UNIVERSITÄT KAISERSLAUTERN

A SIMPLE INTEGRAL REPRESENTATION FOR THE SECOND MOMENT OF ADDITIVE RANDOM VARIABLES ON STOCHASTIC POLYHEDRA

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FACHBEREICH MATHEMATIK

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A Simple Integral Representation for the Second Moments of Additive Random Variables on Stochastic Polyhedra

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Summary:

Let a_i, $i := 1, \ldots, m$, be an i.i.d. sequence taking values in $\mathbb{R}^n$, whose convex hull is interpreted as a stochastic polyhedron $\mathcal{P}$. For a special class of random variables, which decompose additively relative to their boundary simplices, eg. the volume of $\mathcal{P}$, simple integral representations of its first two moments are given in case of rotationally symmetric distributions in order to facilitate estimations of variances or to quantify large deviations from the mean.

1. Introduction

In their pioneering articles "Über die konvexe Hülle von n zufällig gewählten Punkten I-II"[7,8], Renyi and Sulanke analyzed the expectation values of the number of edges, the area and the circumference of random convex polytopes generated by the convex hull of i.i.d. vectors on $\mathbb{R}^2$ for the first time. Their ideas and their techniques while treating random variables on stochastic polyhedra led to a variety of generalizing publications in different fields of research like the stochastic approximation theory of bounded convex sets, cf. Gruber[3] or Buchta[2] for a survey, or the stochastic complexity theory of algorithms, cf. Borgwardt[1]. Most of these articles exclusively deal with expectation values and don't give information about the concentration of the random variable's value around the mean. In many cases experimental evidence suggests that the distribution of the random variable is highly concentrated. For that reason it is of interest to obtain higher moments, especially the variance, of the variable investigated in order to quantify large deviations from the mean.

The aim of this paper is to give integral representations for the first two moments of a class of random variables on stochastic bounded polyhedra on $\mathbb{R}^n$, $n \geq 2$, in order to facilitate estimations of variances.

To explain our stochastic model precisely, let $a_1, \ldots, a_m$, $m \geq n$, be an i.i.d. sequence taking values in $\mathbb{R}^n$ distributed by an arbitrary rotationally symmetric distribution not concentrated in 0 and let $F(t) := P(\|x\|_2 \leq t)$ be the associated radial distribution function (RDF). If there is a function $f \in \mathcal{L}^1(\mathbb{R}^n)$ such that $F(r) = \int\limits_{\|x\|_2 \leq r} f(x)\, dx$, $r > 0$, we call f the density function of the underlying distribution. All theorems of the present paper are formulated for distributions with density functions in order to avoid a too difficult notation. (This is no restriction of generality because the set of distributions with density

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function is pointwise dense in the class of all rotationally symmetric distributions and so the general result is an easy gained limiting case, cf. Natanson[5].) In the above described respect

$$(1.1) \qquad\qquad \mathcal{P} := convhull\{a_1, \ldots, a_m\}$$

is a randomly generated polyhedron on $\mathbb{R}^n$.

For an axiomatic definition of the random variables we are dealing with we need some more notation. For any set of indices $I := \{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$ with $i_j \neq i_k$ for $i \neq k$, let $A_I := (a_{i_1}, \ldots, a_{i_n})$ and $S_I := convhull\{a_i \mid i \in I\}$. Furthermore, if the vectors a_i, $i \in I$, are linearly independent, i.e. S_I is a simplex, $H(A_I) := affhull\{S_I\}$ denotes the hyperplane containing S_I and if additionally $0 \notin H(A_I)$, $H^{(1)}(A_I)$ denotes the closed halfspace generated by the hyperplane $H(A_I)$ containing the origin whereas $H^{(2)}(A_I)$ represents its closed complement. (The special case $0 \in H(A_I)$ doesn't matter because $P(0 \in H(A_I)) = 0$ for all distributions of our model.) $h(A_I)$ denotes the distance of $H(S_I)$ to the origin.

We call a simplex S_I, whose associated hyperplane $H(A_I)$ doesn't contain the origin, a boundary simplex of $\mathcal{P}$ of the first kind, if $a_j \in H^{(1)}(A_I)$ for $j \in \overline{I}$, and we call it a boundary simplex of $\mathcal{P}$ of the second kind, if $a_j \in H^{(2)}(A_I)$ for $j \in \overline{I}$, $\overline{I}$ being the complementary set to I as a subset of $\{1, \ldots, m\}$. We relate the last definitions to the characteristic functions

$$(1.2) \qquad \chi_i(A_I) := \begin{cases} 1 & \text{if } S_I \text{ is a boundary simplex of } \mathcal{P} \text{ of the } i\text{-th kind} \\ 0 & \text{else} \end{cases} .$$

Now we are able to state the definition of the class of random variables investigated:

Definition:

A random variable $\tilde{Z}$ of a randomly generated bounded polyhedron $\mathcal{P}$ is of additive type iff for almost all $\mathcal{P}$:

$$(1.3) \qquad \tilde{Z}(\mathcal{P}) = \sum_{\substack{I \subset \{1, \ldots, m\} \\ |I| = n}} [\chi_1(A_I) + \sigma \chi_2(A_I)] Z(A_I),$$

where $\sigma \in \mathbb{R}$ is a fixed constant and $Z : \mathbb{R}^{n \times n} \to \mathbb{R}$ is a continous function invariant under orthogonal transformations and under exchanging columns of the argument.

Roughly speaking, a random variable on a polyhedron is of additive type if it can be additively decomposed resp. its boundary simplices.

The examples treated by Renyi and Sulanke in [5,6] are easily recognized as special cases, by taking $\sigma = 1$ and $Z(A_I) = 1$ for the polyhedron's number of boundary simplices, taking $\sigma = -1$ and $Z(A_I) = |convhull(0, S_I)|$ for the volume and finally by taking $\sigma = 1$ and $Z(A_I) = |S_I|$ for the surface area of the polyhedron, $|\cdot|$ representing Lebesgue-measure of appropriate dimension.

