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THE LOCAL ISOMETRIC IMBEDDING IN $\mathbb{R}^3$ OF TWO-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH GAUSSIAN CURVATURE CHANGING SIGN ARBITRARILY

Werner Greub[†] and Dan Socolescu

DEDICATED TO THE MEMORY OF PROFESSOR ALEXANDRU FRODA

1. GEOMETRIC PRELIMINARIES AND STATEMENT OF THE IMBEDDING PROBLEM

Let M be an oriented Riemannian two-dimensional manifold and let $\phi : M \to \mathbb{R}^3$ be locally an isometric imbedding, i.e. having a non-degenerate differential $d\phi$. Let F_x denote the oriented plane in $\mathbb{R}^3$ given by

$$(1) \qquad x \in M : \quad F_x = (d\phi)_x T_x M ,$$

where $T_x M$ is the tangent space. Then there is a unique vector $n(x) \in \mathbb{R}^3$ such that the oriented plane F_x together with $n(x)$ induces the given orientation of $\mathbb{R}^3$. The correspondence $x \to n(x)$ determines a smooth map $n : M \to \mathbb{R}^3$ called the normal field of ϕ.

Recall that the second fundamental form for ϕ is the symmetric tensor field of degree two on M given by

$$(2) \qquad \Lambda(x;h,k) = -\langle (d\phi)_x h, (dn)_x k \rangle , \quad x \in M , \quad h,k \in T_x M .$$

$\langle \cdot , \cdot \rangle$ denotes here the scalar product in $\mathbb{R}^3$. Thus Λ determines a selfadjoint tensor field Γ of type (1,1), called the Weingarten tensor, such that

$$(3) \qquad g(x;\Gamma(x)h,k) = \Lambda(x;h,k) , \quad x \in M , \quad h,k \in T_x M ,$$

where g denotes the Riemannian metric. We note that Γ is a Gauss-Codazzi field, i.e. a selfadjoint tensor field of type (1,1) on a Riemannian two-dimensional manifold satisfying (cf. [3])

$$(4) \qquad \nabla_X(\Gamma(Y)) - \nabla_Y(\Gamma(X)) = \Gamma([X,Y]) , \quad X,Y \in \mathcal{X}(M) .$$

Here $\Sigma(M)$ is the ring of smooth functions on M, $\mathcal{X}(M)$ is the $\Sigma(M)$-module of vector fields on M. ∇_X denotes covariant differentiation in the direction of the vector field X with respect to the corresponding Levi-Cività connection, and [X,Y] is the Lie bracket of X and Y. Recall further that the mean curvature and the Gaussian curvature of ϕ are defined by

(5) $$H = \frac{1}{2}\operatorname{tr}\Gamma \ ,$$

and respectively

(6) $$K = \det \Gamma \ .$$

Let denote now

(7) $$\Theta = \Gamma - H\,I \ ,$$

where I is the identity and, according to (5),

(8) $$\operatorname{tr}\Theta = 0 \ .$$

Using the selfadjointness of Γ, from (7) we infer that Θ is selfadjoint too, and hence

(9) $$\Gamma = \begin{pmatrix} -U & V \\ V & U \end{pmatrix} + \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = \begin{pmatrix} H-U & V \\ V & H+U \end{pmatrix} \ .$$

By the Cayley-Hamilton theorem we get

(10) $$\Gamma^2 - (\operatorname{tr}\Gamma)\Gamma + (\det\Gamma)\, I = 0 \ ,$$

and hence, by using (5), (6) and (9),

(11) $$K = H^2 - (U^2 + V^2) \ .$$

After these differential geometric preliminaries we are now able to translate the above local isometric imbedding problem, where only a sufficiently small neighbourhood of an arbitrarily chosen point $x_0 \in M$ has to be isometrically imbedded in $\mathbb{R}^3$, into the language of analysis. Let x^1, x^2, x^3 be cartesian coordinates (of a point x) in $\mathbb{R}^3$ and let u^1, u^2 be local coordinates (in a neighbourhood N_0 of x_0) on M. Then the above local isometric problem is equivalent to that of finding a solution to the system

$$(12) \qquad N_0 : \quad \sum_{i=1}^{3} \frac{\partial x^i}{\partial u^k} \frac{\partial x^i}{\partial u^l} = g_{kl} \ , \ k,l = 1,2 \ , \ x = \phi(u) \ ,$$

where g_{kl} are the coefficients of the Riemannian metric g with respect to the coordinates u^1, u^2.

