Locating least-distant lines in the plane

Anita Schöbel *
Fachbereich Mathematik
Universität Kaiserslautern
Germany

November 8, 1996

Abstract: In this paper we deal with locating a line in the plane. If d is a distance measure our objective is to find a straight line l which minimizes

$$f(l) = \sum_{m=1}^{M} w_m d(Ex_m, l)$$

or

$$g(l) = \max_{m=1,\ldots,M} w_m d(Ex_m, l)$$

respectively, where $w_m \geq 0$ are non-negative weights, $Ex_m = (a_{m1}, a_{m2}), m = 1, \ldots, M$ are the M existing facilities represented by points in the plane and $d(Ex_m, l) = \min_{P \in l} d(Ex_m, P)$ is the distance from the existing facility Ex_m to the line l.

We show that for all distance measures d derived from norms, one of the lines minimizing $f(l)$ contains at least two of the existing facilities. For the center objective we always get an optimal line which is at maximum distance from at least three of the existing facilities. If all weights are equal, there is an optimal line which is parallel to one facet of the convex hull of the existing facilities.

1 Introduction

Path location is an extension of classical facility location. As in the usual problems we have a set $\mathcal{E}x = \{Ex_1, Ex_2, \ldots, Ex_M\}$ of existing facilities in the plane with non-negative weights w_m for all $m \in \mathcal{M} := \{1, 2, \ldots, M\}$ representing the

*partially supported by a grant of the Deutsche Forschungsgemeinschaft and grant ERB-CHRXCT930087 of the European HC&M Programme

importance of the existing facilities. Instead of a single point we want to locate a dimensional facility such as a line or a curve in the plane. The objective function is the same as in classical facility location, namely to minimize the distances (average distances or maximum distance) between the existing facilities and the new one. A recent survey about the location of dimensional structures in the plane can be found in [Mes95].

One application is for example the planning of new railways or motorways, where the existing facilities can be cities and the weights represent the number of their inhabitants. Path location can also be used to determine the location of pipelines, drainage or irrigation ditches or in the field of plant layout, see for example [MN80].

The mathematical formulation of the problem is as follows. Given a distance function d and the index set

$$\mathcal{M} = \{1, 2, \ldots, M\}$$

the set of existing facilities

$$\mathcal{E}x = \{Ex_1, Ex_2, \ldots, Ex_M\}, Ex_m = (a_{m1}, a_{m2}) \text{ for all } m \in \mathcal{M}$$

and non-negative weights $w_m \geq 0$ for all $m \in \mathcal{M}$ we are looking for a line

$$l_{P,s} = \{x : x = P + \lambda s, \lambda \in R\}$$

dependent on $P, s \in R^2$ such that we minimize one of the following objective functions:

Median-Problem:

$$f(l) := \sum_{m \in \mathcal{M}} w_m d(Ex_m, l)$$

Center-Problem:

$$g(l) := \max_{m \in \mathcal{M}} w_m d(Ex_m, l)$$

where

$$d(Ex_m, l) := \min_{P \in l} d(Ex_m, P)$$

The optimal lines are called 1–line–median or 1–line–center respectively and are denoted by l^*. We also define $W := \sum_{m \in \mathcal{M}} w_m$ as the sum of all weights.

In the following we use the classification scheme of [HN93] in which our problem can be written as $1l/P/\cdot/d/\sum$ and $1l/P/\cdot/d/\max$, respectively, meaning in short that we want to locate one line ($1l$) in the plane (P) with no special assumptions ($\cdot$), using the distance measure d and minimizing $f(l)$ ($\sum$) or $g(l)$ (max) respectively.

In the next section some known results for Euclidean, rectangular and block norm distances are briefly described. In the main part in Section 3 these results will be extended to arbitrary norms. Section 4 summarizes the results and gives some algorithmic approaches to solve the problem.

2 Results for some special distances

A lot of research has been done for the case that the distance function d is the l_1 or the l_2 metric. In this paper we are dealing with the following statements:

Med1: There exists a 1-line-median passing through at least two of the existing facilities

Med2: Every 1-line-median l^* fulfills

$$\begin{aligned} \sum_{Ex_m \in \mathcal{B}_{l*}^-} w_m &\leq \frac{W}{2} \\ \sum_{Ex_m \in \mathcal{B}_{l*}^+} w_m &\leq \frac{W}{2} \end{aligned}$$

where $\mathcal{B}_{l*}^-$ and $\mathcal{B}_{l*}^+$ are the two halfplanes of R^2 separated by the line l^* (without the line l^* itself).

