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Abstract

Let $a_1, \ldots, a_m$ be i.i.d. vectors uniform on the unit sphere in $\mathbb{R}^n$, $m \geq n \geq 3$, and let $X := \{x \in \mathbb{R}^n | a_i^T x \leq 1\}$ be the random polyhedron generated by. Furthermore, for linearly independent vectors $u, \bar{u}$ in $\mathbb{R}^n$, let $S_{u,\bar{u}}(X)$ be the number of shadow vertices of X in $span(u, \bar{u})$. The paper provides an asymptotic expansion of the expectation value $\mathbf{E}(S_{u,\bar{u}})$ for fixed n and $m \to \infty$. The first terms of the expansion are given explicitly. Our investigation of $\mathbf{E}(S_{u,\bar{u}})$ is closely connected to Borgwardt's probabilistic analysis of the shadow vertex algorithm — a parametric variant of the simplex algorithm. We obtain an improved asymptotic upper bound for the number of pivot steps required by the shadow vertex algorithm for uniformly on the sphere distributed data.

Keywords: Linear programming, simplex algorithm, probabilistic analysis, asymptotic expansion, convex hull, stochastic geometry
Mathematical subject classification: *Primary:* 90C05 *Secondary:* 60D05, 52A22

1 Introduction

Since Klee and Minty showed exponential worst-case behaviour for the first edge algorithm, it has been a challenge for mathematicians to explain the gap between the disappointing worst case and the very good practical performance of many variants of the simplex algorithm. Borgwardt [5] was the first to give a satisfying answer. He showed for a class of distributions of the data that the expected number of pivot steps required by the shadow vertex algorithm – a variant of Gass' and Saaty's parametric algorithm – is polynomially bounded in the problem dimensions. In particular, this means that exponential worst case examples have exponentially small probability. Similar results for an other variant of the simplex algorithm and other classes of distributions are due to Adler, Karp and Shamir [1], Adler and Meggido [2] and Smale [9]. The interested reader is referred to Shamir [8] for a survey.

We concentrate on Borgwardt's approach and investigate linear programming problems of type

$$\max_{x\in X} u^T x, \quad X := \{x \in \mathbb{R}^n \,|\, a_i^T x \le 1,\, i = 1,\dots,m\}, \tag{1}$$

with $a_i, u \in \mathbb{R}^n$ and $m \ge n \ge 2$. Furthermore, we assume the data to be non-degenerate in the following sense: Every subset of $\{a_1,\dots,a_m\}$ with cardinality n be linearly independent and every subset of cardinality $n+1$ be in general position. For this restricted class of linear programming problems the polar polyhedron Y,

$$Y := \{y \in \mathbb{R}^n \,|\, x^T y \le 1;\ x \in X\} = conv(0, a_1, \dots, a_m), \tag{2}$$

is a polytope. We define $n \times n$-matrices $A_I := (a_{i_1} | \dots | a_{i_n})^T$ for any set of indices $I := \{i_1, \dots i_n\} \subset \{1,\dots,m\}$ with $i_1 < i_2 < \dots < i_n$. Then, a vector x_I that satisfies the equation $A_I x = \mathbf{e}$ is a vertex of X if and only if $Y_I := conv(a_i \,|\, i \in I)$ is a facet of Y.

The key to a probabilistic analysis of the number of pivot steps required by the shadow vertex algorithm is the fact that a simplex path from an X-vertex maximizing $\bar{u}^T x$ to an X-vertex maximizing $u^T x$ exclusively consists of X-vertices that are shadow vertices with respect to the plane spanned by the vectors u and $\bar{u}$. A $(u,\bar{u})$-shadow vertex is a vertex of X that remains a vertex of X's image under the orthogonal projection onto $span(u,\bar{u})$. A vertex x_I of X is a $(u,\bar{u})$-shadow vertex if and only if $span(u,\bar{u})$ intersects the corresponding facet Y_I of Y. Let $\chi_1(Y, Y_I)$ be the functional that decides whether Y_I is a facet of Y. More formally, for any pair of polytopes Y_1, Y_2 in $\mathbb{R}^n$ let $\chi_1(Y_1, Y_2) = 1$ if and only if Y_2 is a facet of Y_1 and of $conv(0, Y_1)$, simultaneously. Otherwise, let $\chi_1(Y_1, Y_2)$ be zero. Then, for linearly independent vectors u and $\bar{u}$ the number $s_{u,\bar{u}}$,

$$s_{u,\bar{u}}(X) := \sum_I \chi_1(Y, Y_I)\, \chi_{u,\bar{u}}(Y_I), \tag{3}$$

where

$$\chi_{u,\bar{u}}(Y_I) := \chi(cone(u,\bar{u}) \cap Y_I \ne \emptyset)\, \chi(\mathbb{R}^+ u \cap Y_I = \emptyset) \tag{4}$$

equals the number of pivots that phase II of the shadow vertex algorithm requires to maximize $u^T x$ over X if the algorithm is started with a vertex of X optimal for $\bar{u}^T x$.

Let $S_{u,\bar{u}}(X)$ be the total number of shadow vertices of X with respect to the $(u,\bar{u})$-plane. Then, by definition (3) $S_{u,\bar{u}}(X) = s_{u,\bar{u}}(X) + s_{u,-\bar{u}}(X) + s_{-u,\bar{u}}(X) + s_{-u,-\bar{u}}(X)$.

We define $S(X) := \mathbf{E}(S_{u,\bar{u}}(X))$ averaging on vectors u and $\bar{u}$ that are independent and identically distributed spherically symmetrical in $\mathbb{R}^n$. Spherical symmetry of the distribution gives that $\mathbf{E}(S_{u,\bar{u}}(X)) = 4\,\mathbf{E}(s_{u,\bar{u}}(X))$ for any fixed polyhedron X. Moreover, let the data a_i, $i = 1, \ldots, m$, be independent and identically distributed (i.i.d.) spherically symmetrical in $\mathbb{R}^n$. Then, we obtain

$$\bar{s}_{n,m} := \mathbf{E}(s_{u,\bar{u}}) = \frac{1}{4}\mathbf{E}(S). \tag{5}$$

On the left hand side of (5) we average on the choice of $u, \bar{u}$ and X. The quantity $\bar{s}_{n,m}$ is a measure for the average complexity of phase II of the shadow vertex algorithm under spherically symmetrical distributions. Equation (5) relates the combinatorial functional $s_{u,\bar{u}}$ to the continuous functional S. The functional S has an easy geometrical interpretation on the polar polytope Y, cf. Section 2, which enables the application of methods from stochastic geometry of polytopes.

