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Abstract

Let $a_1, \ldots, a_n$ be independent random points in $\mathbb{R}^d$ spherically symmetrically but not necessarily identically distributed. Let X be the random polytope generated as the convex hull of $a_1, \ldots, a_n$ and for any k-dimensional subspace $L \subseteq \mathbb{R}^d$ let $\mathrm{Vol}_L(X) := \lambda_k(L \cap X)$ be the volume of $X \cap L$ with respect to the k-dimensional Lebesgue measure λ_k, $k = 1, \ldots, d$. Furthermore, let $F^{(i)}(t) := \mathbf{Pr}(\|a_i\|_2 \leq t)$, $t \in \mathbb{R}_0^+$, be the radial distribution function of a_i. We prove that the expectation functional $\Phi_L(F^{(1)}, F^{(2)}, \ldots, F^{(n)}) := \mathbf{E}(\mathrm{Vol_L}(X))$ is strictly decreasing in each argument, i.e. if $F^{(i)}(t) \leq G^{(i)}(t)$, $t \in \mathbb{R}_0^+$, but $F^{(i)} \not\equiv G^{(i)}$, we show $\Phi(\ldots, F^{(i)}, \ldots) > \Phi(\ldots, G^{(i)}, \ldots)$. The proof is done in the more general framework of continuous and f–additive polytope functionals.

Keywords: Stochastic geometry, random polytopes, comparison of expectations
Mathematical subject classification: *Primary:* 60D05 *Secondary:* 52A22

1 Introduction and main results

We consider n random points a_i in $\mathbb{R}^d$, $n > d \geq 1$, with polar representations

$$a_i = r_i\,\omega_i \tag{1}$$

with radii $r_i \in \mathbb{R}_0^+$ and $\omega_i \in \mathcal{S}^{d-1}$, $\mathcal{S}^{d-1}$ being the unit sphere in $\mathbb{R}^d$. We assume each radius r_i to be stochastically independent from ω_i for $i = 1, \ldots, n$. Moreover, let the radii be independent, and each radius r_i, $i = 1, \ldots, n$, may have an arbitrary radial distribution function $F^{(i)}(t) := \mathbf{Pr}(r_i \leq t)$ for $t \in \mathbb{R}_0^+$, which we assume continuous from the right and without mass in zero. More formally, for $i = 1, \ldots, n$ let $F^{(i)} \in \mathcal{F}$ with

$$\mathcal{F} := \left\{ F : \mathbb{R}_0^+ \to [0,1] \;\middle|\; \begin{array}{c} F(0) = 0;\ F(t) \geq F(s),\ t > s \geq 0; \\ \lim\limits_{s \to t+} F(s) = F(t),\ t \geq 0;\ \lim\limits_{t \to \infty} F(t) = 1 \end{array} \right\}. \tag{2}$$

Finally, let the vectors $\omega_i \in \mathcal{S}^{d-1}$ be centrally symmetrically distributed with the additional assumption that their distributions are independent and have densities on $\mathcal{S}^{d-1}$.
We associate each n-tuple $(a_1, \ldots, a_n) = (r_1\omega_1, \ldots, r_n\omega_n)$ the random polytope

$$X := \operatorname{conv}(a_1, \ldots, a_n) \tag{3}$$

generated as the convex hull of $a_1, \ldots, a_n$.
For any k–dimensional subspace $L \subseteq R^d$, $k \in \{1, \ldots, d\}$, we study the polytope functionals

$$\operatorname{Vol}_L(X) := \lambda_k(L \cap X), \tag{4}$$

where we consider λ_k as the k–dimensional Lebesgue measure in $\mathbb{R}^d$. In particular, we are interested in the expectation functionals $\Phi_L : \mathcal{F}^n \to \mathbb{R}_0^+ \cup \{\infty\}$,

$$\Phi_L(F^{(1)}, F^{(2)}, \ldots, F^{(n)}) := \mathbf{E}(\operatorname{Vol}_L(X)). \tag{5}$$

Here, on the right hand side of (5) we average on the choice of r_i having the distribution function $F^{(i)}$ and on the choice of ω_i having an arbitrary but fixed distribution in the above introduced framework.
Many researchers have investigated the asymptotical behaviour of Φ_L for fixed dimension d and $n \to \infty$ in the particular case where the radii r_i are identically distributed and the ω_i are uniformly distributed on S^{d-1}. One of the first results in this field is due to Rényi and Sulanke [5], who investigated Φ_L asymptotically for $k = d = 2$ and $n \to \infty$ under some particular spherically symmetrical distributions like the uniform distribution in the unit disk, the uniform distribution on the unit sphere or the normal distribution in the plane. Their results have been generalized by others for higher dimensions and for classes of distributions: e.g. Wieacker [9] analyzed the expected Lebesgue volume of polytopes under uniform distribution in $\mathbb{R}^d$, Carnal [2] investigated the Lebesgue volume for regularly varying distributions in the unit circle. It would be too much to sketch the history of the development in detail. So, for an extensive survey on asymptotical results about expectations of polytope functionals, we refer the interested reader to the survey papers of Buchta[1], Gruber [3], Schneider [6], or Weil and Wieacker [8].

One natural question in the analysis of Φ_L is: How compare the expectations Φ_L for different distributions? This question is answered for special classes of distributions in the asymptotical case, where d is fixed and $n \to \infty$. For instance, for $R > 0$ let

$$\mathcal{F}_R := \{F \in \mathcal{F} \mid F(R) = 1\} \tag{6}$$

be the class of radial distribution functions supported by the interval $[0, R]$. Moreover, for $\bar{F} \equiv 1 - F$ and $\alpha > 0$ let

$$\mathcal{F}_{R,\alpha} := \left\{ F \in \mathcal{F}_R \mid \lim_{r \to 0} \frac{\bar{F}(R - \lambda r)}{\bar{F}(R - r)} = \lambda^\alpha, \, \lambda > 0 \right\} \tag{7}$$

be the subclass of the near R α–regularly varying radial distribution functions. Carnal proved in [2] for $d = k = 2$ that for $F \in \mathcal{F}_{R,\alpha}$ holds:

$$\Phi_{\mathbb{R}^2}(F) = \pi R^2 - C_{R,\alpha} n^{-2/(1+2\alpha)} + o(n^{-2/(1+2\alpha)}), \; n \to \infty, \tag{8}$$