As known (see [5], [8], [11]), the study of the nonlinear first order system (12) depends essentially on the sign of the corresponding Gaussian curvature K. The local isometric imbedding problem has already been solved not only when $K(x_0) \neq 0$, i.e. in the elliptic case ($K(x_0) > 0$) or in the hyperbolic one ($K(x_0) < 0$), (cf. [5], [8], [10]), but also when $K(x_0) = 0$, i.e. in the parabolic case, provided $\nabla K(x_0) \neq 0$ (see [6], [9]).

In the present paper we are concerned with the above local isometric imbedding problem in the case where $K(x_0) = 0$, $\nabla K(x_0) = 0$. The existence result we obtained (and announced in [4], [12]) is based on a common work with W. H. Greub, who unfortunately died prematurely. I am greatly indebted to him for introducing me to this matter.

Instead of studying the first order nonlinear system (12) we shall study an equivalent one, derived as follows (cf. [2]):

Recall at first that an almost complex structure on M is a tensor field J of type (1,1) such that $J^2 = -I$. We note that a Riemannian metric g on an oriented two-dimensional manifold determines an almost complex structure given by

$$(13) \qquad g(JX,Y) = \Delta_M(X,Y) \ , \ X,Y \in \mathcal{X}(M) \ ,$$

where Δ_M denotes the normed 2-form on M which represents the orientation, i.e. a volume form on M. A vector field Z on M will be called a Cauchy-Riemann field if, for every $X \in \mathcal{X}(M)$,

$$J([Z,X]) = [Z,JX] \tag{14}$$

The local existence of non-trivial Cauchy-Riemann fields is proved by the following

Proposition 1 [3]. *Let $x_0 \in M$ and let $h \in T_{x_0}(M)$ be a nonzero tangent vector. Then there is a Cauchy-Riemann field Z in some neighbourhood N_0 of x_0 such that $Z(x_0) = h$.*

Let now $e_1 \neq 0$ be a Cauchy-Riemann field in a (simply connected) neighbourhood N_0 of a point $x_0 \in M$ and set $e_2 = Je_1$. Using the characterization of a Cauchy-Riemann field given by

Lemma 1 [3]. *A vector field Z on M is a Cauchy-Riemann field if and only if*

$$[Z,JZ] = 0 , \tag{15}$$

we infer that e_2 is again a Cauchy-Riemann field. Moreover, from (13) and (14) it follows

$$|e_1| = |e_2| , \; g(e_1,e_2) = 0 , \; [e_1,e_2] = 0 . \tag{16}$$

Thus e_1, e_2 is an orthogonal frame field in N_0, called a Cauchy-Riemann frame. Next consider the dual frame e^{*1}, e^{*2}. Then

$$de^{*1} = 0 , \; de^{*2} = 0 . \tag{17}$$

Hence, for some $u^i \in \Sigma(N_0)$, i = 1,2,

$$e^{*i} = du^i , \; i = 1,2 . \tag{18}$$

Since the covectors $e^{*1}(x)$ and $e^{*2}(x)$ are linearly independent, it follows that the functions u^1 and u^2 are local coordinates in a neighbourhood $N_1 \subset N_0$ of x_0. In this local coordinate system the metric tensor satisfies $g_{11} = g_{22}$, $g_{12} = 0$ and so (u^1,u^2) is a system of isothermal parameters (see [3] for details).

