Interner Bericht

Free Form Volumes

Definitions, Applications, Visualization Techniques

238/94

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Abstract. Free form volumes in rational Bézier representation are defined via homogeneous coordinates and two applications are pointed out: generation of solid primitives and curve and surface modelling by the way of volume deformation. Visualization techniques are reviewed, too.

Keywords. Bézier volumes, solids, free form deformation, visualization.

I. Introduction

While in the past, CAGD has been mostly concerned with curves and surfaces, more recently, there has been an increasing interest in higher dimensional, trivariate objects such as free form volumes which are suitable to describe inhomogeneous solids. The two most widely used methodes of representing solids are the Constructive Solid Geometry Representation (CSG-Rep.) - a solid based method - and the Boundary Representation (B-Rep.) - a surface based method (see e.g. [Cas 85]). However, free form character of both methods is not very substantial, and they also assume internal homogeneity. On the other hand, free form volumes, of which this paper is dealing with, possess per definition a very high free form characteristic and describe every interior point as well as every point on the boundary surface of the volume uniquely. No assumption on internal homogeneity or structure is done. Beside solid modelling, there are some more applications of free form volumes, for example, the description of spatial movement or deformation of a surface, the description of physical fields, such as temperature or pressure, etc. as functions of several variables, e.g. the positional coordinates, the modification of curves and surfaces through volume deformation, etc.

Free form volumes can be defined by the *tensor* (Cartesian) *product definition*,

$$\mathbf{V}(u,v,w) = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \mathbf{V}_{i,j,k}\, u^i v^j w^k, \quad u,v,w \in [0,1]. \tag{1}$$

The above given definition is based on monomials, in CAGD (*Computer Aided Geometric Design*) *Bernstein polynomials*, see [Far 92], [Hos 92],

$$B_k^n(w) = \binom{n}{k} w^k (1-w)^{n-k}, \quad w \in [0,1], \tag{2}$$

of degree n in w, and analogously for v and u, are very popular. This is, because the expansion in terms of Bernstein polynomials yields, firstly, a numerically very stable behavior of all algorithms. Secondly, a geometric relationship between subject and coefficients of its defining equation. Note, that these properties are not available in case of monomials, Lagrange or Hermite polynomials. Probably this is one of the main reasons why Bézier representations, which are based on the use of Bernstein polynomials, became the de facto industry standard in CAGD during the past years. Thus, a *tensor product Bézier volume* - briefly *TPB volume* - of degree (l,m,n) in (u,v,w) is defined by

$$\mathbf{V}(u,v,w) = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \mathbf{V}_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w), \quad u,v,w \in [0,1]. \tag{3}$$

The coefficients $\mathbf{V}_{i,j,k} \in \mathbb{R}^3$ are called *Bézier points.* They form, connected in the ordering given by their subscripts, a spatial net which is called *Bézier grid*, [Hos 92].

Because of the polynomial character of (3) only polynomials can be represented exactly, sphere (segments) for example, cannot be constructed by (3). Therefore many (primary) elements of CAD systems can not be converted exactly into this kind of free form representation. Rational Bézier representations overcome this disadvantage for the most part: primary elements like conic sections and quadrics, as well as tori and cyclids, for example, can be constructed, thus allowing exact conversion from CSG-Rep. and B-Rep. based solid primitives into rational free form representation.

This paper is concerned with free form volumes in rational Bézier representation which are defined in Section 2. Section 3 discusses two applications of free form volumes: generation of solid primitives defined by rational TPB volumes and curve and surface modelling by free form deformation. Visualization techniques are revied in Section 4.

II. Bézier Volumes

Using homogeneous coordinates r^i, points $\mathbf{R} = (x,y,z)^\top$ of $\mathbb{R}^3$ can be represented by points $\mathsf{R} = (\mathsf{r}^1,\mathsf{r}^2,\mathsf{r}^3,\mathsf{r}^4)^\top$ of $\mathbb{R}^4$ via the projection of $\mathbb{R}^4$ into the hyperplane $\mathsf{r}^4 = 1$ according to (cf. Figure 1)

$$\mathcal{H}(\mathsf{R}) = \begin{cases} \frac{1}{\mathsf{r}^4}(\mathsf{r}^1,\mathsf{r}^2,\mathsf{r}^3)^\top & \text{for} \quad \mathsf{r}^4 \neq 0 \\ (\mathsf{r}^1,\mathsf{r}^2,\mathsf{r}^3)^\top & \text{for} \quad \mathsf{r}^4 = 0\,. \end{cases}$$

The center of projection is the origin of the 4D cartesian coordinate system. A point $\mathbf{R} = (x,y,z)^\top$ is the projection of $\omega(x,y,z,1)^\top$, where $\omega \neq 0$, $\omega \in \mathbb{R}$. The real number ω is called *weight* of the corresponding point. Note, $\mathsf{R} \in \mathbb{R}^4$ and $\varrho\mathsf{R} \in \mathbb{R}^4$, $\varrho \neq 0$, $\varrho \in \mathbb{R}$, describe the same point of $\mathbb{R}^3$. $\mathcal{H}(\mathsf{R})$ is unlimited for $\mathsf{r}^4 \to 0$, $\mathsf{r}^4 \neq 0$. In this way infinite points of $\mathbb{R}^3$ can be described by finite points of $\mathbb{R}^4$ with $\mathsf{r}^4 = 0$. In $\mathbb{R}^3$ these points will be represented by direction vectors, [Pie 87].

