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A Unified Asymptotic Probabilistic Analysis of Polyhedral Functionals: A Survey

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Summary. Let $A := \{a_i \,|\, i := 1, \ldots, m\}$ be an i.i.d. random sample in $\mathbb{R}^n$, which we consider a random polyhedron, either as the convex hull of the a_i or as the intersection of halfspaces $\{x \,|\, a_i^T x \leq 1\}$. We introduce a class of polyhedral functionals we will call "additive-type functionals", which covers a number of polyhedral functionals discussed in different mathematical fields, where the emphasis in our contribution will be on those, which arise in linear optimization theory. The class of additive-type functionals is a suitable setting in order to unify and to simplify the asymptotic probabilistic analysis of first and second moments of polyhedral functionals. We provide examples of asymptotic results on expectations and on variances.

Keywords. Polyhedral functionals, probabilistic analysis

1. Introduction

Systematic probabilistic analysis of polytope functionals like volume, surface area, number of facets or vertices was initiated by the paper "Über die konvexe Hülle von n zufällig gewählten Punkten" by Renyi and Sulanke in 1963, [22], where the authors considered convex hulls of finite random samples in $\mathbb{R}^2$ as random polytopes and analyzed asymptotically expectation values of the functionals under various distributions of the data. Besides, within the discussion of classical problems like Sylvester's four-point problem in stochastic geometry, polytope functionals are of interest in several mathematical fields like approximation theory of convex sets, complexity theory of algorithms or statistics.

Of particular interest to us is the probabilistic analysis of algorithms. An important example is the complexity problem of the simplex algorithm, which is known to be very effective in practice and intractable in worst-case situations. Borgwardt [3,4] was the first to confirm the conjectured polynomiality on the average theoretically for a dual variant of Gass and Saaty's parametric algorithm. He investigates the expectation of a geometrical polyhedral functional, the number of shadow vertices under orthogonal projections into planes, which is linked to the expectation of pivots required. In contrast to Renyi and Sulanke's approach the random objects he uses are halfspaces containing the origin, whose intersection forms a random polyhedron. Borgwardt interprets the polyhedral functional defined on polyhedra generated by intersection of halfspaces as a functional of a polar polytope generated by $\mathbb{R}^n$-samples, which is of similar type as the classical geometrical functionals mentioned above.

Besides the number of pivot steps required by Borgwardt's "shadow vertex algorithm", many other polyhedral random variables of interest in stochastic optimization theory can be interpreted as corresponding "polar" functionals on polar polytopes: the number of a polyhedron's vertices, the indicator functional whether a polyhedron is bounded, the indicator functional whether the maximum value of a linear program is less than a given positive number, the number of shadow vertices in arbitrary subspaces, the rate of a linear program's redundant constraints et cetera.

In our contribution we introduce a class of random variables defined on random finite $\mathbb{R}^n$-subsets, which can be considered random polyhedra, either as convex hulls or as intersections of halfspaces as well. This class of functionals we call "additive-type functionals" covers most polyhedral random variables of interest in the mentioned fields, as we are going to show in the first part of the paper, where the emphasis will be mainly on those functionals which can be interactively interpreted by means of polarisation. Second, after the introduction of the stochastic model we will give a survey on asymptotic results about expectation values. Here, the most important aspect of our presentation should be the unified approach to asymptotic results on moments of polyhedral functionals, which have been treated seperately so far. Finally, we will provide new asymptotic estimations of centralized second moments for the first time in full generality of dimensions.

2. The functionals

A finite $\mathbb{R}^n$-subset A is called nondegenerate if every A-subset of cardinality $\leq n$ is linearly independent and every A-subset of cardinality $\leq n+1$ is in general position. For fixed m, n, $m \geq n \geq 2$ let $\mathcal{A} := \mathcal{A}_{m,n}$,

$$(1) \qquad \mathcal{A}_{m,n} := \{A \,|\, A = \{a_1, \ldots, a_m\} \subset \mathbb{R}^n, \text{ A nondegenerate}\},$$

be the set of nondegenerate $\mathbb{R}^n$-subsets with cardinality m. If ambiguity is excluded we notate $\mathcal{A}$ abbreviating $\mathcal{A}_{m,n}$. Any nondegenerate set $A \in \mathcal{A}$ can be identified with a polyhedron in two ways. The first identification associates any element $a \in A$ with a closed halfspace $H^{(1)}(a) := \{x \in \mathbb{R}^n \,|\, a^T x \leq 1\}$ containing the origin. So, A may be associated wih a polyhedron X_A,

$$(2) \qquad X_A := \bigcap_{a \in A} H^{(1)}(a).$$

On the other hand, A defines the polytope

$$(3) \qquad Y_A := \text{convhull}(A).$$

The assumption of nondegeneracy on A is transferred to polyhedra X_A and Y_A: Both polyhedra are nonvoid and fulldimensional. X_A is simple, that means any k-dimensional face in X_A's face lattice $\mathcal{P}(X_A)$ intersects with exactly k boundary hyperplanes $H(a) := \{x \in \mathbb{R}^n \mid a^T x = 1\}$. Y_A is simplicial, which denotes that any k-dimesional face in Y_A's face-lattice $\mathcal{P}(Y_A)$ is a k-simplex.

Both identifications (2) and (3) are linked by polarisation, cf. Grünbaum [13]. Let

(4) $$\tilde{Y}_A := \text{convhull}(Y_A \cup \{0\}),$$

then $\tilde{Y}_A = X_A^*$, X_A^* being polar to X_A, which means

(5) $$X_A^* := \{y \in \mathbb{R}^n \mid x^T y \le 1, x \in X_A\}.$$

If we define the mapping $\Psi : \mathcal{P}(X_A^*) \to \mathcal{P}(X_A)$ by association of a X_A^*-face P^* with the X_A-face P, $P := \{x \in X_A \mid x^T y = 1, y \in P^*\}$, Ψ is one-to-one and inclusion-reversing. Roughly speaking, if X_A is especially bounded, there is a one-to-one correspondence between the k-dimensional faces of X_A and the $n-k-1$-dimensional faces of X_A^*, $k = 0, \ldots, n-1$.

