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New Integrals for $\zeta(s)\zeta(s+1)$

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0. Introduction

One of the most powerful tools for the study of Dirichlet series are suitable integrals representing them. In many cases such integrals allow the deduction of analytic (e.g. functional equation) or asymptotic properties of the corresponding series. In the case of the Riemann zeta function, which naturally has received the most attention, it is the celebrated *Riemann-Siegel integral formula* serving both purposes. Riemann discovered this particular formula in the middle of the 19th century, but did not publish it. Afterwards it fell into oblivion until it was rediscovered in 1926 by Bessel-Hagen in Riemann's *Nachlaß* and published a few years later by Carl Ludwig Siegel ([3]). The formula reads ($s = \sigma + it$ as usual)

$$\zeta(s) = \int_{0\searrow 1} \frac{x^{-s}e^{-\pi i x^2}}{e^{\pi i x} - e^{-\pi i x}} dx + \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \int_{0\swarrow 1} \frac{x^{s-1}e^{\pi i x^2}}{e^{\pi i x} - e^{-\pi i x}} dx. \tag{0}$$

Here $0 \searrow 1$ (and similarly $0 \swarrow 1$) denotes a straight line from $-\infty e^{-\pi i/4}$ to $\infty e^{-\pi i/4}$ cutting the real axis between 0 and 1. From this representation the functional equation of the zeta function follows immediately. Moreover, employing the saddle point method, the integrals can be evaluated very precisely for $t \to \infty$, thus giving the asymptotic expansion of $\zeta(s)$ (*Riemann-Siegel formula*).

In the present paper we shall derive a similar integral representation for the function $\zeta(s)\zeta(s+1)$ as well as a much more general formula. Apart from showing the possibility of extending the Riemann-Siegel integral formula to other Dirichlet series besides $\zeta(s)$, the results can be used to derive approximate functional equations for $\zeta(s)\zeta(s+1)$ and $|\zeta(1+it)|^2$ as indicated in the last section.

1. Basic formulas

If we let $R(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, the function R satisfies $R(s) = R(1-s)$, which is nothing but the well known functional equation for $\zeta(s)$. Moreover, R is meromorphic, having only simple poles at $s = 0$, $s = 1$, with residues being equal to -1 and 1, respectively. We further define

$$f(s) = \tfrac{1}{2} R(s) R(s+1). \tag{1}$$

From $R(s) = R(1-s)$ we get the functional equation $f(s) = f(-s)$. Also, f is holomorphic in the complex plane, except for simple poles at $s = -1, 1$ and a double pole at $s = 0$. Using the definition of R and the duplication formula for the gamma function, we obtain

$$f(s) = (2\pi)^{-s} \Gamma(s) \zeta(s) \zeta(s+1). \tag{2}$$

Inserting the Dirichlet series

$$\zeta(s)\zeta(s+1) = \sum_{n=1}^{\infty} \sigma_{-1}(n) n^{-s}, \quad \sigma_{-1}(n) = \sum_{d|n} d^{-1}, \tag{3}$$

which converges absolutely for $\sigma = \mathrm{Re}(s) > 1$, and using the gamma integral, yields the basic formula

$$f(s) = \int_0^{\infty} \psi(x) x^{s-1} \mathrm{d}x, \quad \sigma > 1, \tag{4}$$

where

$$\psi(x) = \sum_{n=1}^{\infty} \sigma_{-1}(n) e^{-2\pi n x}, \quad \mathrm{Re}(x) > 0. \tag{5}$$

Obviously, ψ is holomorphic in the half plane $\mathrm{Re}(x) > 0$. This function is well known, since it essentially equals the logarithm of the Dedekind eta function, viz.

$$\log \eta(ix) = -\frac{\pi x}{12} - \psi(x).$$

Some important properties can be deduced by inverting the Mellin transform (4). This gives

$$\psi(x) = \frac{1}{2\pi i} \int_{(c)} f(s) x^{-s} \mathrm{d}s, \quad c > 1.$$

For example, using the functional equation $f(s) = f(-s)$, substituting $-s = w$ in the integral, shifting the contour to the right and evaluating the residues, leads to the formula

$$\psi(x) = \frac{\pi}{12} x^{-1} - \frac{\pi}{12} x + \frac{1}{2} \log x + \psi\left(\frac{1}{x}\right). \tag{6}$$

Here, as usual, $\mathrm{Re}(x) > 0$ and $\log x$ takes its principal value, i.e. $|\arg(x)| < \frac{\pi}{2}$.