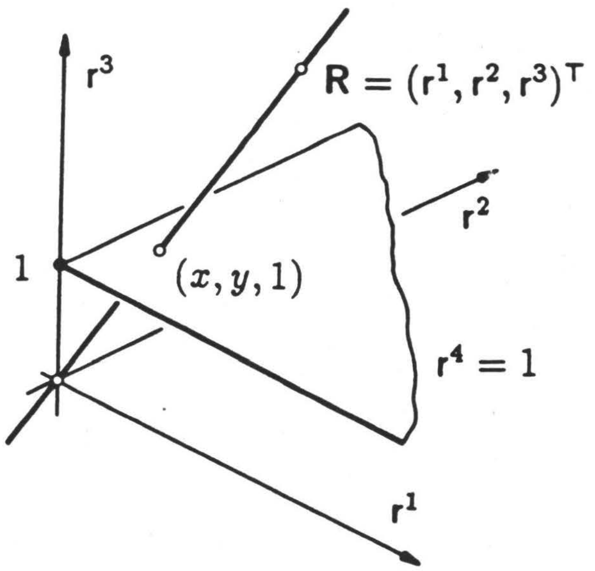

Figure 1. Introducing homogeneous coordinates via projection $\mathcal{H}(\cdot)$ of points of $\mathbb{R}^3$ into the plane $\mathsf{r}^3 = 1$.

Now, a *rational tensor product Bézier volume* - briefly *rational TPB volume* - of degree (l, m, n) in (u, v, w) is defined by, [Kir 89], [Las 90],

$$\mathbf{V}(\mathbf{u}) = \mathcal{H}(\mathsf{V}(\mathbf{u}))\,, \tag{4}$$

where

$$\mathsf{V}(\mathbf{u}) = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \mathsf{V}_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w)\,, \quad u, v, w \in [0,1]\,,$$

and $\mathbf{u} = (u, v, w)$. $\mathsf{V}(\mathbf{u})$ is called homogeneous form of $\mathbf{V}(\mathbf{u})$. Thus, a rational TPB volume of degree (l, m, n) in $\mathbb{R}^3$, $\mathbf{V}(\mathbf{u})$, is defined by a non-rational, polynomial TPB volume of degree (l, m, n) in $\mathbb{R}^4$, $\mathsf{V}(\mathbf{u})$, i.e. is the projection of a polynomial TPB volume into the hyperplane $\mathsf{r}^4 = 1$.

If we write $\mathsf{V}_{i,j,k} = (\mathsf{V}^1_{i,j,k}, \mathsf{V}^2_{i,j,k}, \mathsf{V}^3_{i,j,k}, \mathsf{V}^4_{i,j,k})^\mathsf{T} = (\omega_{i,j,k}\mathbf{V}^\mathsf{T}_{i,j,k}, \omega_{i,j,k})^\mathsf{T} \in \mathbb{R}^4$, where $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^\mathsf{T} \in \mathbb{R}^3$, $\omega_{i,j,k} \in \mathbb{R}$, and assume

$$\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \omega_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w) \neq 0\,, \quad u, v, w \in [0,1]\,,$$

then,

$$\mathbf{V}(\mathbf{u}) = \frac{\displaystyle\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \omega_{i,j,k}\mathbf{V}_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w)}{\displaystyle\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \omega_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w)}\,, \tag{5}$$

and *3D Bézier points* $\mathbf{V}_{i,j,k}$ are projections, $\mathbf{V}_{i,j,k} = \mathcal{H}(\mathsf{V}_{i,j,k})$, of *4D Bézier points* $\mathsf{V}_{i,j,k}$. Bézier points form in their natural ordering, given by their subscripts, the vertices of the *Bézier grid*. Scalars $\omega_{i,j,k} \in \mathbb{R}$ are called *weights*: if we increase one $\omega_{i,j,k}$ the volume will be pulled towards the corresponding $\mathbf{V}_{i,j,k}$.

The assumption of (5) will be fulfilled if weights $\omega_{i,j,k}$ are non-negative and all 8 corner Bézier points of the grid are no infinite control points. If a control point $\mathbf{V}_{i,j,k}$ is a *point at infinity*, i.e. $\omega_{i,j,k} = 0$, we replace, according to definition of $\mathcal{H}(\cdot)$, $\omega_{i,j,k}\mathbf{V}_{i,j,k}$ by $\mathbf{V}_{i,j,k}$ in the numerator of (5). Note, infinite control points, also called *control vectors*, can be eliminated by degree elevation (cf. [Far 92] p. 260).

One of the weights (e.g. $\omega_{0,0,0}$) can be normalized to be of value one and on boundary curves $u = 0$, $v = 0$ and $w = 0$, the parametrization can be chosen so that, for instance $\omega_{l,0,0,} = \omega_{0,m,0} = \omega_{0,0,n} = 1$. All other weights directly influence the shape of the volume. If all weights are equal, (5) yields the non-rational polynomial TPB volume because Bernstein polynomials sum to one (see e.g. [Far 92] p. 42, [Hos 92] p. 116).

Positive weights result in volumes which have all the properties and algorithms we do know from polynomial representations (see e.g. [Hos 92]). Because of definition (4) via the projection $\mathcal{H}(\cdot)$, properties of rational volumes can be deduced from properties of the non-rational volume scheme. This means that relations between Bézier grid and rational volume can be deduced from those of the underlying Bézier curve scheme, and that many constructions in different parameters commute, such as degree raising, de Casteljau construction, and segmentation, [Kir 89].

III. Applications

As it has been pointed out in the introduction, there are various examples of application of free form volumes. Two of which will be outlined in more detail in this section, first, definition of solid primitives, and second, curve and surface modelling by the way of volume distortion.