Borgwardt [5,6] derived upper bounds for $\bar{s}_{n,m}$ independent from the distribution among the class of spherically symmetrical distributions of the data. He proved that there is a positive constant K independent from n, m and the distribution such that for all $m \geq n \geq 2$ holds:

$$\bar{s}_{n,m} \leq K\, n^{\frac{5}{2}}\, m^{\frac{1}{n-1}} \tag{6}$$

This implies polynomiality in expectation for the number of pivots required by phase II of the shadow vertex algorithm.

In earlier papers [3,4] Borgwardt analyzed $\bar{s}_{n,m}$ asymptotically for fixed dimension n and $m \to \infty$ for particular distributions. A typical result is the following. For uniformly on the unit sphere distributed data a_i there are positive constants k_n and K_n, $k_n \leq K_n$, with $\lim_{n\to\infty} k_n = \lim_{n\to\infty} K_n = 1$ such that

$$\frac{1}{2}\, k_n\, n^2 \leq \liminf_{m\to\infty} \frac{\bar{s}_{n,m}}{m^{1/(n-1)}} \leq \limsup_{m\to\infty} \frac{\bar{s}_{n,m}}{m^{1/(n-1)}} \leq \frac{1}{2}\, K_n\, n^2 \tag{7}$$

Asymptotic results are of interest in connection with proper estimations, as they prove the quality of them. The result in (7), for instance, is of particular interest, as it shows that the upper bound (6) cannot be improved in the m order of growth. On the other hand, (7) proves that the order of growth in n of an upper bound for $\bar{s}_{n,m}$ must be at least quadratic.

It is the purpose of our paper to improve the asymptotic analysis of $\bar{s}_{n,m}$ for uniformly distributed data on the unit sphere. While (6) gives an asymptotic estimation for $\bar{s}_{n,m}$ only, we will show for fixed dimension n and $m \to \infty$ how a complete asymptotic expansion of $\bar{s}_{n,m}$ can be evaluated. We provide the first terms of this expansion explicitly. In Section 2, we give a recursion scheme that allows an expansion of arbitrary length. Let ω_n denote the Lebesgue-measure of the unit sphere in $\mathbb{R}^n$, i.e. $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

Theorem 1 *For $n \geq 3$ and uniformly on the unit sphere distributed data a_i there are coefficients $C_k(n)$ such that*

$$\bar{s}_{n,m} = \sum_{k=0}^{N} C_k(n)\, \frac{m!}{(m + \frac{2k-1}{n-1})!} + \mathcal{O}(m^{-(2N+1)/(n-1)}). \tag{8}$$

The coefficients $C_k(n)$ have the form $C_k(n) = R_k(n)\, D_k(n)$, where the functions $R_k(n)$ are rational in n with integer coefficients and the $D_k(n)$ are given by

$$D_k(n) := \frac{(n-1+\frac{2k-1}{n-1})!}{(n-1)!}\left(\frac{\omega_n(n-1)}{\omega_{n-1}}\right)^{\frac{2k-1}{n-1}}\frac{\omega_n^2(n-1)}{2\pi\omega_{n-1}^2}\frac{\mathrm{B}(\frac{1}{2},\frac{n^2-n-2}{2})}{\mathrm{B}(\frac{1}{2},\frac{n^2-2n-2}{2})}. \tag{9}$$

The first two coefficients $C_0(n)$ and $C_1(n)$ are

$$C_0(n) = \frac{(n-1)^2}{2}\frac{n^2-n-2}{n(n-2)}\, D_0(n) \tag{10}$$

$$C_1(n) = -\frac{(n-1)^2}{4}\frac{(n^2-n-2)(n^4-n^3-n^2+3n-3)}{n(n+1)(n^2-n-1)(n^2-2n+2)}\, D_1(n)$$

The coefficients $C_i(n)$ look rather complicated, but the description of their asymptotic behaviour for $n \to \infty$ is easy. As $D_k(n) \sim 1$ for $n \to \infty$ and fixed $k \in \mathbb{N}_0$, we have

$$C_0(n) \sim \frac{1}{2}n^2,\ \ C_1(n) \sim -\frac{1}{4}n^2,\ \ n \to \infty.$$

For $n = 3$ we get

$$C_0(3) = \frac{16}{3\pi^{3/2}},\ C_1(3) = -\frac{34}{5\pi^{3/2}}.$$

The theorem shows that $\bar{s}_{n,m}$ can be expanded in a series of broken products in m. This representation is most appropriate, as the broken products arise in a natural way, cf. Section 2. Of course, it is possible to represent $\bar{s}_{n,m}$ as a series in broken powers of m. As $\frac{(m+\alpha)!}{m!} = m^\alpha(1+\mathcal{O}(m^{-1}))$ for $m \to \infty$ and real α, we have for $n \geq 4$ and $m \to \infty$

$$\bar{s}_{n,m} = C_0(n)\, m^{\frac{1}{n-1}} + C_1(n)\, m^{-\frac{1}{n-1}} + \mathcal{O}(m^{-\frac{2}{n-1}}). \tag{11}$$

(11) is not true for $n = 3$ as the second coefficient is perturbed by the error term of the approximation of $\frac{m!}{(m+\frac{1}{2})!}$. As $C_1(n)$ is always negative, we obtain as a consequence of Theorem 1 an improved upper bound for $\mathbf{E}(S)$ for fixed n and m big enough:

Corollary 1 *For every $n \geq 3$ there exists an $M \geq n$ such that for $m \geq M$:*

$$\bar{s}_{n,m} \leq C_0(n)\, m^{\frac{1}{n-1}} \tag{12}$$

We conjecture that (12) is true for all $m \geq n \geq 3$.