Cen1: There exists a 1-line-center which is at maximum distance from at least three of the existing facilities.

Cen2: If the weights are all equal, there exists a 1-line-center which has the same slope as one of the facets of the convex hull of the existing facilities.

A line l passing through at least two of the existing facilities and fulfilling $\sum_{Ex_m \in \mathcal{B}_l^-} w_m < \frac{1}{2}W$ and $\sum_{Ex_m \in \mathcal{B}_l^+} w_m < \frac{1}{2}W$ is called a halving line, see [KM93].

The main results for locating lines in the plane which are known so far can now be stated.

2.1 Euclidean distances

For the Euclidean distance Med1, Med2, Cen1, and Cen2 are true.

At least three different proofs for Med1 can be found in [MN80], [MT83], [KM90] and [LMW88]. The first three papers also proof Med2. Proofs for Cen1 and Cen2 can be found e.g. in [MN83].

2.2 Rectangular distances

For rectangular distances and a line $l := \{(x_1, x_2) : x_2 = \tilde{s}x_1 + b\} \subset R^2$ (with $\tilde{s}, b \in R$) we get:

$$l_1(Ex_m, l) = \min\{d_{hor}(Ex_m, l), d_{ver}(Ex_m, l)\}$$

where

$$\begin{aligned} d_{hor}(Ex_m, l) &= \frac{1}{|\tilde{s}|}|\tilde{s}a_{m1} - a_{m2} + b| \\ d_{ver}(Ex_m, l) &= |\tilde{s}a_{m1} - a_{m2} + b| \end{aligned}$$

Let us call d_{hor} and d_{ver} the horizontal and vertical distance between the existing facility and the line. They will be important in the next section. Note that which of d_{hor} and d_{ver} is the smaller one is only dependent on the slope $\tilde{s}$ of the line l and is independent of the existing facilities and of the value of b. In the above formulas the case of a vertical line is neglected. If not all existing facilities have the same x_1-coordinate all vertical lines l lead to $d_{ver}(Ex_m, l) = \infty$ for at least one $m \in \mathcal{M}$ and need not be considered any more. On the other hand, it is possible that a vertical line minimizes horizontal distances. We do not discuss that case here, because the usual solution approaches transform the horizontal line problem into the vertical line problem (or vice versa), such that it is enough to study d_{ver}.

The problem $1l/P/\cdot/d_{ver}/\sum$ is known in statistics as L_1–approximation whereas $1l/P/\cdot/d_{ver}/\max$ is called Chebyshev or L_∞–approximation in two variables.

To prove the statements for rectangular distances it is sufficient to show that they hold for horizontal and vertical distances. We formulate that as a lemma (which will be needed later).

Lemma 1 *For d_{ver} the statements* Med1, Cen1, *and* Cen2 *are true.*

Med1 is shown in [MT83]. [MN83] gives a proof for Cen1; an earlier proof for Cen1 and Cen2 can be found in [Sha78].

Lemma 2 *For d_{ver}* Med2 *is true.*

Proof: Suppose $l = \{(x_1, x_2) : x_2 = \tilde{s}x_1 + b\}$ is optimal, but

$$\sum_{Ex_m \in \mathcal{B}_l^+} w_m = \sum_{m : a_{m2} > \tilde{s}a_{m1} + b} w_m > \frac{1}{2}W$$

Then we choose $h \in R$ such that $h > 0$ and

$$\{m : a_{m2} > \tilde{s}a_{m1} + b\} = \{m : a_{m2} > \tilde{s}a_{m1} + b + h\}$$

Evaluating $l_h = \{(x_1, x_2) : x_2 = \tilde{s}x_1 + b + h\}$ leads to

$$\begin{aligned}
f(l_h) &= \sum_{m: a_{m2} > \tilde{s}a_{m1}+b} w_m |\tilde{s}a_{m1} - a_{m2} + b + h| + \sum_{m: a_{m2} \leq \tilde{s}a_{m1}+b} w_m |\tilde{s}a_{m1} - a_{m2} + b + h| \\
&= \sum_{m: a_{m2} > \tilde{s}a_{m1}+b} w_m (|\tilde{s}a_{m1} - a_{m2} + b| - h) + \sum_{m: a_{m2} \leq \tilde{s}a_{m1}+b} w_m (|\tilde{s}a_{m1} - a_{m2} + b| + h) \\
&= f(l) + h \left(\sum_{m: a_{m2} \leq \tilde{s}a_{m1}+b} w_m - \sum_{m: a_{m2} > \tilde{s}a_{m1}+b} w_m \right) < f(l)
\end{aligned}$$

contradicting the optimality of l. q.e.d.