III.1 Solid Primitives

Compared with solid definitions such as CSG or B-Rep, TPB volumes possess a very high free form characteristic, describe every interior point as well as every point on the boundary surface of the volume uniquely and no assumptions are done on internal homogeneity or structure, on the other side, they also allow the exact representation of solid primitives. Bézier volumes can be constructed in various ways: by specification of control points (and weights), by interpolation or approximation of digitized points, or by performing sweep, spin, loft, etc. transformations on *profile surfaces*, [Cas 85], [Sai 87]. Rational volumes are in particular very useful for giving exact descriptions of solid primitives such as sphere, cylinder, torus. Following the idea of [Faro 85] Bézier points of volumes $\mathbf{V}(\mathbf{u})$ can be calculated by performing a continuum of transformations $\mathbf{M}(w)$ on a profile surface $\mathbf{F}(u, v)$. Considerations should be based on rationals and homogeneous coordinates. Therefore, $\mathbf{M}(w)$ might be given by the following 4×4 matrix

$$\mathbf{M}(w) = \begin{pmatrix} m^{1,1} & m^{1,2} & m^{1,3} & t^1 \\ m^{2,1} & m^{2,2} & m^{2,3} & t^2 \\ m^{3,1} & m^{3,2} & m^{3,3} & t^3 \\ p^1 & p^2 & p^3 & s \end{pmatrix},$$

where $m^{\alpha,\beta}$ indicates rotation, reflection, scaling and shear, t^{β} indicates translation, p^{α} perspective projection, and s an overall scaling. We start by looking at *Sweep volumes* which result by moving a given surface $\mathbf{F}(u,v)$ along a prescribed curve $\mathbf{K}(w)$ which is sometimes referred to as *directrix*. Translation volumes are special sweep volumes and we are going to discuss them first. *Translation volumes* are specified by

Theorem 1. Let $\mathsf{K}(w)$ be homogeneous form of a rational Bézier curve $\mathbf{K}(w)$ of degree n with 4D Bézier points $\mathsf{K}_k = (\mathsf{K}_k^1, \mathsf{K}_k^2, \mathsf{K}_k^3, K_k^4)^{\mathsf{T}} = (\beta_k \mathbf{K}_k^{\mathsf{T}}, \beta_k)^{\mathsf{T}}$, where $\mathbf{K}_k = (x_k, y_k, z_k)^{\mathsf{T}}$ and weights β_k (w.l.o.g. $\beta_0 = \beta_n = 1$).
Let $\mathsf{F}(u,v)$ be homogeneous form of a rational TPB surface $\mathbf{F}(u,v)$ of degree (l,m) with 4D Bézier points $\mathsf{F}_{i,j} = (\mathsf{F}_{i,j}^1, \mathsf{F}_{i,j}^2, \mathsf{F}_{i,j}^3, \mathsf{F}_{i,j}^4)^{\mathsf{T}} = (\beta_{i,j} \mathbf{F}_{i,j}^{\mathsf{T}}, \beta_{i,j})^{\mathsf{T}}$, where $\mathbf{F}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})^{\mathsf{T}}$ and weights $\beta_{i,j}$ (w.l.o.g. $\beta_{0,0} = \beta_{l,0} = \beta_{0,m} = 1$).
4D Bézier points $\mathsf{V}_{i,j,k} = (\mathsf{V}_{i,j,k}^1, \mathsf{V}_{i,j,k}^2, \mathsf{V}_{i,j,k}^3, \mathsf{V}_{i,j,k}^4)^{\mathsf{T}} = (\omega_{i,j,k} \mathbf{V}_{i,j,k}^{\mathsf{T}}, \omega_{i,j,k})^{\mathsf{T}}$, with weights $\omega_{i,j,k}$ and $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^{\mathsf{T}}$ of a rational Bézier solid, $\mathbf{V}(\mathbf{u})$, represented in homogeneous form $\mathsf{V}(\mathbf{u})$, and generated by moving of $\mathbf{F}(u,v)$ along $\mathbf{K}(w)$, such that $\mathbf{V}(u,v,0) = \mathbf{F}(u,v)$, $\mathbf{V}(0,0,w) = \mathbf{K}(w)$, are given by

$$\mathsf{V}_{i,j,k} = \mathsf{M}_k \mathsf{F}_{i,j}\,, \tag{6}$$

where $\mathsf{M}_0 = \mathsf{I}_4$ and

$$\mathsf{M}_k = \frac{1}{(\mathsf{K}_0^4)^2} \begin{pmatrix} \mathsf{K}_0^4\mathsf{K}_k^4 & 0 & 0 & \mathsf{K}_0^4\mathsf{K}_k^1 - \mathsf{K}_0^1\mathsf{K}_k^4 \\ 0 & \mathsf{K}_0^4\mathsf{K}_k^4 & 0 & \mathsf{K}_0^4\mathsf{K}_k^2 - \mathsf{K}_0^2\mathsf{K}_k^4 \\ 0 & 0 & \mathsf{K}_0^4\mathsf{K}_k^4 & \mathsf{K}_0^4\mathsf{K}_k^3 - \mathsf{K}_0^3\mathsf{K}_k^4 \\ 0 & 0 & 0 & \mathsf{K}_0^4\mathsf{K}_k^4 \end{pmatrix}.$$

Remark. Although $\mathsf{V}(\mathbf{u})$ was generated by moving a surface along a w line, parametric surfaces of constant u are congruent as well and the same holds for isoparametric surfaces of constant v.

For $n = 1$ the trivariate analogy of a cylindrical surface is created for which Bézier points in w direction define parallel lines,

$$\mathbf{V}_{i,j,0} = \mathbf{F}_{i,j}\,, \qquad \mathbf{V}_{i,j,1} = \mathbf{F}_{i,j} + \mathbf{T}\,,$$

where $\mathbf{T} = \mathbf{K}_1 - \mathbf{K}_0$ is translation vector and $\omega_{i,j,k} = \beta_{i,j}$, for all i,j and $k = 0,1$.