2 An asymptotic expansion with unknown coefficients

The aim of this section is to derive an asymptotic expansion of $\mathbf{E}(S)$ for fixed $n \geq 3$ and $m \to \infty$, if the data a_i are uniformly distributed on the unit sphere. By Equation (5) this expansion immediately yields an expansion of $\bar{s}_{n,m}$. We first interpret $S(X)$ geometrically on the polar polytope Y. From the definition of S we get for non-degenerate data a_i that

$$S(X) = \sum_I \chi_1(Y, Y_I)\, \tilde{W}(A_I) \tag{13}$$

with $\tilde{W}(A_I) := \mathbf{Pr}(span(u, \bar{u}) \cap Y_I \neq \emptyset)$. As u and $\bar{u}$ are assumed to be i.i.d., spherically symmetrical in $\mathbb{R}^n$, and the halfplane $cone(u, -\bar{u}, \bar{u})$ intersects the simplex Y_I in exactly two of its facets with probability one, we have

$$\tilde{W}(A_I) = \sum_{k=1}^{n} W(A_{I_k}), \;\; W(A_{I_k}) := \mathbf{Pr}(Y_{I_k} \cap cone(u, -\bar{u}, \bar{u}) \neq \emptyset), \tag{14}$$

with $I_k = I \setminus \{i_k\}$ and $Y_{I_k} := conv(a_i | i \in I_k)$. A_{I_k} results from A_I if we delete the k-th row. Let $\mathcal{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and $\omega_n := |\mathcal{S}^{n-1}|$, where $|\cdot|$ denotes Lebesgue-measure of appropriate dimension. As the normal vector of the plane $span(u, \bar{u})$ is spherically symmetrically distributed also if u and $\bar{u}$ are, it is elementary to see that

$$W(A_{I_k}) = \frac{1}{\omega_{n-1}} |cone(Y_{I_k}) \cap \mathcal{S}^{n-1}|. \tag{15}$$

Thus, $W(A_{I_k})$ equals the spherical angle generated by Y_{I_k} and $\tilde{W}(A_I)$ equals the sum of the spherical angles generated by the facets of Y_I. Taking expectations in (19) we obtain with $Y_A := Y_{\tilde{I}}$ for $\tilde{I} := \{1, \ldots, n\}$

$$\mathbf{E}(S) = \binom{m}{n} \mathbf{E}(\mathbf{Pr}(Y_A \text{ is a facet of } Y)\tilde{W}(A)), \tag{16}$$

as the data a_i are identically distributed. For any regular $n \times n$-matrix $A = (a_1 | \ldots | a_n)^T$ let $h(A)$ be the distance of the hyperplane that supports the points a_i, $i = 1, \ldots, n$. Let $G(h)$ be the probability that a vector a lies beyond a fixed hyperplane at distance h from the origin. Then, as the vectors a_i are i.i.d., we have

$$\mathbf{Pr}(Y_A \text{ is a facet of } Y) = (1 - G(h(A)))^{m-n}. \tag{17}$$

In the light of the equations (16) and (17) it is useful to establish $h = h(A)$ as independent variable. If we average on all events A that lie on a fixed hyperplane at distance h first and afterwards on the choice of the hyperplane, Fubini's theorem gives

$$\mathbf{E}(S) = \binom{m}{n} \int_0^1 (1 - G(h'))^{m-n}\, \mathbf{E}(\tilde{W}(A) | h(A) = h')\, d\{\mathbf{Pr}(h(A) \leq h')\}. \tag{18}$$

For spherically symmetrically distributed data $a_1, \ldots, a_n$ it is well known that the distribution function $\mathbf{Pr}(h(A) \leq h')$ is absolutely continuous. Hence, there is a density function p such that

$$\mathbf{E}(S) = \binom{m}{n} \int_0^1 (1 - G(h'))^{m-n} \Lambda_{\tilde{W}}(h') dh', \;\; \Lambda_{\tilde{W}}(h') := \mathbf{E}(\tilde{W}(A) | h(A) = h')\, p(h'). \tag{19}$$

The integral in (19) enables the application of Laplace's asymptotical method for $m \to \infty$. Before we can do that we need information about the asymptotic behaviour of G and $\Lambda_{\tilde{W}}$ near 1, as $1-G(h)$ attains its maximum for $h = 1$. First, we provide explicit representations of these functions. Geometrically, the value $G(h)$ equals the part of the surface area of the sphere $\mathcal{S}^{n-1}$ that is cut off by an hyperplane at distane h from the origin. It holds

$$G(h) = \frac{\omega_{n-1}}{\omega_n} \int_h^1 \sqrt{1-x^2}^{\,n-3} dx. \tag{20}$$

The evaluation of a suitable explicit representation of $\Lambda_{\tilde{W}}(h')$ is more involved. Up to the normalizing constant $p(h')$, $\Lambda_{\tilde{W}}(h')$ equals the expectation of $\tilde{W}(A)$ under the condition $h(A) = h'$. For a particular subset of matrices A that fulfil $h(A) = h'$, $W(A_k)$ can be written as a surface integral that enables asymptotic expansions for $h \to \infty$. Here, A_k arises from A if we delete the k-th row. Let the matrix A_k have the form $A_k = (tC|r\mathbf{e}|h\mathbf{e})$ with $C = (c_1 | \dots | c_{n-1})^T$, $c_i \in \mathcal{S}^{n-3}$, and positive constants r, h and $t = \sqrt{1-r^2-h^2} \le 1$. Then, from (15) we get denoting $Y_C := conv(c_i | 1 \le i \le n-1)$:

$$W(tC|r\mathbf{e}|h\mathbf{e}) = \frac{\sqrt{1-t^2}\, t^{n-2}}{\omega_{n-1}} \int_{Y_C} \frac{dc}{[1-t^2(1-\|c\|^2)]^{(n-1)/2}}. \tag{21}$$

Obviously, the right hand side of (21) depends on t and C only. Thus, we may define $w(t,C) := W(tC|r\mathbf{e}|h\mathbf{e})$. If we exploit this fact and average first on all events A_k that fulfil $A_k = (tC|r\mathbf{e}|h\mathbf{e})$, we obtain an explicit representation of $\Lambda_{\tilde{W}}$ ready for an asymptotic expansion near 1:

Lemma 1 *For uniformly on $\mathcal{S}^{n-1}$ distributed data a_i and $n \ge 3$ holds:*

$$\Lambda_{\tilde{W}}(h) = \left(\frac{n-2}{2\pi}\right)^n n\,\omega_n\omega_{n-1} \sqrt{1-h^2}^{\,n^2-2n-1} \int_0^1 \mu_w(r)\, \lambda_w(\sqrt{(1-h^2)(1-r^2)})dr \tag{22}$$

with

$$\mu_w(r) := \sqrt{1-r^2}^{\,n^2-3n} \int_{-1}^1 |r-x| \sqrt{1-x^2}^{\,n-4} dx \tag{23}$$

and

$$\lambda_w(t) := \mathbf{E}(\det{}^2(C|\mathbf{e})\, w(t,C)), \tag{24}$$

where the average is taken on $(n-1)\times(n-2)$-matrices C, whose rows are i.i.d. uniform on the unit sphere $\mathcal{S}^{n-3}$.

