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On the Approximation of a Ball by Random Polytopes

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Summary. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of identically and independently distributed random vectors drawn from the d-dimensional unit ball B^d and let $X_n := \text{convhull}(a_1, \ldots, a_n)$ be the random polytope generated by $a_1, \ldots, a_n$. Furthermore, let $\Delta(X_n) := \text{Vol}(B^d \setminus X_n)$ be the deviation of the polytope's volume from the volume of the ball. For uniformly distributed a_i and $d \geq 2$, we prove that the limiting distribution of $\frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_n))}$ for $n \to \infty$ satisfies a 0-1-law. Especially, we provide precise information about the asymptotic behaviour of the variance of $\Delta(X_n)$. We deliver analogous results for spherically symmetric distributions in B^d with regularly varying tail.

Keywords. Stochastic approximation, convex hull, variance, limiting distribution

1. Introduction

Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of independently and identically distributed random vectors drawn from the d-dimensional unit ball B^d, $d \geq 1$, and X_n, $n \geq d+1$, be the polytope generated as convex hull of the vectors $a_1, \ldots, a_n$. The main subject of the paper will be the behaviour of the deviation of volume $\Delta(X_n)$, $\Delta(X_n) := \text{Vol}(B^d \setminus X_n)$, for large n.

There is much known about the asymptotic behaviour of the expectation value of $\Delta(X_n)$ in case of uniformly distributed a_i. One of the first results on this topic is due to Rényi and Sulanke [11], who investigated the planary case $d = 2$. Their result was generalized to arbitrary dimensions by Wieacker [15], who obtained that

$$\mathrm{E}(\Delta(X_n)) = C_d\, n^{-\frac{2}{d+1}} + \mathcal{O}(n^{-\frac{3}{d+1}}), n \to \infty, \tag{1.1}$$

with a positive constant C_d depending on d. For an extensive survey upon related results about the expectation of polytope variables and upon the history of their development we refer to the articles of Buchta [3], Gruber [7], Schneider [12], Weil and Wieacker [13] and especially to the work of Bárány and Larman [1,2].

Little is known so far about higher moments of polytope variables in general. For $d = 2$ Groeneboom [6] analyzed the limiting distribution of the X_n's vertex process, which was generalized by Hueter [8] to arbitrary dimensions. Using Efron's identity Groeneboom's results about the vertex process of X_n enable estimates for the variance of $\Delta(X_n)$ in case of $d = 2$ and uniformly distributed a_i but not for $d \geq 3$ or more general distributions.

In the present paper we are going to estimate the variance of $\Delta(X_n)$ asymptotically from above for uniformly distributed a_i in B^d, $d \geq 1$. Furthermore, as an interesting consequence of the analysis of expectation and variance, we will show that the limiting distribution of $\frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_n))}$ satisfies a 0-1-law for $d \geq 2$, which means $\Delta(X_n) = \mathrm{E}(\Delta(X_n))(1 + o(1))$ for $n \to \infty$ almost surely. The structure of the paper is the following. Section 2 contains the main results. In section 3 we provide additional notation and auxiliary lemmata necessary for the proofs in sections 4 and 5. Generalizations of the results in section 2 are added in section 6. We state analogous results for spherically symmetric distributions with regularly varying tail in the ball. All proofs of the paper are formulated in a manner, which allows a generalization to spherically symmetric distributions with regularly varying tail without much further work. The restriction to uniformly distributed a_i in the main part of the paper has been done to make reading easier for non-specialists and in order to keep necessary technicalities at a minimum.

2. Main results

Our first result provides an asymptotic upper bound for the variance of $\Delta(X_n)$ in case of a uniformly distributed sample:

Theorem 1: *For uniformly distributed a_i in B^d, $d \geq 1$:*

$$\mathrm{Var}(\Delta(X_n)) = \mathcal{O}(n^{-\frac{d+3}{d+1}}),\ n \to \infty. \tag{2.1}$$

The order of the asymptotic bound in (2.1) cannot be improved in general, as it is an easy exercise to confirm for $d = 1$ that $\mathrm{Var}(\Delta(X_n)) = \Theta(n^{-2})$, $n \to \infty$. It is also possible to establish $\mathrm{Var}(\Delta(X_n)) = \Theta(n^{-5/3})$, $n \to \infty$, for $d = 2$. We conjecture that (2.1) is sharp in order for $d \geq 3$ too, as it seems natural that the variance tends to zero more slowly in higher dimensions. But, this question remains still open. It is no surprise that $\mathrm{Var}(\Delta(X_n))$ becomes small for large n, but it is an interesting observation that the quotient $\frac{\mathrm{Var}(\Delta(X_n))}{\mathrm{E}^2(\Delta(X_n))}$ tends to zero also as n tends to infinity for $d \geq 2$. In the light of Chebychev's inequality this means that even small relative deviations from the mean are very unlikely for large n. But even more is true. Denoting $f(t) \sim g(t)$, $t \to t_0$, for $f(t) = g(t)(1 + o(1))$, $t \to t_0$, we have:

Theorem 2: *For uniformly distributed a_i in B^d, $d \geq 2$, almost everywhere holds:*

(2.2) $$\Delta(X_n) \sim \mathrm{E}(\Delta(X_n)),\ n \to \infty.$$

In other words, the random variables $\liminf_{n\to\infty} \frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_n))}$, $\limsup_{n\to\infty} \frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_n))}$ and 1 are equal up to sets of measure zero. This means the limiting distribution of $\frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_n))}$ satisfies a 0-1-law: If we consider a special, randomly generated sequence $(a_i)_{i\in\mathbb{N}}$ we know that with probability one $\Delta(X_n)$ has the asymptotic behaviour we expect if we average on all events $(a_i)_{i\in\mathbb{N}}$. For $d \geq 2$:

(2.3) $$\Pr\left(\lim_{n\to\infty} \frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_n))} \leq s\right) = \begin{cases} 0, & s < 1 \\ 1, & s \geq 1 \end{cases}.$$

Surprisingly, (2.3) is not true if $d = 1$. Here, the limiting distribution is given by

(2.4) $$\Pr\left(\lim_{n\to\infty} \frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_n))} \leq s\right) = 1 - (1 + 2s)\exp(-2s).$$

3. Definitions and auxiliary lemmata

As the random variable $\Delta(X_n)$ is not very handy for an asymptotic evaluation of moments, in most papers, which deal with the expectation of $\Delta(X_n)$, the random variable $\mathrm{Vol}(X_n)$ is studied instead. For the analysis of variances we go another way and approximate $\Delta(X_n)$ by a related random variable, which has the same asymptotic behaviour. Let $\tilde{X}_n := \mathrm{convhull}(X_n \cup \{0\})$ and

(3.1) $$\tilde{\Delta}(X_n) := \mathrm{Vol}((\mathrm{cone}(\tilde{X}_n) \setminus \tilde{X}_n) \cap B^d).$$

Then, $\tilde{\Delta}(X_n)$ is equal to $\Delta(X_n)$ if and only if $0 \in \mathrm{int}(X_n)$. If $0 \notin \mathrm{int}(X_n)$, we have $\frac{1}{2}\mu_d \leq \Delta(X_n) - \tilde{\Delta}(X_n) \leq \mu_d$, where $\mu_d := \mathrm{Vol}(B^d)$. Thus, for any $k \in \mathbb{N}$ and any spherically symmetric distribution there exists a constant $\eta \in [\frac{1}{2}\mu_d, \mu_d]$, such that

(3.2) $$\mathrm{E}(\Delta^k(X_n) - \tilde{\Delta}^k(X_n)) = \eta^k \Pr(0 \notin \mathrm{int}(X_n)) = \eta^k 2^{-n+1} \sum_{j=0}^{d-1} \binom{n-1}{j}.$$

The identity on the right hand side of (3.2), which is independent from the choice of the spherically symmetric distribution, is due to Schlaefli and was rediscovered by Wendel [14]. Especially, (3.2) means that the moments of $\Delta(X_n)$ and $\tilde{\Delta}(X_n)$ are equal up to terms of exponentially decreasing order in n. Hence, if $\mathrm{E}(\tilde{\Delta}^k(X_n))$ does not decrease exponentially in n, $\mathrm{E}(\Delta^k(X_n)) \sim \mathrm{E}(\tilde{\Delta}^k(X_n))$, $n \to \infty$, and therefore, if $\mathrm{Var}(\tilde{\Delta}(X_n)) = \mathcal{O}(n^\alpha)$ for an $\alpha \in \mathbb{R}$, we know $\mathrm{Var}(\Delta(X_n)) = \mathcal{O}(n^\alpha)$ for $n \to \infty$. It is the main advantage of investigating $\tilde{\Delta}$ instead of Δ that $\tilde{\Delta}$ can be additively represented in terms of functionals of $\tilde{X}_n$'s boundary simplices. Before we explain this fact, we introduce some notation, which is basic for our considerations in the following sections.

