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A Note on Approximation Algorithms for the Multicriteria Δ-TSP

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Abstract

The Tree and Christofides heuristic are well known 1- and $\frac{1}{2}$- approximate algorithms for the Δ-TSP. In this note their performance for the multicriteria case is described, depending on the norm in $\mathbb{R}^Q$ in case of Q criteria.

1 Algorithms

Let G be a complete graph on n nodes and $w : E(G) \rightarrow \mathbb{R}_+^Q$ be a Q–criteria weight function. We assume that the triangle inequality is fulfilled, i.e. $w(ik) \leq w(ij) + w(jk)$ for all nodes i, j, k of G. Furthermore we assume that $||.|| : \mathbb{R}^Q \rightarrow \mathbb{R}$ is a monotonous norm on $\mathbb{R}^Q$. Hence $||a|| \leq ||b||$ whenever $a \leq b$ for $a, b \in \mathbb{R}^Q$, where the order on $\mathbb{R}^Q$ is the commonly used componentwise order. We first state the extensions of the two algorithms for the case of Q criteria and will then investigate their peformance. In the following text we will always abbreviate feasible Travelling Salesman tours by TS-tours. The weight of a TS-tour T is $w(T) = (w_1(T), \dots w_Q(T))$ where $w_i(T) = \sum_{e \in E(T)} w_i(e)$.

Tree algorithm

Step 1 Let $\bar{T} \in$ argmin $\{||w(T)|| | T$ is a spanning tree of $G\}$
Step 2 Let $G' = (V(G), E')$, where E' consists of two copies of each edge of T
Step 3 Find an Eulertour ET in G' and a TS-tour HT embedded in ET.

As $\bar{T}$ is a spanning tree of G it follows that G' is Eulerian. It is then possible to find a TS-tour in $\mathcal{O}(n^2)$ time.

Christofides algorithm

Step 1 Let $\bar{T} \in$ argmin $\{||w(T)|| | T$ is a spanning tree of $G\}$
Step 2 Find all nodes in T of odd degree
Define $G^* := (V^*, E^*)$ where
$V^* = \{v \in V(G) | v$ has odd degree in $\bar{T}\}$, $E^* = \{(u, v) | u, v \in V^*\}$
Step 3 Let $\bar{M} \in$ argmin $\{||w(M)|| | M$ is a perfect matching of $G^*\}$
Step 4 Let $G'' = (V(G), E(\bar{M}) \cup E(\bar{T}))$
Step 5 Find an Euler tour ET in G'' and the TS-tour HT embedded in ET

Note that G^* is a complete graph on a subset of nodes of G with even cardinality. Thus it contains a perfect matching. Again the resulting graph G'' is Eulerian. Validity of both heuristics, i.e. that they produce a TS-tour, is shown as in the one criterion case and can be found in [5]. The Christofides algorithm was first published in[1]. Note that the triangle-inequality is used when constructing a TS-tour from an Euler tour in G' and G'', respectively.

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2 Results

The concept of optimality we use in this note is that of pareto optimal TS-tours. A TS-tour PT is said to be pareto optimal if there does not exist another TS-tour T such that $w_i(T) \leq w_i(PT)$ for all $i = 1, \ldots Q$, with strict inequality in at least one case. In general there obviously exist several pareto optimal tours for an instance of the multicriteria Δ-TSP
We will give two definitions of ϵ-approximate tours, the first is as follows.

Definition 1 *Let HT be a heuristic TS-tour and PT be a pareto optimal TS-tour, then HT is an ϵ-approximate tour if*

$$\frac{|||w(HT)|| - ||w(PT)|||}{||w(PT)||} \leq \epsilon$$

Theorem 1 *The Tree algorithm provides a tour HT, which is a 1-approximate tour for any pareto optimal tour PT.*

Proof:
We have to show that $-||w(PT)|| \leq ||w(HT)|| - ||w(PT)|| \leq ||w(PT)||$. Since the first inequality is trivial we look at the second. From the algorithm $w(HT) \leq 2w(\bar{T}) = w(G')$, hence

$$||w(HT)|| \leq 2||w(\bar{T})|| \tag{1}$$

since the norm is monotonous. By the choice of T:

$$||w(\bar{T})|| \leq ||w(PT)|| \tag{2}$$

since removing one edge from PT yields a spanning tree of G. By (1) and (2) we have

$$||w(HT)|| \leq 2||w(PT)|| \tag{3}$$

and the claim holds.