The proof of the lemma is delayed to Section 4.1. Now, we expand the integrand from (21) and $\sqrt{1-t^2}$ in Taylor series around $t = 0$ and obtain the corresponding series representation

$$\lambda_w(t) = \frac{1}{\omega_{n-1}} \sum_{k=0}^{\infty} \alpha_k\, t^{n-2+2k} \tag{25}$$

for $\lambda_w(t)$, $t \in [0,1)$, with coefficients α_k,

$$\alpha_k := (-1)^k \sum_{j=0}^{k} \sum_{i=0}^{j} \binom{-\frac{n-1}{2}}{j} \binom{\frac{1}{2}}{k-j} \binom{j}{i} (-1)^i K_i, \tag{26}$$

where the constants K_i are defined by

$$K_i := \mathbf{E}(\det{}^2(C|\mathbf{e}) \int_{Y_C} \|c\|^{2i} dc). \tag{27}$$

Now, we insert (23) and (25) into (22) and get the following expansion of $\Lambda_{\tilde{W}}(h)$ in powers of $\sqrt{1-h^2}$:

Corollary 2 *For $h \in (0,1]$ holds:*

$$\Lambda_{\tilde{W}}(h) = \left(\frac{n-2}{2\pi}\right)^{n-1} \frac{n\,\omega_n}{2\pi} \sum_{k=0}^{\infty} \alpha_k \beta_k \sqrt{1-h^2}^{\,n^2-n-3+2k}, \tag{28}$$

where the coefficients β_k are given by

$$\beta_k := \frac{n^2-n-2+2k}{n^2-2n+2k}\, \mathrm{B}\left(\frac{n^2-n-2}{2}+k, \frac{1}{2}\right). \tag{29}$$

The coefficients β_k are calculated by use of their integral representations

$$\beta_k = (n-2) \int_0^1 \mu_w(r) \sqrt{1-r^2}^{\,n-2+2k} dr, \tag{30}$$

which we integrate by parts. (30) arises if we insert the series (25) into (22). Now, we are ready for an asymptotic expansion of $\mathbf{E}(S)$. If we substitute $G(h) = x$ in (19) and replace $\Lambda_{\tilde{W}}(h)$ by its series representation (28), we obtain a series representation for $\mathbf{E}(S)$. It holds

$$\mathbf{E}(S) = \gamma_n \sum_{k=0}^{\infty} \alpha_k \beta_k \, e_k(m,n), \tag{31}$$

where the functions e_k are defined by

$$e_k(m,n) := n\binom{m}{n} \int_0^{\frac{1}{2}} (1-x)^{m-n} \sqrt{1-h^2}^{\,n^2-2n+2k} dx \tag{32}$$

and with constants

$$\gamma_n := \left(\frac{n-2}{2\pi}\right)^{n-1} \frac{\omega_n^2}{2\pi\omega_{n-1}}. \tag{33}$$

As the expansion of $G(h)$ in powers of $\sqrt{1-h^2}$ takes the form

$$G(h) = \delta_n \sum_{k=0}^{\infty} \tau_k \sqrt{1-h^2}^{\,n-1+2k}, \quad \tau_k := \binom{-\frac{1}{2}}{k} (-1)^k \frac{n-1}{n-1+2k}, \tag{34}$$

for all $h \in (0,1]$ with $\delta_n := \frac{\omega_{n-1}}{\omega_n(n-1)}$, $\sqrt{1-h^2}$ has an asymptotic expansion of type

$$\sqrt{1-h^2} = \sum_{j=0}^{N-1} \eta_j \left(\frac{x}{\delta_n}\right)^{(2j+1)/(n-1)} + \mathcal{O}\left(x^{(2N+1)/(n-1)}\right) \tag{35}$$

in broken powers of $x = G(h)$ for $x \to 0+$. It is not hard to show with Cauchy's method of dominated series that the expansion (35) is convergent for small positive x. We calculate the coefficients η_k recursively and obtain

$$\eta_0 = 1,\ \eta_1 = -\frac{1}{2(n+1)},\ \eta_2 = -\frac{2n^2-n-9}{8(n+1)^2(n+3)},\ \dots \tag{36}$$

If we replace the powers of $\sqrt{1-h^2}$ in (32) by their expansions and apply Watson's lemma, we obtain for $m \to \infty$

$$e_k(m,n) = \sum_{j=0}^{N-1} \zeta_{k,j} \frac{(n-1+\frac{2k+2j-1}{n-1})!}{(n-1)!} \frac{m!}{(m+\frac{2k+2j-1}{n-1})!} + \mathcal{O}(m^{-(2N+2k-1)/(n-1)}) \tag{37}$$

with appropriate coefficients $\zeta_{k,j}$. Hence, $e_k(m,n)$ asymptotically takes the form of a series of broken products. The coefficients $\zeta_{k,j}$ can be calculated recursively with the aid of (35). In particular, we have

$$\zeta_{0,0} = \delta_n^{-n+1+\frac{1}{n-1}},\ \zeta_{1,0} = \zeta_{0,0}\,\delta_n^{-\frac{2}{n-1}},\ \zeta_{0,1} = -\frac{n(n-2)}{2(n+1)}\zeta_{0,0}\,\delta_n^{-\frac{2}{n-1}},\ \dots \tag{38}$$

The expansion of $\mathbf{E}(S)$ is of similar type as (37). We obtain from (31) and (37) for fixed n and $m \to \infty$

$$\mathbf{E}(S) = \sum_{k=0}^{N} \xi_k \frac{m!}{(m+\frac{2k-1}{n-1})!} + \mathcal{O}(m^{-(2N+1)/(n-1)}) \tag{39}$$

with coefficients

$$\xi_k = \gamma_n \frac{(n-1+\frac{2k-1}{n-1})!}{(n-1)!} \sum_{i=0}^{k} \zeta_{i,k-i}\alpha_i\beta_i. \tag{40}$$

In particular, we obtain

$$\xi_0 = \frac{(n-1-\frac{1}{n-1})!}{(n-1)!}\,\delta_n^{-n+1+\frac{1}{n-1}}\gamma_n\beta_0\,K_0 \tag{41}$$

$$\xi_1 = \frac{(n-1+\frac{1}{n-1})!}{(n-1)!}\,\delta_n^{-n+1-\frac{1}{n-1}}\gamma_n\left[\left(\frac{n-2}{2}\beta_1 - \frac{n(n-2)}{2(n+1)}\beta_0\right)K_0 - \frac{n-1}{2}\beta_1 K_1\right].$$

As $\bar{s}_{n,m} = \frac{1}{4}\,\mathbf{E}(S)$ the coefficients $C_i(n)$ from Theorem 1 are given by $C_i(n) = \frac{1}{4}\xi_i$. So far, the only open question is that we do not know the coefficients K_i from (27). As it seems a hard problem to evaluate these directly from their definition for $n \geq 4$, we go another way. In the next section we will introduce a polytope functional Z that is similar to S. The expectation value of this functional is exactly known. On the other hand, we will evaluate the expectation $\mathbf{E}(Z)$ in terms of the K_i. Then, by comparison of coefficients we will give recursive formulae for the coefficients K_i. The first three K_i will be calculated exactly. Hence, the beginning of an asymptotic expansion for $\mathbf{E}(S)$ and for $\bar{s}_{n,m}$ will be given.