Besides this well known function ψ, we need a new one, ψ_1, which we define through

$$\psi_1(x) = \sum_{n=1}^{\infty} \sigma_{-1}(n) n^{-1} e^{-2\pi n x}, \quad \mathrm{Re}(x) > 0. \tag{7}$$

Its chief properties are summarized in

Theorem 1: *For complex x with positive real part let $\psi_1(x)$ be defined by (7). Then the function ψ_1 is holomorphic in the right half plane* $\mathrm{Re}(x) > 0$ *and $\psi_1' = -2\pi\psi$. Moreover,*

$$\psi_1(x) = 2\pi \int_x^\infty \psi(u)\mathrm{d}u,$$

where the path of integration may be any rectifiable curve extending to infinity and lying in a sector $|\arg(u)| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$ ($\delta > 0$ fixed). Finally, ψ_1 satisfies the functional equation

$$\psi_1(x) = A\log x + B + Cx\log x + Dx + Ex^2 - 2\pi x \sum_{n=1}^{\infty} \sigma_{-1}(n) E_2\left(\frac{2\pi n}{x}\right). \tag{8}$$

Here $A, \ldots, E$ are suitable complex numbers not depending on x, and the function E_2 (generalized exponential integral) is given by

$$E_2(z) = \int_1^\infty e^{-zt} t^{-2} \mathrm{d}t, \quad \mathrm{Re}(z) \ge 0.$$

Proof: All statements are easy deductions from the definitions (5) and (7) of ψ and ψ_1, except the last two concerning the functional equation. To show these, one may proceed as in the case of $\psi(x)$, using the Mellin transform

$$\int_0^\infty \psi_1(x) x^{s-1} \mathrm{d}x = (2\pi)^{-s}\Gamma(s)\zeta(s+1)\zeta(s+2) = \frac{2\pi}{s} f(s+1),$$

and its reciprocal

$$\frac{1}{2\pi}\psi_1(x) = \frac{1}{2\pi i}\int_{(c)} f(s+1)\frac{x^{-s}}{s}\mathrm{d}s, \quad c > 0.$$

Applying the functional equation for f, shifting the contour appropriately, taking into account the double poles at $s = -1, 0$, as well as the simple one at $s = 1$, leads to

$$\psi_1(x) = A\log x + B + Cx\log x + Dx + Ex^2 - 2\pi x \frac{1}{2\pi i}\int_{(c)} f(s)\frac{x^s}{s+1}\mathrm{d}s, \; c > 1.$$

Inserting (2), (3) and noting

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s)\frac{z^s}{s+1}\mathrm{d}s = E_2\left(\frac{1}{z}\right),$$

proves the assertion. Another, and possibly more direct, way to obtain the modular relation (8) consists in using the corresponding equation (6) for ψ and integrating. For,

$$\psi_1(x) = 2\pi\int_x^\infty \psi(u)\mathrm{d}u = 2\pi\int_x^1 \psi(u)\mathrm{d}u + \psi_1(1).$$

If $\mathrm{Re}(x) > 0$, we get from (6)

$$\int_x^1 \psi(u)\mathrm{d}u = \int_x^1 \left[\frac{\pi}{12}u^{-1} - \frac{\pi}{12}u + \frac{1}{2}\log u + \psi\left(\frac{1}{u}\right)\right] \mathrm{d}u$$
$$= -\frac{\pi}{12}\log x - \frac{\pi}{24}(1-x^2) + \frac{1}{2}(-1 - x\log x + x) + \int_1^{x^{-1}} \psi(u)u^{-2}\mathrm{d}u.$$

Write the last integral as

$$\int_1^\infty \psi(u)u^{-2}\mathrm{d}u - \int_{x^{-1}}^\infty \psi(u)u^{-2}\mathrm{d}u = \int_1^\infty \psi(u)u^{-2}\mathrm{d}u - x\int_1^\infty \psi\left(\frac{u}{x}\right)u^{-2}\mathrm{d}u,$$

where

$$\int_1^\infty \psi\left(\frac{u}{x}\right)u^{-2}\mathrm{d}u = \sum_{n=1}^\infty \sigma_{-1}(n)E_2\left(\frac{2\pi n}{x}\right).$$

This concludes the derivation of (8) and gives the values

$$A = -\frac{\pi^2}{6},\ C = -\pi,\ D = \pi,\ E = \frac{\pi^2}{12},$$

and

$$B = \psi_1(1) + 2\pi\int_1^\infty \psi(u)u^{-2}\mathrm{d}u - \frac{\pi^2}{12} - \pi.$$

Hence, Theorem 1 is completely proved.