where $C_{R,\alpha}$ is a positive constant that depends on R and α only. $\Phi_{\mathbb{R}^2}(F)$ is a shorthand for $\Phi_{\mathbb{R}^2}(F, \ldots, F)$. Asymptotically, it is natural to compare the rate of decay for $n \to \infty$ of $\pi R^2 - \Phi_{\mathbb{R}^2}(F)$: If $F \in \mathcal{F}_{R,\alpha}$ and $G \in \mathcal{F}_{R,\beta}$ with $\alpha > \beta$ then

$$\lim_{n \to \infty} \frac{\pi R^2 - \Phi_{\mathbb{R}^2}(G)}{\pi R^2 - \Phi_{\mathbb{R}^2}(F)} = 0, \tag{9}$$

i.e. for large n and fixed d the expected approximation of the unit ball is much better for the radial distribution function G as it is for F. Moreover, it is possible to prove that there is an *extremal* distribution in $\mathcal{F}_R$ with an optimal rate of decay: Let F_r, $r > 0$, be defined by

$$F_r(t) := \begin{cases} 0; & t < r \\ 1; & t \geq r \end{cases}. \tag{10}$$

Geometrically, the distribution functions F_r are the extremal points of the convex set $\mathcal{F}$, i.e. F_r cannot be represented as proper convex combinations of other distribution functions in $\mathcal{F}$. The rate of decay for $n \to \infty$ of $\pi R^2 - \Phi_{\mathbb{R}^2}(F_R)$ is the best within the class $\mathcal{F}_R$, i.e. there is no $G \in \mathcal{F}_R$ such that (9) holds with $F \equiv F_R$.
Analogous asymptotical results for other classes of distributions can be stated in a similar way like, for instance, Carnal did for the class of distributions in the plane with exponential tail in [2].
In contrast to the asymptotical case, very little is known in the non–asymptotical case up to now, because it seems very hard to calculate Φ_L for fixed n and d even for "easy" distributions like the uniform distribution in the unit ball or the normal distribution in $\mathbb{R}^d$.
It is the objective of this paper to analyze how the expected volumes $\Phi_L(\ldots, F^{(i)}, \ldots)$ and $\Phi_L(\ldots, G^{(i)}, \ldots)$ compare for fixed d and n if $F^{(i)}$ and $G^{(i)}$ are pointwise comparable radial distribution functions:

Definition: *Let $F, G \in \mathcal{F}$. We call $F \prec G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}_0^+$ and if $F \not\equiv G$.*

The relation "$\prec$" is a partial ordering of the set $\mathcal{F}$. We call two distributions $F, G \in \mathcal{F}$ *comparable* with respect to "$\prec$" if either $F \prec G$ or $G \prec F$. Geometrically, $F^{(i)} \prec G^{(i)}$ means for the distribution of the a_i that inside the ball of any radius $t > 0$ the distribution with radial distribution function G has more mass than the distribution with radial distribution function F.

Theorem 1: *Let $F^{(i)}, G^{(i)} \in \mathcal{F}$ with $F^{(i)} \prec G^{(i)}$ for an $i \in \{1, \ldots, n\}$. Then, for any $k \in \{1, \ldots, n\}$ and any k-dimensional subspace L holds*

$$\Phi_L(\ldots, F^{(i)}, \ldots) > \Phi_L(\ldots, G^{(i)}, \ldots), \tag{11}$$

if the right hand side is finite. That means Φ_L is strictly decreasing in each argument.

Though the claim of Theorem 1 seems plausible, the proof cannot be based on the simple observation of single events X. For instance, let X be a random triangle in the plane with $0 \notin X$, cf. Figure 1. Obviously, $\mathrm{Vol}_{\mathbb{R}^2}(X(a_1, \tilde{a}_2, a_3))$ is smaller than $\mathrm{Vol}_{\mathbb{R}^2}(X(a_1, a_2, a_3))$, whereas the radius of $\tilde{a}_2$ is bigger than that of a_2.

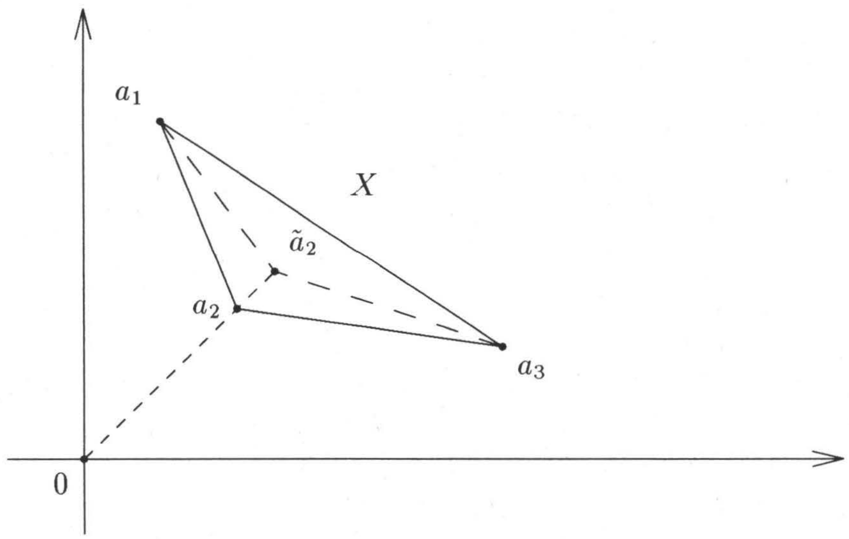

Figure 1: *The area of $X = X(a_1, a_2, a_3)$ is not increasing in the radii*

As a direct consequence of Theorem 1 we obtain that if the distributions of the vectors a_i are concentrated in the ball of radius R, $R > 0$, the value of the expectation functional Φ_L is maximal if $r_i = R$ for $i = 1, \ldots, n$ with probability one, i.e. the mass of the distributions of the a_i is concentrated on the sphere $R\,\mathcal{S}^{d-1}$. Obviously, for all $F \in \mathcal{F}_R$ holds $F_R \prec F$. Henceforth, we obtain from Theorem 1 the following result:

Corollary: *(Maximality of distributions on the R–sphere)*

$$\max_{F^{(i)} \in \mathcal{F}_R,\ i=1,\ldots,n} \Phi_L(F^{(1)}, \ldots, F^{(n)}) = \Phi_L(F_R). \tag{12}$$