3 The determination of the coefficients

For all polytopes $Y = conv(0, a_1, \ldots, a_m)$, cf. (2), we define the functional Z by

$$Z(Y) := \mathbf{Pr}(u \in cone(Y)), \tag{42}$$

where u is assumed to be spherically symmetrically distributed in $\mathbb{R}^n$. For non-degenerate data a_i, Z has the representation

$$Z(Y) := \sum_I \chi_1(Y, Y_I)\, \tilde{V}(A_I) \tag{43}$$

with $\tilde{V}(A_I) := \mathbf{Pr}(u \in cone(Y_I))$. Hence, Z's representation in (43) is completely analogous to S's in (13). Therefore, with the same arguments as before we get for Z's expectation

$$\mathbf{E}(Z) = \binom{m}{n} \int_0^1 (1 - G(h'))^{m-n}\, \Lambda_{\tilde{V}}(h')\, dh' \tag{44}$$

where $\Lambda_{\tilde{V}}(h') := \mathbf{E}(\tilde{V}(A)|h(A) = h')\, p(h')$. We evaluate a Taylor series for $\Lambda_{\tilde{V}}(h)$ with similar methods as for $\Lambda_{\tilde{W}}(h)$. If the matrix A has the form $A = (\sqrt{1-h^2}B|h\mathbf{e})$ with $B = (b_1|b_2|\ldots|b_n)^T$, $b_i \in \mathcal{S}^{n-2}$, and $h \in (0,1]$, $\tilde{V}(A)$ can be written as a surface integral by

$$\tilde{V}(\sqrt{1-h^2}B|h\mathbf{e}) = \frac{h\sqrt{1-h^2}^{\,n-1}}{\omega_n} \int_{Y_B} \frac{db}{[1-(1-h^2)(1-\|b\|^2)]^{n/2}}. \tag{45}$$

As Y_B has dimension $n-1$ and Y_C in (21) has dimension $n-2$ a direct expansion of (45) would lead to coefficients for the resulting expansion of $\Lambda_{\tilde{V}}(h)$ that we cannot compare with K_i. So, we dissect Y_B in n sub-simplices Y_{B_i} where the matrix B_i of generating vectors arises from B if we delete the i-th row. For any pair of polytopes Y_1 and Y_2 let $\chi_2(Y_1, Y_2) = 1$ if and only if Y_2 is a facet of Y_1 and not of $conv(0, Y_1)$. Otherwise, let $\chi_2(Y_1, Y_2)$ be zero. Then, for non-degenerate matrices B holds

$$\tilde{V}(\sqrt{1-h^2}B|h\mathbf{e}) = \sum_{i=1}^{n} (\chi_1(Y_B, Y_{B_i}) - \chi_2(Y_B, Y_{B_i}))\, V(B_i, h), \tag{46}$$

where

$$V(B_i, h) := \tilde{V}(\sqrt{1-h^2}\begin{pmatrix} B_i \\ 0 \end{pmatrix}|h\mathbf{e}). \tag{47}$$

Geometrically, equation (46) dissects the sphercial angle $\tilde{V}(A)$ generated by Y_A for matrices of the form $A = (\sqrt{1-h^2}B|h\mathbf{e})$ as a signed sum of spherical angles $V(B_i, h)$ that are generated by Y_A's facets Y_{B_i} and $(0, \ldots, 0, h)$. $V(B_i, h)$ can be written as a surface integral over Y_{B_i} if B_i has the form $B_i = (\sqrt{1-r^2}C|r\mathbf{e})$ with $C = (c_1|, \ldots, |c_{n-1})^T$, $c_i \in \mathcal{S}^{n-3}$ and $0 \le r \le 1$. Furthermore, let $v(r, h, C) := V((\sqrt{1-r^2}C|r\mathbf{e}), h)$. Then, an application of Cavalieri's principle yields

$$v(r, h, C) = \frac{h\sqrt{1-h^2}^{\,n-1} r\sqrt{1-r^2}^{\,n-2}}{\omega_n} \int_{Y_C} \int_0^1 \frac{x^{n-2}\, dx\, dc}{[1-(1-h^2)(1-x^2(r^2+(1-r^2)\|c\|^2))]^{n/2}}. \tag{48}$$

Because of the expectation's linearity we can investigate each summand of (46) separately. For each summand we average first on those events B_i of the form $B_i = (\sqrt{1-r^2}C|r\mathbf{e})$ and afterwards on the choice of the supporting linear manifold. We get:

Lemma 2 *For uniformly on $\mathcal{S}^{n-1}$ distributed data a_i and $n \geq 3$, $\Lambda_{\tilde{V}}(h)$ has the representation*

$$\Lambda_{\tilde{V}}(h) = \left(\frac{n-2}{2\pi}\right)^{n-1} n\,\omega_{n-1}^2 \sqrt{1-h^2}^{n^2-2n-1} \int_0^1 \mu_v(r)\lambda_v(r,h)dr \tag{49}$$

with

$$\mu_v(r) := r\sqrt{1-r^2}^{n^2-3n} \tag{50}$$

and

$$\lambda_v(r,h) := \mathbf{E}(\det^2(C|\mathbf{e})v(r,h,C)), \tag{51}$$

where the average is taken on $(n-1)\times(n-2)$-matrices C, whose rows are i.i.d. uniform on the unit sphere $\mathcal{S}^{n-3}$.