Note that the infinite series involving E_2 is absolutely convergent. This follows from

$$E_2(z) = O(e^{-z}), \quad z \to \infty,\ |\arg(z)| < \pi.$$

2. Integrals representing $f(s)$ and $\zeta(s)\zeta(s+1)$

On the basis of the preceding investigations, we are now going to derive a new formula for $f(s)$, as given by the following result.

Theorem 2: *For $s \in \mathbf{C}$, except $s = \pm 1$ and $s = 0$*

$$f(s) = \frac{s-1}{2\pi}\int_0^\infty \psi_1(x)(x+i)^{s-2}\mathrm{d}x - \frac{s+1}{2\pi}\int_0^\infty \psi_1(x)(x-i)^{-s-2}\mathrm{d}x + H(s),$$

where H is defined by

$$H(s) = -\frac{\pi i}{12}e^{\frac{\pi i s}{2}}\left(\frac{1}{s-1} + \frac{1}{s+1}\right) + \frac{e^{\frac{\pi i s}{2}}}{2s}\left(\frac{\pi i}{2} - \frac{1}{s}\right).$$

Proof: Let w be a complex number having positive real part. If $\arg(w) = \varphi$ with $|\varphi| < \frac{\pi}{2}$, we may turn the line of integration in (4) around the origin to get

$$f(s) = \int_0^{\infty e^{i\varphi}} \psi(x) x^{s-1} \mathrm{d}x,$$

provided $\sigma > 1$, as we shall assume temporarily. Splitting the integral at $x = w$ and applying the functional equation (6) to the finite part (i.e. from 0 to w), we thus obtain

$$f(s) = \int_w^{\infty e^{i\varphi}} \psi(x) x^{s-1} \mathrm{d}x + \int_{w^{-1}}^{\infty e^{-i\varphi}} \psi(x) x^{-s-1} \mathrm{d}x + H(s,w), \tag{9}$$

with

$$H(s,w) = \frac{\pi}{12}\left(\frac{w^{s-1}}{s-1} - \frac{w^{s+1}}{s+1}\right) + \frac{w^s}{2s}\left(\log w - \frac{1}{s}\right).$$

As usual, $\log w$ denotes the principal branch of the logarithm, where $|\operatorname{Im}(\log w)| < \frac{\pi}{2}$. Integrating by parts in (9) yields by Theorem 1

$$f(s) = \frac{1}{2\pi}\psi_1(w) w^{s-1} + \frac{s-1}{2\pi}\int_w^{\infty e^{i\varphi}} \psi_1(x) x^{s-2} \mathrm{d}x +$$
$$+ \frac{1}{2\pi}\psi_1(w^{-1}) w^{s+1} - \frac{s+1}{2\pi}\int_{w^{-1}}^{\infty e^{-i\varphi}} \psi_1(x) x^{-s-2} \mathrm{d}x + H(s,w).$$

Since $\psi_1(x)$ decays exponentially as $\operatorname{Re}(x) \to \infty$, the paths of integration can again be rotated, so as to run in a direction parallel to the positive real axis. Thus

$$f(s) = \frac{w^s}{2\pi}\left[\psi_1(w) w^{-1} + \psi_1(w^{-1}) w\right] + \frac{s-1}{2\pi}\int_w^{w+\infty} \psi_1(x) x^{s-2} \mathrm{d}x -$$
$$- \frac{s+1}{2\pi}\int_{w^{-1}}^{w^{-1}+\infty} \psi_1(x) x^{-s-2} \mathrm{d}x + H(s,w). \tag{10}$$

Up to this point w was aribitrary, subject only to the condition $\operatorname{Re}(w) > 0$. Now we specialize to $w = e^{i\varphi}$, where $\varphi = \frac{\pi}{2} - \delta$ and $0 < \delta < \frac{\pi}{2}$. We are going to show that we may let δ tend to 0, i.e. $w \to i$ in (10). This is, of course, obvious for $H(s,w)$. To discuss the integrals, observe that

$$w = ie^{-i\delta} = \sin\delta + i\cos\delta = i + \delta + O(\delta^2),$$
$$w^{-1} = -ie^{i\delta} = \sin\delta - i\cos\delta = -i + \delta + O(\delta^2),$$

for $\delta \to 0$. Hence, setting $w = i+u$ and $w^{-1} = -i+u'$, we find $u = \delta + O(\delta^2)$, $u' = \delta + O(\delta^2)$. Moreover, $\operatorname{Re}(u) > 0$, $\operatorname{Re}(u') > 0$ and $|\arg(u)|, |\arg(u')| \le \frac{\pi}{4}$ (say), provided δ is small enough. Applying this to the first integral in (10) yields

$$\int_w^{w+\infty} \psi_1(x) x^{s-2} \mathrm{d}x = \int_u^{u+\infty} \psi_1(x+i)(x+i)^{s-2} \mathrm{d}x = \int_u^{u+\infty} \psi_1(x)(x+i)^{s-2} \mathrm{d}x.$$