We profit from this fact for we are able to interpret many combinatorial functionals on polyhedra of type X_A as functionals of polytopes Y_A as well and vice versa.

Within our considerations on the one hand functionals of polyhedra defined by intersections of halfspaces like (2) are of special interest, because this setting is the common situation we meet in linear optimization. On the other hand functionals of polytopes defined as convex hulls of points like (3) are easier to handle in the framework of stochastic geometry. Thus, if possible we analyze the polar interpretation of an X_A-functional defined on Y_A instead of the primal interpretation defined on X_A. This idea is due to Liebling [17], who used the polar link-up between X_A-vertices and X_A^*-facets in order to interpret results by Renyi and Sulanke [22], Carnal [7] and Raynaud [21] about the expected number of Y_A-facets as results on the number of X_A-vertices as well.

We are going to study a special class of functionals on sets $A \in \mathcal{A}$, which can be interpreted alternately on polyhedra of type X_A or Y_A. The functionals on A we are interested in are defined in terms of A-subsets of cardinality n. For any set of indices $I \subset \{1, \ldots, m\}$, $|I| = n$, let

(6) $$A_I := \{a_i \in A \mid i \in I\}, \quad S_I := \text{convhull}(A_I), \quad \tilde{S}_I := \text{convhull}(S_I \cup \{0\}).$$

By assumption of nondegeneracy S_I is an $n-1$-simplex for any I. In our considerations only subsets A_I are of interest, whose corresponding simplices S_I are boundary simplices (facets) of Y_A. We distinguish between two kinds of Y_A-boundary simplices. S_I is called a Y_A-boundary simplex of the first kind, if S_I is a facet of Y_A and $\tilde{S}_I \cap Y_A \neq S_I$, that means S_I is a boundary simplex of both Y_A and $\tilde{Y}_A$. S_I is called a Y_A-boundary simplex of the second kind, if S_I is a boundary simplex of Y_A and $S_I \cap \tilde{S}_I = S_I$, which means that S_I is a facet of Y_A and not of $\tilde{Y}_A$. We define corresponding indicator functionals $\chi_j(A, A_I)$ by

(7) $$\chi_j(A, A_I) := \begin{cases} 1 & S_I \text{ is a } Y_A\text{-boundary simplex of } j\text{-th kind} \\ 0 & \text{else} \end{cases}, \quad j = 1, 2.$$

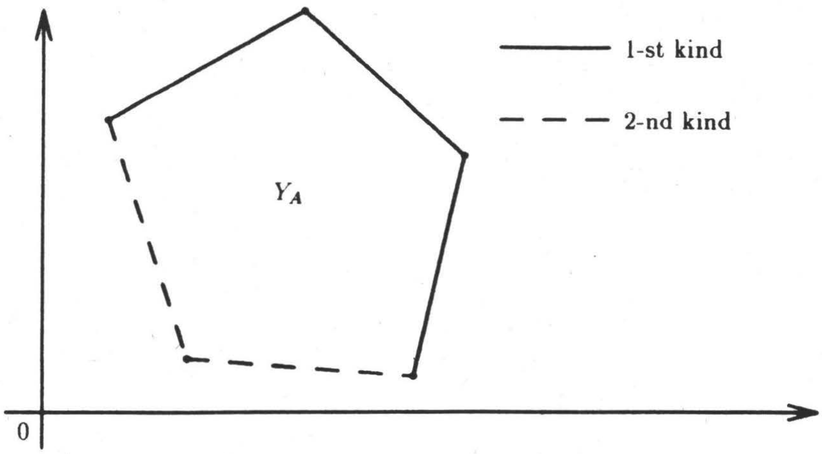

Fig. 1. Boundary simplices of two kinds

Now we are enabled to state our central definition of functionals we investigate:

Definition *A functional* $\tilde{Z} = (Z, \sigma)$ *is called an additive-type functional if for all* $A \in \mathcal{A}$:

$$\tilde{Z}(A) := \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=n}} [\chi_1(A, A_I) + \sigma\chi_2(A, A_I)]\, Z(A_I), \tag{8}$$

where $\sigma \in \{-1, 0, 1\}$ *and* Z *is a positive functional defined on* $\mathbb{R}^n$*-subsets of cardinality* n.

The first factor in sum (8) indicates whether a subset A_I generates a Y_A-boundary simplex of first or second kind. The sign σ decides whether the contribution of second kind boundary simplices has to be considered either negatively, positively or not at all. The functional Z defined on the Y_A-boundary simplices, say, and σ characterize the functional $\tilde{Z}$ completely. That means an additive-type functional decomposes additively relative to the boundary simplices of Y_A. The term additive-type functional is above all a technical one, which covers most of the polyhedral functionals of interest. We introduce it mainly in order to unify and to simplify formulations of results on polyhedral functionals rather than for axiomatic reasons.

Though many polytope functionals can be analyzed deterministically too, c.f. Grünbaum [13], McMullen and Shephard [19], Croft, Falconer and Guy [8] for surveys and further references, probabilistic analysis is often more appropriate. If sets $A \in \mathcal{A}$ are generated at random within a suitably chosen stochastic model, functionals of type (8) become random variables. Before we state precisely the stochastic model we use, we present some examples from different fields of application covered by definition (8) supplied with references on their probabilistic analysis. The first two examples are polytope functionals, which have no natural interpretations on polyhedra of type (2).

