c < c^{1/2}$ implies the estimate

$$P(\alpha_n = 0) \le C \exp(-\lambda n) \qquad (C, \lambda > 0). \qquad (4)$$

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Now, it follows from (4) that

$$P(\prod_{i=[n/2]}^{n-1} \alpha_i = 0) \leq C \exp(-\lambda n) \tag{5}$$

with some new $C, \lambda > 0$. Denote $\prod_{i=[n/2]}^{n-1} \alpha_i = \alpha(n-1)$.

VI. We have

$$P(X_n \in \Gamma_n | X_{-\infty}^0 = x) - P(Y_n \in \Gamma_n | Y_{-\infty}^0 = y)$$

$$= P(X_n \in \Gamma_n | X_{-\infty}^0 = x) - P(\tilde{Y}_n \in \Gamma_n | Y_{-\infty}^0 = y)$$

$$= E[I(X_n \in \Gamma_n) - I(\tilde{Y}_n \in \Gamma_n) | X_{-\infty}^0 = x, Y_{-\infty}^0 = y]$$

$$= E[I(X_n \in \Gamma_n) - I(\tilde{Y}_n \in \Gamma_n) I(\alpha(n-1) = 0 | X_{-\infty}^0 = x, Y_{-\infty}^0 = y]$$

$$= E[I(X_n \in \Gamma_n) - I(\tilde{Y}_n \in \Gamma_n) I(\alpha(n-1) = 1 | X_{-\infty}^0 = x, Y_{-\infty}^0 = y]$$

$$= A_1 + A_2.$$

VII. (5) and assumption 1 imply the bound

$$A_1 \leq C \exp(-\lambda n) E[I(\tilde{Y}_n \in \Gamma) | Y_{-\infty}^0 = y; F_{[n/2]}] \tag{6}$$

$$\leq C \exp(-\lambda n) E[I(Y_n \in \Gamma) | Y_{-\infty}^0 = y]$$

(with new $C > 0$!).

VIII.

$$A_2 = E[I(\alpha(n-1) = 1) E[I(X_n \in \Gamma)$$

$$-I(\tilde{Y}_n \in \Gamma)) | Y_{-\infty}^0 = y; X_{-\infty}^0 = x; F_{n-1}]]$$

$$= E[I(\alpha(n-1) = 1) E[I(X_n \in \Gamma)$$

$$-I(\tilde{Y}_n \in \Gamma)) | \alpha(n-1) = 1; Y_{-\infty}^0 = y; X_{-\infty}^0 = x; F_{n-1}]].$$

It follows from assumptions that

$$E[I(X_n \in \Gamma) - I(\tilde{Y}_n \in \Gamma)) | \alpha(n-1) = 1; Y_{-\infty}^0 = y; X_{-\infty}^0 = x; F_{n-1}]$$

$$\leq (1 + C \exp(-\lambda n/2)) E[I(\tilde{Y}_n \in \Gamma)) | \alpha(n-1) = 1; Y_{-\infty}^0 = y; X_{-\infty}^0 = x; F_{n-1}].$$

Thus,

$$\frac{P(X_n \in \Gamma_n | X_{-\infty}^0 = x)}{P(Y_n \in \Gamma_n | Y_{-\infty}^0 = y)} \leq 1 + C \exp(-\lambda n). \tag{7}$$

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IX. We have for $\Gamma = \Gamma_n \times \Gamma_{n+1} \times \cdots$

$$\frac{P(X_n^\infty \in \Gamma | X_{-\infty}^0 = x)}{P(Y_n^\infty \in \Gamma | Y_{-\infty}^0 = y)} \tag{8}$$

$$= \frac{P(X_n \in \Gamma_n | X_{-\infty}^0 = x)}{P(Y_n \in \Gamma_n | Y_{-\infty}^0 = y)} \times$$

$$\frac{P(X_{n+1} \in \Gamma_{n+1} | X_{-\infty}^0 = x; X_n \in \Gamma_n)}{P(Y_{n+1} \in \Gamma_{n+1} | Y_{-\infty}^0 = y; Y_n \in \Gamma - n)} \cdots$$

Similarly to (7) we obtain estimates

$$\frac{P(X_{n+1} \in \Gamma_{n+1} | X_{-\infty}^0 = x; X_n \in \Gamma_n)}{P(Y_{n+1} \in \Gamma_{n+1} | Y_{-\infty}^0 = y; Y_n \in \Gamma - n)} \leq 1 + C \exp(-\lambda(n+1)),$$

and so on. So it follows from (8) that

$$\frac{P(X_n^\infty \in \Gamma | X_{-\infty}^0 = x)}{P(Y_n^\infty \in \Gamma | Y_{-\infty}^0 = y)} \leq \prod_{k=n}^\infty (1 + C \exp(-\lambda k))$$

$$\leq 1 + C \exp(-\lambda n)$$

with some new $C > 0$. Theorem 2 is proved.

The proof of theorem 1 is similar.

REFERENCES

Berbee, H., Bradley, R.C. (1984) A limitation of Markov representation for stationary processes. Stoch. Proc. Appl. Vol. 18. 33-45.

Bradley, R.C. (1980) On the ϕ-mixing condition for stationary random sequences. Duke Math. J. Vol. 47. 421-433.

Davydov, Yu.A. (1973) Mixing conditions for Markov chains. Teoria Veroyatnostey i Primen. Vol. 18 (2). P. 321-338. (in Russian)

Doob, J.L. (1953) Stochastic processes. - N.Y.: John Wiley & Sons, London: Chapman & Hall.

Gulinsky, O.V., Veretennikov, A.Yu. (1992) Large deviations for discrete-time stochastic processes with averaging. (to appear)

Ibragimov, I.A. (1961) Spectral functions of certain classes of Gaussian stationary processes. DAN SSSR. Vol. 137 (5). P. 1046-1048. (in Russian)

Ibragimov, I.A. (1962) Stationary Gaussian sequences that satisfy strong mixing condition. DAN SSSR. Vol. 147 (6). P. 1282-1284. (in Russian)

Kolmogorov, A.N., Rozanov, Yu.A. (1960) On strong mixing conditions for Gaussian stationary process. Teoria Veroyatnostey i Primen. Vol. 5. No. 2. P. 222-227 (in Russian)

Nummelin, E. (1984) General irreducible Markov chains and non-negative operators. Cambridge et al.: Cambridge Univ. Press.

Veretennikov, A.Yu. (1988). Bounds for the mixing rate in the theory of stochastic equations. Teor. Probab. Rel. Fields. Vol. 32(2). P. 273-281.

Veretennikov, A.Yu. (1991) On the estimates of the mixing rate for Markov processes. Lietuvos Matem. Rink. Vol. 31(1). P. 40-49 (in Russian)

Veretennikov, A.Yu. (1992) On asymptotic properties of stationary processes: large deviations and exponential mixing. Proc. VI Soviet–Japan Symp. on Probab. Theory and Mathem. Stat. (Kiev, 1991) (to appear)