But Theorem 1 shows that $\psi_1(x) = O(\log x)$ for x tending to 0 in a sector $|\arg(x)| \leq \frac{\pi}{4}$. Consequently, the last integral converges as $u \to 0$, i.e. $\delta \to 0$, $w \to i$. The same argument applies to

$$\int_{w^{-1}}^{w^{-1}+\infty} \psi_1(x) x^{-s-2} \mathrm{d}x = \int_{u'}^{u'+\infty} \psi_1(x)(x-i)^{-s-2} \mathrm{d}x.$$

It remains to discuss the bracketed term in (10). With the same notation as before, namely $w = i + u$, $w^{-1} = -i + u'$, we have using Theorem 1

$$\begin{aligned}\psi_1(w)w^{-1} + \psi_1(w^{-1})w &= \psi_1(u)w^{-1} + \psi_1(u')w \\ &= \big[A\log u + B + O(u\log u)\big]w^{-1} + \big[A\log u' + B + O(u'\log u')\big]w \\ &= (A\log\delta + B)(w^{-1} + w) + O(\delta\log\delta) = O(\delta\log\delta),\end{aligned}$$

since $w^{-1} + w = u + u' = O(\delta)$. Thus, as $\delta \to 0$ (equivalently $w \to i$), the bracketed term in (10) vanishes, giving

$$f(s) = \frac{s-1}{2\pi}\int_{i}^{i+\infty} \psi_1(x) x^{s-2} \mathrm{d}x - \frac{s+1}{2\pi}\int_{-i}^{-i+\infty} \psi_1(x) x^{-s-2} \mathrm{d}x + H(s,i).$$

The restriction $\sigma > 1$ made at the beginning of the proof can now be dropped by analytic continuation, excluding only the poles of $H(s,i)$. Substitution of the variable of integration and defining $H(s) = H(s,i)$ finally concludes the proof of Theorem 2.

The formula given by this result seems to be new. Its validity depends essentially on the properties of ψ_1, which permitted the limiting value $w = i$ in (10). This has some interesting consequences, viz. the analogue of the Riemann-Siegel integral formula for $\zeta(s)\zeta(s+1)$, as we are now going to show. As a preliminary step, we divde both sides of the formula of Theorem 2 by $(2\pi)^{-s}\Gamma(s)$ to get

Theorem 3: *For any $s \in \mathbf{C}$ except $s = 0, 1, 2, \ldots$*

$$\zeta(s)\zeta(s+1) = T(s) + X(s)\overline{T(-\overline{s})} + (2\pi)^s\Gamma(s)^{-1}H(s),$$

where

$$T(s) = (2\pi)^{s-1}\Gamma(s-1)^{-1}\int_0^\infty \psi_1(x)(x+i)^{s-2}\mathrm{d}x, \quad X(s) = (2\pi)^{2s}\frac{\Gamma(-s)}{\Gamma(s)},$$

and $H(s)$ is defined as in Theorem 2.

Proof: This is obvious from Theorem 2 and formula (2). The points $s = 0, 1, 2, \ldots$ excluded are poles of $X(s)$ and $H(s)$.

To proceed further, we are going to transform $T(s)$ into a loop integral around the positive imaginary axis. We require some preliminary considerations.

With ψ_1 as in (7) we define a new function F by

$$F(z) = 2\pi \int_0^\infty e^{2\pi xz}\psi_1(x)\mathrm{d}x, \quad \mathrm{Re}(z) < 1. \tag{11}$$

It follows from (7) that $\psi_1(x) = O(e^{-2\pi x})$ as $x \to \infty$. Moreover $\psi_1(x) = O(\log x)$ as x tends to 0 by Theorem 1. Consequently, the above integral converges absolutely and uniformly provided $\mathrm{Re}(z) \le 1-\varepsilon < 1$. This shows that F is a holomorphic function in the left half plane $\mathrm{Re}(z) < 1$. Furthermore we have

$$|F(z)| \le 2\pi \int_0^\infty e^{2\pi x\,\mathrm{Re}(z)}\psi_1(x)\mathrm{d}x = O\big((1-\mathrm{Re}(z))^{-1}\big)$$

if $\mathrm{Re}(z) < 1$ and this implies that F is uniformly bounded in any half plane $\mathrm{Re}(z) \le 1-\varepsilon$ if $\varepsilon > 0$ is fixed. Now assume $\sigma < 2$ and consider