Lemma 1 and Lemma 2 are also true for horizontal distances d_{hor} such that we can conclude:

For rectangular distance Med1, Med2, Cen1, and Cen2 are true.

2.3 Block norm distances

The statements Med1 and Cen1 were extended to block norm distances in [Sch96]. With similar proofs, Med2 and Cen2 can be shown to be true for block norms.

3 Locating a line with arbitrary norms

Let B be a convex compact set in the plane containing the origin in its interior. Moreover let B be symmetric with respect to the origin and let x be a point in the plane. The gauge

$$\gamma_B(x) := \min\{|\lambda| : x \in \lambda B\}$$

then defines a norm with the unit circle B. On the other hand we know from [Min67] that all norms can be characterized by their unit circles.

We repeat that the distance between a point Ex_m in the plane and a line $l = l_{P^0,s} := \{x \in R^2 : x = \lambda s + P^0, \lambda \in R\}$ is defined as

$$d(Ex_m, l) = \min_{P \in l} d(Ex_m, P)$$

In [MT82] it is shown that to locate r lines in the plane is NP-hard for all distances derived from norms. In the following we will prove that for $r = 1$ this problem is polynomially solvable.

As a first result we note, that to determine the distance between a point Ex_m and a line l we can increase the unit circle around Ex_m until it touches the line.

Lemma 3 *For any norm d with unit circle B, any line l, and any point Ex_m in the plane it holds that*

$$d(Ex_m, l) = \min\{|\lambda| : (Ex_m + \lambda B) \cap l \neq \emptyset\}$$

Proof:

$$\begin{aligned} d(Ex_m, l) &= \min_{P \in l} d(Ex_m, P) \\ &= \min_{P \in l} \min\{|\lambda| : P - Ex_m \in \lambda B\} \\ &= \min_{P \in l} \min\{|\lambda| : P \in \lambda B + Ex_m\} \\ &= \min\{|\lambda| : \exists P \in l \text{ such that } P \in \lambda B + Ex_m\} \\ &= \min\{|\lambda| : (Ex_m + \lambda B) \cap l \neq \emptyset\} \qquad \text{q.e.d.} \end{aligned}$$

Let d be any metric derived from a norm. The problem that we want to solve in this paper can now be written as:
Find a line l minimizing

$$f(l) = \sum_{m \in \mathcal{M}} w_m d(Ex_m, l)$$

or

$$g(l) = \max_{m \in \mathcal{M}} w_m d(Ex_m, l)$$

respectively.
In the classification scheme of [HN93] (see Section 1) we can write the problems as $1l/P/\cdot/norm/\sum$ and $1l/P/\cdot/norm/\max$, respectively.
To prove Med1, Med2, Cen1, and Cen2 for these location problems we first need to define another location problem, which can easily be solved.

Definition 1 *Let $t \in R^2$ be a given direction. For two points x, y in the plane we then define*

$$d_t(x, y) := \gamma_t(y - x)$$

where

$$\gamma_t(x) := \begin{cases} |\alpha| & \text{if } x = \alpha t \\ \infty & \text{else} \end{cases}$$

Note that γ_t can be infinity and therefore does not define a norm, but fulfills for all $x, y \in R^2$ and all $\alpha \in R$ that $\gamma_t(x) \geq 0$, $\gamma_t(x) = 0 \iff x = 0$, $\gamma_t(\alpha x) = |\alpha|\gamma_t(x)$ and the triangle inequality. In the following we define the distance between a point Ex_m and a line l. That definition is derived from Definition 1 which will formally be stated in Lemma 4.

Definition 2 *For $Ex_m \in R^2, t \in R^2$ and any line $l \subset R^2$ define*

$$d_t(Ex_m, l) := \min\{|\lambda| : Ex_m + \lambda t \in l\}$$

where $\min \emptyset := \infty$.