The result and the proof of Theorem 1 can be embedded in the more general context of continuous polytope functionals of f–additive type, which we introduce in Section 2. The main tool of the proof of Theorem 1 will be a symmetrization technique of elementary events.
An interesting open question for further research is the following: For any polytope $X \subset \mathbb{R}^d$ let $f_k(X)$ be the number of X's faces of dimension k, $k \in \{0, \ldots, d-1\}$. Moreover, let $\Phi_{f_k}(F^{(1)}, \ldots, F^{(n)}) := \mathbf{E}(f_k(X))$ be the expected number of X's k–faces within the above introduced probability model. Φ_{f_k} cannot be strictly decreasing in the radial distribution functions as f_k is homogeneous with exponent 0, i.e. $f_k(tX) = f_k(X)$ for positive t. But nevertheless, we conjecture that the weaker claim (12) is true for f_k as well, which would have interesting applications in the probabilistic analysis of linear programming problems. Unfortunately, f_k is not continuous. So, our symmetrization method from Section 2 fails for Φ_{f_k}.

2 f–additive functionals and symmetrization

Let $(a_1, \ldots, a_n)$ be an n-tuple of vectors a_i in $\mathbb{R}^d$, $1 \leq d < n$, with polar representations $a_i = r_i \omega_i$, $r_i \in \mathbb{R}_0^+$, $\omega_i \in \mathcal{S}^{d-1}$. We call such an n-tuple $(a_1, \ldots, a_n)$ *non–degenerate* if the entries of each sub–tuple of cardinality d are linearly independent and if the entries of each sub–tuple of cardinality $d+1$ are in general position. Let

$$\mathcal{A}_n^d := \{(a_1, \ldots, a_n) \in \mathbb{R}^{d \times n} \,|\, (a_1, \ldots, a_n) \text{ is non–degenerate}\} \tag{13}$$

be the set of non–degenerate n–tuples. We associate each n–tuple $(a_1, \ldots, a_n)$ the polytope $X = X(a_1, \ldots, a_n) := \text{conv}(a_1, \ldots, a_n)$. If $(a_1, \ldots, a_n)$ is non–degenerate $X(a_1, \ldots, a_n)$ is a *simplicial polytope*, i.e. each face of X is a simplex. Let

$$\mathcal{P}_n^d := \{X = X(a_1, \ldots, a_n) \subset \mathbb{R}^d \,|\, (a_1, \ldots, a_n) \in \mathcal{A}_n^d\} \tag{14}$$

be the set of all simplicial polytopes in $\mathbb{R}^d$ generated by n-tuples $(a_1, \ldots, a_n)$. If we choose $(a_1, \ldots, a_n)$ at random under the assumptions on the distribution made in Section 1, we know that $(a_1, \ldots, a_n) \in \mathcal{A}_n^d$ with probability one and hence $X(a_1, \ldots, a_n) \in \mathcal{P}_n^d$ with probability one. For every simplicial polytope in $\mathbb{R}^d$ there exists an n such that $X \in \mathcal{P}_n^d$. Finally, we denote the set of all (not necessarily simplicial) polytopes in $\mathbb{R}^d$ with $\mathcal{P}^d$.

Now, in generalization of the functionals (4) we consider real functionals $\phi : \mathcal{P}^d \to \mathbb{R}$ with the property that $\phi(X)$ can be represented as a sum of functionals of ϕ's facets *if* X is simplicial. More precisely, let

$$\mathcal{I} := \{I := \{i_1, \ldots, i_d\} \,|\, 1 \leq i_1 < i_2 < \ldots < i_d \leq n\} \tag{15}$$

be the set of all sets I of indices $1, \ldots, n$ with cardinality d. We associate any set of indices $I := \{i_1, \ldots, i_d\} \in \mathcal{I}$ the d-tuple $(a_{i_1} | \ldots | a_{i_d})$ and the polytope $X_I := X(a_{i_1}, \ldots, a_{i_d})$. If $(a_{i_1} \ldots, a_{i_d}) \in \mathcal{A}_d^d$, each set X_I is a simplex of dimension $d-1$ and a candidate for being a boundary simplex of the simplicial polytope X. In our context of centrally symmetrically distributed vectors a_i it seems quite natural to differentiate between two types of boundary simplices of X. We call X_I a *boundary simplex of the first kind* of X if X_I is a boundary simplex of X as well as a boundary simplex of $\text{conv}(X \cup \{0\})$. X_I is called a *boundary simplex of the second kind* if X_I is a boundary simplex of X but not of $\text{conv}(X \cup \{0\})$. Obviously, every polytope $X \in \mathcal{P}_n^d$ has boundary simplices of the first kind but not necessarily boundary simplices of the second kind. More precisely, X has boundary simplices of the second kind if and only if the origin is an interior point of X, cf. Figure 2.

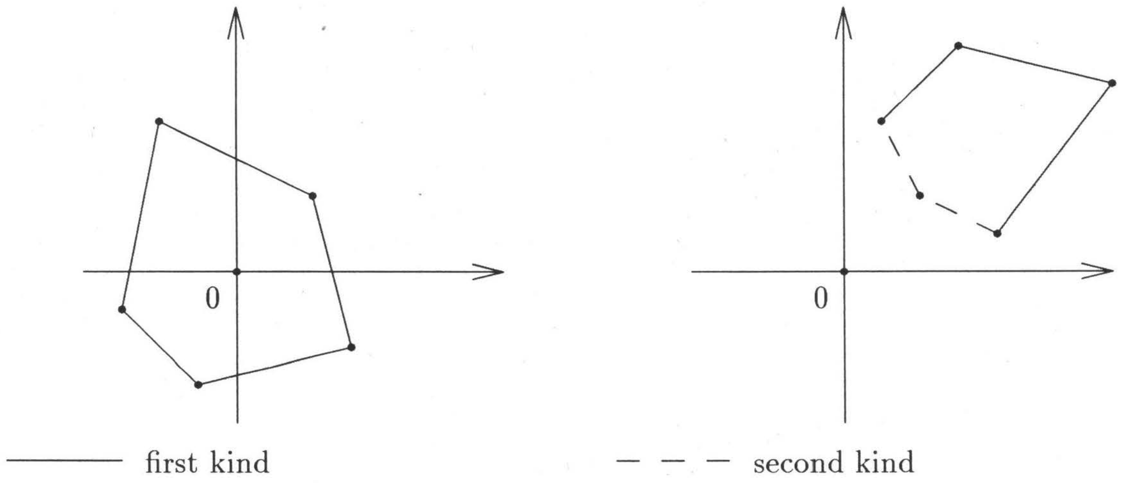

Figure 2: *Boundary simplices of first and second kind*

We call $\sigma : \mathcal{P}_n^d \times \mathcal{I} \to \{-1, 0, 1\}$ a *sign functional of the boundary simplices* of X if and only if

$$\sigma(X, I) := \chi_1(X, I) - \tau\, \chi_2(X, I), \tag{16}$$

where $\tau \in \{-1, 0, 1\}$ is fixed and $\chi_j(X, I)$ is the indicator functional for the event "X_I is a boundary simplex functional of X of the j–th kind". The value of τ depends only on the polytope functional under consideration and not on the particular choice of X or I.