$$I(s) = \int_0^\infty e^{-2\pi u}F(iu)u^{1-s}\mathrm{d}u,$$

which is absolutely and uniformly convergent in any strip $\sigma_0 \le \sigma \le \sigma_1 < 2$. Note that $F(iu) = O(1)$ as stated above. Using the definition of F, we get

$$\begin{aligned} I(s) &= 2\pi \int_0^\infty e^{-2\pi u}u^{1-s} \int_0^\infty e^{2\pi xiu}\psi_1(x)\mathrm{d}x\mathrm{d}u \\ &= 2\pi \int_0^\infty \psi_1(x) \int_0^\infty e^{-u(2\pi-2\pi ix)}u^{1-s}\mathrm{d}u\mathrm{d}x, \end{aligned}$$

where the interchange of the order of integration is permitted by absolute convergence. The inner integral takes the value $(2\pi)^{s-2}(1-ix)^{s-2}\Gamma(2-s)$, hence

$$e^{\frac{\pi i}{2}(s-2)}I(s) = (2\pi)^{s-1}\Gamma(2-s)\int_0^\infty \psi_1(x)(x+i)^{s-2}\mathrm{d}x. \tag{12}$$

Next we consider

$$J(s) = \frac{1}{2\pi i}\int_\Lambda e^{2\pi iz}z^{1-s}F(z)\mathrm{d}z, \tag{13}$$

where Λ is a loop around the positive imaginary axis. To be more precise, we cut the z-plane from 0 to $i\infty$ and define $z^{-s} = e^{-s\log z}$ with $\log z = \log|z| + i\arg(z)$ and $\arg(z)$ taking its principal value. Then Λ is defined by letting z run from $\infty e^{-\frac{3\pi i}{2}}$ to $\varepsilon e^{-\frac{3\pi i}{2}}$ (where $0 < \varepsilon < 1$ is fixed), then along the circle $z = \varepsilon e^{i\varphi}$, $-\frac{3\pi}{2} \le \varphi \le \frac{\pi}{2}$, and finally returning form $\varepsilon e^{\frac{\pi i}{2}}$ to $\infty e^{\frac{\pi i}{2}}$. Using the properties of F, we conclude that J is an entire function of s. It is now easy to relate $J(s)$ to $T(s)$. Let us assume that $\sigma < 2$. Then we may let tend ε to 0 in our parametrization of Λ and thus we obtain

$$\begin{aligned} J(s) &= \frac{1}{2\pi i}\left(e^{-\frac{3\pi i}{2}(2-s)}\int_\infty^0 e^{-2\pi u}u^{1-s}F(iu)\mathrm{d}u + e^{\frac{\pi i}{2}(2-s)}\int_0^\infty e^{-2\pi u}u^{1-s}F(iu)\mathrm{d}u\right) \\ &= \frac{1}{2\pi i}e^{\frac{\pi i}{2}(s-2)}I(s)(e^{-\pi is} - e^{\pi is}) \\ &= -\frac{1}{\pi}(2\pi)^{s-1}\sin\pi s\Gamma(2-s)\int_0^\infty \psi_1(x)(x+i)^{s-2}\mathrm{d}x \end{aligned}$$

by (12). Since $\Gamma(s-1)\Gamma(2-s) = -\pi/\sin \pi s$, we get

$$J(s) = (2\pi)^{s-1}\Gamma(s-1)^{-1}\int_0^\infty \psi_1(x)(x+i)^{s-2}\mathrm{d}x = T(s).$$

The restriction of σ to values less than 2 can now be removed by analytic continuation. We have thus proved

Theorem 4: *If Λ denotes a loop around the positive imaginary axis as described above, then*

$$T(s) = \frac{1}{2\pi i}\int_\Lambda e^{2\pi i z} z^{1-s} F(z)\mathrm{d}z$$

for any complex s. The function F is defined by (11).

3. Generalizations of the integral formula

We shall now derive much more general formulas than those given in Theorems 2 to 4. The method will be the same, but some further preparations are necessary. If z is complex and $|z| < 1$ let

$$g(z) = -\log \prod_{m=1}^{\infty}(1 - z^m) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n}z^{mn}.$$

This function is related to $\psi(x)$ and to the logarithm of the Dedekind eta function and has been studied in a classical work by Rademacher [2]. In our notation $g(e^{-2\pi x}) = \psi(x)$ if $\mathrm{Re}(x) > 0$. Let h, k be integers such that $k > 0$, $(h,k) = 1$, and denote by h' any solution of the congruence $hh' \equiv -1 \bmod k$. Then, as Rademacher has shown ([2], p. 317, (1.45))

$$g\left(e^{-\frac{2\pi x}{k}+\frac{2\pi i h}{k}}\right) = g\left(e^{-\frac{2\pi}{xk}+\frac{2\pi i h'}{k}}\right) - \frac{\pi}{12k}(x - x^{-1}) + \frac{1}{2}\log x - \pi i S(h',k). \tag{14}$$