Example 1: *Volume of a polytope*

The functional "volume" of a polytope is of great interest in stochastic approximation theory of convex sets. Given a convex and compact $\mathbb{R}^n$-subset $\mathcal{C}$, choose some m vectors a_i in $\mathcal{C}$ and take their convex hull Y_A as approximation for $\mathcal{C}$. The deviation of volume $d(\mathcal{C}, Y_A) := \lambda_n(\mathcal{C} \setminus Y_A)$, λ_n being the Lebesgue-measure of dimension n, may serve as the error of approximation. The question is: How does this error depend on the geometry of $\mathcal{C}$? As it is very hard to find best approximations for general $\mathcal{C}$, probabilistic analysis seems more appropriate.

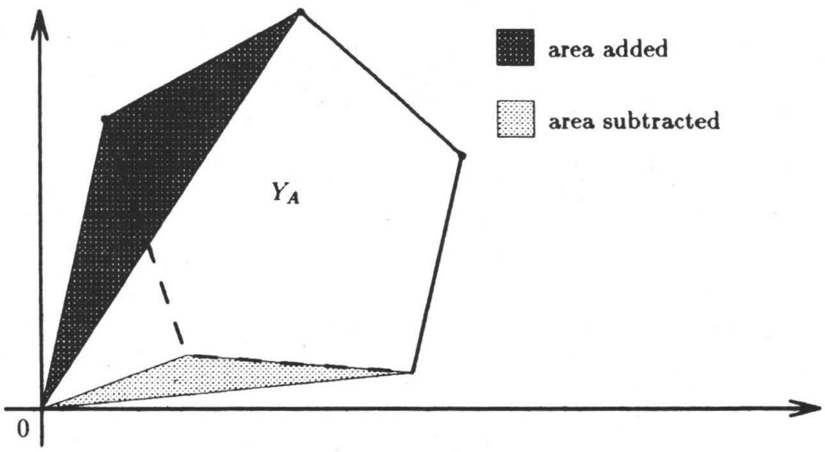

Fig. 2. Illustration of the functional "volume"

Most papers deal with the expectation $\overline{d}(\mathcal{C}, m) := \mathrm{E}_A(d(\mathcal{C}, Y_A))$ of the volume deviation, where sets A consist of random vectors a_i, which are independently, uniformly on $\mathcal{C}$ distributed random vectors. Almost all results are of asymptotic character and link the rate of $\overline{d}$'s decreasing for large m with smoothness properties of $\mathcal{C}$'s boundary. As the list of references on this topic is very long, we refer the reader to the survey articles of Gruber [12], Buchta [6], Schneider [23] and Weil and Wieacker [25]. A widely open problem is the question for the distribution of the volume deviation or for its higher moments. The only result so far is due to Groeneboom [11], who announced an asymptotic analysis of Y_A's limiting distribution for large m in case of $n = 2$.

The Lebesgue-volume λ_n of a polytope is an additive-type polytope functional in the sense of our definition:

$$\lambda_n(Y_A) = (Z(A_I), \sigma), \quad Z(A_I) = \lambda_n(\tilde{S}_I), \quad \sigma = -1. \tag{9}$$

An easy generalization of the functional "volume" is at hand, if we replace the Lebesgue-measure λ_n in (9) by an arbitrary measure μ. In case of measure μ being a probability measure on $\mathbb{R}^n$ the resulting functional equals the "probability content" of Y_A. This interpretation finds interesting statistical applications, cf. Efron [10] or Meilijson [20].

Example 2: *Surface area of a polytope*

A second variable of interest in stochastic geometry and stochastic approximation theory of convex sets is the surface area of a polytope. The interested reader is referred to the survey papers cited above. Let ∂B be the boundary of an $\mathbb{R}^n$-subset B and λ_{n-1} the Lebesgue-measure of dimension $n-1$, then the Lebesgue surface area satisfies

$$(10) \qquad \lambda_{n-1}(\partial Y_A) = (Z(A_I), \sigma), \quad Z(A_I) = \lambda_{n-1}(S_I), \quad \sigma = 1.$$

Examples 3 and 4 deal with combinatorial functionals of polytopes and polyhedra of type (2).

Example 3: *Number of a polytope's (first kind) boundary-simplices*

The number of a polytope's boundary simplices (facets) is a polytope functional which was investigated by many mathematicians beginning with Euler, who stated the famous relation between number of facets, edges and vertices in the three-dimensional case. Research about the combinatorial theory of convex polytopes has been renewed by complexity analysis in linear programming. Motzkin's question "What is the maximum number of facets of a polytope, when the number of vertices is given?" and his "upper-bound conjecture" were a challenge for a number of researchers, cf. McMullen and Shephard [19] for a survey. The probabilistic analysis of expectations of the functional "number of facets" was initiated by Renyi and Sulanke [22], who investigated random polygones, generated by random samples under special distributions in $\mathbb{R}^2$. Carnal [7] and Raynaud [21] generalized their results to classes of distributions and arbitrary dimensions respectively. Groeneboom [11] and Aldous et al. [1] gave asymptotic limiting distributions for large random samples in $\mathbb{R}^2$. All results about the number of facets of the polytope $\tilde{Y}_A$ are polarly interpretable as results on the polyhedron X_A's vertices as well.

Let $f_{n-1}(Y_A)$ denote the number of boundary simplices of Y_A, then obviously by the definition of additive-type functionals:

$$(11) \qquad f_{n-1}(Y_A) = (Z(A_I), \sigma), \quad Z(A_I) = 1, \quad \sigma = 1.$$

Defining $f_{n-1}^{(1)}(Y_A)$ as the number of Y_A-boundary simplices of the first kind, (11) holds for $f_{n-1}^{(1)}(Y_A)$ too if we set $\sigma = 0$. It is well known, cf. Liebling [17], that $f_{n-1}^{(1)}(Y_A) = v(X_A)$, $v(X_A)$ being the number of X_A's vertices. Thus, $f_{n-1}^{(1)}(Y_A) = (Z(A_I), \sigma)$, $Z(A_I) = 1$, $\sigma = 0$, is interpretable on Y_A and X_A.