□

Lemma 1 *Let PT be a pareto TS-tour. Then there exists some $\delta \in [0,1]$ such that $||w(PT)|| \geq (1+\delta)||w(\bar{M})||$ where $\bar{M}$ is the perfect matching of Step 3 in the Christofides algorithm.*

Proof:
Let $\{i_1, \ldots i_{2m}\}$ be the odd-degree nodes of the spanning tree $\bar{T}$ as theyappear in PT, i.e.

$$PT = \alpha_0 i_1 \alpha_1 i_2 \ldots \alpha_{2m-1} i_{2m} \alpha_{2m}$$

where α_i are possibly empty sequences of nodes. Let $M_1 = \{[i_1, i_2], [i_3, i_4], \ldots [i_{2m-1}, i_{2m}]\}$ and $M_2 = \{[i_2, i_3], [i_4, i_5], \ldots [i_{2m}, i_i]\}$. Then by the triangle-inequality $w(PT) \geq w(M_1) + w(M_2)$. Now if $w(\bar{M}) \leq w(M_1)$, $w(\bar{M}) \leq w(M_2)$ it follows that $||w(PT)|| \geq ||w(M_1) + w(M_2)|| \geq 2||w(\bar{M})||$. Otherwise at least $||w(M_1)|| \geq ||w(\bar{M})||$ and $||w(M_2)|| \geq ||w(\bar{M})||$ and hence $||w(M_1) + w(M_2)|| \geq \max\{||w(M_1)||, ||w(M_2)||\} \geq ||w(\bar{M})||$.

□

Given a pareto optimal TS-tour PT we denote the maximal $\delta \in [0,1]$ for which $||w(PT)|| \geq (1+\delta)||w(M)||$ holds by $\delta(PT)$.

Theorem 2 *Let PT be a pareto optimal TS-tour. Then the TS-tour HT of the Christofides algorithm is a $\frac{1}{1+\delta(PT)}$-approximate tour.*

Proof:
We have that $||w(HT)|| \leq ||w(G'')|| = ||w(\bar{T}) + w(\bar{M})|| \leq ||w(\bar{T})|| + ||w(\bar{M})||$. By Lemma 1

$$||w(\bar{M})|| \leq \frac{1}{1+\delta(PT)}||w(PT)|| \tag{4}$$

Hence (2), which holds here, too, and (4) imply

$$||w(HT)|| \leq ||w(PT)|| \left(1 + \frac{1}{1+\delta}\right). \tag{5}$$

□

Theorems 1 and 2 show that the bounds guaranteed by the two heuristics carry over from the one criterion to the multicriteria case, with a weaker result for the Christofides algorithm, when Definition 1 is used.
Nevertheless a more intuitive definition of ϵ-approximate solutions for multicriteria problems is the following.

Definition 2 *Let HT be a heuristic TS-tour and PT be a pareto optimal TS-tour, then HT is an ϵ-approximate tour if*

$$\frac{||w(HT) - w(PT)||}{||w(PT)||} \leq \epsilon$$

Note that if HT is an ϵ-approximate tour in the sense of Definition 2 it is so in the sense of of Definition 1.
We will now restrict ourselves to l_p-norms, i.e.$||x|| = \left(\sum_{i=1}^{Q} |x_i|^p\right)^{\frac{1}{p}}$ for $x \in \mathbb{R}^Q$, which of course are monotonous norms.

Lemma 2 *If $a_i, b_i \geq 0 \quad i = 1 \ldots Q \quad p \geq 1$ then*

$$\left(\sum_{i=1}^{Q} |a_i - b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{Q} (a_i^p + b_i^p)\right)^{\frac{1}{p}}$$

Proof:
Without loss of generality we may assume $a_i \geq b_i \quad i = 1, \ldots Q$. Hence $|a_i - b_i| = a_i - b_i \leq a_i$ and $|a_i - b_i|^p \leq a_i^p \leq a_i^p + b_i^p$.

□

Theorem 3 *For the second definition of ϵ-approximate solution the following hold.*

1. *The TS-tour HT of the Tree algorithm is a $(2^p + 1)^{\frac{1}{p}}$-approximate tour for all pareto optimal TS-tours PT.*

2. *For any pareto TS-tour PT the Christofides algorithm gives a TS-tour HT which is a $\left(\left(1 + \frac{1}{1+\delta(PT)}\right)^p + 1\right)^{\frac{1}{p}}$-approximate tour.*

Proof:

1.

$$\frac{||w(HT) - w(PT)||}{||w(PT)||} = \frac{\left(\sum_{i=1}^{Q} |w_i(HT) - w_i(PT)|^p\right)^{\frac{1}{p}}}{\left(\sum_{i=1}^{Q} (w_i(PT))^p\right)^{\frac{1}{p}}}$$

$$\leq \frac{\left(\sum_{i=1}^{Q}\left((w_i(HT))^p + (w_i(PT))^p\right)\right)^{\frac{1}{p}}}{\left(\sum_{i=1}^{Q}(w_i(PT))^p\right)^{\frac{1}{p}}}$$

$$= \left(\frac{||w(HT)||^p + ||w(PT)||^p}{||w(PT)||^p}\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{2^p||w(PT)||^p + ||w(PT)||^p}{||w(PT)||^p}\right)^{\frac{1}{p}}$$

$$= (2^p + 1)^{\frac{1}{p}}$$

where the first inequality follows from Lemma 2 and the second from (3).