Further examples of interest included are the weighted volume of $\mathcal{P}$, to which the number of vertices can be reduced, the characteristic function $\chi(cone(\mathcal{P}) = \mathbb{R}^n)$ or the average

number of (u, v) shadow vertices of $\mathcal{P}$, which is closely related to the analysis of average complexity of the simplex algorithm, cf. Borgwardt[1].

In the next section we give an integral representation for the first moment of a random variable of type (1.3), section 3 deals with the second moment. For concrete estimations of variances under slightly sharpened propositions we refer to a further article of the author being published.

2. Expectation values

The results in this short section are mainly due to Raynaud[6] and Borgwardt[1], who treated special cases of additive variables under rotationally symmetric distributions on $\mathbb{R}^n$. Let

$$(2.1) \qquad\qquad G_1(h) := P(x^{(n)} \leq h), \, h \in \mathbb{R},$$

$$(2.2) \qquad\qquad G_2(h) := P(x^{(n)} > h) = 1 - G_1(h), \, h \in \mathbb{R},$$

$x^{(n)}$ being the n-th coordinate of the vector x, we state without proof, which can be given analogously to Borgwardt's[1]:

Theorem 2.1:

For random variables $\tilde{Z}$ of additive type on bounded stochastic polyhedra $\mathcal{P}$ generated by a rotationally symmetric distribution with density function f holds while $m \geq n$ and $n \geq 2$:

$$(2.3) \qquad\qquad E(\tilde{Z}) = \binom{m}{n} \int\limits_0^\infty [G_1^{m-n}(h) + \sigma G_2^{m-n}(h)] \Lambda_Z(h) dh$$

with

$$(2.4) \qquad \Lambda_Z(h) = |\omega_n| \int\limits_{\mathbb{R}^{n-1}}^{(n)} Z \begin{pmatrix} \bar{b}_1 & \cdots & \bar{b}_n \\ h & \cdots & h \end{pmatrix} |det(B)| \prod_{i=1}^n f((\bar{b}_i, h)^T) d\bar{b}_i,$$

where

$$(2.5) \qquad B := \begin{pmatrix} \bar{b}_1 & \cdots & \bar{b}_n \\ 1 & \cdots & 1 \end{pmatrix}, \, \bar{b}_i := (b_i^{(1)}, \ldots, b_i^{(n-1)})^T, \, i = 1, \ldots, n$$

and

$$(2.6) \qquad\qquad |\omega_n| := \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}.$$

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Remarks on 2.1:

i) The functions $G_i(h)$ are easily evaluated by

$$(2.7) \qquad G_2(h) = \int\limits_h^\infty g_0(t)dt, \; g_0(t) := \int\limits_{\mathbb{R}^{n-1}} f((\bar{b},t)^T)d\bar{b}$$

for distributions with density function f.

ii) The function $\Lambda_Z(h)$ can be interpreted in the language of conditional expectation values by

$$(2.8) \qquad \Lambda_Z(h) = E(Z(A_I) \mid h(A_I) = h)\, p(h)$$

where

$$(2.9) \qquad p(h) := |\omega_n| \int\limits_{\mathbb{R}^{n-1}}^{(n)} |det(B)| \prod_{i=1}^{n} f((\bar{b}_i, h)^T)d\bar{b}_i,$$

is the marginal density function of the distribution function $P(h(A_I) \le h)$ for some set of indices I.

3. Second moments

Because of the nonlinearity of the functional $E(\tilde{Z}^2)$ it is much more complicated to give an integral formula for second moments of an additive type random variable $\tilde{Z}$ corresponding to (2.4). So, we deduce our result step by step and introduce further notation when necessary.

Let $\tilde{Z}$ be a random variable of additive type and $m \ge 2n \ge 4$, then by (1.3)

$$(3.1) \qquad E(\tilde{Z}^2) = \sum_{i,j=1}^{2} \sigma^{i+j-2} \sum_{I,J \subset \{1,\dots,m\}} \tilde{e}_Z(I,J,i,j),$$

where $I = \{i_1,\dots,i_n\}$ and $J = \{j_1,\dots,j_n\}$ are arbitrary sets of indices $1,\dots,m$ with cardinality n and

$$(3.2) \qquad \tilde{e}_Z(I,J,i,j) := \int\limits_{\mathbb{R}^n}^{(m)} \chi_i(A_I)\chi_j(A_J)Z(A_I)Z(A_J) \prod_{\ell=1}^{m} f(a_\ell)da_\ell,$$

f being the density function of the underlying rotationally symmetric distribution. For reasons of symmetry $\tilde{e}_Z(I,J,i,j)$ obviously depends only on the cardinality k of the set $I \setminus J$ resp. the variables i and j. Denoting q_k for the number of pairs (I,J), $I,J \subset \{1,\dots,m\}$, with $|I \setminus J| = k$, we define

$$(3.3) \qquad e_Z(k,i,j) := \tilde{e}_Z(I,J,i,j) \text{ if } |I \setminus J| = k$$

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for $i, j \in \{1, 2\}$ and $k \in \{0, \ldots, n\}$. Now, we get from (3.1) by changing summations:

$$(3.4) \qquad E(\tilde{Z}^2) = \sum_{k=0}^{n} q_k \sum_{i,j=1}^{2} \sigma^{i+j-2} e_Z(k, i, j),$$

where

$$(3.5) \qquad q_k = \binom{m}{n} \binom{n}{k} \binom{m-n}{k}, \ k = 0, \ldots, n,$$

which is obtained by simple combinatorial arguments.

In order to evaluate the terms $e_Z(k, i, j)$ we define for $i, j \in \{1, 2\}$ and $k = 0, \ldots, n$:

$$(3.6) \qquad p_{i,j}(A_0, A_k) := \int_{\mathbb{R}^n}^{(m-n-k)} \chi_i(A_0) \chi_j(A_k) \prod_{\ell=n+k+1}^{m} f(a_\ell) da_\ell$$

with

$$(3.7) \qquad A_\ell := (a_{\ell+1}, \ldots, a_{\ell+n}), \ \ell = 0, \ldots, n.$$

$p_{i,j}(A_0, A_k)$ equals the probability of A_0 being a boundary simplex of the i-th kind of $\mathcal{P}$ and A_k being a boundary simplex of the j-th kind of $\mathcal{P}$ at the same time.