Example 4: *Number of shadow vertices*

If we project a polyhedron orthogonally into a subspace L , its image is again a polyhedron. We call a vertex of the polyhedron "L-shadow vertex" if the image of the vertex remains a vertex under the projection. A natural question is: "What is the number of the polyhedron's shadow vertices in L?" For polyhedra X_A the number of shadow vertices is an additive-type functional:

For fixed $k \in \{1, \dots, n\}$ let $U := \{u_1, \dots, u_k\}$ be a linear independent $\mathbb{R}^n$-subset. Hence, $L :=$linhull(U) is a linear $\mathbb{R}^n$-subspace of dimension k. For any set $A \in \mathcal{A}_{m,n}$, which satisfies $A \cup U \in \mathcal{A}_{m+k,n}$, let $S_U(X_A)$ be an abbreviation for the number of X_A's shadow vertices under an orthogonal projection into the subspace L, we have

(12) $$S_U(X_A) = (Z(A_I),\sigma),\ \ Z(A_I) = \chi(L \cap S_I \neq \emptyset),\ \ \sigma = 0.$$

This is due to the observation that the shadow vertices of X_A in the projection space L correspond one-to-one to the first kind boundary simplices of Y_A intersected by L. For $k = n$ the number of shadow vertices equals the number of vertices. Thus, example 4 is a generalization of example 3. For $k = 2$ the number of X_A-shadow vertices has an interesting application in complexity theory of linear optimization:

Example 5: *The number of pivots taken by the simplex algorithm*

As mentioned in the introduction, Borgwardt's probabilistic complexity analysis of the shadow vertex algorithm, a dual interpretation of Gass and Saaty's parametric simplex variant, is based on geometrical functionals on a polytope.

Let $A \in \mathcal{A}_{m,n}$ be such that $A \cup \{u,v\} \in \mathcal{A}_{m+2,n}$ is a nondegenerate set for fixed linear independent $\mathbb{R}^n$-vectors u and v. The simplex path taken by phase II of the shadow vertex algorithm in order to solve the linear programming problem $\max_{x \in X_A} v^T x$, when we start with an X_A-vertex x_0, which satisfies $u^T x_0 = \max_{x \in X_A} v^T x$, is uniquely determined. We will denote the number of pivots of the simplex path by $s_{u,v}(X_A)$. Functional $s_{u,v}$ is a functional of additive type, because

(13) $$\begin{aligned} s_{u,v}(X_A) &= (Z(A_I),\sigma),\ \ \sigma = 0,\\ Z(A_I) &= \chi(\mathrm{cone}(u,v) \cap S_I \neq \emptyset)\,\chi(\mathbb{R}^+ u \cap S_I = \emptyset). \end{aligned}$$

In the language of the polyhedron Y_A, the number of pivots $s_{u,v}(X_A)$ is equal to the boundary simplices S_I of Y_A, which are intersected by the cone generated by u and v and which are not intersected by the ray generated by u. (Representation (13) is an immediate consequence of (2.1.6) in Borgwardt(1987).)

The name shadow vertex algorithm is due to the fact that all vertices of the simplex path are shadow vertices of the polyhedron X_A under an orthogonal projection into the plane linhull(u,v). In Borgwardt's stochastic model the expected number of pivots required and the expected number of shadow vertices are the same up to a constant factor, cf. Borgwardt [3,4,5]. So, Borgwardt analyzes the expectation of the functional "number of shadow vertices" $S_{u,v}$, cf. example 4, instead of functional $s_{u,v}$'s expectation.

The remaining examples of additive-type functionals are indicator functionals (0-1-functionals), that means they indicate whether a statement is true, which we denote by 1, or false, which is denoted by 0. Expectation values of such indicator functionals lead to probability distributions in a natural way.

Example 6: $\chi(a \in Y_A)$

If we choose $A \in \mathcal{A}_{m,n}$ such that $A \cup \{a\} \in \mathcal{A}_{m+1,n}$ for a fixed a in $\mathbb{R}^n$, then

(14) $$\chi(a \in Y_A) = (Z(A_I),\sigma),\ \ Z(A_I) = \chi(a \in S_I),\ \ \sigma = -1.$$

The functional defined by (14) can be interpreted as follows: a is no vertex of $Y_{A \cup \{a\}}$ if and only if $a \in Y_A$. Hence, the number $v(Y_A)$ of vertices of the polytope Y_A can be represented by the equation

$$(15)\qquad v(Y_A) = \sum_{i=1}^{m} (1 - \chi(a_i \in Y_{A\setminus\{a_i\}})),$$

which is due to Efron [10]. Thus, the functional "number of vertices" of the polytope Y_A is not additive-type itself, but it is a sum of additive-type functionals.