In this formula S denotes the well-studied Dedekind sum

$$S(h',k) = \sum_{\mu=1}^{k}\frac{\mu}{k}\left(\frac{h'\mu}{k} - \left[\frac{h'\mu}{k}\right] - \frac{1}{2}\right),$$

with $[t]$ being the greatest integer not exceeding t. To employ our function ψ, we replace x by $\overline{x}$ in (14) and take complex conjugates on both sides. The result is

$$\psi\left(\frac{x}{k} + i\frac{h}{k}\right) = \psi\left(\frac{1}{xk} + i\frac{h'}{k}\right) - \frac{\pi}{12k}(x - x^{-1}) + \frac{1}{2}\log x + \pi i S(h',k). \tag{15}$$

The formula corresponding to Theorem 1 is given by

Theorem 5: *Let h, k be integers, $k > 0$, $(h,k) = 1$ and let h' be a solution of $hh' \equiv -1 \bmod k$. Then, if* $\mathrm{Re}(x) > 0$

$$\psi_1\left(\frac{x}{k} + i\frac{h}{k}\right) = A_k \log x + B_{hk} + C_k x \log x + D_{hk} x + E_k x^2 - \\ - \frac{2\pi x}{k} \sum_{n=1}^{\infty} \sigma_{-1}(n) e^{-\frac{2\pi i n h'}{k}} E_2\left(\frac{2\pi n}{xk}\right).$$

Here A_k, B_{hk}, C_k, D_{hk}, E_k are complex numbers not depending on x. In fact

$$A_k = -\frac{\pi^2}{6k^2},\ C_k = -\frac{\pi}{k},\ D_{hk} = \frac{\pi}{k} - \frac{2\pi^2 i}{k} S(h',k),\ E_k = \frac{\pi^2}{12k^2},$$

and

$$B_{hk} = -\frac{\pi}{k} - \frac{\pi^2}{12k^2} + \frac{2\pi^2 i}{k} S(h',k) + \frac{2\pi}{k} \int_1^{\infty} \psi\left(\frac{u}{k} + i\frac{h'}{k}\right) u^{-2} du + \psi_1\left(\frac{1}{k} + i\frac{h}{k}\right).$$

Proof: Using the definition of ψ_1 we have

$$\psi_1\left(\frac{x}{k} + i\frac{h}{k}\right) = 2\pi \int_{\frac{x}{k} + i\frac{h}{k}}^{\infty} \psi(u) du.$$

The integral equals

$$\begin{aligned}\psi_1\left(\frac{x}{k} + i\frac{h}{k}\right) &= 2\pi \int_{\frac{x}{k} + i\frac{h}{k}}^{\frac{1}{k} + i\frac{h}{k}} \psi(u) du + \psi_1\left(\frac{1}{k} + i\frac{h}{k}\right) \\ &= \frac{2\pi}{k} \int_x^1 \psi\left(\frac{u}{k} + i\frac{h}{k}\right) du + \psi_1\left(\frac{1}{k} + i\frac{h}{k}\right).\end{aligned}$$

As in the proof of Theorem 1 the assertion follows from (15) by integration, q.e.d.

The analogue of Theorem 2 is provided by

Theorem 6: *Let h, k be integers, $k > 0$, $(h,k) = 1$. Then*

$$\begin{aligned} f(s) = \frac{s-1}{2\pi} &\int_0^{\infty} \psi_1\left(x + i\frac{h}{k}\right)\left(x + i\frac{h}{k}\right)^{s-2} dx - \\ &- \frac{s+1}{2\pi} \int_0^{\infty} \psi_1\left(x - i\frac{h}{k}\right)\left(x - i\frac{h}{k}\right)^{-s-2} dx \\ &+ e^{\frac{\pi i s}{2}} \left(\frac{h}{2\pi k}\right) iD(h,k) + H\left(s, i\frac{h}{k}\right),\end{aligned}$$

where

$$D(h,k) = \frac{\pi^2}{6hk}\log\frac{h}{k} - \frac{k}{h}B_{hk} + \frac{h}{k}B_{kh},$$

and B_{hk} is defined as in Theorem 5. Moreover, $H(s,w)$ is given by (9).