Using definitions (3.3) and (3.6) we obtain

$$(3.8) \qquad \begin{aligned} e_Z(k, i, j) &= \int_{\mathbb{R}^n}^{(m)} \chi_i(A_0) \chi_j(A_k) Z(A_0) Z(A_k) \prod_{\ell=1}^{m} f(a_\ell) da_\ell \\ &= \int_{\mathbb{R}^n}^{(n+k)} p_{i,j}(A_0, A_k) Z(A_0) Z(A_k) \prod_{\ell=1}^{n+k} f(a_\ell) da_\ell \end{aligned}$$

proving the

Proposition 3.1:

For random variables $\tilde{Z}$ of additive type, rotationally symmetric distributions with density function f and $m \geq 2n \geq 4$ holds:

$$(3.9) \qquad E(\tilde{Z}) = \sum_{k=0}^{n} q_k \sum_{i,j=1}^{2} \sigma^{i+j-2} e_Z(k, i, j)$$

with

$$(3.10) \qquad q_k = \binom{m}{n} \binom{n}{k} \binom{m-n}{k}$$

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and

$$(3.11) \qquad e_Z(k,i,j) = \int_{\mathbb{R}^n}^{(n+k)} p_{i,j}(A_0, A_k) Z(A_0) Z(A_k) \prod_{\ell=1}^{n+k} f(a_\ell) da_\ell,$$

where

$$(3.12) \qquad p_{i,j}(A_0, A_k) = P(A_0 \text{ is boundary simplex of } \mathcal{P} \text{ of the } i\text{-th kind} \wedge$$
$$A_k \text{ is boundary simplex of } \mathcal{P} \text{ of the } j\text{-th kind}).$$

After this first step of evaluation we look at $p_{i,j}$ more closely. By use of the obvious equations

$$(3.13) \qquad \chi_i(A_\ell) = \prod_{j=0}^{k} \chi(a_j \in H^{(i)}(A_\ell)) \prod_{j=n+k+1}^{m} \chi(a_j \in H^{(i)}(A_\ell))$$
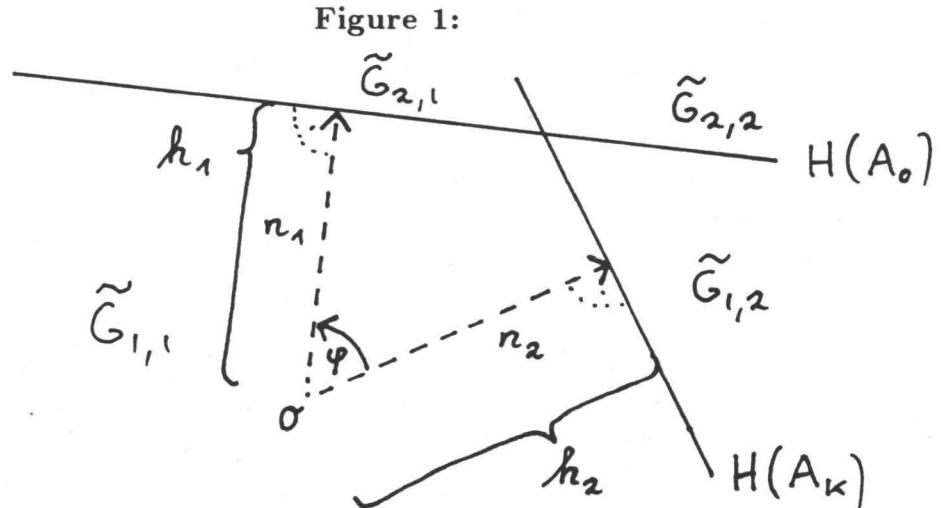
for $i = 1, 2$ and $\ell = 0, \ldots, n$ we gain

$$(3.14) \qquad p_{i,j}(A_0, A_k) = (\tilde{G}_{i,j}(A_0, A_k))^{m-n-k} \chi(S_k \subset H^{(i)}(A_0)) \chi(S_0 \subset H^{(j)}(A_k))$$

with

$$(3.15) \qquad \tilde{G}_{i,j}(A_0, A_k) := P(a \in (H^{(i)}(A_0) \cap H^{(j)}(A_k))).$$

and S_ℓ defined analogously to (3.7).

The quantities $\tilde{G}_{i,j}$ can be illustrated geometrically as weighted areas dividing the measure space $(\mathbb{R}^n, P)$ in four parts, cf. figure 1.

Figure 1:

More formally holds:

$$(3.16) \qquad \sum_{i,j=1}^{2} \tilde{G}_{i,j}(A_0, A_k) = 1.$$

Elementary geometric insight makes clear that $G_{i,j}(A_0, A_k)$ for $k = 1, \ldots, n$ depends only on the distance $h_1 = h(A_0)$ of the hyperplane $H(A_k)$ to the origin, the distance $h_2 = h(A_k)$ of the hyperplane $H(A_k)$ to the origin and on the angle $\varphi(A_0, A_k)$ enclosed by the normal vectors n_1 of $H(A_0)$ and n_2 of $H(A_k)$, cf. figure 1. Clearly, $\|n_1\|_2 = h_1$ and $\|n_2\|_2 = h_2$. The natural objective now is to represent $e_Z(k, i, j)$ for $k = 1, \ldots, n$ by an integral of the form

$$(3.17) \qquad e_Z(k, i, j) = \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\pi G_{i,j}^{m-n-k}(h_1, h_2, \varphi) \Lambda_{i,j,k,Z}(h_1, h_2, \varphi) d\varphi \, dh_1 \, dh_2$$

with appropriately chosen functions $\Lambda_{i,j,k,Z}$ and

$$(3.18) \qquad G_{i,j}(h_1, h_2, \varphi) := \tilde{G}_{i,j}(A_0, A_k).$$

For $k = 0$ we obtain for $i, j \in \{1, 2\}$:

$$(3.19) \qquad \tilde{G}_{i,j}(A_0, A_0) = \begin{cases} G_i(h(A_0)) & i = j \\ 0 & i \neq j \end{cases}$$

G_i being defined by (2.1) resp. (2.2), which leads to

$$(3.20) \qquad e_Z(0, i, j) = \begin{cases} \int\limits_0^\infty G_i^{m-n}(h) \Lambda_{Z^2}(h) dh & i = j \\ 0 & i \neq j \end{cases} \quad , i, j \in \{1, 2\}$$

with Λ_{Z^2} as in (2.4).