In the same way we can handle the functional "number of facets" of the polyhedron X_A, which is polar to $v(Y_A)$: a lies in the polytope $\tilde{Y}_A$ if and only if the additional constraint $a^T x \leq 1$ does not change the polyhedron X_A, that means $a^T x \leq 1$ is a redundant equation for the polyhedron $X_{A\cup\{a\}}$. The indicator functional $\chi(a^T x \leq 1$ is redundant for $X_A)$ can be defined by (14) if we replace $\sigma = -1$ by $\sigma = 0$. So, the number of facets $f_{n-1}(X_A)$ and the rate of redundancy redrate(X_A) of the polyhedron X_A is described by

$$(16)\qquad f_{n-1}(X_A) = \sum_{i=1}^{m} (1 - \chi(a_i \in \tilde{Y}_{A\setminus\{a_i\}})) = m(1 - \text{redrate}(X_A)).$$

The number of vertices of the polyhedron X_A, cf. example 3, and the rate of redundancy of X_A have to be discussed together with the choice of the stochastic model for the probabilistic analysis of the simplex method, cf. May and Smith [18], Shamir [24] or Borgwardt [5], as the mass of highly redundant systems or the mass of systems with very few vertices should not be too high in the probability space, otherwise the results about the expected number of pivot steps are not significant for real problems. The next functional may be discussed in the same context:

Example 7: $\chi(\text{cone}(Y_A) = \mathbb{R}^n)$

The indicator functional deciding whether total $\mathbb{R}^n$ is spanned by the convex cone of a polytope was first discussed by Schläfli and independently by Wendel [26]. The polar interpretation on X_A is as follows: If $\text{cone}(Y_A) = \mathbb{R}^n$ then X_A is bounded and vice versa. Let Ω_n be the n-dimensional unit ball. The introduced functional is given as additive-type functional by

(17)
$$\chi(\text{cone}(Y_A) = \mathbb{R}^n) = (Z(A_I), \sigma), \quad Z(A_I) = \frac{\lambda_n(\text{cone}(A_I) \cap \Omega_n)}{\lambda_n(\Omega_n)}, \quad \sigma = -1.$$

Example 8: *The maximum value of a linear programming problem*

Another question of interest in stochastic analysis of linear programming is the distribution function of the objective function's optimal value. For polyhedra of type (3) this question can be reduced to the calculation of the expectation of the additive-type functional introduced as follows:

Let $A \in \mathcal{A}_{m,n}$ such that $A \cup \{v\} \in \mathcal{A}_{m+1,n}$ is nondegenerate for a fixed $v \in \mathbb{R}^n$. Furthermore, for any set A_I let $w(A_I)$ be the normal vector of the hyperplane $H(A_I)$ of unit length, $h(A_I)$ be the distance of the hyperplane $H(A_I)$ to the origin. Then, the indicator functional $\chi(\max_{x \in X_A} v^T x < s)$ depending on the parameter $s \in \mathbb{R}^+$ can be additively defined by

$$(18) \qquad \chi(\max_{x \in X_A} v^T x < s) = (Z(A_I), \sigma), \quad \sigma = 0,$$

$$Z(A_I) = \chi(\mathbb{R}^+ v \cap S_I \neq 0)\chi(\frac{|(v, w(A_I))|}{h(A_I)} < s).$$

As identity (18) is not obvious, we shall prove it: If $\max_{x \in X_A} v^T x$ is finite, then, by assumption of nondegeneracy, the maximum is attained in a uniquely determined vertex x_0, say, whose set of active constraints may be A_I. On the other hand, we know vertex x_0 with active constraints A_I is optimal for the functional $v^T x$, if the ray $\mathbb{R}^+$v intersects S_I being a first kind boundary-simplex of Y_A. This establishes the first factor of $Z(A_I)$'s definition. If y_0 is the point of intersection of $\mathbb{R}^+$v and S_I there exists a positive ρ with $v = \rho y_0$ implying $v^T x_0 = \rho$. A simple geometric observation delivers $\rho h(A_I) = |(v, w(A_I))|$, cf. figure 3. So, $v^T x_0 < s$, if and only if $|(v, w(A_I))| < sh(A_I)$, which completes the proof of (18).

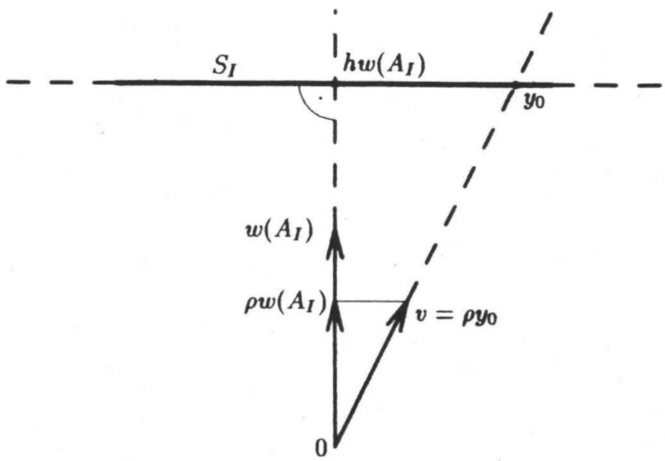

Fig. 3. Illustration of (18)

Besides the mentioned examples many other functionals of polytopes or polyhedra of type (2) and (3) can be denoted as additive-type functionals. For example, if we average on the choice of subspace $L = L(U)$ in example 4, on the choice of a in example 6 or on the choice of v in example 8, new functionals of additive type arise. More generally, any weighted sum or integral of additive-type functionals is an additive-type functional again. Of course, this is not true for sums (15) or (16) as m is a limit of the summation index there.

3. The stochastic model

The stochastic model which we use for generation of random sets A is the "rotation invariance model", which was introduced by Renyi and Sulanke [22] and which has been used in most contributions on probabilistic analysis of polyhedral functionals. We choose random samples $A := \{a_1, \ldots, a_m\}$ consisting of m identically, indepently, distributed random vectors $a_i \in \mathbb{R}^n \setminus \{0\}$ by a rotation invariant distribution. That means the polar representations of a_i, $i = 1, \ldots, m$,

$$(19) \qquad a_i = r_i w_i, \quad r_i \in \mathbb{R}^+, \quad w_i \in \omega_n,$$

consist of stochastically independent radial parts r_i and directional parts w_i. The radial part r_i is arbitrarily distributed by a radial distribution function (RDF) F, which we assume to be continuous from the right without loss of generality. Often, we use the tail $\overline{F} := 1 - F$ of the distribution F instead of F. The directional part w_i is uniformly distributed on the unit sphere ω_n in $\mathbb{R}^n$. The sets A generated in the described way are nondegenerate with probability one, i.e. $\Pr(A \in \mathcal{A}) = 1$. Hence, it is possible to omit the discussion of degenerate events A.