Proof: We start from equation (10) which was proved for any complex w with $Re(w) > 0$ and for $\sigma > 1$. The last condition can be dropped, avoiding only the singularities of $H(s,w)$. Let δ be real, $0 < \delta \leq \frac{\pi}{2}$ and set $w = i\frac{h}{k}e^{-i\delta}$. We shall show that $\delta \to 0$, i.e. $w \to i\frac{h}{k}$ is permissible in (10). We write $w = \frac{h}{k}(i+u)$, $w^{-1} = \frac{k}{h}(-i+u')$. Hence $\mathrm{Re}(u) > 0$, $\mathrm{Re}(u') > 0$ and $u = \delta + O(\delta^2)$, $u' = \delta + O(\delta^2)$ as $\delta \to 0$. The first integral in (10) is equal to

$$\int_{\frac{h}{k}u}^{\frac{h}{k}u+\infty} \psi_1\left(x + i\frac{h}{k}\right)\left(x + i\frac{h}{k}\right)^{s-2} \mathrm{d}x,$$

which converges absolutely for $u \to 0$ since $\psi_1(x + i\frac{h}{k})$ grows only logarithmically at $x = 0$ by the previous theorem. The other integral is treated similarly. Thus it remains to show that

$$\lim_{\delta\to 0}\left[\psi_1(w)w^{-1} + \psi_1(w^{-1})w\right]$$

exists. From the definition of w, u, u' and Theorem 5 we obtain

$$\begin{aligned}\psi_1(w)w^{-1} + \psi_1(w)w &= \psi_1\left(u\frac{h}{k} + i\frac{h}{k}\right)w^{-1} + \psi_1\left(u'\frac{h}{k} + i\frac{h}{k}\right)w\\ &= -\frac{\pi^2}{6}\left[w^{-1}k^{-2}\log(uh) + wh^{-2}\log(u'k)\right] +\\ &\quad + B_{hk}w^{-1} + B_{kh}w + O(\delta\log\delta)\end{aligned}$$

as δ tends to 0. Further

$$\begin{aligned}w^{-1}k^{-2}\log(uh) + wh^{-2}\log(u'k) &= \frac{\log(uh)}{hk}(-i+u') + \frac{\log(u'k)}{hk}(i+u)\\ &= \frac{1}{hk}\left[i\log u' - i\log u + i\log\frac{k}{h} + O(\delta\log\delta)\right].\end{aligned}$$

This expression has the limit $i(hk)^{-1}\log\frac{k}{h}$ for $\delta \to 0$. Consequently, we may let tend $w \to i\frac{h}{k}$ in (10) to get

$$\begin{aligned}f(s) = \frac{w^s}{2\pi}&\left[-\frac{\pi^2 i}{6hk}\log\frac{k}{h} + B_{hk}w^{-1} + B_{kh}w\right] +\\ &+ \frac{s-1}{2\pi}\int_0^\infty \psi_1(x+w)(x+w)^{s-2}\mathrm{d}x -\\ &- \frac{s+1}{2\pi}\int_0^\infty \psi_(x+w^{-1})(x+w^{-1})^{-s-2}\mathrm{d}x + H(s,w),\end{aligned}$$

form which the assertion follows immediately.

The next two results give generalizations of Theorems 3 and 4. As the proofs do not involve any new idea, we omit them.

Theorem 7: *Let h, k be positive integers, $(h,k) = 1$. Then*

$$\zeta(s)\zeta(s+1) = T\left(s, \frac{h}{k}\right) + \overline{T\left(-\overline{s}, \frac{k}{h}\right)}X(s) + \\ + i(2\pi)^{s-1}\frac{e^{\frac{\pi i s}{2}}}{\Gamma(s)}\left(\frac{h}{k}\right)^s D(h,k) + (2\pi)^s\Gamma(s)^{-1}H\left(s, i\frac{h}{k}\right),$$

with the function T defined by

$$T\left(s, \frac{h}{k}\right) = (2\pi)^{s-1}\Gamma(s-1)^{-1}\int_0^\infty \psi_1\left(x + i\frac{h}{k}\right)\left(x + i\frac{h}{k}\right)^{s-2} \mathrm{d}x,$$

$X(s)$, $D(h,k)$ as in Theorems 3 and 6, respectively, and finally

$$H\left(s, i\frac{h}{k}\right) = -\frac{\pi i}{12}e^{\frac{\pi i s}{2}}\left(\frac{h}{k}\right)^{s-1}\left(\frac{1}{s-1} + \frac{h^2k^{-2}}{s+1}\right) + \frac{e^{\frac{\pi i s}{2}}}{2s}\left(\frac{h}{k}\right)^s\left(\frac{\pi i}{2} + \log\frac{h}{k} - \frac{1}{s}\right).$$