Definition 2: *A polytope functional $\phi : \mathcal{P}^d \to \mathbb{R}$ is called an f–additive polytope functional if and only if for all $n > d \geq 1$ there exists a sign functional $\sigma : \mathcal{P}_n^d \times \mathcal{I} \to \{-1, 0, 1\}$ and if there exists a simplex functional $\varphi : \mathcal{P}_d^d \to \mathbb{R}_0^+$ such that for any X in $\mathcal{P}_n^d$ holds:*

$$\phi(X) = \sum_{I \in \mathcal{I}} \sigma(X, I)\, \varphi(X_I). \tag{17}$$

We denote an additive type functional ϕ by $\phi = (\varphi, \sigma)$.

The term f–additive is an abbreviation for *facet–additive*. The non–symmetrical definition of the sign functional σ contributes to the above mentioned fact, that every polytope in $\mathcal{P}_n^d$ has boundary simplices of the first kind but not necessarily boundary simplices of the second kind. The contribution of boundary simplices of the first kind is always considered

positive, while the contribution of the boundary simplices of the second kind may be either positive, negative or irrelevant. Many polytope functionals beyond the emphasized volume functionals Vol_L are f–additive. The interested reader is referred to Küfer[4] for a survey. The setting of f–additivity warrants positivity of expectations of ϕ if the vectors a_i are chosen randomly under the assumptions on the distributions stated in Section 1. This is due to the fact that the expectation of the sign functional σ is always positive or zero, as will be shown in the proof of Lemma 2. In general, the dissection property (17) of ϕ does not make sense for non–simplicial polytopes. In our stochastic model random polytopes are simplicial almost surely. So, we know that the event "$\phi(X)$ has a representation of the form (17) for some n" has probability one. Hence, we are allowed to restrict our attention to simplicial polytopes X.
In order to avoid misleading interpretations it is worth noticing that additivity of polytope functionals in stochastic geometry is defined in a more set-theoretical sense, cf. Schneider and Weil [6]: Let $\phi : \mathcal{P}^d \to \mathbb{R}$ be a real polytope functional and let X, Y and $X \cup Y$ be in $\mathcal{P}^d$. ϕ is called *set–additive* if $\phi(X) + \phi(Y) = \phi(X \cap Y) + \phi(X \cup Y)$. Obviously, this definition differs from ours, but the volume functionals Vol_L from (4) fulfill both definitions: It is geometrically clear that Vol_L is set–additive. The f–additivity of Vol_L is established defining $\sigma(X, I) = -1$ if X_I is a boundary simplex of the second kind and with the simplex functional

$$\varphi_L(X_I) := \lambda_k(\mathrm{conv}(X_I \cup \{0\}) \cap L). \tag{18}$$

In generalization of the functionals Vol_L defined in Section 1 we study expectation functionals

$$\Phi(F^{(1)}, \ldots, F^{(n)}) := \mathbf{E}(\phi(X(A))) \tag{19}$$

for a class of *continuous* f–additive functionals $\phi = (\varphi, \sigma)$ under the assumptions on the distribution of the vectors a_i stated in Section 1.

Definition 3: *A polytope functional $\phi : \mathcal{P}^d \to \mathbb{R}$ is called continuous if and only if for all $X \in \mathcal{P}^d$ and any sequence $(X_k)_{k \in \mathbb{N}} \subset \mathcal{P}^d$ with $\lim_{k\to\infty} d_H(X_k, X) = 0$ holds: $\lim_{k\to\infty} \phi(X_k) = \phi(X)$.*

Here, d_H is the well–known Hausdorff metric defined by $d_H(X, Y) := \max_{x \in X} \min_{y \in Y} \|x - y\|_2$. The volume functionals Vol_L are wellknown to be continuous with respect to the Hausdorff metric, cf. Schneider and Weil [7].

Definition 4: *A simplex functional $\varphi : \mathcal{P}_d^d \to \mathbb{R}_0^+$ is called (strictly) increasing if and only if $\varphi(X(r_1\omega_1, \ldots, r_d\omega_d))$ is (strictly) increasing in the radii r_i, $i \in 1, \ldots, d$, for all non–degenerate d–tuples $(\omega_1, \ldots, \omega_d)$, $\omega_i \in \mathcal{S}^{d-1}$.*

We know that Vol_L has the simplex functional φ_L defined in (18). It is easily verified that these simplex functionals are increasing in case of $k = d$. However, this is not generally true if $k < d$. Here, $\varphi_L(X(r_1\omega_1, \ldots, r_d\omega_d))$ is strictly increasing for non–degenerate $(\omega_1, \ldots, \omega_n)$ if and only if $L \cap X^0(r_1\omega_1, \ldots, r_d\omega_d) \neq \emptyset$. X^0 is the relative interior of X. Though φ_L

is not strictly increasing for subspaces L with $\dim L = k < d$, we can state Theorem 1 in its strict form, as we can construct an f-additive functional Vol_k, whose expectation functional Φ_k fulfills $\Phi_k \equiv \Phi_L$ and whose simplex functional φ_k is strictly increasing:
For $k = 1, \ldots, d$ define

$$\mathrm{Vol}_k(X) := \mathbf{E}(\mathrm{Vol}_L(X)), \tag{20}$$

where the average in (20) is taken on all k-dimensional subspaces $L \subseteq \mathbb{R}^d$. We assume the distribution of the subspaces L invariant under rotations around the origin. Obviously, Vol_k is f-additive with sign functional $\sigma(X, I) = -1$ if X_I is a boundary simplex of X of the second kind and with simplex functional $\varphi_k : \mathcal{P}_d^d \to \mathbb{R}_0^+$,

$$\varphi_k(X) := \mathbf{E}(\varphi_L(X)), \tag{21}$$

with φ_L as in (18). Under the distribution assumptions for a_i of Section 1, the expectation

$$\Phi_k(F^{(1)}, \ldots, F^{(n)}) := \mathbf{E}(\mathrm{Vol}_k(X)) \tag{22}$$

must be the same as the expectation $\Phi_L(F^{(1}, \ldots, F^{(n)})$ for a fixed subspace L with dimension k. On the other hand, the simplex functional φ_k from (21) is strictly increasing, which we prove in the following preposition:

Preposition: *For $k \in \{1, \ldots, d\}$ the simplex functional $\varphi_k : \mathcal{P}_d^d \to \mathbb{R}_0^+$ is strictly increasing.*

Proof: For any simplex $X \in \mathcal{P}_d^d$, $X = X(r_1\omega_1, \ldots, r_d\omega_d)$ holds

$$\mathbf{Pr}(X^0 \cap L \neq \emptyset) > 0, \tag{23}$$

as $\dim(\mathrm{conv}(X \cup \{0\})) = d$. The probability $\mathbf{Pr}(X^0 \cap L \neq \emptyset)$ is independent from the choice of the radii $r_i \in (0, \infty)$. Hence, by the law of total probability, we have for all $X \in \mathcal{P}_d^d$:

$$\varphi_k(X) = \mathbf{E}(\varphi_L(X) \mid L \cap X^0 \neq \emptyset)\, \mathbf{Pr}(L \cap X^0 \neq \emptyset) + \mathbf{E}(\varphi_L(X^0) \mid L \cap X^0 = \emptyset)\, \mathbf{Pr}(L \cap X = \emptyset). \tag{24}$$

The first conditioned expectation in (24) is strictly increasing in the radii $r_1, \ldots, r_d$, whereas the second is only increasing. Hence, as the probabilities of the conditions in (24) do not depend on the radii and (23) holds, φ_k must be strictly increasing. □

Theorem 2: *Let $\phi = (\varphi, \sigma)$ be a continuous f-additive polytope functional. In addition, let the simplex functional φ be (strictly) increasing. Then, $\Phi(F^{(1)}, \ldots, F^{(n)})$ is (strictly) decreasing in $F^{(i)}$ for all $i \in \{1, \ldots, n\}$.*

As Vol_k is a continuous f-additive polytope functional with strictly increasing simplex functionial φ_k, Theorem 2 is a generalization of Theorem 1.
The proof of Theorem 2 is dissected in two stages each stage formulated as a lemma. But before we can start, we need some more notation.

Let $(a_1, \ldots, a_n) = (r_1\omega_1, \ldots, r_n\omega_n)$ be an elementary event. As ω_i and r_i are stochastically independent, the functionals Φ from (19) can be written as

$$\Phi(F^{(1)}, \ldots, F^{(n)}) = \int\limits_{(0,\infty)^n} K_\phi(r_1, \ldots, r_n) dF^{(1)}(r_1) \ldots dF^{(n)}(r_n) \tag{25}$$

with

$$K_\phi(r_1, \ldots, r_n) := \mathbf{E}(\phi(X(r_1\omega_1, \ldots, r_n\omega_n))). \tag{26}$$

On the right hand side of (26) we average on the choice of $\omega_i \in \mathcal{S}^{d-1}$. K_ϕ is the *kernel* of the multilinear functional Φ. It is easily checked that

$$K_\phi(r_1, \ldots, r_n) = \Phi(F_{r_1}, \ldots, F_{r_n}), \tag{27}$$

where F_{r_i} is defined as in (10). Our first lemma links the monotonicity of Φ with that of K_ϕ:

Lemma 1: $\Phi(\ldots, F^{(i)}, \ldots)$ *is (strictly) decreasing in* $F^{(i)}$ *for an* $i \in \{1, \ldots, n\}$ *if and only if* $K_\phi(\ldots, r_i, \ldots)$ *is (strictly) increasing in* r_i.

Proof: We prove the lemma only in the strict form. The weak form is easier and can easily be derived from the given proof.
Necessity: Let us assume that there is an r_i and an s_i in $\mathbb{R}^+$ with $0 < r_i < s_i < \infty$, such that $K_\phi(\ldots, r_i, \ldots) > K_\phi(\ldots, s_i, \ldots)$. Using equation (27) and $F_{s_i} \prec F_{r_i}$ we obtain

$$\Phi(F_{r_1}, \ldots, F_{r_{i-1}}, F_{s_i}, F_{r_{i+1}}, \ldots, F_{r_n}) < \Phi(F_{r_1}, \ldots, F_{r_{i-1}}, F_{r_i}, F_{r_{i+1}}, \ldots, F_{r_n}), \tag{28}$$

which contradicts the assumption that Φ is strictly decreasing in the radial distribution functions.
Sufficiency: Let $F^{(i)}, G^{(i)} \in \mathcal{F}$ with $G^{(i)} \prec F^{(i)}$, where $\Phi(F^{(1)}, \ldots, F^{(i)}, \ldots, F^{(n)}) < \infty$ and $\Phi(F^{(1)}, \ldots, F^{(i-1)}, G^{(i)}, F^{(i+1)}, \ldots, F^{(n)}) < \infty$. From (25) we have

$$\Phi(F^{(1)}, \ldots, F^{(i)}, \ldots, F^{(n)}) = \int\limits_{(0,\infty)^{n-1}} \int\limits_0^\infty K_\phi(r_1, \ldots, r_i, \ldots, r_n) dF^{(i)}(r_i) \prod_{\substack{j=1 \\ j\neq i}}^n dF^{(j)}(r_j). \tag{29}$$

As $\Phi(F^{(1)}, \ldots, F^{(i)}, \ldots, F^{(n)})$ is finite, the inner integral in (29) must be finite as well. Let $0 < R < \infty$. Integration by parts of the $(0, R)$–part of the inner integral in (29) delivers

$$\int\limits_0^R K_\phi(\ldots, r_i, \ldots) dF^{(i)}(r_i) = \int\limits_0^R (F^{(i)}(R) - F^{(i)}(r_i)) dK_\phi(\ldots, r_i, \ldots). \tag{30}$$