In the following we will establish (3.17) by three successive simultaneous rotations of the vectors a_ℓ, $\ell = 1, \ldots, n + k$.

First Rotation:

The normal vector n_1 of the hyperplane $H(A_0)$ have the polar representation

$$(3.21) \qquad (h_1, \psi_1, \ldots, \psi_{n-1})^T,$$

where $h_1 \in \mathbb{R}_0^+$, $\psi_1 \in [0, 2\pi)$, $\psi_\ell \in [0, \pi)$ for $\ell = 2, \ldots, n-1$.
Furthermore, let $\tilde{d}_1, \ldots, \tilde{d}_n$ be an orthonormal basis of $\mathbb{R}^n$, with $\tilde{d}_n = n_1^0$. This basis can be chosen as

$$(3.22) \qquad \tilde{d}_n^{(\ell)} = \begin{cases} \sin\psi_1 \sin\psi_2 \ldots \sin\psi_{n-1} & \text{if } \ell = 1 \\ \cos\psi_{\ell-1} \sin\psi_\ell \ldots \sin\psi_{n-1} & \text{if } \ell = 2, \ldots, n-1 \\ \cos\psi_{n-1} & \text{if } \ell = n \end{cases}$$

$$\tilde{d}_\ell = \frac{\sin\psi_1 \ldots \sin\psi_\ell}{\sin\psi_1 \ldots \sin\psi_{\ell-1}} \frac{\partial \tilde{d}_n}{\partial \psi_\ell}, \quad \ell = 1, \ldots, n-1 \,.$$

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Now define the orthogonal matrix $R_1 := (\tilde{d}_1, \ldots, \tilde{d}_n)$ and the new variables $b_\ell := R_1^{-1} a_\ell$ for $\ell = 1, \ldots, n + k$ with $b_\ell := ((\bar{b}_\ell, h_1)^T)$.

The Jacobian Φ_1 of this transformation satisfies

$$(3.23) \qquad |det(\Phi_1)| = \sin \psi_2 (\sin \psi_3)^2 \ldots (\sin \psi_{n-1})^{n-2} |det(B)|,$$

as one easily proves like Raynaud[6] with B defined in (2.5).

Using the formula

$$(3.24) \qquad \int\limits_0^{2\pi} \int\limits_0^{\pi} \ldots \int\limits_0^{\pi} |det(\Phi_1)| \sin \psi_2 \sin \psi_3^2 \ldots \sin \psi_{n-1}^{n-2} d\psi_{n-1} \ldots d\psi_2 d\psi_1 = |\omega_n| |det(B)| ,$$

we obtain for $e_Z(k, i, j)$ the representation:

$$(3.25) \qquad e_Z(k, i, j) := \int\limits_0^{\infty} \int\limits_{\mathbb{R}^n}^{(k)} p_{i,j}(\tilde{A}_0, \tilde{A}_k) \tilde{\Lambda}_{k,Z}(\tilde{A}_0, \tilde{A}_k) \prod_{i=n+1}^{n+k} f((\bar{b}_i, h_1)^T) d\bar{b}_i \, dh_1 ,$$

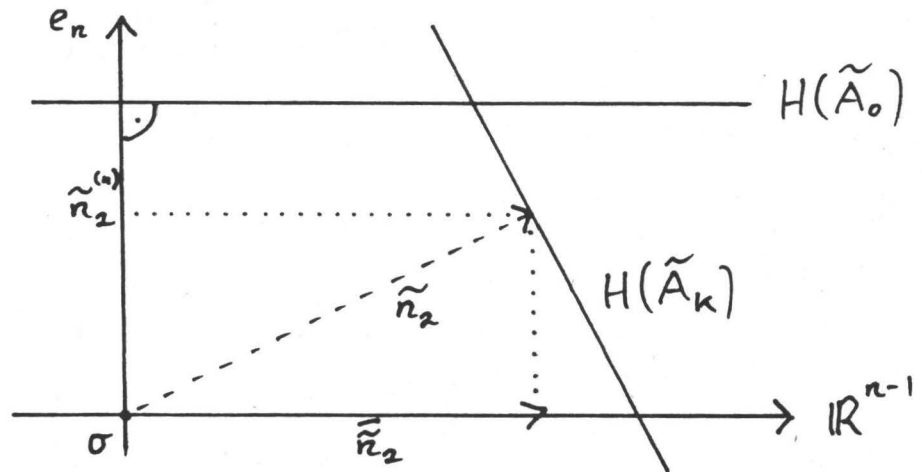where now defining $\tilde{A}_0 := R_1^{-1} A_0$ and $\tilde{A}_k := R_1^{-1} A_k$ holds:

$$(3.26) \quad p_{i,j}(\tilde{A}_0, \tilde{A}_k) = G_{i,j}^{m-n-k}(h_1, h_2(\tilde{A}_k), \varphi(\tilde{A}_0, \tilde{A}_k)) \chi_(\tilde{A}_k \subset H^{(i)}(\tilde{A}_0) \wedge \tilde{1}_0 \subset H^{(j)}(\tilde{A}_k))$$

and where

$$(3.27) \qquad \tilde{\Lambda}_{k,Z}(\tilde{A}_0, \tilde{A}_k) = |\omega_n| \int\limits_{\mathbb{R}^{n-1}}^{(n)} Z(\tilde{A}_0) Z(\tilde{A}_k) |det(B)| \prod_{\ell=1}^{n} f((\bar{b}_\ell, h_1)^T) d\bar{b}_\ell .$$

Geometrically, by this first rotation the hyperplane $H(A_0)$ containing the boundary simplex A_0 was rotated into a hyperplane normal to the unit vector e_n, cf. figure 2.