2. Analogously

$$\frac{||w(HT) - w(PT)||}{||w(PT)||} \leq \left(\frac{\left(\frac{2+\delta}{1+\delta}\right)^p ||w(PT)||^p + ||w(PT)||^p}{||w(PT)||^p}\right)^{\frac{1}{p}}$$

$$= \left(\left(\frac{2+\delta}{1+\delta}\right)^p + 1\right)^{\frac{1}{p}}$$

where we made use of (5) and again of Lemma 2.

□

Thus for $p \to \infty$ the Tree algorithm gives a 2-approximate tour, the Christofides algorithm a $\left(1 + \frac{1}{1+\delta}\right)$-approximation. These values could be calculated dirctly using $||x||_\infty = \max_{i=1\dots Q}|x_i|$ in Theorem 3.
The first part of Theorem 3 as well as Theorem 1 are from a thesis of the second author [2]. Figure 1 shows the values of $\epsilon(p) = (2^p + 1)^{\frac{1}{p}}$ and $\epsilon(p) = (\frac{3}{2}^p + 1)^{\frac{1}{p}}$, i.e. $\delta = 1$. For $\delta = 0$ both values are the same.

3 Remarks

Another possibility to define ϵ-approximate solutions would be to require

$$\frac{|w_i(HT) - w_i(PT)|}{|w_i(PT)|} \leq \epsilon \quad i = 1, \dots Q$$

But there is no known procedure to guarantee that even for a given pareto TS-tour PT. Note that equations (2), (4) do not hold componentwise in general.
Another remark is on the problem of finding a norm-minimizing spanning tree or perfect matching, which are essential steps in the two algorithms.
In case $||x|| = ||x||_1 = \sum_{i=1}^{Q}|x_i|$ we have that for $S \subseteq E$ $||w(S)|| = \sum_{i=1}^{Q}\sum_{e\in S} w_i(e) = \sum_{e\in S}\left(\sum_{i=1}^{Q} w_i(e)\right)$ and can solve the tree and matching problems as one criterion problems with $w'(e) = \sum_{i=1}^{Q} w_q(e)$, and thus in polynomial time.
In case $||x|| = ||x||_\infty = \max|x_i|$, however, $||w(S)|| = \max_{i=1\dots Q}\sum_{e\in S} w_i(e)$ and to find a norm minimizing spanning tree or matching is the Max-Ordering spanning tree and Max-Ordering matching problem. Both of these problems are known to be NP-hard, see [3] and [4].

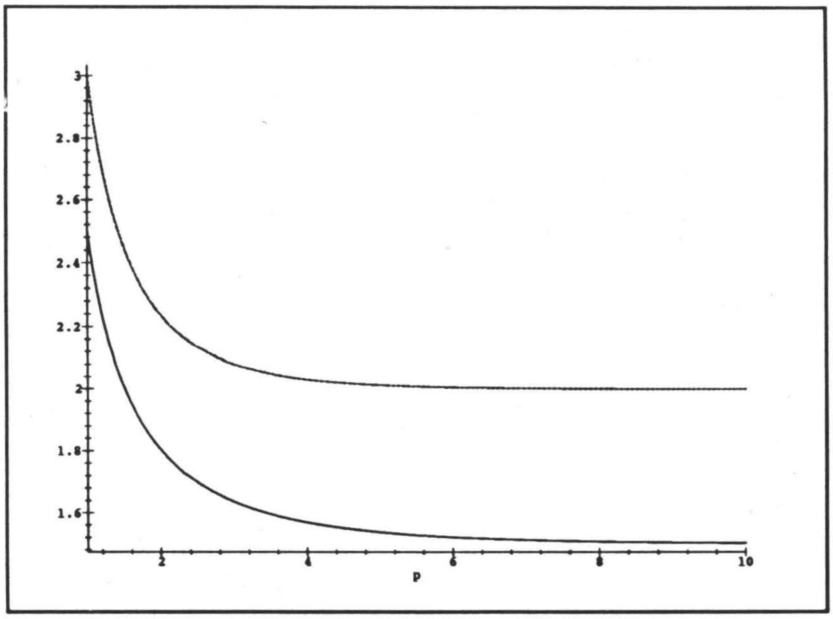

Figure 1: Performance guarantee for l_p norms

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