As $K_\phi(\ldots, r_i, \ldots)$ is strictly increasing in r_i, the Riemann-Stieltjes integral on the right hand side of (30) is well defined. As $\Phi(\ldots, F^{(i)}, \ldots) < \infty$, we are allowed to pass over to limits $R \to \infty$ in (30). Hence, we have

$$\int\limits_0^\infty K_\phi(\ldots, r_i, \ldots) dF^{(i)}(r_i) = \int\limits_0^\infty \bar{F}^{(i)}(r_i) dK_\phi(\ldots, r_i, \ldots). \tag{31}$$

As $\bar{F}^{(i)} \prec \bar{G}^{(i)}$, we know $\bar{F}^{(i)}(t) < \bar{G}^{(i)}(t)$ for $t \in T$, where T is a subset of $(0,\infty)$ with positive Lebesgue measure. On the other hand, $K_\phi(\dots,r_i,\dots)$ is strictly increasing. Thus, we get

$$\int_0^\infty \bar{F}^{(i)}(r_i)dK_\phi(\dots,r_i,\dots) < \int_0^\infty \bar{G}^{(i)}(r_i)dK_\phi(\dots,r_i,\dots) \tag{32}$$

and hence by (31):

$$\int_0^\infty K_\phi(\dots,r_i,\dots)dF^{(i)}(r_i) < \int_0^\infty K_\phi(\dots,r_i,\dots)dG^{(i)}(r_i). \tag{33}$$

If we insert the estimate (33) into (29) we obtain the desired estimate

$$\Phi(\dots,F^{(i)},\dots) < \Phi(\dots,G^{(i)},\dots) \tag{34}$$

and the lemma is completely proven. □

Lemma 1 allows us to restrict our considerations to the kernel K_ϕ. Hence, by the aid of Lemma 1, Theorem 2 is an immediate consequence of the following lemma.

Lemma 2: *Under the prepositions of Theorem 2, $K_\phi(r_1,\dots,r_n)$ is (strictly) increasing in r_i, $i = 1,\dots,n$.*

Proof: By definition (26), we have

$$K_\phi(r_1,\dots,r_n) = \mathbf{E}(\phi(X(r_1\omega_1,\dots,r_n\omega_n))). \tag{35}$$

Unfortunately, the functional $\phi(X(r_1\omega_1,\dots,r_n\omega_n))$ is not necessarily (strictly) increasing in r_i for all non-degenerate elementary events $(a_1,\dots,a_n) = (r_1\omega_1,\dots,r_n\omega_n)$. For instance, take $\phi \equiv \mathrm{Vol}_L$ and the polytope X from Figure 1 as counterexamples. So, we try to replace ϕ by another functional $\bar{\phi}$ having the same expectation as ϕ if we average on ω_i and being (strictly) increasing in r_i for $i \in \{1,\dots,n\}$. Let $(a_1,\dots,a_n)$=$(r_1\omega_1,\dots,r_n\omega_n)$ be a non-degenerate event. Then, for any signature $s = (s_1,\dots,s_n)$, $s_i \in \{-1,1\}$, the event $(s_1a_1,\dots,s_na_n)$ is non-degenerate and has the same density, as the vectors ω_i have centrally symmetrical distributions in $\mathbb{R}^d$. Thus, if s and $\tilde{s}$ are two arbitrary but different signatures, we have

$$\mathbf{E}(\phi(X(s_1a_1,\dots,s_na_n))) = \mathbf{E}(\phi(X(\tilde{s}_1a_1,\dots,\tilde{s}_na_n))). \tag{36}$$

We define

$$\bar{\phi}(a_1,\dots,a_n) := 2^{-n} \sum_{\substack{s_i\in\{-1,1\}\\ i=1,\dots,n}} \phi(X(s_1a_1,\dots,s_na_n)) \tag{37}$$

and we conclude from (35) and (36) that

$$K_\phi(r_1,\dots,r_n) = \mathbf{E}(\bar{\phi}(r_1\omega_1,\dots,r_n\omega_n)). \tag{38}$$

The only matter left is to prove that $\bar{\phi}(a_1, \dots, a_n) = \bar{\phi}(r_1\omega_1, \dots, r_n\omega_n)$ is (strictly) increasing in all radii $r_i \in [0, \infty)$. By the definition of f-additive polytope functionals and from (37) we obtain

$$\bar{\phi}(a_1, \dots, a_n) = 2^{-d} \sum_{I=\{i_1,\dots,i_d\}\in\mathcal{I}} \sum_{\substack{s_i\in\{-1,1\}\\ i\in I}} \mathbf{E}(\sigma(X(s_1a_1, \dots s_na_n), I))\varphi(X(s_{i_1}a_{i_1}, \dots, s_{i_d}a_{i_d})) \tag{39}$$

with

$$\mathbf{E}(\sigma(X(s_1a_1, \dots s_na_n), I)) := 2^{-n+d} \sum_{\substack{s_i\in\{-1,1\}\\ i\notin I}} \sigma(X(s_1a_1, \dots, s_na_n), I). \tag{40}$$

For any non-degenerate event $(a_1, \dots, a_n)$ the right hand side of (40) is the expectation of $\sigma(X(s_1a_1, \dots, s_na_n), I)$ in the discrete centrally symmetrical probability space

$$\Omega(a_1, \dots, a_n) := \{(s_1a_1, \dots, s_na_n) \mid s_i \in \{-1, 1\} \text{ for } i = 1, \dots, n\} \tag{41}$$

illustrated in Figure 3. The probability space $\Omega(a_1, \dots, a_n)$ is a *symmetrization* of the single event $(a_1, \dots, a_n)$.