A generalization of the translation volume is the *blending volume* (*loft volume*, [Sai 87]): here two non-congruent surfaces $\mathsf{F}^0(u,v)$ and $\mathsf{F}^n(u,v)$ defined by Bézier points $\mathsf{F}_{i,j}^0$ and $\mathsf{F}_{i,j}^n$ are connected appropriately. For polynomial blending, n has to be specified and, in case of $n > 1$, Bézier points of *intermediate nets* $\mathsf{F}_{i,j}^k$, $k = 1\,(1)\,n-1$. Then,

$$\mathsf{V}_{i,j,k} = \mathsf{F}_{i,j}^k\,. \tag{7}$$

Bézier points $\mathsf{F}_{i,j}^k$ of intermediate nets can be provided in different ways. For example, via interpolation of $n-1$ intermediate surfaces (*skinning*, *lofting*), via continuity constructions, if $\mathsf{F}_{i,j}^0$ and $\mathsf{F}_{i,j}^n$ are supposed to be boundary surfaces of connecting volume segments or, via interactive input procedures.

$n = 1$ gives a very special blending volume, the *linear blending volume.* It is the trivariate generalization of a ruled surface and thus is sometimes called *ruled volume* (see e.g. [Cas 85]). Defined by taking a convex combination,

$$\mathbf{V}(\mathbf{u}) = (1-w)\,\mathbf{F}^0(u,v) + w\,\mathbf{F}^n(u,v)\,, \quad w \in [0,1]\,,$$

of surfaces $\mathbf{F}^0(u,v)$ and $\mathbf{F}^n(u,v)$, $n = 1$, Bézier points are given by (7).

Volumes of revolution also called *spin volumes* are sweep volumes as well and can be created easily as stated by, [Kir 89], [Las 90],

Theorem 2. Let $\mathsf{F}(u,v)$ be the homogeneous form of a rational TPB surface $\mathbf{F}(u,v)$ of degree (l,m) with 4D Bézier points $\mathsf{F}_{i,j} = (\mathsf{F}^1_{i,j}, \mathsf{F}^2_{i,j}, \mathsf{F}^3_{i,j}, \mathsf{F}^4_{i,j})^\mathsf{T} = (\beta_{i,j}\mathbf{F}^\mathsf{T}_{i,j}, \beta_{i,j})^\mathsf{T}$, where $\mathbf{F}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})^\mathsf{T}$ and weights $\beta_{i,j}$.
The rational TPB volume $\mathbf{V}(\mathbf{u})$ of degree (l,m,n), $n = 2$, with 4D Bézier points

$$\mathsf{V}_{i,j,k} = \mathsf{M}_k\,\mathsf{F}_{i,j}\,, \tag{8}$$

where $\mathsf{M}_0 = \mathsf{I}_4$,

$$\mathsf{M}_1 = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} & 0 & 0 \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\alpha}{2} & 0 \\ 0 & 0 & 0 & \cos\frac{\alpha}{2} \end{pmatrix}, \qquad \mathsf{M}_2 = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

represents a rotation of $\mathbf{F}(u,v)$ through the arc α, with $|\alpha| \le 180^0$, round the z-axis such that $\mathbf{V}(u,v,0) = \mathbf{F}(u,v)$.

Note, Bézier points $\mathsf{V}_{i,j,1} = \mathsf{M}_1\mathsf{F}_{i,j}$ become infinite control points for $|\alpha| = 180^0$.

Rotations about the x-axis and the y-axis can be described analogously.

Figures 2 and 3 show two examples of volumes of revolution, in both cases solid definition involves several infinite control points.

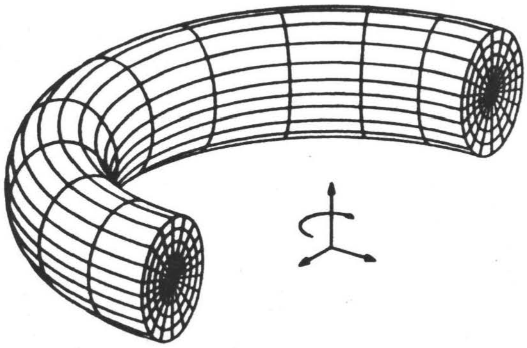

Figure 2. Solid half-torus described by a rational TPB volume of degree $(1,2,2)$.

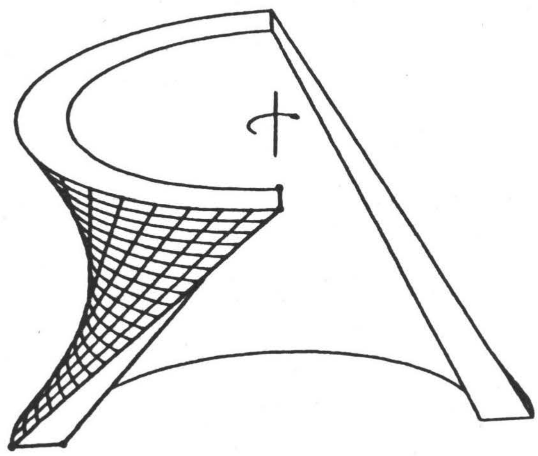

Figure 3. Part of a solid hyperboloid of revolution of one sheet described by a rational TPB volume of degree $(1,1,2)$.