Let now $x_0 \in M$ and let e_1, e_2 be a Cauchy-Riemann frame in a neighbourhood N_1 of x_0. From (7) and (9) it follows then

$$(19) \qquad N_1 : \begin{cases} \Theta(e_1) = -Ue_1 + Ve_2 , \\ \Theta(e_2) = Ve_1 + Ue_2 . \end{cases}$$

On the other hand, according to (4) and (7), the selfadjoint, trace free tensor field Θ has to satisfy the equation

$$(20) \qquad N_1 : \quad \nabla_X\Theta(Y) - \nabla_Y\Theta(X) + X(H)Y - Y(H)X = \Theta([X,Y]) , \qquad \forall\ X,Y \in \mathcal{X}(M) .$$

Using now the formula [3]

$$(21) \qquad \nabla_X Z = X(\ln|Z|)Z - JX(\ln|Z|)JZ , \ X \in \mathcal{X}(M) ,$$

characterizing those vector fields Z without zeros, which are Cauchy-Riemann fields, and taking $X = e_1$, $Y = e_2$, from (19) we get

$$(22_1) \qquad N_1 : \quad \nabla_{e_1}\Theta(e_2) = \frac{\partial V}{\partial u^1}e_1 + \frac{\partial U}{\partial u^1}e_2 + V\nabla_{e_1}e_1 + U\nabla_{e_1}e_2 ,$$

$$(22_2) \qquad N_1 : \quad \nabla_{e_2}\Theta(e_1) = -\frac{\partial U}{\partial u^2}e_1 + \frac{\partial V}{\partial u^2}e_2 - U\nabla_{e_2}e_1 + V\nabla_{e_2}e_2 ,$$

$$(23_1) \qquad N_1 : \quad \nabla_{e_1}e_1 = -\nabla_{e_2}e_2 = \frac{\partial G}{\partial u^1}e_1 - \frac{\partial G}{\partial u^2}e_2 ,$$

$$(23_2) \qquad N_1 : \quad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \frac{\partial G}{\partial u^2}e_1 + \frac{\partial G}{\partial u^1}e_2 ,$$

where $G = \ln|e_1| = \ln|e_2|$. Inserting (22_1), (22_2), (23_1) and (23_2) into (20) we obtain finally the following nonlinear first order system of equations

$$(24) \qquad N_1 : \begin{cases} \dfrac{\partial U}{\partial u^1} - \dfrac{\partial V}{\partial u^2} + 2\dfrac{\partial G}{\partial u^1}U - 2\dfrac{\partial G}{\partial u^2}V + \dfrac{\partial H}{\partial u^1} = 0 , \\ \dfrac{\partial V}{\partial u^1} + \dfrac{\partial U}{\partial u^2} + 2\dfrac{\partial G}{\partial u^2}U + 2\dfrac{\partial G}{\partial u^1}V - \dfrac{\partial H}{\partial u^2} = 0 . \end{cases}$$

Consequently we seek a solution (U,V,H) of (11) and (24).

Remark 1. *As we shall see in Section 2, this equivalent formulation of the local isometric imbedding problem is very convenient in the case where* x_0 *is a parabolic point (independently of the vanishing or nonvanishing of* $\nabla K(x_0)$*) as well as in the case where* x_0 *is an elliptic point.*

2. EXISTENCE OF A SOLUTION OF THE LOCAL ISOMETRIC IMBEDDING PROBLEM IN THE CASE WHERE THE GAUSSIAN CURVATURE CHANGES SIGN ARBITRARILY

The existence of a solution to the quasilinear system (11) and (24) is based on the theory of quasilinear accretive systems of partial differential equations developed in [7]. In order to formulate the existence theorem for the local isometric imbedding problem we need the following results:

Definition 1 [7]. *Let us consider the quasilinear system of equations*

$$(25) \qquad \sum_{i=1}^{2} A^i(u,\tilde{U})\partial_{u^i}\tilde{U} + B(u,\tilde{U}) = 0 ,$$

where $\tilde{U} = (U,V,H)$ *is the unknown vector,* $\tilde{U} = \tilde{U}(u)$, $u = (u^1,u^2) \in \mathbb{R}^2$, $A^i(u,y)$, $i=1,2$, *is a* 3×3 *symmetric matrix,* $y = (y^1,y^2,y^3)$ *belongs to some neighbourhood of the origin in* $\mathbb{R}^3$, $B(u,y)$ *is a vector in* $\mathbb{R}^3$. *With (25) we associate the following generalized linear system*

$$(26) \qquad \sum_{i=1}^{2} A^i(u,w)\partial_{u^i}\tilde{U} + B_y(u,w)\tilde{U} = f(u,w) ,$$

where $w(u) = (w^1(u),w^2(u),w^3(u))$ *is a given vector function of* $u \in \mathbb{R}^2$ *belonging to the same neighbourhood of the origin in* $\mathbb{R}^3$ *as* y, *and* $B_y(u,w)$, $f(u,w)$ *are respectively defined by*

$$B_y(u,w) = (\partial_y^j B^{(i)}(u,y)|_{y=w})_{i,j=1,2} \, , \tag{27}$$

$$f(u,w) = B_y(u,w) - B(u,w) \, . \tag{28}$$

Differentiating (26) s-times formally, where s is a given positive integer, we obtain again a first order system for the unknown $\tilde{U}$ *and its derivatives* $D^\sigma \tilde{U}$, $1 \leq |\sigma| \leq s$. *The last system can be expressed in matrix form as*

$$\sum_{i=1}^{2} \hat{A}^i(u,w)\partial_{u^i}\tilde{U}^s + P_0(u,w)\tilde{U}^s + P_1(u,W^s)\tilde{U}^s = f^{(s)}(u,W^s) \, . \tag{29}$$

Here $\tilde{U}^s$, W^s *and* $f^{(s)}$ *denote the vectors formed by taking* $\tilde{U}$, w *and* f *and their distinct derivatives up to order s respectively.* $\hat{A}^i(u,w)$ *is the p×p matrix formed by taking all the diagonal elements to be equal to* A^i, *and setting the non-diagonal elements equal to zero, where*

$$p = 3\binom{s+2}{2} \, . \tag{30}$$

The p×p matrix P_0 *depends only on u and w, while the p×p matrix* P_1 *has the property to vanish when* W^s *is zero.*

The linear system (26) is said to be accretive in the generalized sense if the p×p matrix Q(u,y), defined by

$$Q(u,y) = P_0(u,y) + P_0^{tr}(u,y) - \sum_{i=1}^{2} \partial_{u^i}\hat{A}^i(u,y) \, , \tag{31}$$

where $P_0^{tr}(u,y)$ *denotes the transpose matrix of* $P_0(u,y)$, *is uniformly positive definite for all* $u \in \mathbb{R}^2$ *and all y lying in a small neighbourhood* N_θ, *of radius* θ, *of the origin in* $\mathbb{R}^3$, *i.e.*

$$\sum_{i,j=1}^{p} Q_{ij}(u,y)\xi^i\xi^j \geq \gamma|\xi|^2 \, , \; \forall\, \xi \in \mathbb{R}^p \, , \tag{32}$$

holds for some positive constant γ *which is independent of u and y, for all* $u \in \mathbb{R}^2$ *and all* $y \in N_\theta \subset \mathbb{R}^3$.

The quasilinear system (25) is said to be accretive, if its associated linear system (26) is accretive in the generalized sense.

Existence theorem for the quasilinear system (25) [7]. *Let s be a positive integer, $s > 2$, and assume that the symmetric matrix A^i and the vector B possess bounded continuous derivatives up to a certain order, specifically for $i = 1,2$,*

$$D_u^\alpha D_y^{\beta+2} B,\ D_u^\alpha D_y^\beta A^i \in C_b^0(\mathbb{R}^2 \times N_1)\ ,\ |\alpha| + |\beta| \leqq s\ , \tag{33}$$

$$\begin{cases} \sup\limits_{\substack{u\in R^2 \\ |y|<1}} \sum\limits_{\substack{|\alpha|+|\beta|\leqq s+2 \\ |\alpha|\leqq s}} |D_u^\alpha D_y^\beta B(u,y)| \leqq C_1\ , \\ \\ \sup\limits_{\substack{u\in R^2 \\ |y|<1}} \sum\limits_{|\alpha|+|\beta|\leqq s} |D_u^\alpha D_y^\beta A^i(u,y)| \leqq C_2\ . \end{cases} \tag{34}$$