We concentrate on two important subclasses of rotationally invariant distributions. The first is the class of distributions with algebraic tails concentrated in the n-dimensional unit ball, given by

$$(20) \qquad \mathcal{F}_{\mathrm{alg},\Omega_n} := \{F \mid \overline{F}(1-r) \sim Cr^\alpha,\ r \to 0+,\ C, \alpha > 0\}.$$

The sign "$\sim$" denotes asymptotic equivalence. $\mathcal{F}_{\mathrm{alg},\Omega_n}$ serves as an example for a class of distributions, whose masses are concentrated in a compact subset of $\mathbb{R}^n$. The uniform distribution in the ball is a special case in $\mathcal{F}_{\mathrm{alg},\Omega_n}$ setting $C = n$ and $\alpha = 1$. The uniform distribution on the unit sphere can be considered a pointwise limiting case for α tending to 0. The second class consists of distributions, which have no bounded support, and is characterized by exponential tails:

$$(21) \qquad \mathcal{F}_{\mathrm{exp},\mathbb{R}^n} := \{F \mid \overline{F}(r) \sim Cr^\eta \exp(-\beta r^\gamma),\ r \to \infty,\ \eta \in \mathbb{R},\ \mathrm{C}, \beta, \gamma > 0\}.$$

Here the Gaussian normal distribution is a special case if we set $\eta = n - 2$, $\beta = 1$, $\gamma = 2$ and $C = n/\Gamma(n/2)$. The investigation of classes (20) and (21) in the context of polytope functionals has been initiated by Carnal [7] for $n = 2$. Instead of $\mathcal{F}_{\mathrm{alg},\Omega_n}$ Carnal studied the wider class of distributions with regularly varying tails. Though our results can be generalized to this class too, cf. Küfer [14], we deal with the smaller class (20) for simplicity of notation. Besides the mentioned classes we use $\mathcal{F}_{\Omega_n}$ as abbreviation for the class of all distributions, whose support is the n-dimensional unit ball Ω_n, more formally $\mathcal{F}_{\Omega_n} := \{F \mid F(1) = 1,\ F(r) < 1,\ r < 1\}$. $\mathcal{F}_{\mathbb{R}^n}$ denotes the family of distributions, which is not concentrated in any compact subset of $\mathbb{R}^n$.

4. Asymptotic results on expectations

If random samples A are generated by the rotation-invariance model, expectation values $\mathrm{E}(\tilde{Z})$ of additive-type functionals $\tilde{Z}$ satisfy

$$\mathrm{E}(\tilde{Z}) = \binom{m}{n} \int_0^\infty [G_1^{m-n}(h) + \sigma G_2^{m-n}(h)] \Lambda_Z(h) dh. \tag{22}$$

$G_1(h')$ equals the probability that a vector $a \in A \setminus A_I$ belongs to the halfspace $H^{(1)}(A_I)$ under the condition that the distance $h(A_I)$ of the generating hyperplane $H(A_I)$ is h'. More formally: $G_1(h') := \Pr(a \in H^{(1)}(A_I) \,|\, h(A_I) = h')$. $G_2 = 1 - G_1$ is the complementary distribution. $\Lambda_Z(h)$ is closely related to a conditioned expectation value of the boundary simplex functional Z. We have: $\Lambda_Z(h') := E(Z(A_I) \,|\, h(A_I) = h')p(h')$, where $p(h')$ is the density function of the distribution $\Pr(h(A_I) \leq h')$. If $Z(A_I)$ is additionally invariant under simultanious rotations of its arguments, (22) is proven by Fubini's theorem and $\Lambda_Z(h)$ can be presented explicitly.

For special functionals and special distributions formula (22) was first developed by Renyi and Sulanke [22] for $n = 2$ and by Raynaud [21] for arbitrary n. Integrals of type (22) become Laplacian-type integrals if we substitute $G_2(h) = e^{-t}$. These integrals can be asymptotically evaluated for large m by use of Watson-type results on Laplacian-type integrals if we know the asymptotic behaviour of $\Lambda_Z(h)$ near the boundary of the underlying distribution's domain. On the other hand, the asymptotic behaviour of $\Lambda_Z(h)$ depends on the growth of $Z(A_I)$ near the boundary of the distribution's domain and on the tail of the distribution's RDF.

A basic class of additive-type functionals $\tilde{Z}$ is described by boundary-simplex functionals Z, which are asymptotically essentially equivalent to powers of absolute values of determinants. In this situation an asymptotic evaluation of $\Lambda_Z(h)$ is possible for classes (20) and (21) of rotationally invariant distributions, which results in asymptotic equivalents for the expectation values $\mathrm{E}(\tilde{Z})$. For ease of notation, we will use $|\det(A_I)|$ in the following, where the set A_I is interpreted as a matrix. The term is well defined, as the absolute value of a determinant is independent under permutations of columns.