Since matrices $\mathbf{M}_k$ (and *start point* $\mathbf{K}_0$) define directrix $\mathbf{K}(w)$, different curves can be described by changing these matrices and all w parameter lines of $\mathbf{V}(\mathbf{u})$ will adapt the geometry introduced by the $\mathbf{M}_k$. For example, if we choose parameter s_2 in

$$\mathbf{K}(w) = \sum_{k=0}^{2} \hat{\mathbf{M}}_k \mathbf{K}_0 \, B_k^2(w) \tag{9}$$

with $\hat{\mathbf{M}}_k = s_k \mathbf{M}_k$, $\mathbf{M}_k$ according to (8), $s_0 = s_2 = 1$, appropriately, we obtain an arc of an ellipse, parabola or hyperbola, [Kir 89], [Las 90]: a *parabola* results for $s_2 = s^* \equiv 1/\cos\frac{\alpha}{2}$, because in that case $\mathbf{K}(w)$ is a non-rational, polynomial quadratic Bézier curve (all weights are equal!), for $s_2 > s^*$ a *hyperbola* results, for $s_2 < s^*$ an *ellipse* and, for $s_2 = 1$ we have Theorem 2, i.e. $\mathbf{K}(w)$ defines a circle, see Figure 4.

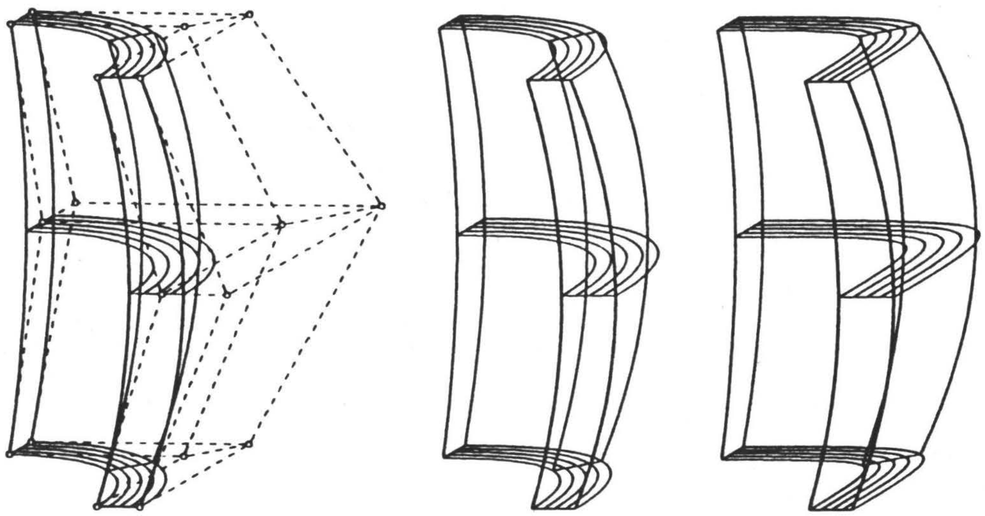

Figure 4. Three rational TPB volumes with arcs of circles ($s_2 = 1$, left), parabolas ($s_2 = s^*$, middle) and hyperbolas ($s_2 = 3s^*$, right) as w parameter lines.

III.2 Curve and Surface Modelling

Free form volumes can be interpreted as deformation of the associated parameter domain, i.e. the unit cube, under the mapping induced by their defining equation. In general, the position of every point in the interior as well as those on the boundary surfaces is altered. It is of particular interest for designers to be able to embed curves, surfaces, and volumes in the parameter domain of a free form volume, since this allows concentrating on well-defined subsets of the parameter space. In simple cases, these subsets are in fact isoparametric lines and planes, which under the deformation are mapped to isoparametric curves and surfaces of the free form volume. The more interesting subsets, however, are non-isoparametric subsets, which might be given by any kind of representation. In context of these more general non-isoparametric subsets, the term FFD (*Free Form Deformation*) is commonly used, [Sed 86], [Cha 89], [Coq 90].

In order to carry out an FFD modelling process, for every point of the object we must first find the associated parameter value in the parameter space of the volume. For this, we define a box-shaped local coordinate system, the deformation domain, which includes all parts of the object to be deformed. It might be subdivided into sub-boxes corresponding to a segmented TPB volume. By comparing coordinates, we find the sub-box $\mathbf{Q}_0$, defined by

$\mathbf{P}_0$, $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ as in Figure 5 which contains the given point $\mathbf{P}$ of the object. We have

$$\mathbf{P} = \mathbf{P}_0 + u\,\mathbf{U} + v\,\mathbf{V} + w\,\mathbf{W}\,, \tag{10}$$

where local coordinates u, v, w are calculated by

$$u = \frac{\mathbf{V}\times\mathbf{W}\cdot(\mathbf{P}-\mathbf{P}_0)}{\mathbf{V}\times\mathbf{W}\cdot\mathbf{U}}\,, \qquad v = \frac{\mathbf{U}\times\mathbf{W}\cdot(\mathbf{P}-\mathbf{P}_0)}{\mathbf{U}\times\mathbf{W}\cdot\mathbf{V}}\,, \qquad w = \frac{\mathbf{U}\times\mathbf{V}\cdot(\mathbf{P}-\mathbf{P}_0)}{\mathbf{U}\times\mathbf{V}\cdot\mathbf{W}}\,.$$

Now we have to prescribe the polynomial degree, corresponding weights and a control point grid which covers the deformation domain. Figure 6 illustrates this process in case of a volume segment of degree $(3,2,2)$. The control points of the grid are given by

$$\mathbf{V}_{i,j,k} = \mathbf{P}_0 + \frac{i}{l}\mathbf{U} + \frac{j}{m}\mathbf{V} + \frac{k}{n}\mathbf{W}\,, \tag{11}$$

and initialy weights $\omega_{i,j,k}$ are chosen to be equal to one.