Assume furthermore that the $p\times p$ matrix $Q_0(u) := Q(u,0)$ is positive definite, uniformly with respect to $u \in \mathbb{R}^2$. Then there exists a positive number η_1 such that for any prescribed ball B_η^s, $\eta < \eta_1$, about the origin in the Sobolev space $H_s(\mathbb{R}^2)$, there is a positive number $\alpha = \alpha(\eta)$ such that for $||B(\cdot,0)||_s < \alpha$, the quasilinear accretive system (25) possesses a unique solution u lying in B_η^s.

We shall prove now the following

Existence theorem for the local isometric imbedding problem [4]. *Suppose that the Gaussian curvature K of an oriented two-dimensional manifold M is a C^s-function, $s \geqq 3$, and that $K(x_0) = 0$, $\nabla K(x_0) = 0$, where $x_0 \in M$; then the local isometric imbedding problem (11) and (24) can be solved in a neighbourhood N_1 of x_0 by a $C^{s-2,\lambda}$-**Weingarten** tensor field Γ, $0 < \lambda < 1$.*

Proof. Without loss of generality we choose N_1 to be the disk of radius ε and center $x_0 = (0,0)$, and introduce the new local coordinates

$$N_1 :\quad v^i = \varepsilon^{-1} u^i\ ,\ i = 1,2\ . \tag{35}$$

Taking now into account on one hand that the Gaussian curvature K and its gradient vanish at x_0, on the other hand that by Proposition 1 the function $G = \ln|e_1| = \ln|e_2|$ can be taken to vanish at x_0, we introduce also the new functions

(36) $$N_1 : \quad \tilde{K}(v^1,v^2) = \varepsilon^{-2}K(u^1,u^2) \ , \ \tilde{G}(v^1,v^2) = \varepsilon^{-1}G(u^1,u^2) \ .$$

Using (35) and (36), we give the quasilinear system (11) and (24) the equivalent form

(37) $$N_1 : \quad A^1\partial_{v^1}\tilde{U}^* + A^2\partial_{v^2}\tilde{U}^* + \varepsilon C\,\tilde{U}^* + \varepsilon^3 D = 0 \ ,$$

where $\tilde{U}^* = \tilde{U}^*(v)$ is the unknown vector in the new local coordinates,

(38_1) $$N_1 : \quad A^1_{11} = A^1_{13} = A^1_{22} = 1 \ , \ A^1_{12} = A^1_{21} = A^1_{23} = A^1_{31} = A^1_{32} = A^1_{33} = 0 \ ,$$

(38_2) $$N_1 : \quad A^2_{11} = A^2_{13} = A^2_{22} = A^2_{31} = A^2_{32} = A^2_{33} = 0 \ , \ A^2_{12} = A^2_{23} = -1 \ , \ A^2_{21} = 1 \ ,$$

(39) $$N_1 : \quad C_{11} = C_{22} = 2\partial_{v^1}\tilde{G} \ , \ C_{21} = -C_{12} = 2\partial_{v^2}\tilde{G} \ , \ C_{13} = C_{23} = 0 \ , \ C_{31} = -U^* \ , \ C_{32} = -V^* \ , \ C_{33} = H^* \ ,$$

(40) $$N_1 : \quad D^1 = D^2 = 0 \ , \ D^3 = -\tilde{K} \ .$$

In order to apply the above existence result we add the first two scalar equations (37) to the third one and obtain the following equivalent quasilinear system