Lemma 4 *For all $t \in R^2$ and for all $Ex_m \in R^2$ we get that*

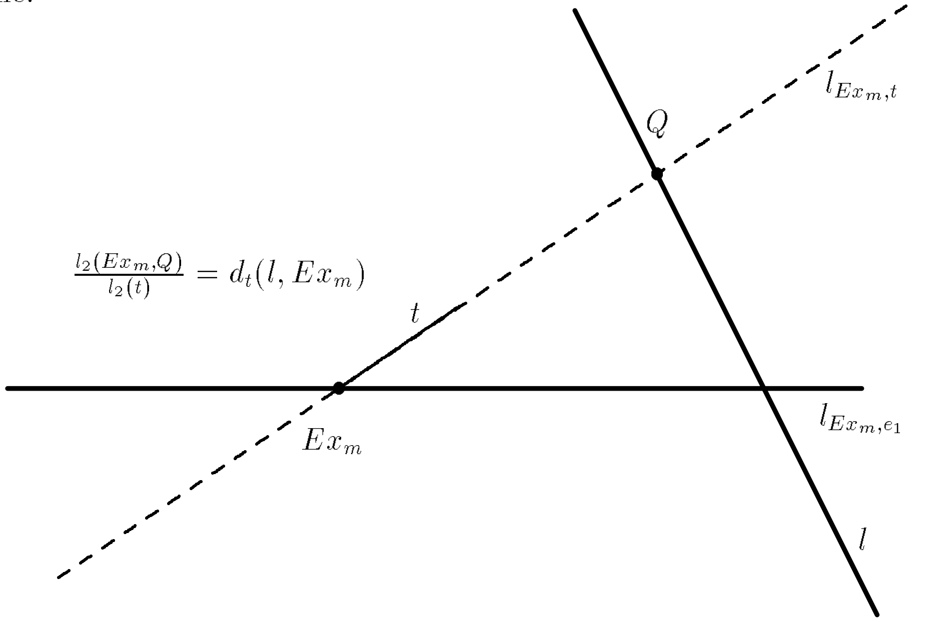
$$d_t(Ex_m, l) = \min_{P \in l} d_t(Ex_m, P)$$

Proof: The result follows from the proof of Lemma 3 with $B := \{X = \lambda t : |\lambda| \leq 1\}$. q.e.d.

If $l = l_{P,s}$ is a straight line and $l_2(t)$ the Euclidean length of $t \in R^2$ we get

$$d_t(Ex_m, l) := \begin{cases} \frac{l_2(Ex_m,Q)}{l_2(t)} \text{ with } Q \in l \cap l_{Ex_m,t} & \text{if } |l \cap l_{Ex_m,t}| = 1 \\ 0 & \text{if } |l \cap l_{Ex_m,t}| = \infty \\ \infty & \text{if } l \cap l_{Ex_m,t} = \emptyset \end{cases}$$

Note that $0 < d_t(Ex_m, l) < \infty$ if and only if s and t are linearly independent vectors. The next figure shows the meaning of Definition 2 for the case that l is a straight line.



As examples we get that the length of the horizontal line from Ex_m to l then is $d_{e_1}(Ex_m, l) = d_{hor}(Ex_m, l)$ and the length of the vertical line is $d_{e_2}(Ex_m, l) = d_{ver}(Ex_m, l)$, where e_1 and e_2 are the unit vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can now define the following location problems ($1l/P/\cdot/d_t/\sum$ and $1l/P/\cdot/d_t/\max$, respectively): For a given $t \in R^2$ find a line l_t minimizing

$$f(l_t) = \sum_{m \in \mathcal{M}} w_m d_t(Ex_m, l_t)$$

or

$$g(l_t) = \max_{m \in \mathcal{M}} w_m d_t(Ex_m, l_t)$$

respectively.

To extend the results of Section 2 to arbitrary norms, we will prove that for any distance d derived from a norm γ and any line with fixed slops s there exists a $t \in R^2$ such that

$$d(Ex_m, l) = d_t(Ex_m, l) \text{ for all } m \in \mathcal{M}.$$

That means when evaluating the objective functions $f(l)$ or $g(l)$ respectively, we can replace d by d_t. The next lemma shows how to find an optimal line l_t^* when the used distance is of the type d_t.

Lemma 5 *Let $p, q \in R^2$ and D be a linear transformation with*

1. $D(p) = q$
2. $\det(D) \neq 0$

Then we have for all $Ex_m \in R^2$

$$d_q(D(Ex_m), D(l)) = d_p(Ex_m, l)$$

where $D(l) := \{D(P) : P \in l\}$.