The details of Lemma 2's proof are delayed to Section 4.2. We expand the integral representation (48) of $v(r,h,C)$ and h in a Taylor series for small $\sqrt{1-h^2}$ and obtain a series for $\lambda_v(r,h)$ that converges for $h \in (0,1]$:

$$\lambda_v(r,h) = \frac{1}{\omega_n}\sum_{\ell=0}^{\infty} \sqrt{1-h^2}^{n-1+2\ell}\,\tilde{\rho}_\ell(r) \tag{52}$$

with

$$\tilde{\rho}_\ell(r) := (-1)^\ell \sum_{k=0}^{\ell} \binom{\frac{1}{2}}{\ell-k}\binom{-\frac{n}{2}}{k}\tilde{\sigma}_k(r), \tag{53}$$

where

$$\tilde{\sigma}_k(r) := \sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\binom{j}{i}\frac{(-1)^j}{n-1+2j}K_i\, r^{2(j-i)+1}\sqrt{1-r^2}^{n-2+2i}. \tag{54}$$

We insert (52) and (50) into (49) and get a series representation for $\Lambda_{\tilde{V}}(h)$:

Corollary 3 *For $h \in (0,1]$ holds:*

$$\Lambda_{\tilde{V}}(h) = \left(\frac{n-2}{2\pi}\right)^{n-1} n\frac{\omega_{n-1}^2}{\omega_n}\sum_{\ell=0}^{\infty}\sqrt{1-h^2}^{n^2-n-2+2\ell}\rho_\ell \tag{55}$$

with

$$\rho_\ell := (-1)^\ell \sum_{k=0}^{\ell}\binom{\frac{1}{2}}{\ell-k}\binom{-\frac{n}{2}}{k}\sigma_k, \tag{56}$$

where

$$\sigma_k := \frac{1}{2}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\binom{j}{i}\frac{(-1)^j}{n-1+2j}\mathrm{B}\left(j-i+\frac{3}{2}, i+\frac{n^2-2n}{2}\right)K_i. \tag{57}$$

Hence, we have established a series representation for $\Lambda_{\tilde{V}}(h)$ that is very similar to $\Lambda_{\tilde{W}}(h)$'s in Corollary 2, and which contains the same undetermined coefficients K_i. By its definition (36) it is obvious that $\mathbf{E}(Z)$ is independent from the choice of the distribution within the class of spherically symmetrical distributions. Using a result due to Wendel [10], we can give $\mathbf{E}(Z)$ explicitly. It holds:

$$1 - \mathbf{E}(Z) = 2^{-m}\left[\binom{m-1}{n-1} + \sum_{k=0}^{n-2} 2\binom{m-1}{k}\right]. \tag{58}$$

Thus, if we substitute $G(h) = x$ in (44) we obtain for $m \to \infty$:

$$\mathbf{E}(Z) = \binom{m}{n} \int_0^{\frac{1}{2}} (1-x)^{m-n} \frac{\Lambda_{\tilde{V}}(h)}{-G'(h)} dx = 1 - \mathcal{O}(m^{n-1}2^{-m}). \tag{59}$$

From (34) and (55), we know that the functions G' and $\Lambda_{\tilde{V}}$ have expansions in 0 in powers of $\sqrt{1-h^2}$ for $h \in (0,1]$. Thus, $\Lambda_{\tilde{V}}(h)/(-G'(h))$ has an expansion in powers of $\sqrt{1-h^2}$. By (35), $\sqrt{1-h^2}$ has an expansion in broken powers of x, which converges for small positive x. Thus, the quotient $\Lambda_{\tilde{V}}(h)/(-G'(h))$ has an expansion in broken powers of x that converges for small positive x, as well. If we replace $\Lambda_{\tilde{V}}(h)/(-G'(h))$ in (59) by this expansion and apply Watson's lemma we obtain by comparison of coefficients that

$$\Lambda_{\tilde{V}}(h)/(-G'(h)) = n\, x^{n-1} \tag{60}$$

for small x. By the identity theorem of real power series, (60) must be true for all $h \in (0,1]$ as G and G' have expansions in powers of $\sqrt{1-h^2}$ that converge for $h \in (0,1]$. As $\Lambda_{\tilde{V}}$, G and G' are continous in 0, we have:

Lemma 3 *For $h \in [0,1]$ holds:*

$$\Lambda_{\tilde{V}}(h) = -n\, G'(h)\, G^{n-1}(h). \tag{61}$$

If we expand (61) in powers of $\sqrt{1-h^2}$ using (34) and compare the coefficients with the series (55), we obtain equations for the coefficients ρ_k, from which the K_i can be evaluated recursively. We have for $k \in \mathbb{N}_0$:

$$\rho_\ell = \left(\frac{\omega_{n-1}}{\omega_{n-2}(n-1)}\right)^{n-1} \frac{1}{\omega_{n-1}} P_\ell \tag{62}$$

with

$$P_\ell := \sum_{j_1+\ldots+j_{n-1}=\ell} \prod_{i=1}^{n-1} \tau_{j_i}.$$

We substitute

$$K_\ell := \left(\frac{\omega_{n-1}}{\omega_{n-2}(n-1)}\right)^{n-1} \frac{1}{\omega_{n-1}} \frac{1}{\mathrm{B}(\frac{3}{2}, \frac{n^2-2n}{2})} \tilde{K}_\ell \tag{63}$$

in (57) and obtain the following recursion scheme for $\tilde{K}_i$ from (62):

$$\tilde{K}_\ell = \frac{2P_\ell - \sum_{i=0}^{\ell-1} Q_{i,\ell}\tilde{K}_i}{q_{\ell,\ell}} \tag{64}$$

where the $Q_{i,\ell}$ are defined by

$$Q_{i,\ell} := \sum_{j=i}^{\ell} p_{j,\ell}\, q_{i,j}$$

with

$$p_{j,\ell} := (-1)^\ell \sum_{k=j}^{\ell} \binom{k}{j}\binom{\frac{1}{2}}{\ell-k}\binom{-\frac{n}{2}}{k}$$

and

$$q_{i,j} := (-1)^j \frac{1}{n-1+2j}\,\frac{\binom{j-i+\frac{1}{2}}{j-i}\binom{i+\frac{n^2-2n-2}{2}}{i}}{\binom{j+\frac{n^2-2n+1}{2}}{j}}.$$

We calculate the first coefficients $\tilde{K}_i$ and get:

$$\tilde{K}_0 = 2(n-1),\ \tilde{K}_1 = \frac{2n-3}{n^2}\tilde{K}_0,\ \tilde{K}_2 = \frac{(2n^4+5n^3-32n^2+57n-36)(2n-1)}{n^2(n+1)^2(n+2)(n^2-2n+2)}\tilde{K}_0,\ \ldots \tag{65}$$

By (62), all $\tilde{K}_i$ are rational functions in n. We conjecture that $\tilde{K}_i = \mathcal{O}(n^{-i}\tilde{K}_0)$ for $n \to \infty$. In the particular case $n = 3$ it is easy to caclulate the coefficients K_i directly from its definition (27). We get $K_i = 4/(2i+1)$ for all $i \in \mathbb{N}_0$, which we also obtain from (62) and (63) for $i = 0, 1, 2$. Now, we insert the above calculated values of K_i into (41) and obtain ξ_0 and ξ_1 explicitly.