Theorem 1 *For any additive-type functional $\tilde{Z} = (Z, \sigma)$, which satisfies*

$$Z(A_I) \sim C(1 - h(A_I))^\tau |\det(A_I)|^q, \; h(A_I) \to 1-, \tag{23}$$

for some $C, \tau > 0$, $q \in \mathbb{N}_0$ and any distribution with RDF $F \in \mathcal{F}_{\mathrm{alg},\Omega_n}$:

$$\mathrm{E}(\tilde{Z}) \sim \tilde{C}\, m^{[(1-q)(n-1)-\tau]/(n-1+2\alpha)}, \; m \to \infty. \tag{24}$$

Theorem 2 *For any additive-type functional $\tilde{Z} = (Z, \sigma)$, which satisfies*

$$Z(A_I) \sim C h(A_I)^\delta |\det(h^{-1}(A_I) A_I)|^q, \; h(A_I) \to \infty, \tag{25}$$

for some $C > 0$, $\delta \in \mathbb{R}$, $q \in \mathbb{N}_0$ and any distribution with RDF $F \in \mathcal{F}_{\mathrm{exp},\mathbb{R}^n}$:

$$\mathrm{E}(\tilde{Z}) \sim \tilde{C} \log(m)^{\delta/\gamma + (1-q)(n-1)/2}, \; m \to \infty. \tag{26}$$

Theorems 1 and 2 show, how we can profit from the setting of additive-type functionals. The given theorems and modifications of them allow a unified presentation of many well-known and new results about expectations of polyhedral functionals.

Volume and surface area of a polytope Y_A, cf. examples 1 and 2, become special cases of the functionals $\tilde{Z}$ in both theorems by setting $\tau = 0$, $q = 1$ and $\delta = n$ or $\tau = 0$, $q = \delta = 1$ respectively, as is easily established using (9) and (10). Sharpening and generalizing theorem 1, one can show that for all RDF $F \in \mathcal{F}_{\Omega_n}$ the expected volume of Y_A tends to the volume of the ball and the expected surface area of Y_A tends to the surface area of the unit sphere.

Much more interesting than these plausible asymptotic equivalents for volume and surface area is the investigation of the asymptotic behaviour of the deviation of volume and surface area from the volume and the surface area of the unit ball. We only deal with the deviation of the volume, while the surface area can be treated in an analogous way.

Unfortunately, the deviation of the volume $d(\Omega_n, Y_A)$ is not additive-type. Thus, instead of $d(\Omega_n, Y_A)$ we investigate $\tilde{Z}(A){:=}d(\Omega_n \cap \text{cone}(Y_A), Y_A)$, which is additive-type by definition

(27) $$\tilde{Z} = (Z, \sigma), \quad Z(A_I) := \lambda_n(\text{cone}(S_I) \cap \Omega_n) - \lambda_n(\tilde{S}_I), \quad \sigma = -1.$$

We observe that both functionals are identical for nondegenerate A if Y_A has no boundary-simplices of the second kind. On the other hand, Y_A has no boundary simplices of the second kind if and only if $\text{cone}(Y_A) = \mathbb{R}^n$ for nondegenerate sets A. Schläfli and independently Wendel [26] showed for all distributions of our model, cf. example 6:

(28) $$\Pr(\text{cone}(Y_A) \neq \mathbb{R}^n) = \sum_{k=0}^{n-1} \binom{m-1}{k} 2^{-m+1},$$

which implies:

(29) $$\mathrm{E}(\lambda_n(\Omega_n \setminus \text{cone}(Y_A))) = \mathcal{O}(m^n 2^{-m}), \quad m \to \infty.$$

Hence, $d(\Omega_n \cap \text{cone}(Y_A), Y_A)$ is a good approximation for $d(\Omega_n, Y_A)$ if m is large. $d(\Omega_n \cap \text{cone}(Y_A), Y_A)$ satisfies the prepositions of theorem 1 with $\tau = 1$ and $q = 1$. Here, in case of uniform distribution theorem 1 meets Renyi and Sulanke's [22], Raynaud's [21] and Meilijson's [20] results. For an asymptotic evaluation of the expected deviation $d(C, Y_A)$ in case of general convex and compact sets C the interested reader is referred to Bárány [2].

For the Gaussian normal distribution in $\mathbb{R}^2$ result (26) on area and circumference is due to Renyi and Sulanke [22]. The generalization to class (21) in $\mathbb{R}^2$ was done by Carnal [7], while Raynaud [21] achieved the corresponding result for the volume in case of the normal distribution in $\mathbb{R}^n$ for arbitrary n.

The number of Y_A's facets or polarly spoken X_A's vertices, cf. example 3, are special cases of the theorems 1 and 2 too with $q = 0$, $\tau = 0$ and $\delta = 0$, for which the same authors achieved results in case of the mentioned special distributions and dimensions.

The additive-type functional "number of shadow vertices $S_U(X_A)$ of X_A in the k-dimensional subspace $L(U)$", cf. example 4, does not fulfill the assumptions of the theorems 1 and 2. Here, for $k < n$ the asymptotic behaviour of the boundary simplex functional Z is of more general but similar type as those in (25) or (26). We omit the details and refer to a forthcoming paper of the author. The results are announced in Küfer [16]. Asymptotic estimations of expectations in the special case $k = 2$ and for the related functional "number of pivot steps required by the shadow vertex algorithm" have been done by Borgwardt [3,4,5]. Asymptotic equivalents for classes (20) and (21) are due to the author and can be found in Küfer [14,15]. A typical result is the following one: Let $F \in \mathcal{F}_{\mathrm{alg},\Omega_n}$, then

$$\mathrm{E}(S_U) \sim \tilde{C} m^{(k-1)/(n-1+2\alpha)}, \quad m \to \infty. \tag{30}$$

To the end of this section we provide two typical results for expectation values of functionals introduced in examples 7 and 8:

For every RDF $F \in \mathcal{F}_{\Omega_n}$ redrate(X_A) tends to 1 as m tends to infinity. In case of $F \in \mathcal{F}_{\mathrm{alg},\Omega_n}$ the rate of converge can be given exactly:

$$1 - \mathrm{redrate}(X_A) = \mathcal{O}(m^{-2\alpha/(n-1+2\alpha)}), \quad m \to \infty. \tag{31}$$