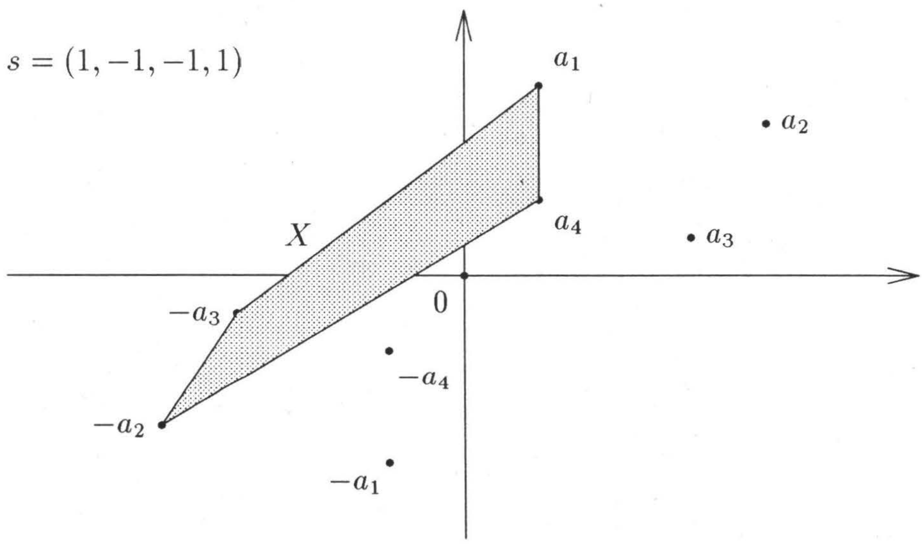

Figure 3: *The discrete probability space $\Omega(a_1, a_2, a_3, a_4)$*

The average in (40) is taken on the choice of s_j with $s_j \in \{-1, 1\}$ and $j \notin I$. s_j takes the value 1 and -1 resp. with probability 1/2. As $(a_1, \dots, a_n) \in \mathcal{A}_n^d$ is non-degenerate, it is easily checked that degeneracy has probability zero in $\Omega(a_1, \dots, a_n)$. We remark that $\Omega(a_1, \dots, a_n)$ does not satisfy the assumptions of the stochastic model from Section 1, as the probability of each elementary event in $\Omega(a_1, \dots, a_n)$ equals 2^{-n} and not zero.

Let us analyze the expectation $\mathbf{E}(\sigma(X, I))$ in a more general framework for ease of notation:

From (16), we know that for a fixed $\tau \in \{-1, 0, 1\}$ holds:

$$\mathbf{E}(\sigma(X(a_1, \dots, a_n), I)) = \mathbf{E}(\chi_1(X(a_1, \dots, a_n), I)) - \tau\,\mathbf{E}(\chi_2(X(a_1, \dots, a_n), I)). \tag{42}$$

The average in (42) is taken on centrally symmetrically distributed and stochastically independent a_i, $i \notin I$. In addition, we assume that degeneracy of $(a_1, \ldots, a_n)$ has probability zero. This setting covers the discrete probability spaces $\Omega(a_1, \ldots, a_n)$ as well as the stochastic model from Section 1.

It is our first objective to prove that $\mathbf{E}(\sigma(X(a_1, \ldots, a_n), I)) \geq 0$ for all $I \in \mathcal{I}$:

For any $I \in \mathcal{I}$ and any X in $\mathcal{P}_n^d$ let $H_I(X) := \text{aff}(X_I)$ be the hyperplane generated as affine hull of X_I and $\mathcal{H}_I^{(j)}(X)$, $j = 1, 2$, be the closed halfspaces generated by $H_I(X)$ with $0 \in \mathcal{H}_I^{(1)}(X)$ and $\mathcal{H}_I^{(1)}(X) \cup \mathcal{H}_I^{(2)}(X) = \mathbb{R}^d$, cf. Figure 4.

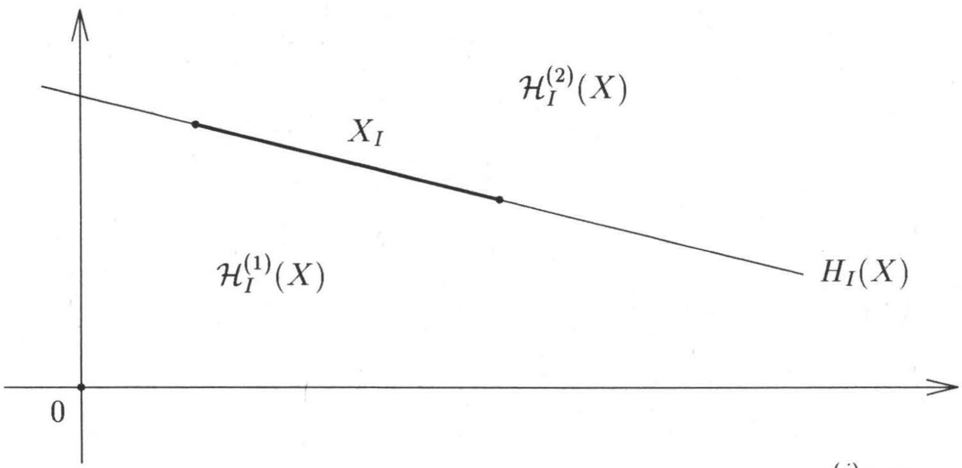

Figure 4: *The hyperplane $H_I(X)$ and the halfspaces $\mathcal{H}_I^{(j)}(X)$*

Elementary geometrical insight delivers that

$$\chi_j(X, I) = \prod_{i \in I} \chi(a_i \in \mathcal{H}_I^{(j)}(X)). \tag{43}$$

Thus, by independence of the a_i we get

$$\mathbf{E}(\chi_j(X, I)) = \prod_{i \in I} \mathbf{Pr}(a_i \in \mathcal{H}_I^{(j)}(X)). \tag{44}$$

For any centrally symmetrical probability space, elementary geometrical insight delivers that for all $I \in \mathcal{I}$ and all X in $\mathcal{P}_n^d$ holds:

$$\mathbf{Pr}(a \in \mathcal{H}_I^{(1)}(X)) \geq \frac{1}{2} \geq \mathbf{Pr}(a \in \mathcal{H}_I^{(2)}(X)). \tag{45}$$

Hence, using (42) and (44) we conclude that

$$\mathbf{E}(\sigma(X(a_1, \ldots, a_n), I)) \geq 0. \tag{46}$$

Next, we will show that in our discrete probability spaces $\Omega(a_1, \ldots, a_n)$ there are always certain $J \in \mathcal{I}$ and signatures $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n)$, such that $\mathbf{E}(\sigma(X(s_1a_1, \ldots, s_na_n), J)) = 1$ if $s_j = \tilde{s}_j$ for $j \in J$:

We take a set of indices $J = \{j_1, \ldots, j_d\} \in \mathcal{I}$ and a signature $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n)$ in such a way that the simplex $X(\tilde{s}_{j_1}a_{j_1}, \ldots, \tilde{s}_{j_d}a_{j_d})$ is a boundary simplex of the centrally symmetrical simplicial polytope $X(-a_1, \ldots, -a_n, a_1, \ldots, a_n)$. Then, for all $s_j \in \{-1, 1\}$ and $j \in \{1, \ldots, n\}$ holds

$$s_ja_j \in \mathcal{H}_J^{(1)}(X(\tilde{s}_{j_1}a_{j_1}, \ldots, \tilde{s}_{j_d}a_{j_d})), \tag{47}$$

which yields

$$\mathbf{E}(\sigma(X(s_1a_1, \ldots, s_na_n), J) \mid s_j = \tilde{s}_j \text{ for } j \in J) = 1. \tag{48}$$

Now, we prove that $\mathbf{E}(\sigma(X(s_1a_1, \ldots, s_na_n), I))$ is locally constant:

For any event $(a_1, \ldots, a_n) = (r_1\omega_1, \ldots, r_n\omega_n) \in \mathcal{A}_n^d$, any set of indices I and any signature

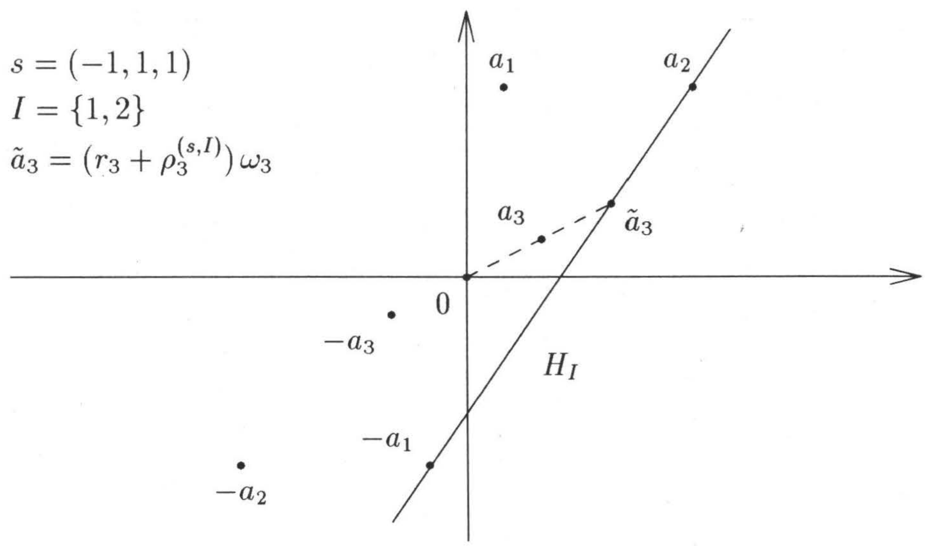

Figure 5: *The interval* $[r_j, r_j + \rho_j^{(s,I)})$

s we look at the functionals $e_j^{(s,I)} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$,

$$e_j^{(s,I)}(r) := \mathbf{E}(\sigma(X(s_1r_1\omega_1, \ldots, s_jr\,\omega_j, \ldots, s_nr_n\omega_n), I)), \tag{49}$$

for $j = 1, \ldots, n$. For any j there exists an interval $[r_j, r_j + \rho_j^{(s,I)})$ of maximal length $\rho_j^{(s,I)} > 0$, such that

$$e_j^{(s,I)}(r) = e_j^{(s,I)}(r_j) \text{ for } r \in [r_j, r_j + \rho_j^{(s,I)}). \tag{50}$$

Indeed, if we start with $r = r_j$, the functional $e_j^{(s,I)}$ does not change its value as long as the n–tuples $(s_1r_1\omega_1, \ldots, s_jr\,\omega_j, \ldots, s_nr_n\omega_n)$ remain non–degenerate for all $s_j \in \{-1, 1\}$ and

$j \in \{1, \ldots, n\}$. Geometrically, $(s_1 r_1 \omega_1, \ldots, s_j r\, \omega_j, \ldots, s_n r_n \omega_n)$ becomes degenerate for the smallest $r = r_j + \rho_j^{(s,I)}$ with

$$r\, s_j \omega_j \in H_I(\mathrm{conv}(s_i r_i \omega_i,\ i \in I,\ j \notin I)), \tag{51}$$

for some signature s and some set of indices I, cf. Figure 5 for an example.
As we consider only a finite number of points $s_j r_j \omega_j$, $\rho_j^{(s,I)}$ must be strictly positive for all choices of s and I.

Now, we are ready to check the monotonicity of $\bar{\phi}$: On the one hand, we have assumed that the simplex functional φ is (strictly) increasing. On the other hand, the expectation value $\mathbf{E}(\sigma(X(s_1 r_1 \omega_1, \ldots, s_n r_n \omega_n), I))$ is not zero for all s and I, as we have shown above. Thus, as $\mathbf{E}(\sigma(X(\cdot), I))$ is locally constant in the radii, cf. (50), we conclude that the functional $\bar{\phi}(r_1\omega_1, \ldots, r_{j-1}\omega_{j-1}, r\,\omega_j, r_{j+1}\omega_{j+1}, \ldots, r_n\omega_n)$ is (strictly) increasing for $r \in [r_j, r_j + \rho_j)$, where we define

$$\rho_j := \min_{s,I} \rho_j^{(s,I)} > 0 \tag{52}$$

for each $j = 1, \ldots, n$ and any event $(a_1, \ldots, a_n) = (r_1\omega_1, \ldots, r_n\omega_n) \in \mathcal{A}_n^d$. Furthermore, as ϕ is a continuous polytope functional, we obtain from (37) that $\bar{\phi}(r_1\omega_1, \ldots, r_n\omega_n)$ is continuous for all $r_i \in [0, \infty)$. Thus, $\bar{\phi}(r_1\omega_1, \ldots, r_n\omega_n)$ must be globally (strictly) increasing for all radii $r_i \in [0, \infty)$, which completes the proof of Lemma 2. □

Concluding remark: The surface area of a polytope $X \in \mathcal{P}_n^d$ with respect to the $(d-1)$-dimensional Lebesgue measure is a continuous and f-additive polytope functional with sign functional $\sigma(X, I) = 1$ and simplex functional $\varphi(X) = \lambda_{d-1}(X)$. On the other hand, it is easily checked that the simplex functional is not increasing. So, the functional "surface area" is not covered by Theorem 2. Nevertheless, the claim of Theorem 1 is true for the surface area as well. The proof can be done in a very similar way using a slightly different symmetrization technique.

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