4 Proofs of the auxiliary lemmata

The only matter left is to prove Lemmata 1 and 2 from Section 3. To calculate explicite representations of $\Lambda_{\tilde{W}}$ and $\Lambda_{\tilde{V}}$ we use an affine variant of Blaschke's and Petkantschin's transformation formula, cf. Schneider and Weil [7]. Unfortunately, the uniform distribution on the sphere has no radial density function, which leads to difficulties in notation. So, we transform the expectations first for arbitrary distributions with radial densities and consider the uniform distribution on the sphere as pointwise limit.

Before we start with the proofs we introduce some notation. Let the data a_i be i.i.d., spherically symmetrical in the unit ball $\mathcal{B}^n$ with radial distribution function F, $F(x) = \mathbf{Pr}(\|a\| \le x)$. Let F have a density function. That means, there is a positive function f that satisfies $F(x) = \int_{x\mathcal{B}^n} f(a)da$. As the distribution is spherically symmetrical, we know $F(x) = \omega_n \int_0^x y^{n-1}\hat{f}(y)dy$ with $\hat{f}(y) = f(ya)$ for $a \in \mathcal{S}^{n-1}$. Let A be a $k \times n$-matrix with row vectors $a_1, \ldots, a_k$, $a_i \in \mathbb{R}^n$. Moreover, let $\Pi_f(A)$ be defined by $\Pi_f(A) := \prod_{i=1}^{k} f(a_i)$ and let $dA := da_1 \ldots da_k$.

4.1 Proof of Lemma 1

For an arbitrary spherically symmetrical distribution with density f we obtain from (16) and (17)

$$\mathbf{E}(S) = \binom{m}{n} \int\limits_{(\mathcal{B}^n)^n} (1 - G(h(A)))^{m-n}\, \tilde{W}(A)\, \Pi_f(A)\, dA \tag{66}$$

We average first on all matrices A, whose row vectors lie on a hyperplane at distance h from the origin, and afterwards on the choice of the hyperplane. By spherical symmetry all hyperplanes with distance h are equally likely. So, applying Blaschke's and Petkantschin's formula we get

$$\mathbf{E}(S) = \binom{m}{n} \int\limits_0^1 (1 - G(h(A)))^{m-n}\, \Lambda_{\tilde{W}}(h)\, dh \tag{67}$$

with

$$\Lambda_{\tilde{W}}(h) = \omega_n \int\limits_{(\sqrt{1-h^2}\mathcal{B}^{n-1})^n} |\det(B|\mathbf{e})|\, \tilde{W}(B|h\mathbf{e})\, \Pi_f(B|h\mathbf{e})\, dB. \tag{68}$$

Now, by the definition of $\tilde{W}$ and the identical distribution of the data, $\Lambda_{\tilde{W}}$ satisfies

$$\Lambda_{\tilde{W}}(h) = n\omega_n \int\limits_{\sqrt{1-h^2}\mathcal{B}^{n-1}} L(b,h)\, f(b|h)\, db \tag{69}$$

with

$$L_W(b,h) := \int\limits_{(\sqrt{1-h^2}\mathcal{B}^{n-1})^{n-1}} |\det(\begin{pmatrix} B_n \\ b \end{pmatrix} |\mathbf{e})|\, W(B_n|h\mathbf{e})\, \Pi_f(B_n|h\mathbf{e})\, dB_n. \tag{70}$$

Here, B_n arises from B if we delete the n-th row. We consider the function $L_W(b,h)$ and average first on all events B_n that lie on a fixed hyperplane in $\mathbb{R}^{n-1}$ at distance r from the origin. All hyperplanes at distance r are equally likely by spherical symmetry of the distribution. Thus, a second application of Blaschke's and Petkantschin's formula to $L_W(b,h)$ integral yields

$$L_W(b,h) = \omega_{n-1} \int\limits_0^{\sqrt{1-h^2}} |r - b^{(n-1)}|\, l_w(\sqrt{1-h^2-r^2})\, dr \tag{71}$$

with

$$l_w(t) := t^{n^2-n-2} \int\limits_{(\mathcal{B}^{n-2})^{n-1}} \det^2(C|\mathbf{e})\, w(t,C) \prod_{i=1}^{n-1} \hat{f}((1 - t^2(1 - \|c_i\|^2))^{1/2})\, dc_i. \tag{72}$$

$w(t,C)$ is defined as in Section 2. We dissect the domain of integration on the right of (72) in radial and spherical parts and obtain

$$l_w(t) = t^{n^2-3n} \left(\frac{\omega_{n-2}}{\omega_n}\right)^{n-1} \int\limits_{[\sqrt{1-t^2},1]^{n-1}} \mathbf{E}(\det^2(X_tC|\mathbf{e})\, w(t,X_tC))\, \det^{n-4}(X_t) \prod_{i=1}^{n-1} \frac{dF(x_i)}{x_i^{n-2}}, \tag{73}$$

where $X_t := diag((1-(1-x_1^2)/t^2)^{1/2}, \ldots, (1-(1-x_{n-1}^2)/t^2)^{1/2})$. The inner expectation in (73) is taken on i.i.d. vectors c_i uniform on $\mathcal{S}^{n-3}$. As the class of distributions with densities in the ball lies pointwise dense in the class of all spherically symmetrical distribution in the ball, (73) holds true for all spherically symmetrical distributions. In case of uniformly distributed data, (73) is equal to

$$l_w(t) = \left(\frac{\omega_{n-2}}{\omega_n}\right)^{n-1} t^{n^2-3n} \lambda_w(t) \tag{74}$$

with λ_w as in (24). $L_W(b,h)$ depends only from $b^{(n-1)}$ and h. Thus, we obtain from (68) and (71) with $t = \sqrt{1-h^2-r^2}$:

$$\Lambda_{\tilde{W}}(h) = n\omega_{n-1}\omega_{n-2} \int\limits_0^{\sqrt{1-h^2}} \int\limits_{-\sqrt{1-h^2}}^{\sqrt{1-h^2}} |r-x| \int\limits_{\sqrt{s^2+h^2}}^{1} \frac{(y^2-s^2-h^2)^{(n-4)/2}}{y^{n-2}} dF(y) ds\, l_w(t) dr. \tag{75}$$

For uniformly distributed data we get from (74) and (75) for the uniform distribution on the unit sphere

$$\Lambda_{\tilde{W}}(h) = n\omega_{n-1}\omega_n \left(\frac{\omega_{n-2}}{\omega_n}\right)^n \sqrt{1-h^2}^{\,n^2-2n-1} \int\limits_0^1 \mu_w(r)\lambda_w(\sqrt{(1-h^2)(1-r^2)})\, dr \tag{76}$$

with μ_w as in (23). Observing that $\frac{\omega_{n-2}}{\omega_n} = \frac{n-2}{2\pi}$, Lemma 1 is completely proven.