If A is generated randomly by a distribution with an RDF $F \in \mathcal{F}_{\Omega_n}$ and v is distributed independently from A by an arbitrary distribution with RDF F_v, we have:

$$\Pr(\max_{x \in X_A} v^T x < s) \sim F_v(s), \quad s \in \mathbb{R}^+, \ m \to \infty. \tag{32}$$

5. Asymptotic results on variances

Corresponding to expectation values, integral representations of second moments of additive-type functionals $\tilde{Z} = (Z, \sigma)$ can be asymptotically handled as generalized Laplacian-type integrals. But here the situation is much more complicated for reasons of nonlinearity. Central object is the asymptotic analysis of threefold integrals of type

$$\int_0^\infty \int_0^\infty \int_0^\pi \tilde{G}^m(h_1, h_2, \varphi) \tilde{\Lambda}_Z(h_1, h_2, \varphi) d\varphi dh_1 dh_2 \tag{33}$$

for large m. We refer to the authors dissertation [12], where an integral formula for second moments of additive-type polyhedral functionals is achieved in case of $\tilde{Z}$ being invariant under simultaneous rotations of its arguments. We state two results on variances, which correspond to theorems 1 and 2 on expectations:

Theorem 3 *For any additive-type functional $\tilde{Z} = (Z, \sigma)$, which satisfies*

$$Z(A_I) \sim C(1 - h(A_I))^\tau |\det(A_I)|^q, \quad h(A_I) \to 1-, \tag{34}$$

for some $C, \tau > 0$, $q \in \mathbb{N}_0$ and any distribution with RDF $F \in \mathcal{F}_{\mathrm{alg},\Omega_n}$:

$$\frac{\mathrm{Var}(\tilde{Z})}{\mathrm{E}^2(\tilde{Z})} = \mathcal{O}(m^{-(n-1)/(n-1+2\alpha)}), \quad m \to \infty. \tag{35}$$

The asymptotic order in (35) is sharp up to a few pathological cases of distributions, where the rate of decreasing is better. Under the same assumptions on the boundary simplex functional Z theorem 3 can be generalized to the class $\mathcal{F}_{\Omega_n}$: For every $F \in \mathcal{F}_{\Omega_n}$ the quotient on the left hand side in (35) tends to zero as m tends to infinity. That means in the light of Chebychev's inequality: Even a small deviation from the mean value becomes rare if m is large for every fixed dimension n.

Theorem 3 generalizes a result of Groeneboom [11], who obtained information about the variance (and the limiting distribution) of the number of a polytope Y_A's facets in $\mathbb{R}^2$ for the uniform distribution in the unit ball. Theorem 3 is the first result on higher moments of polytope functionals like volume or surface area in full generality of dimensions. For general rotationally invariant distributions holds:

Theorem 4 *For any additive-type functional $\tilde{Z} = (Z, \sigma)$, which satisfies*

$$(36) \qquad Z(A_I) \sim Ch(A_I)^{\delta} |\det(h^{-1}(A_I)A_I)|^q, \; h(A_I) \to 1-,$$

for some $C > 0$, $\delta \in \mathbb{R}$, $q \in \mathbb{N}_0$ and any distribution with RDF $F \in \mathcal{F}_{\mathbb{R}^n}$:

$$(37) \qquad \frac{\mathrm{Var}(\tilde{Z})}{\mathrm{E}^2(\tilde{Z})} \leq \tilde{C}, \quad m \geq n \geq 2.$$

Theorem 4 meets a result of Aldous et al. [1], who investigated the limiting distribution for $m \to \infty$ for the number of a polytope Y_A's facets in $\mathbb{R}^2$ under rotationally invariant distributions with slowly varying tail. In this situation the expected number of facets tends to a constant, a fact which we know from Carnal [7]. The same is true for the variance. Thus, theorem 4 can not be improved for general rotationally invariant distributions. But, we conjecture that for RDFs $F \in \mathcal{F}_{\exp,\mathbb{R}^n}$ one can proof that the quotient of variance and the squared expectation tends to zero if m tends to infinity.

Theorem 3 and 4 cover the additive-type functionals introduced by examples 1,2,3 and 6 as mentioned in the previous section. For indicator type functionals the discussion of higher moments is not meaningful.
More involved is the investigation of a polyhedron X_A's number of shadow vertices in an arbitrary subspace $L(U)$, because the boundary simplex functional Z is not invariant under simultatious rotations of its arguments. We obtained as a first result for an RDF $F \in \mathcal{F}_{\mathrm{alg},\Omega_n}$, $k \in \{1, \ldots, n\}$, cf. Küfer [16]:

$$(38) \qquad \frac{\mathrm{Var}(S_U)}{\mathrm{E}^2(S_U)} = \mathcal{O}(m^{-(k-1)/(n-1+2\alpha)}), \quad m \to \infty.$$

A related paper, we should mention here, is due to Devroye [9], who estimated higher moments for a class of functionals on random samples A. Devroye's functionals are defined as cardinal numbers of certain A-subsets B, which are selected by a property remaining true for all subsets of B too. For instance let B be the set of all $a \in A$, which are vertices of the polytope Y_A. Obviously, every subset of B consists of Y_A-vertices too. Devroye assumed the vectors $a \in A$ i.i.d. in $\mathbb{R}^n$ under an arbitrary (not necessarily rotation invariant) distribution and proved that for the cardinality number $N(B)$ of sets having the "subset-property" the

p-th moment is bounded by the p-th power of the expectation times a constant depending on p only. Unfortunately, only few polyhedral functionals in discussion have this property.

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