Proof: To prove that result we first show that

$$d_q(D(x), D(y)) = d_p(x, y) \text{ for points } x, y \in R^2$$

Case 1: $d_p(x, y) = \overline{\alpha} < \infty$: That means, $x - y = \alpha p$ with $|\alpha| = \overline{\alpha}$ and we get

$$\begin{aligned} d_q(D(x), D(y)) &= \gamma_q(D(y) - D(x)) \\ &= \gamma_q(D(y - x)) \\ &= \gamma_q(D(\alpha p)) \\ &= \gamma_q(\alpha D(p)) \\ &= \gamma_q(\alpha q) = |\alpha| = \overline{\alpha} \end{aligned}$$

Case 2: $d_p(x, y) = \infty$: Then we know that $x - y$ and t are linearly independent, which means that also $D(x - y)$ and $D(t)$ are linearly independent (because $\det(D) \neq 0$) and we get

$$d_q(D(x), D(y)) = \infty$$

With Lemma 4 we now can conclude that for a line l and any point Ex_m

$$\begin{aligned} d_q(D(Ex_m), D(l)) &= \min_{P\in l} d_q(D(Ex_m), D(P)) \\ &= \min_{P\in l} d_p(Ex_m, P) = d_p(Ex_m, l) \qquad \text{q.e.d.} \end{aligned}$$

We can use that result to transform $1l/P/\cdot/d_t/\sum$ and $1l/P/\cdot/d_t/\max$ to the corresponding vertical line problems, which have been described in Section 2. As a consequence we get:

Theorem 1 *For all distances d_t* Med1, Med2, Cen1, *and* Cen2 *are true.*

<u>Proof:</u> Using Lemma 5 with $p = t$ and $q = e_2$ we get that

$$d_{e_2}(D(Ex_m), D(l)) = d_t(Ex_m, l) \qquad (*)$$

such that we have for the objective functions $f(l)$ and $g(l)$

$$l \text{ is optimal for } d_t \text{ and } \mathcal{E}x \Longleftrightarrow D(l) \text{ is optimal for } d_{ver} \text{ and } D(\mathcal{E}x)$$

For d_{ver} we know from Lemma 2 and Lemma 1 that Med1, Med2, Cen1, and Cen2 are true. It remains to show that those properties are not changed by applying the transformation D. Due to the continuity of D we get

$$D(\text{boundary}(A)) = \text{boundary}(D(A)) \qquad (**)$$

Med1: Because of $(*)$ we have that

$$Ex_m \in l \Longleftrightarrow D(Ex_m) \in D(l)$$

that means, $D(l)$ is passing through at least two different points $D(Ex_k)$ and $D(Ex_l)$ if and only if l is passing through at least two of the existing facilities.

Med2: Because $\det(D) \neq 0$ we know that

$$Ex_m \in \mathcal{B}_l^- \Longleftrightarrow D(Ex_m) \in D(\mathcal{B}_l^-)$$

With $(**)$ we conclude that $D(\mathcal{B}_l^-) \in \{\mathcal{B}_{D(l)}^+, \mathcal{B}_{D(l)}^-\}$. That means two points Ex_m, Ex_k lie on the same side of l, if and only if $D(Ex_m)$ and $D(Ex_k)$ lie on the same side of $D(l)$.

Cen1: An immediate consequence of $(*)$ is that

$$d_t(Ex_k, l) \geq d_t(Ex_m, l) \Longleftrightarrow d_{e_2}(D(Ex_k), D(l)) \geq d_{e_2}(D(Ex_m), D(l))$$

enforcing that l is at maximum distance from at least three of the existing facilities if and only if $D(l)$ is at maximum distance from at least three points $\in D(\mathcal{E}x)$.

Cen2: From $(**)$ it follows that f is a facet of a set $\mathcal{A}$ if and only if $D(f)$ is a facet of $D(\mathcal{A})$. Because of $(*)$ it holds that $d(Ex_k, l) = d(Ex_m, l)$ if and only if $d(D(Ex_k), D(l)) = d(D(Ex_m), D(l))$ such that we can conclude:

l has the same slope as one of the facets (say between Ex_k and Ex_m) of the convex hull of the existing facilities if and only if $D(l)$ has the same slope as one of the facets (between $D(Ex_k)$ and $D(Ex_m)$) of the convex hull of the transformed existing facilities $D(\mathcal{E}x)$. q.e.d.

To solve $1l/P/\cdot/d_t/\sum$ we can proceed as follows: We choose a matrix D with $D(t) = e_2$ and $\det(D) \neq 0$. Then we define $Dx_m := D(Ex_m)$ for all $m \in \mathcal{M}$. With the new set of existing facilities $Dx_1, \ldots, Dx_m$ we solve the vertical line problem. Denote the optimal solution by l^*. Then $D^{-1}(l^*)$ is an optimal solution of $1l/P/\cdot/d_t/\sum$. The same can certainly be done for $1l/P/\cdot/d_t/\max$.