Figure 2:

Next, by a second simultaneous rotation of the vectors b_ℓ we will establish h_2 und φ as free quantities. This transformation is only required for $n \geq 3$, for that reason let $n \geq 3$ in the following.

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Second Rotation:

The normal vector $\tilde{n}_2$ of the hyperplane $H(\tilde{A}_k)$ may be written in the form (cf. figure 2):

$$(3.28) \qquad \tilde{n}_2 = (\bar{\tilde{n}}_2, \tilde{n}_2^{(n)})^T .$$

The truncated vector $\bar{\tilde{n}}_2 \in \mathbb{R}^{n-1}$ have the polar coordinates $(\|\bar{\tilde{n}}_2\|_2, \hat{\psi}_1, \ldots, \hat{\psi}_{n-2})^T$ with $\hat{\psi}_1 \in [0, 2\pi)$, $\hat{\psi}_\ell \in [0, \pi)$ for $\ell = 2, \ldots, n-2$. Additionally $\hat{d}_1, \ldots, \hat{d}_{n-1}$ be an orthonormal basis of $\mathbb{R}^{n-1}$, where $\hat{d}_{n-1} = \bar{\tilde{n}}_2^0$. Like (3.22) we define

$$(3.29) \qquad \hat{d}_{n-1}^{(\ell)} = \begin{cases} \sin \hat{\psi}_1 \sin \hat{\psi}_2 \ldots \sin \hat{\psi}_{n-2} & \text{if } \ell = 1 \\ \cos \hat{\psi}_{\ell-1} \sin \hat{\psi}_\ell \ldots \sin \hat{\psi}_{n-2} & \text{if } \ell = 2, \ldots, n-2 \\ \cos \hat{\psi}_{n-2} & \text{if } \ell = n-1 \end{cases}$$

$$\hat{d}_\ell = \frac{\sin \hat{\psi}_1 \ldots \sin \hat{\psi}_\ell}{\sin \hat{\psi}_1 \ldots \sin \hat{\psi}_{\ell-1}} \frac{\partial \hat{d}_{n-1}}{\partial \hat{\psi}_\ell}, \quad \ell = 1, \ldots, n-2 .$$

We define the matrix $R_2 := (\hat{d}_1, \ldots, \hat{d}_{n-1})$ and the new variables $\hat{c}_\ell$ by $\hat{c}_\ell := (\bar{\hat{c}}_\ell, \hat{c}_\ell^{(n)})^T$ with $\bar{\hat{c}}_\ell := R_2^{-1} \bar{b}_\ell$ and by $\hat{c}_\ell^{(n)} := b_\ell^{(n)}$ for $\ell = 1, \ldots n+k$. In the following $\hat{c}_\ell$ often will be notated by $\hat{c}_\ell = (\bar{\bar{\hat{c}}}_\ell, \hat{c}_\ell^{(n-1)}, \hat{c}_\ell^{(n)})^T$.

For $\ell = k+1, \ldots, n+k$ the vector $\hat{c}_\ell^{(n-1)}$ satisfies the linear equation

$$(3.30) \qquad \begin{pmatrix} \sin \varphi \\ \cos \varphi \end{pmatrix} \begin{pmatrix} \hat{c}_\ell^{(n-1)} \\ \hat{c}_\ell^{(n)} \end{pmatrix} - h_2 = 0$$

for $\varphi \in [0, \pi]$. Especially for $\varphi \in (0, \pi)$ holds, cf. figure 3:

$$(3.31) \qquad \hat{c}_\ell^{(n-1)} = \frac{h_2 - \cos \varphi \hat{c}_\ell^{(n)}}{\sin \varphi} .$$

The evaluation of the second rotation's Jacobian Φ_2 requires lengthy calculations omitted here. Φ_2 satisfies the relation

$$(3.32) \qquad |det \Phi_2| = \frac{1}{\sin^2 \varphi} \prod_{i=2}^{n-2} \sin^{i-1} \psi_i |det(C)|,$$
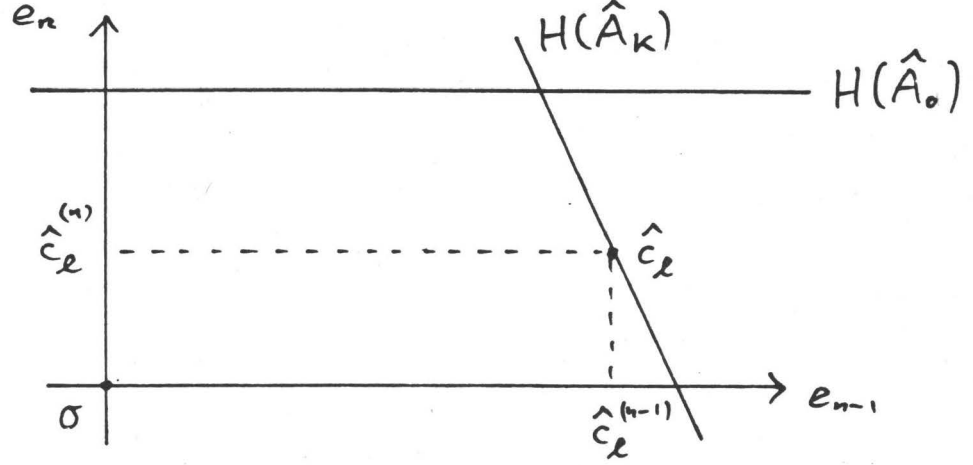
which leads to

$$(3.33) \qquad \int_0^{2\pi} \int_0^\pi \ldots \int_0^\pi |det(\Phi_2)| d\hat{\psi}_1 \ldots d\hat{\psi}_{n-2} = \frac{|\omega_{n-1}|}{\sin^2 \varphi} |det(\hat{C})|$$

corresponding to (3.24), where

$$(3.34) \qquad \hat{C} := \begin{pmatrix} \bar{\bar{\hat{c}}}_{k+1} & \ldots & \bar{\bar{\hat{c}}}_{n+k} \\ \frac{\hat{c}_{k+1}^{(n)} - \cos \varphi h_2}{\sin \varphi} & \ldots & \frac{\hat{c}_{n+k}^{(n)} - \cos \varphi h_2}{\sin \varphi} \\ 1 & \ldots & 1 \end{pmatrix} .$$

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For detailed proofs of (3.32) and (3.33) the interested reader is referred to Küfer[4]. The objective of the second rotation was to move both normal vectors n_1 and n_2 of the hyperplanes $H(\tilde{A}_0)$ and $H(\tilde{A}_k)$ into the (e_{n-1}, e_n)-plane, cf. figure 3.