Theorem 8: *The function T from Theorem 7 admits the integral representation*

$$T\left(s, \frac{h}{k}\right) = \frac{1}{2\pi i}\int_\Lambda e^{2\pi i\frac{h}{k}z}z^{1-s}F\left(z, \frac{h}{k}\right)\mathrm{d}z,$$

where

$$F\left(z, \frac{h}{k}\right) = 2\pi\int_0^\infty e^{2\pi x z}\psi_1\left(x + i\frac{h}{k}\right)\mathrm{d}x.$$

Moreover, F is holomorphic and uniformly bounded in every half plane $\mathrm{Re}(z) \le 1 - \varepsilon$ *(where $\varepsilon > 0$ is fixed).*

4. Some remarks on the integral formula

In this concluding section we indicate how to apply the formulas to obtain some kind of approximate functional equation for $\zeta(s)\zeta(s+1)$. We shall return to this matter on a later occasion ([1]), so we restrict ourselves to a few short remarks.

Theorem 3 shows that the study of $\zeta(s)\zeta(s+1)$ is reduced to that of $T(s)$, since $X(s)$ and $H(s)$ may be considered as elementary functions. Together with the formula of Theorem 6 we have an analogue of the Riemann-Siegel formula (0). Thus one might expect to derive also an asymptotic expansion for $\zeta(s)\zeta(s+1)$ if $|t| \to \infty$, i.e. an approximate functional equation or, even better, a complete asymptotic series like the Riemann-Siegel formula.

It is plain from Theorem 6 that a closer examination of the integral

$$T(s) = \frac{1}{2\pi i} \int_\Lambda e^{2\pi i z} z^{1-s} F(z) \mathrm{d}z, \tag{16}$$

requires a detailed study of $F(z)$. From its definition (11) we get, after insertion of the series (7), by integration

$$F(z) = \sum_{n=1}^{\infty} \sigma_{-1}(n) n^{-1} \frac{1}{n-z}.$$

This gives the analytic continuation of F over the whole complex plane and shows that F is holomorphic there, except for simple poles at $z = n$ with the residue being equal to $-\sigma_1 n^{-1}$. Now assume $t \geq 1$ and let $\eta = \frac{t}{2\pi}$ be the saddle point of the function $e^{2\pi i z - it \log z}$ in (16). Deforming the path of integration suitably, we may show

$$T(s) = \sum_{n \leq \eta} \sigma_1(n) n^{-s} + \frac{1}{2\pi i} \int_{\eta\nearrow} e^{2\pi i z - it \log z} z^{1-\sigma} F(z) \mathrm{d}z + O(e^{-ct}),$$

for some constant $c > 0$. Here $\int_{\eta\nearrow}$ denotes integration along the line $z = \eta + re^{\frac{\pi i}{4}}$, $-\infty \leq r \leq \infty$ (assuming temporarily that η is not an integer). The integral can be estimated to be of order $O(t^{1-\sigma} \log t)$, hence

$$T(s) = \sum_{n \leq \eta} \sigma_1(n) n^{-s} + O(t^{1-\sigma} \log t). \tag{17}$$

In conjunction with Theorem 3 we get the approximate functional equation for $\zeta(s)\zeta(s+1)$.

The most interesting case of the function $\zeta(s)\zeta(s+1)$ certainly occurs for $\sigma = 0$. For, the functional equation of the zeta function yields

$$|\zeta(1+it)|^2 = \zeta(1+it)\zeta(1-it) = 2(2\pi)^{it} \cosh \tfrac{\pi t}{2} \Gamma(it)\zeta(it)\zeta(1+it).$$

We then get from Theorem 3

$$|\zeta(1+it)|^2 = 2\cosh \tfrac{\pi t}{2} \left[(2\pi)^{-it}\Gamma(it)T(it) + (2\pi)^{it}\Gamma(-it)\overline{T(it)}\right] + 2\cosh \tfrac{\pi t}{2} H(it),$$

where

$$H(it) = e^{-\frac{\pi t}{2}} \left(\frac{\pi}{4t} + \frac{1}{2t^2} - \frac{\pi t}{6(t^2+1)} \right).$$

This formula can be used to give an approximate functional equation for $|\zeta(1+it)|^2$. In this case, however, a more refined asymptotic expansion than (17) is necessary to yield nontrivial results.

More general versions of the above formulas can be derived from Theorem 8. Here the saddle point occurs at $z = \frac{t}{2\pi}\frac{k}{h}$. Proceeding as before, we get an unsymmetric form of the approximate functional equation in complete analogy with the situation for $\zeta(s)$ itself.

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