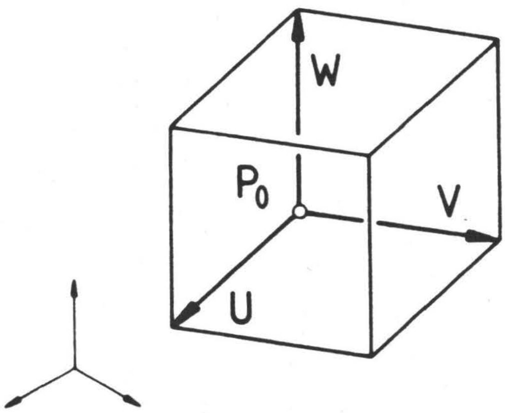

Figure 5. Local coordinate system.

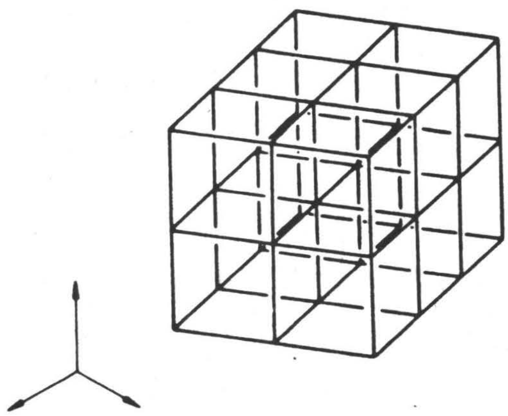

Figure 6. Control point grid.

The actual deformation of the object involves translation of points, $\mathbf{V}_{i,j,k} \mapsto \overline{\mathbf{V}}_{i,j,k}$, changing of weights, $\omega_{i,j,k} \mapsto \overline{\omega}_{i,j,k}$, and evaluation of the FFD defining equation with coefficients $\overline{\mathbf{V}}_{i,j,k}$, weights $\overline{\omega}_{i,j,k}$, and parameter values u, v, w of the object point $\mathbf{P}$ calculated above.

From a mathematical point of view, the FFD construction involves composing two mappings, the mappings of object and of volume definition, what has been investigated in [Las 91], [DeR 93]. It should be remarked, however, that the polynomial degree of FFDs can grow very quickly. Thus, the exact description of FFDs via functional composition might not always be the best approach.

IV. Visualization Techniques

For simple patch configurations and/or small polynomial degrees, we can visualize free form volumes by displaying *a set of isoparametric surfaces.* Figure 8 shows several parametric surfaces corresponding to constant w values for the TPB volume of Figure 7.

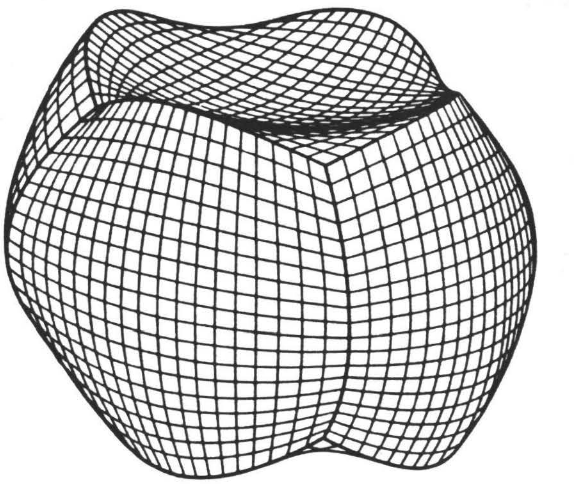

Figure 7. Triquadratic free form volume.

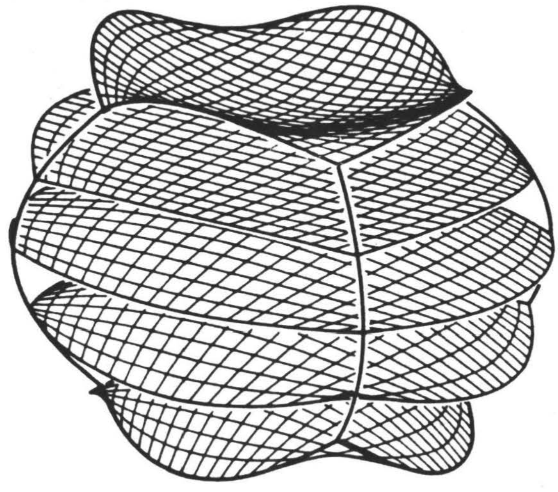

Figure 8. Representation of a family of isoparametric surfaces.

Another way to get a good idea of the interior of a volume is to display several different images. For example, we can look at three images displaying *sets of isoparametric surfaces* corresponding to constant u, v, and w, respectively (see Figure 9).