4.2 Proof of Lemma 2

Like for the functional S an application of Blaschke's and Petkantschin's formula to $\mathbf{E}(Z)$ in (44) yields

$$\Lambda_{\tilde{V}}(h) = \omega_n \int\limits_{(\sqrt{1-h^2}\mathcal{B}^{n-1})^n} |\det(B|\mathbf{e})|\, \tilde{V}(B|h\mathbf{e})\, \Pi_f(B|h\mathbf{e})\, dB \tag{77}$$

for any spherically symmetrical distribution with density function f in the unit ball. We factorize B exploiting symmetries of $|\det(B|\mathbf{e})|$ and $\tilde{V}(\sqrt{1-h^2}B|h\mathbf{e})$. We know from (46) that for non-degenerate matrices B

$$\tilde{V}(B|h\mathbf{e}) = \sum_{i=1}^{n} (\chi_1(Y_B, Y_{B_i}) - \chi_2(Y_B, Y_{B_i}))\, \tilde{V}(\begin{pmatrix} B_i \\ 0 \end{pmatrix} |h\mathbf{e}). \tag{78}$$

A similar dissection is possible for $|\det(B|\mathbf{e})|$. It holds

$$|\det(B|\mathbf{e})| = \sum_{i=1}^{n} (\chi_1(Y_B, Y_{B_i}) - \chi_2(Y_B, Y_{B_i}))\, |\det(B_i)| \tag{79}$$

We define sign-functions $\mathrm{sig}_{i,j}$ by

$$\mathrm{sig}_{i,j}(B) := (\chi_1(Y_B, Y_{B_i}) - \chi_2(Y_B, Y_{B_i}))\, (\chi_1(Y_B, Y_{B_j}) - \chi_2(Y_B, Y_{B_j})) \tag{80}$$

for $1 \le i,j \le n$. Inserting (78) and (79) into (77) we obtain

$$\Lambda_{\tilde{V}}(h) = \omega_n \sum_{i,j=1}^{n} \int\limits_{(\sqrt{1-h^2}\mathcal{B}^{n-1})^n} \mathrm{sig}_{i,j}(B)\, |\det(B_i)|\, \tilde{V}(\begin{pmatrix} B_j \\ 0 \end{pmatrix} |h\mathbf{e})\, \Pi_f(B|h\mathbf{e})\, dB \tag{81}$$

We consider $\mathrm{sig}_{i,j}$ as functions of B's row vectors $b_1, \ldots, b_n$. Geometrical insight delivers that for non-degenerate matrices B $\mathrm{sig}_{i,j}$ is odd in the argument vectors b_i and b_j for $1 \le i,j \le n$. Using this observation and the fact that the vectors b_i are identically, spherically symmetrically distributed, we obtain

$$\Lambda_{\tilde{V}}(h) = n\omega_n \int\limits_{\sqrt{1-h^2}\mathcal{B}^{n-1}} f(b|h)\, db\; L_V(h) \tag{82}$$

with

$$L_V(h) := \int\limits_{(\sqrt{1-h^2}\mathcal{B}^{n-1})^{n-1}} |\det(B_n)|\, \tilde{V}(\begin{pmatrix} B_n \\ 0 \end{pmatrix} h\mathbf{e})\, \Pi_f(B_n|h\mathbf{e})\, dB_n. \tag{83}$$

Now, we apply Blaschke's and Petkantschin's transformation formula to $L_V(h)$ and obtain after normalization of the domain of integration

$$L_V(h) = \omega_{n-1}(1-h^2) \int_0^1 r\, l_v(r,h) dr \tag{84}$$

where $\lambda_v(r,h)$ is defined by

$$l_v(r,h) := t^{n^2-n-2} \int\limits_{(\mathcal{B}^{n-2})^{n-1}} \det{}^2(C|\mathbf{e})\, v(r,h,C) \prod_{i=1}^{n-1} \hat{f}((1-(1-h^2)(1-r^2-(1-r^2)\|c_i\|^2))^{\frac{1}{2}}) dc_i \tag{85}$$

for $t = \sqrt{1-h^2}\sqrt{1-r^2}$: The function $v(r,h,C)$ in (85) is defined as in (48). We split off the domain of integration in spherical and radial parts and obtain with X_t as above for all spherically symmetrical distributions in the ball:

$$\lambda_v(r,h) = t^{n^2-3n} \left(\frac{\omega_{n-2}}{\omega_n}\right)^{n-1} \int\limits_{[\sqrt{1-t^2},1]^{n-1}} \mathbf{E}(\det(X_t C|\mathbf{e})\, v(r,h,X_t C)) \det{}^{n-4}(X_t) \prod_{i=1}^{n-1} \frac{dF(x_i)}{x_i^{n-2}}. \tag{86}$$

In case of uniformly distributed data on the unit sphere, (86) is equal to

$$l_v(r,h) = \left(\frac{\omega_{n-2}}{\omega_n}\right)^{n-1} t^{n^2-3n}\, \lambda_v(r,h) \tag{87}$$

with $\lambda_v(r,h)$ as in (51). Thus, $L_V(h)$ satisfies

$$L_V(h) = \omega_{n-1} \left(\frac{n-2}{2\pi}\right)^{n-1} \sqrt{1-h^2}^{\,n^2-3n+2} \int_0^1 \mu_v(r)\, \lambda_v(r,h)\, dr \tag{88}$$

with $\mu_v(r)$ as in (50). Dissecting the domain of integration in (82) in radial and spherical parts, we obtain for all spherically symmetrical distributions in the ball

$$\Lambda_{\tilde{V}}(h) = n\omega_{n-1} \int_h^1 \frac{\sqrt{y^2 - h^2}^{n-3}}{y^{n-2}} dF(y)\; L_V(h). \tag{89}$$

We insert $L_V(h)$ and obtain for uniformly on the sphere distributed data the desired formula

$$\Lambda_{\tilde{V}}(h) = n\omega_{n-1}^2 \left(\frac{n-2}{2\pi}\right)^{n-1} \sqrt{1-h^2}^{n^2-2n-1} \int_0^1 \mu_v(r)\, \lambda_v(r,h)\, dr. \tag{90}$$

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