Figure 3:

Having performed the second rotation we come to the desired form (3.17) for the quantities $e_Z(k, i, j)$. It holds for $k = 1, \ldots, n$ and $i, j \in \{1, 2\}$:

$$(3.35) \quad \Lambda_{i,j,k,Z}(h_1, h_2, \varphi) = \frac{|\omega_n||\omega_{n-1}|}{\sin^2\varphi} \int_{\mathbb{R}^{n-2}}^{(n+k)} \int_{\hat{K}_{1,i,j}}^{(k)} \int_{\hat{K}_{2,i,j}}^{(k)} \hat{\lambda}_{k,Z}(\dot{c}_1 \ldots , \ddot{c}_{n+k}) \, d\hat{\mu}_k$$

where

$$(3.36) \quad \hat{K}_{1,i,j} := \begin{cases} (-\infty, h_1] & \text{if } i = 1 \\ [h_1, \infty) & \text{if } i = 2 \end{cases}, \quad \hat{K}_{2,i,j} := \begin{cases} (-\infty, d_1] & \text{if } j = 1 \\ [d_1, \infty) & \text{if } j = 2 \end{cases},$$

$$(3.37) \quad \hat{\lambda}_{k,Z}(\hat{c}_1, \ldots, \hat{c}_{n+k}) := |det(\hat{C}_k)|Z(\hat{A}_k)|det(\hat{C}_0)|Z(\hat{A}_0),$$

and

$$(3.38) \quad d\hat{\mu}_k := \prod_{\ell=1}^{k+n} f(\hat{c}_\ell) \prod_{\ell=1}^{k} d\hat{c}_\ell^{(n-1)} \prod_{\ell=n+1}^{n+k} d\hat{c}_\ell^{(n)} \prod_{\ell=1}^{n+k} d\bar{\bar{c}}_\ell,$$

the matrices $\hat{C}_0$ and $\hat{C}_k$ taking forms

$$(3.39) \quad \hat{C}_0 = \begin{pmatrix} \bar{\bar{c}}_1 & \cdots & \bar{\bar{c}}_k & \bar{\bar{c}}_{k+1} & \cdots & \bar{\bar{c}}_n \\ \hat{c}_1^{(n-1)} & \cdots & \hat{c}_k^{(n-1)} & d_1 & \cdots & d_1 \\ 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}$$

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and

$$(3.40) \qquad \hat{C}_k = \begin{pmatrix} \bar{\bar{\hat{c}}}_{k+1} & \ldots & \bar{\bar{\hat{c}}}_n & \bar{\bar{\hat{c}}}_{n+1} & \ldots & \bar{\bar{\hat{c}}}_{n+k} \\ d_2 & \ldots & d_2 & \frac{\hat{c}_{n+1}^{(n)} - \cos\varphi h_2}{\sin\varphi} & \ldots & \frac{\hat{c}_{n+k}^{(n)} - \cos\varphi h_2}{\sin\varphi} \\ 1 & \ldots & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Furthermore, $\hat{A}_0 := R_2^{-1}\tilde{A}_0$ and $\hat{A}_k := R_2^{-1}\tilde{A}_k$. The quantities d_ℓ, $\ell = 1, 2$, defined by

$$(3.41) \qquad d_1 := \frac{h_2 - \cos\varphi h_1}{\sin\varphi}; \quad d_2 := \frac{h_1 - \cos\varphi h_2}{\sin\varphi}$$

can be interpreted geometrically. $|d_i|$ equals the distance between the intersection point of the hyperplanes $H(\hat{A}_0)$ and $H(\hat{A}_k)$ in the (e_{n-1}, e_n)-plane and the intersection point of the ray $\mathbb{R}^+\hat{n}_i$ and the hyperplane $H(\hat{A}_0)$ for $i = 1$ resp. the hyperplane $H(\hat{A}_k)$ for $i = 2$, cf. figure 4.

Figure 4:
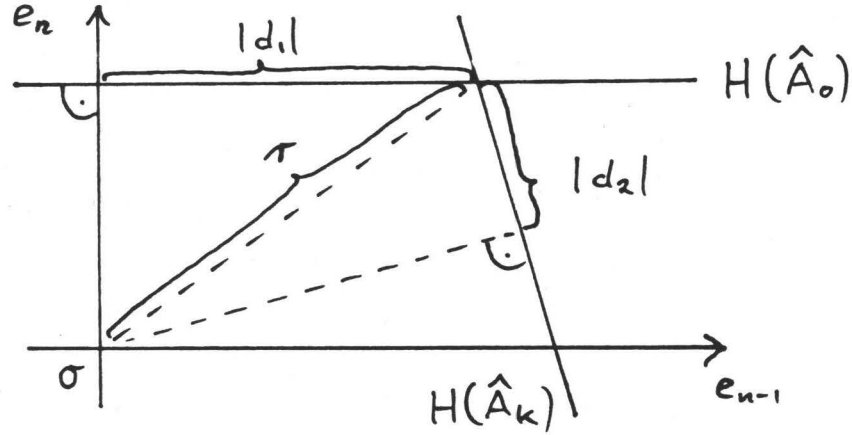
Third Rotation:

Intuitively it is clear that functions $\Lambda_{i,j,k,Z}(h_1, h_2, \varphi)$ are to be symmetric in the arguments h_1 und h_2. In order to realize this suggestion formally by the representation of the functions $\Lambda_{i,j,k,Z}(h_1, h_2, \varphi)$ we rotate the vectors $\hat{c}_{k+1}, \ldots \hat{c}_{n+k}$ simultaneously resp. both of its last coordinates by the angle $\pi - \varphi$ counterclockwise in the (e_{n-1}, e_n)-plane. Because of (1.3) $Z(\hat{A}_k)$ remains unc'anged while performing the described rotation. Defining the matrix

$$(3.42) \qquad R_3 := \begin{pmatrix} Id_{n-2} & 0 \\ 0 & \tilde{R} \end{pmatrix},$$

where Id_{n-2} represents the identity in $\mathbb{R}^{n-2}$ and

$$(3.43) \qquad \tilde{R} = \begin{pmatrix} -\cos\varphi & \sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

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holds:

$$
(3.44) \qquad R_3 \hat{c}_\ell = \begin{cases} (\bar{\bar{\hat{c}}}_\ell, d_2, h_2) & \text{for } \ell = k+1, \dots, n \\ (\bar{\bar{\hat{c}}}_\ell, \frac{\hat{c}_\ell^{(n)} - \cos\varphi h_2}{\sin\varphi}, h_2) & \text{for } \ell = n+1, \dots, n+k \end{cases} .
$$

We substitute

$$
(3.45) \qquad \hat{c}_\ell^{(p)} = \begin{cases} c_\ell^{(p-1)} \sin\varphi + \cos\varphi h_2 & p = n; \ \ell = n+1, \dots, n+k \\ c_\ell^{(p)} & p = 1, \dots, n-2; \ \ell = n+1, \dots, n+k \\ c_\ell^{(p)} & p = 1, \dots, n; \ \ell = 1, \dots, n \end{cases}
$$

and finally reach a symmetric form of $\Lambda_{i,j,k,Z}(h_1, h_2, \varphi)$, which we summarize in

Theorem 3.2:

Under the prepositions of Proposition 3.1 $e_Z(k,i,j)$ satisfies (3.20) for $k = 0$ and (3.17) for $k = 1, \dots, n$, where in the latter case

(3.46)

$$
\Lambda_{i,j,k,Z}(h_1, h_2, \varphi) = \frac{|\omega_n||\omega_{n-1}|}{\sin^{2-k}\varphi} \int_{\mathbb{R}^{n-2}}^{(n+k)} \int_{K_{1,i,j}}^{(k)} \int_{K_{2,i,j}}^{(k)} \lambda_{k,Z}(c_1, \dots, c_{n+k}) \ d\mu_k ,
$$

with

$$
(3.47) \qquad K_{1,i,j} := \begin{cases} (-\infty, d_2] & \text{if } i = 1 \\ [d_2, \infty) & \text{if } i = 2 \end{cases} , \quad K_{2,i,j} := \begin{cases} (-\infty, d_1] & \text{if } j = 1 \\ [d_1, \infty) & \text{if } j = 2 \end{cases} ,
$$

$$
(3.48) \qquad \lambda_{k,Z}(c_1, \dots, c_{n+k}) := \left| \det \begin{pmatrix} \overline{C}_0 \\ \underline{e}^T \end{pmatrix} \right| Z \begin{pmatrix} \overline{C}_0 \\ h_1 \underline{e}^T \end{pmatrix} \left| \det \begin{pmatrix} \overline{C}_k \\ \underline{e}^T \end{pmatrix} \right| Z \begin{pmatrix} \overline{C}_k \\ h_2 \underline{e}^T \end{pmatrix} ,
$$

and

$$
(3.49) \qquad d\mu_k := \prod_{\ell=1}^{k+n} f(c_\ell) \prod_{\ell=1}^{k} dc_\ell^{(n-1)} \prod_{\ell=n+1}^{n+k} dc_\ell^{(n-1)} \prod_{\ell=1}^{n+k} d\bar{\bar{c}}_\ell
$$

The matrices $\overline{C}_\ell$ take the form

$$
(3.50) \qquad \overline{C}_0 = \begin{pmatrix} \bar{c}_1 & \dots & \bar{c}_k & \bar{\bar{c}}_{k+1} & \dots & \bar{\bar{c}}_n \\ & & & d_1 & \dots & d_1 \end{pmatrix}
$$

and

$$
(3.51) \qquad \overline{C}_k = \begin{pmatrix} \bar{\bar{c}}_{k+1} & \dots & \bar{\bar{c}}_n & \bar{c}_{n+1} & \dots & \bar{c}_{n+k} \\ d_2 & \dots & d_2 & & & \end{pmatrix}
$$

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whereas $\underline{e} = (1, \ldots, 1)^T \in \mathbb{R}^n$.

In the remaining part of the paragraph we concentrate on the quantities $G_{i,j}(h_1, h_2, \varphi)$ in order to obtain integral formulas like (2.7). Obviously, because of (3.16) and the easily proved equations

$$(3.52) \qquad G_{1,i}(h_1, h_2, \varphi) + G_{2,i}(h_1, h_2, \varphi) = G_i(h_2),$$

$$G_{i,1}(h_1, h_2, \varphi) + G_{i,2}(h_1, h_2, \varphi) = G_i(h_1)$$

for $h_1, h_2 \in [0, \infty)$, $\varphi \in (0, \pi)$ and $i \in \{1, 2\}$ it is enough to investigate $G_{1,1}$. Before we formulate our result on this issue we introduce a new variable r by

$$(3.53) \qquad r = \sqrt{d_1^2 + h_1^2} = \sqrt{d_2^2 + h_2^2} = \frac{\sqrt{h_1^2 + h_2^2 - 2h_1 h_2 \cos \varphi}}{\sin \varphi},$$

Geometrically, r equals the distance between the origin and the intersection point of the hyperplanes containing the boundary simplices we are dealing with, cf. figure 4.