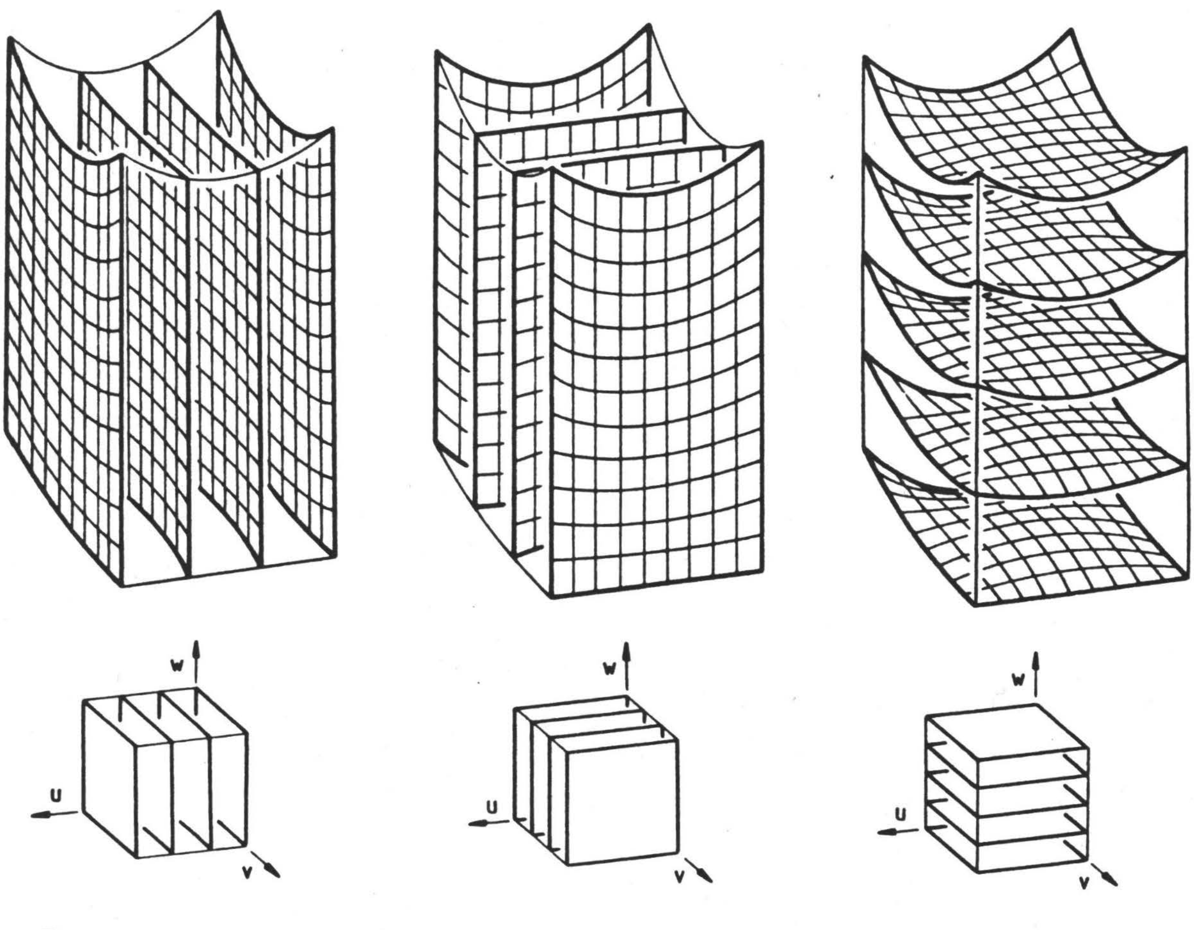

Figure 9. Visualization of all three families of isoparametric surfaces of a free form volume. The parameter space domains are shown below.

Such images can become complicated and confusing because of the limitations of two dimensional graphics, especially for objects consisting of many pieces or of pieces of high degree. In these cases we could proceed as follows: we display only the *three parametric surfaces* $\mathbf{V}(u_i, v, w)$, $\mathbf{V}(u, v_j, w)$, and $\mathbf{V}(u, v, w_k)$ which pass through the point $\mathbf{V}(u_i, v_j, w_k)$ in order to get a better impression of the interior structure of the volume. This can be particular useful if we let the point (and thus the corresponding three associated parametric surfaces) move around in the object (see Figure 10).

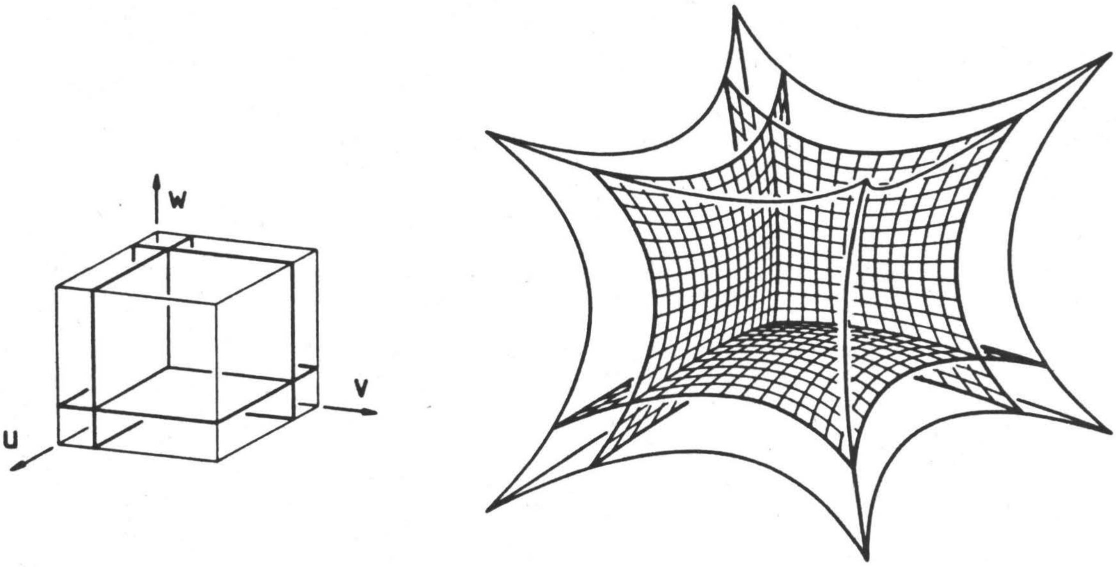

Figure 10. Displaying three parametric surfaces of a free form volume passing through the point $\mathbf{V}(u_i, v_j, w_k)$. Parameter space domain and corresponding isoparametric planes are shown on the left.