(37') $$N_1 : \quad \bar{A}^1\partial_{v^1}\tilde{U}^* + (\bar{A}^2 + \bar{B})\partial_{v^2}\tilde{U}^* + \varepsilon\bar{C}\,\tilde{U}^* + \varepsilon^3\bar{D} = 0 \ ,$$

where

(41_1) $$N_1 : \quad \bar{A}^1_{11} = \bar{A}^1_{13} = \bar{A}^1_{22} = \bar{A}^1_{31} = \bar{A}^1_{33} = 1 \ , \ \bar{A}^1_{12} = \bar{A}^1_{21} =$$

$$\bar{A}^1_{23} = \bar{A}^1_{32} = 0 ,$$

(41_2) $\quad N_1 :$ $\bar{A}^2_{11} = \bar{A}^2_{13} = \bar{A}^2_{22} = \bar{A}^2_{31} = \bar{A}^2_{33} = 0 , \bar{A}^2_{12} = \bar{A}^2_{21} =$

$$\bar{A}^2_{23} = \bar{A}^2_{32} = -1 ,$$

(42) $\quad N_1 :$ $\bar{B}_{11} = \bar{B}_{12} = \bar{B}_{13} = \bar{B}_{22} = \bar{B}_{23} = \bar{B}_{31} = \bar{B}_{32} = \bar{B}_{33}$

$$= 0 , \bar{B}_{21} = 2 ,$$

(43) $\quad N_1 :$ $\bar{C}_{11} = \bar{C}_{22} = 2\partial_{v^1}\tilde{G} , \bar{C}_{21} = -\bar{C}_{12} = 2\partial_{v^2}\tilde{G} , \bar{C}_{13}$

$$= \bar{C}_{23} = 0 , \bar{C}_{31} = -U^* + 2\partial_{v^1}\tilde{G} , \bar{C}_{32} = -V^* -$$

$$2\partial_{v^2}\tilde{G} , \bar{C}_{33} = H^* ,$$

(44) $\quad N_1 :$ $\bar{D}^1 = 0 , \bar{D}^2 = 0 , \bar{D}^3 = -\tilde{K} .$

Next we scale up (or down) $\tilde{U}^*$ and $\tilde{K}$ such that

(45) $$\min_{\bar{N}_1} \tilde{K}(v^1,v^2) = -1 ,$$

and demand that $\tilde{U}^*$ satisfy the conditions

(46) $$U^*(0,0) = 1 , V^*(0,0) = 1 , H^*(0,0) = \sqrt{2} .$$

Taking account of (46) we introduce a new unknown $\tilde{U}' = \tilde{U}'(v)$, such that

(47) $\quad N_1 :$ $U'(v) = U^*(v) - 1 , V'(v) = V^*(v) - 1 , H'(v)$

$$= H^*(v) - \sqrt{2} .$$

Using now the following

Extension theorem [13]. *Let f be a scalar or vector function defined in the disk D(0,r) of center 0 and radius r. Assume that*

f belongs to C^s, $s \in N_0$, in $D(0,r)$ and $|f(v) - c| \leq \eta$ there, where c is a constant. Then there exists an extension $\tilde{f}$ of f from $D(0,r/2)$ to $\mathbb{R}^2$, such that

$$\tilde{f} \in C^s(\mathbb{R}^2) \ , \ |\tilde{f}(v) - c| \leq \eta \ , \tag{48}$$

as well as (47), from (37') we obtain finally the following e-quivalent quasilinear first order system

$$\mathbb{R}^2 : \quad \bar{A}^1 \partial_{v^1} \tilde{U}' + (\bar{A}^2 + \bar{B}) \partial_{v^2} \tilde{U}' + \bar{\bar{C}} \tilde{U}' = -\varepsilon^3 \bar{\bar{D}} - \varepsilon E \ , \tag{37''}$$