Lemma 6 *Let a slope $s \in R^2$ be given. Let γ be a norm or $\gamma = \gamma_t$ for some vector $t \in R^2$, with t and s linearly independent and let $d(x,y) = \gamma(y-x)$ be the corresponding distance. Then there exists a constant $C := C(s, d, l_2)$ such that for all $P \in R^2$ and all $Ex_m \in R^2$*

$$d(Ex_m, l_{P,s}) = C\ l_2(Ex_m, l_{P,s})$$

<u>Proof:</u> Consider at first $Ex_m = 0$. For a fixed point P_0, P_0 and s linearly independent, we know that $l_2(0, l_{P_0,s}) \neq 0$ and $0 \neq d(0, l_{P_0,s}) < \infty$ and therefore we find a real number $C \neq 0$ such that $d(0, l_{P_0,s}) = Cl_2(0, l_{P_0,s})$. Now we look at a line $l = l_{P,s} \neq l_{P_0,s}$. Because P_0 and s are linearly independent l can be written as $l_{P,s}$ with $P = \beta P_0$ for a real number β.

$$\begin{aligned} \beta d(0, l_{P_0,s}) &= \beta \min_{Q \in l_{P_0,s}} d(0, Q) = \min_{\alpha \in R} \beta d(0, \alpha s + P_0) \\ &= \min_{\alpha \in R} \beta\gamma(\alpha s + P_0) = \min_{\alpha \in R} \gamma(\beta\alpha s + \beta P_0) \\ &= \min_{\alpha' \in R} \gamma(\alpha' s + P) = d(0, l_{P,s}) \end{aligned}$$

$$\Longrightarrow d(0, l_{P,s}) = \beta d(0, l_{P_0,s}) = \beta C l_2(0, l_{P_0,s}) = C l_2(0, l_{P,s})$$

using the above equation for both d and l_2. For any point $Ex_m \neq 0$, $Ex_m \in R^2$ we finally get:

$$\begin{aligned} d(Ex_m, l_{P,s}) &= \min_{Q \in l_{P,s}} d(Ex_m, Q) \\ &= \min_{\alpha \in R} d(Ex_m, \alpha s + P) \\ &= \min_{\alpha \in R} d(0, \alpha s + P - Ex_m) \\ &= d(0, l') \text{ with } l' = \{x : x = \alpha s + P - Ex_m\} \\ &= Cl_2(0, l') \text{ because the slope of } l' \text{ remains } s \\ &= Cl_2(Ex_m, l_{P,s}) \quad \text{q.e.d.} \end{aligned}$$

Note that in Lemma 6 the properties of the Euclidean distance l_2 have not been used, such that l_2 can be replaced by any other distance derived from a norm or by distances derived from γ_t with t and s linearly independent. If d_1, d_2, and d_3 are such distances we get that

$$C(s, d_1, d_2) = \frac{C(s, d_1, d_3)}{C(s, d_2, d_3)}$$

If $l = \{(x_1, x_2) : x_2 = \tilde{s}x_1 + b\}, \tilde{s}, b \in R,\ \tilde{s} \neq 0$ we get as example (see Section 2) that:

$$C(\begin{pmatrix} 1 \\ \tilde{s} \end{pmatrix}, d_{hor}, d_{ver}) = |\tilde{s}|$$

Theorem 2 *For a given slope $s \in R^2$ all 1-line-medians with slope s, that means the straight lines $l_{P*,s}$ minimizing*

$$\min_{P \in R^2} f(l_{P,s})$$

are the same for all distances d derived from norms and distances d_t. The result also holds for the 1-line-centers with fixed slope s.

<u>Proof:</u> The result follows directly from Lemma 6. q.e.d.

The connection between any norm γ (with the derived metric $d(x, y) = \gamma(y - x)$) and the distances d_t becomes clear in the following lemma.

Lemma 7 *Let γ be a norm and d the corresponding distance. Let $Ex_m \in R^2$ and l be a straight line. Then we have:*

$$d(Ex_m, l) = \min_{t \in R^2, \gamma(t)=1} d_t(Ex_m, l)$$

<u>Proof:</u> Because of Lemma 3 we know that

$$d(Ex_m, l) = \min\{|\lambda| : Ex_m + \lambda B \cap l \neq \emptyset\} =: \lambda^0$$