Another way to render solids is to display *slices* (e.g. *cross* or *longitudinal sections*) of the object (see Figures 11 and 12). One way to do this is to choose a *series of parallel slices*, but as was done above with parametric surfaces, sometimes it can be useful to instead display three mutually *orthogonal slices* through a given point $\mathbf{V}(u_i, v_j, w_k)$, therefore creating a *core section*, Figure 13. More general slices (*paddlewheel probes*) can also be useful. Some alternative visualization techniques of volumetric data are described in [Nie 90].

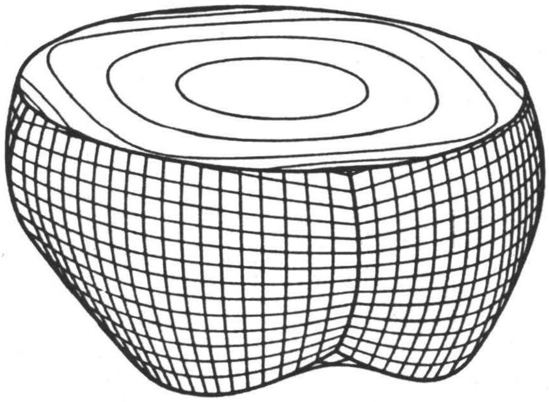

Figure 11. Cross section.

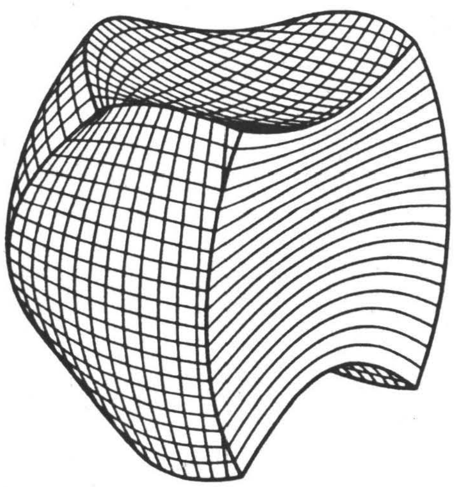

Figure 12. Transversal slice.

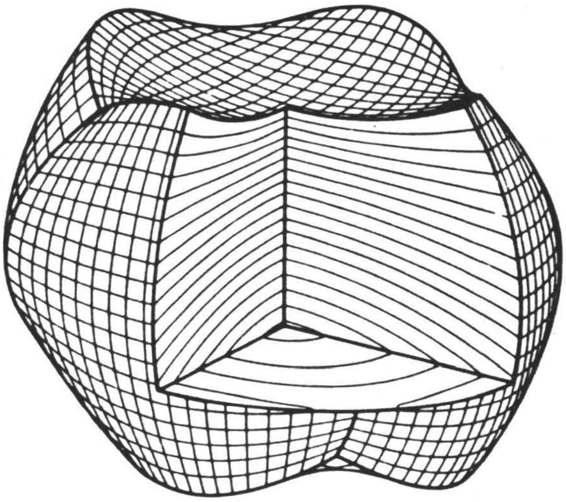

Figure 13. Core section.

References

[Cas 85] M.S. Casale, E.L. Stanton : An overview of analytic solid modeling. IEEE Computer Graphics & Applications 5 (1985) 45-56.

[Cha 89] J.E. Chadwick, D.R. Haumann, R.E. Parent: Layered construction for deformable animated characters. ACM Computer Graphics 23 (1989) 243-252.

[Coq 90] S. Coquillart: Extended free-form deformation: A sculpturing tool for 3D geometric modeling. ACM Computer Graphics 24 (1990) 187-196.

[DeR 93] T.D. DeRose, R.N. Goldman, H. Hagen, St. Mann: Functional composition algorithms via blossoming. ACM Transactions on Graphics 12 (1993) 113-135.

[Far 92] G. Farin: Curves and surfaces for computer aided geometric design. A practical guide. 3. ed. Academic Press 1992.

[Faro 85] R.T. Farouki, J.K. Hinds: A hierarchy of geometric forms. IEEE Computer Graphics & Applications 5 (1985) 51-78.

[Hos 92] J. Hoschek, D. Lasser: Grundlagen der Geometrischen Datenverarbeitung. 2. ed. Teubner 1992.

[Kir 89] P. Kirchgeßner: Rationale Bézier-Volumina. Diplomarbeit, Darmstadt 1989.

[Las 90] D. Lasser, P. Kirchgeßner: Definition of Solid Primitives by Rational Bézier Volumes. Interner Bericht 205/90, Fachbereich Informatik, Universität Kaiserslautern 1990.

[Las 91] D. Lasser: Composition of tensor product Bézier representations. Interner Bericht 213/91, Fachbereich Informatik, Universität Kaiserslautern 1991.

[Nie 90] G.M. Nielson, B. Hamann: Techniques for the visualization of volumetric data. in A. Kaufman (ed.): Visualization '90. IEEE Computer Society Press (1990) 45-50.

[Pie 87] L. Piegl: On the use of infinite control points in CAGD. Computer Aided Geometric Design 4 (1987) 155-166.

[Sai 87] A. Saia, M.S.Bloor, A. DePennington: Sculptured solids in a CSG based geometric modelling system. in R.R. Martin (ed.): The Mathematics of Surfaces II. Oxford University Press (1987) 321-341.

[Sed 86] Th.W.Sederberg, S.R. Parry: Free-form deformation of solid geometric models. ACM 20 (1986) 151-160.