Theorem 3.3:

For $n \geq 2$, rotationally symmetric distributions with RDF F, $0 \leq h_1, h_2 < \infty$ and $\varphi \in (0, \pi)$ holds:

$$(3.54) \quad G_{1,1}(h_1, h_2, \varphi) = 1 - \tfrac{1}{2} G_2(h_1) - \tfrac{1}{2} G_2(h_2) - A_1(h_1, h_2, \varphi) - A_2(h_1, h_2, \varphi),$$

where for $\ell \in \{1, 2\}$

$$(3.55) \qquad A_\ell(h_1, h_2, \varphi) := \text{sign}(d_\ell) \int_{h_\ell}^{r} g_{0,2}(t) \frac{h_\ell dt}{t \sqrt{t^2 - h_\ell^2}}$$

with $r = r(h_1, h_2, \varphi)$ and

$$(3.56) \qquad g_{0,2}(t) := \frac{|\omega_{n-2}|}{|\omega_n|(n-2)} \int_{t}^{\infty} \frac{(s^2 - t^2)^{(n-2)/2}}{s^{n-2}} dF(s).$$
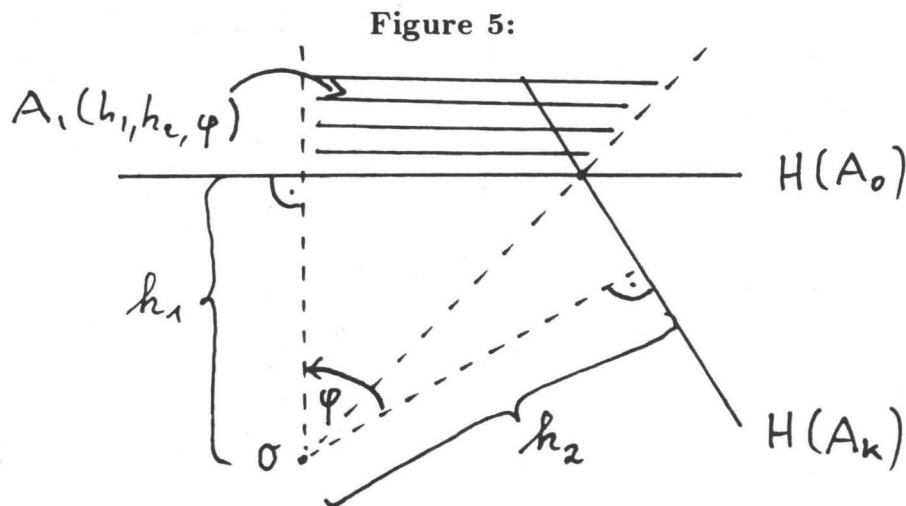
Proof:

We prove the theorem for $n \geq 3$ the result remaining valid for the easy special case $n = 2$. The weighted area $A_1(h_1, h_2, \varphi)$, cf. figure 5, satisfies:

$$(3.57) \qquad A_1(h_1, h_2, \varphi) = \int_{0}^{\arcsin \frac{d_1}{r}} \int_{\frac{h_1}{\cos \psi}}^{\infty} g_{0,0}(t) t \, dt \, d\psi$$

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with

$$(3.58) \qquad g_{0,0}(t) := \frac{|\omega_{n-2}|}{|\omega_n|} \int\limits_t^\infty \frac{(s^2 - t^2)^{(n-4)/2}}{s^{n-2}} dF(s)$$

being the radial density of $G_{1,1}$ for a point with polar coordinates (t, ψ) in the (e_n, e_{n-1})-plane.

Figure 5:

The variable r is seen as a function of variables h_1, h_2, φ. Using

$$(3.59) \qquad \frac{d}{dt} g_{0,2}(t) = -t g_{0,0}(t) \ , \quad g_{0,2}(\infty) = 0 \ ,$$

we have

$$(3.60) \qquad A_1(h_1, h_2, \varphi) = \int\limits_0^{\arcsin \frac{d_1}{r}} g_{0,2}(\frac{h_1}{\cos \psi}) d\psi \ .$$

By substituting

$$(3.61) \qquad \frac{h_1}{\cos \psi} = t \ , \ d\psi = \frac{h_1 dt}{t\sqrt{t^2 - h_1^2}} dt \ ,$$

we reach the desired

$$(3.62) \qquad A_1(h_1, h_2, \varphi) := \text{sign}(d_1) \int\limits_{h_1}^r g_{0,2}(t) \frac{h_1 dt}{t\sqrt{t^2 - h_1^2}} \ .$$

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The corresponding representation for the weighted area $A_2(h_1, h_2, \varphi)$ is established analogously. By use of (3.52) the statement (3.53) follows.

∎

Finally we give a useful

Corollary 3.4:

  Under the prepositions of Theorem 3.3 holds:

(3.63) $1 - G_2(h_1) - G_2(h_2) \leq G_{1,1}(h_1, h_2, \varphi) \leq \min(G_1(h_1), G_1(h_2))$

  and

(3.64) $G_{1,1}(h_1, h_2, \varphi_1) \leq G_{1,1}(h_1, h_2, \varphi_2)$ *for* $\varphi_1 \geq \varphi_2$

Proof:
While (3.63) being obvious we have:

(3.65) $$\frac{d}{d\varphi} G_{1,1}(h_1, h_2, \varphi) = -\frac{d}{d\varphi} \sum_{\ell \in \{1,2\}} A_\ell(h_1, h_2, \varphi)$$

$$= -\frac{d}{d\varphi}(\arcsin \frac{d_1}{r} + \arcsin \frac{d_2}{r}) g_{0,2}(r) = -g_{0,2}(r) \leq 0 \ .$$

∎

Concluding remarks:

In a further article being published we will exploit the above developed theory on the first two moments of additive type variables systematically under slightly sharpened prepositions concerning the class of distributions and the function Z in (1.3). As a first example we state without proof:

Theorem 3.5:

 i) *For uniform distribution in the n-dimensional unit ball, $n \geq 2$, holds:*

(3.66) $Var(Vol(\mathcal{P})) = O(m^{-\frac{n-1}{n+1}})$, $m \to \infty$.

 ii) *for uniform distribution on the n-dimensional unit sphere, $n \geq 2$, holds:*

(3.67) $Var(Vol(\mathcal{P})) = O(m^{-1})$, $m \to \infty$.

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