That means there exists $t^0 \in R^2$ with $\gamma(t^0) = 1$ such that $Ex_m + \lambda^0 t^0 \in l$. (Note that $\gamma(t) = 1$ if and only if $t \in$ boundary(B).) Using the definition of d_{t^0}, that means:

$$d(Ex_m, l) = d_{t^0}(Ex_m, l)$$

For all t' with $\gamma(t') = 1$ we can calculate that

$$\begin{aligned} d_{t'}(Ex_m, l) &= \min\{|\lambda| : Ex_m + \lambda t' \in l\} \\ &\geq \min\{|\lambda| : Ex_m + \lambda B \cap l \neq \emptyset\} \\ &= d(Ex_m, l) \text{ using Lemma 3 again} \qquad \text{q.e.d.} \end{aligned}$$

Lemma 8 *Let γ be a norm and d the corresponding distance. Let l be a straight line with slope $s \in R^2$. Then there exists $t \in R^2$ such that*

$$d(Ex_m, l) = d_t(Ex_m, l) \text{ for all } Ex_m \in R^2$$

<u>Proof:</u> Let Ex_m be one fixed existing facility. According to Lemma 7 we can find $u \in R^2$ such that $\gamma(u) = 1$ and

$$d(Ex_m, l) = d_u(Ex_m, l) \leq d_t(Ex_m, l) \text{ for all t} \in R^2$$

Now suppose there exists $k \in \mathcal{M}$ and $v \in R^2$ with $\gamma(v) = 1$ and

$$d(Ex_k, l) = d_v(Ex_k, l) < d_u(Ex_k, l)$$

Note that u and s (and v and s) are linearly independent, because $d(Ex_m, l) \neq \infty$ ($d(Ex_k, l) \neq \infty$). With Lemma 6 we know that there exists $C := C(s, d_u, d_v)$ such that

$$d_u(Ex_k, l) = C d_v(Ex_k, l) \text{ and } d_u(Ex_m, l) = C d_v(Ex_m, l)$$

yielding $C > 1$ in the first case and $C \leq 1$ in the second case, which is impossible. q.e.d.

Theorem 3 *The statements* Med1*,* Med2*,* Cen1*, and* Cen2 *hold for all distances d derived from norms.*

<u>Proof:</u>

Med1: Suppose l^* is an optimal line, but does not pass through at least two of the existing facilities. Choose t^* such that $d(Ex_m, l^*) = d_{t^*}(Ex_m, l^*)$ for all $m \in \mathcal{M}$ according to Lemma 8.

Because of Theorem 1 we know that Med1 holds for $1l/P/\cdot/d_{t^*}/\sum$, such that we can choose l^0 minimizing the distance d_{t^*} and passing through at least two of the existing facilities.

Now let t^0 such that $d(Ex_m, l^0) = d_{t^0}(Ex_m, l^0)$ for all $m \in \mathcal{M}$ according to Lemma 8 again. Then we get:

$$\begin{aligned}
f(l^*) &= \sum_{m \in \mathcal{M}} w_m d(Ex_m, l^*) \\
&= \sum_{m \in \mathcal{M}} w_m d_{t^*}(Ex_m, l^*) \\
&\geq \sum_{m \in \mathcal{M}} w_m d_{t^*}(Ex_m, l^0) \\
&\geq \sum_{m \in \mathcal{M}} w_m d_{t^0}(Ex_m, l^0) \text{ because of Lemma 7} \\
&= \sum_{m \in \mathcal{M}} w_m d(Ex_m, l^0) \\
&= f(l^0) \geq f(l^*) \text{ because of the optimality of } l^*
\end{aligned}$$

That means l^0 is an optimal line, too, which passes through at least two of the existing facilities, which completes the proof.

Cen1 and Cen2: The proof of Cen1 and Cen2 is the same as the proof for Med1. By replacing $\sum$ by max in the above formulas and using the property *to be at maximal distance from at least three of the existing facilities* for Cen1, and *to pass through one facet of the convex hull of the existing facilities* for Cen2 instead of *passing through at least two of the existing facilities* in the above proof we analogously construct a line l^0 for that we get $g(l^*) \geq g(l^0) \geq g(l^*)$.

Med2: We assume that there is a 1-line-median l^* with $\sum_{Ex_m \in \mathcal{B}_{l^*}^-} w_m > \frac{W}{2}$. With the same notation as in the proof of Med1 we know from Theorem 1 that any line l^0 minimizing d_{t^*} fulfills

$$\sum_{Ex_m \in \mathcal{B}_{l^0}^-} w_m \leq \frac{W}{2} \text{ and } \sum_{Ex_m \in \mathcal{B}_{l^0}^+} w_m \leq \frac{W}{2}$$

Therefore we get:

$$\begin{aligned} f(l^*) &= \sum_{m \in \mathcal{M}} w_m d_{t^*}(Ex_m, l^*) \\ &> \sum_{m \in \mathcal{M}} w_m d_{t^*}(Ex_m, l^0) \\ &\geq f(l^0) \end{aligned}$$

contradicting the optimality of l^*. q.e.d.