where

$$\mathbb{R}^2 : \quad \bar{\bar{C}}_{11} = \bar{\bar{C}}_{22} = 2\eta_1(v) \partial_{v^1} \tilde{G} + \eta_2(v) \ , \ \bar{\bar{C}}_{21} = -\bar{\bar{C}}_{12} = 2\eta_1(v) \partial_{v^2} \tilde{G} \ , \ \bar{\bar{C}}_{13} = \bar{\bar{C}}_{23} = 0 \ , \ \bar{\bar{C}}_{31} = -2 + 2\eta_1(v) \times \partial_{v^1} \tilde{G} - U' \ , \ \bar{\bar{C}}_{32} = -2 - 2\eta_1(v) \partial_{v^2} \tilde{G} - V' \ , \ \bar{\bar{C}}_{33} = 2\sqrt{2} + H' \ , \tag{49}$$

$$\mathbb{R}^2 : \quad \bar{\bar{D}}^1 = 0 \ , \ \bar{\bar{D}}^2 = 0 \ , \ \bar{\bar{D}}^3 = -\eta_1(v) \tilde{K} \ , \tag{50}$$

$$\mathbb{R}^2 : \quad E^1 = 2\eta_1(v) \partial_{v^1} \tilde{G} \ , \ E^2 = -2\eta_1(v) \partial_{v^2} \tilde{G} \ , \ E^3 = 0 \ , \tag{51}$$

for appropriately chosen C^s-bump **functions** $\eta_1(v)$, $\eta_2(v)$, $\eta_1(\mathbb{R}^2) = [0,1]$, $\eta_2(\mathbb{R}^2) = [0,1]$, i.e. for $N_2 \supset N_1$

$$\mathbb{R}^2 : \quad \eta_1(v) = \begin{cases} 1 \ , & v \in N_1 \ , \\ 0 \ , & v \in \mathbb{R}^2 \setminus N_2 \ , \end{cases} \qquad \eta_2(v) = \begin{cases} 0 \ , & v \in N_1 \ , \\ 1 \ , & v \in \mathbb{R}^2 \setminus N_2 \end{cases} \tag{52}$$

and where (41_1), (41_2) and (42) are now valid in $\mathbb{R}^2$. It is a straight forward calculus to show that the system (37'') satisfies the conditions (32), (33) and (34) for a suitably chosen neighbourhood N_θ of the origin in $\mathbb{R}^3$. However, since the matrix $\bar{B}$ is no longer a symmetric one, we have to verify that the existence theorem proved in [7] still holds. Fortunately, it does! Consequently there exists a unique solution $\tilde{U}'$ of (37'') belonging to a small ball B_η^s of $H_s(\mathbb{R}^2)$, provided ε is sufficiently small. Taking now into account the following

Sobolev imbedding theorem [1]. *Let Ω be a domain in $\mathbb{R}^2$ having the strong local Lipschitz property. Then $H_s(\Omega)$ can be imbedded continuously in $C^{s-2,\lambda}(\bar{\Omega})$, $0 < \lambda < 1$,*

the local existence of a $C^{s-2,\lambda}$-Weingarten tensor field Γ then follows.

Remark 2. *For the proof of (32) we need the positiveness of $\partial_{v^1}G$. For the sake of completeness we note that this condition can always be satisfied. Indeed, otherwise we introduce a new Riemannian metric $\tilde{g}$ on M by setting*

$$\tilde{g} = e^{2\lambda}g \ , \ \lambda \in \Sigma(M) \ . \tag{53}$$

The corresponding Gaussian curvature $\tilde{K}$ is then given by

$$e^{2\lambda}\tilde{K} = K + \Delta_g\lambda \ , \tag{54}$$

where Δ_g denotes the Laplace-Beltrami operator with respect to the metric g. For an appropriate choice of λ the positiveness of $\partial_{v^1}G$ then follows.

Final observations. *1) Since the vanishing of $\nabla K(x_0)$ does not play an essential role in the above proof (in the case where $K(x_0) = 0$, $\nabla K(x_0) \neq 0$ we have to replace ε^{-2} by ε^{-1} in the first relation (36)), the result we obtain improves also Lin's result in* [9].

2) It is easy to see also that the argument we used applies with minor changes in the case where x_0 is an elliptic point.

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