4 Algorithms and Conclusions

For l_2 and l_1 metric a lot of algorithms have been developed. The 1-line-median for l_2 can be found in $O(M^2 log M)$ or in $O(M^2)$ time (see [MT83] and [LC85]). For l_1 an $O(M log^2 M)$ algorithm is proposed in [MT83], whereas the algorithm given in [Zem84] runs in linear time. The l-line-center problem can be solved in $O(M log M)$ time in the Euclidean case ([HIIR89]) and for l_1 in $O(M)$ time via linear programming ([Meg84]).

For distances d derived from block norms with G extreme points [Sch96] proposes an $O(GM)$-algorithm for the median problem and also an $O(GM)$-algorithm for finding a 1-line-center.

In this paper we dealt with the line-location-problem where the distance function is derived from an arbitrary norm and could extend some properties to that case. One result is that there always exists a 1-line-median, passing through at least two of the existing facilities. That easily proposes an O(M^3) algorithm, checking all pairs of existing facilities. The complexity of that algorithm certainly can be reduced, using the fact that the 1-line-median has to be a halving line, such that not all pairs of existing facilities have to be evaluated.

For center problems there always exists an optimal line which is at maximum distance from at least three of the existing facilities, such that it is possible to find a 1-line-center in O(M^4) time by enumerating all triples of existing facilities. If all weights are equal, one of the optimal lines has the same slope as one facet of the convex hull of the existing facilities. That result means that it is enough to determine the convex hull of the set of existing facilities which leads to an O($MlogM$) algorithm.

References

[HIIR89] M.E. Houle, H. Imai, K. Imai, and J.-M. Robert. Weighted orthogonal linear L_∞-approximation and applications. *Lecture Notes Computer Science*, 382:183–191, 1989.

[HN93] H.W. Hamacher and S. Nickel. Multicriteria planar location problems. Technical Report 243, Universität Kaiserslautern, 1993. accepted by EJOR.

[KM90] N.M. Korneenko and H. Martini. Approximating finite weighted point sets by hyperplanes. *Lecture Notes Computer Science*, 447:276–286, 1990.

[KM93] N.M. Korneenko and H. Martini. Hyperplane approximation and related topics. In J. Pach, editor, *New Trends in Discrete and Computational Geometry*, chapter 6, pages 135–162. Springer-Verlag, 1993.

[LC85] D.T. Lee and Y.T. Ching. The power of geometric duality revisited. *Inform. Process. Letters*, 21:117–122, 1985.

[LMW88] R.F. Love, J.G. Morris, and G.O. Wesolowsky. *Facilities Location*, chapter 3.3, pages 51–60. North-Holland, 1988.

[Meg84] N. Megiddo. Linear programming in linear time when the dimension is fixed. *Journal of the ACM*, 31:114–127, 1984.

[Mes95] J.A. Mesa. Continuous location of dimensional structures. Workshop on Locational Analysis, 1995.

[Min67] H. Minkowski. *Gesammelte Abhandlungen, Band 2.* Chelsea Publishing Company, New York, 1967.

[MN80] J.G. Morris and J.P. Norback. A simple approach to linear facility location. *Transportation Science*, 14(1):1–8, 1980.

[MN83] J.G. Morris and J.P. Norback. Linear facility location - solving extensions of the basic problem. *EJOR*, 12:90–94, 1983.

[MT82] N. Megiddo and A. Tamir. On the complexity of locating linear facilities in the plane. *Operations Research Letters*, 1(5):194–197, 1982.

[MT83] N. Megiddo and A. Tamir. Finding least-distance lines. *SIAM J. on Algebraic and Discrete Methods*, 4(2):207–211, 1983.

[Sch96] A. Schöbel. Locating least-distant lines with block norms. In S. Nickel H.W.Hamacher, K. Klamroth and F. Plastria, editors, *Studies in Locational Analysis*, 1996. to appear.

[Sha78] M.I. Shamos. *Computational Geometry*. PhD thesis, Department of Computer Science, Yale University, New Haven, 1978.

[Zem84] E. Zemel. An O(n) algorithm for the linear multiple choice knapsack problem and related problems. *Inform. Process. Letters*, 18:123–128, 1984.