Let $A_n := \{a_1, \ldots, a_n\}$, $n \geq d + 1$, be nondegenerate. We call a set A_n nondegenerate, if any A_n-subset of cardinality $\leq d$ is linearly independent and any A_n-subset of cardinality $\leq d+1$ is affinely independent. Geometrically, nondegeneracy of A_n means that every facet of $\tilde{X}_n$ is a simplex. In case of spherically symmetric distributions with $\Pr(a=0)=0$, A_n is nondegenerate with probability one. For any set I of indices $1,\ldots,n$ with cardinality d let $A_I := \{a_i | i \in I\}$, $S_I :=$ convhull(A_I) and $\tilde{S}_I :=$ convhull$(S_I \cup \{0\})$. $H(A_I)$ be the hyperplane supporting S_I and $H^{(1)}(A_I)$ be the closed halfspace generated by $H(A_I)$, which contains the origin. For any A_n-subset A_I with cardinality d and any A_n-subset B let $\chi_1(B, A_I) := \chi(B \subset H^{(1)}(A_I))$ be the indicator functional deciding whether B belongs to $H^{(1)}(A_I)$. Especially, if $B = A_n$, $\chi_1(A_n, A_I)$ indicates whether S_I is a boundary simplex of $\tilde{X}_n$ or not. Finally, for $d \geq 1$, κ_d denote the surface area of S^{d-1}. For nondegenerate sets A_n we have the following representation of $\tilde{\Delta}(X_n)$:

$$(3.3)\quad \tilde{\Delta}(X_n) = \sum_I \chi_1(A_n, A_I)\tilde{\delta}(A_I) \text{ with } \tilde{\delta}(A_I) := \mathrm{Vol}((\mathrm{cone}(\tilde{S}_I) \setminus \tilde{S}_I) \cap B^d).$$

In order to estimate the variance of $\tilde{\Delta}(X_n)$ asymptotically we need a bound for $\tilde{\delta}(A_I)$, which is good if S_I lies near the boundary of B^d, as X_n exhausts the ball more and more if n becomes large with high probability. Let $h(A_I)$ be $H(A_I)$'s distance from the origin. Then:

Lemma A: *If A_I is a linearly independent set,*

$$(3.4)\qquad \tilde{\delta}(A_I) = \mathcal{O}((1-h')^{\frac{d+1}{2}}),\, h' \to 1-,$$

uniformly for all A_I with $h(A_I) = h'$. Moreover, $\tilde{\delta}(A_I)$ is globally bounded by $\frac{1}{2}\mu_d$.

Proof: By geometrical insight, we have in case of linearly independent A_I with $h(A_I) = h'$:

$$\tilde{\delta}(A_I) \leq \mathrm{Vol}(\frac{1}{h'}\tilde{S}_I) - \mathrm{Vol}(\tilde{S}_I) = \frac{1-h'^d}{d\,h'^{d-1}}\mathrm{Area}(S_I).$$

As S_I is a simplex in a $d-1$-dimensional ball with radius $\sqrt{1-h'^2}$, a rough bound for its area is given by $\mathrm{Area}(S_I) \leq \mu_{d-1}(1-h'^2)^{(d-1)/2}$. Therefore, we obtain

$$\tilde{\delta}(A_I) \leq 2^{\frac{d-1}{2}}\mu_{d-1}(1-h')^{\frac{d+1}{2}}h'^{-d+1} = \mathcal{O}((1-h')^{\frac{d+1}{2}}),\; h' \to 1-.$$

The global bound is obvious by the definition of $\tilde{\delta}(A_I)$.

Besides the geometrical quantities introduced above, we need some probabilistic quantities. Let $F(h') := \Pr(\|a\|_2 \geq h')$ be the probability that a random point lies outside a sphere of radius h' centered at the origin. For uniformly in B^d distributed vectors, F is given by $F(h') = 1 - h'^d$. $G(h') := \Pr(a^{(1)} \geq h')$ be the probability that a random point lies beyond a fixed hyperplane at distance h' from the origin and $g(h')$ be $G(h')$'s density function. Another quantity often used in the following sections is the density function $p(h')$ of $h(A_I)$'s distribution function $P(h') := \Pr(h(A_I) \le h')$. Raynaud [10] proved that for uniformly distributed a_i:

$$(3.5) \qquad G(h') = \frac{\mu_{d-1}}{\mu_d} \int_{h'}^{1} (1-t^2)^{\frac{d-1}{2}} dt, \; p(h') = \frac{2}{d} \frac{\kappa_{d^2+1}}{\kappa_{d^2+2}} (1-h'^2)^{\frac{d^2-1}{2}}.$$

For our asymptotic estimations we widely do not need explicit representations of these quantities but some relations between them, which we summarize in the following lemma:

Lemma B: *In case of uniformly distributed a_i in B^d there exist positive constants α_d, β_d and γ_d such that for $d \ge 2$:*

$$(3.6) \qquad G(h') \sim \alpha_d (1-h')^{\frac{d-1}{2}} F(h'), \; h' \to 1-,$$

$$(3.7) \qquad p(h') \sim \beta_d G^{d-2}(h') g(h') F(h'), \; h' \to 1-$$

$$(3.8) \qquad p(h') \le \gamma_d g(h'), \; h' \in [0,1].$$

For $d = 1$, we have $p(h') = 2g(h')$ and $P(h') = F(h') = 2G(h')$ for $h \in [0,1]$.

Lemma B is an easily established consequence of (3.5) and therefore we do not proof it. Mainly, we state the lemma because its claims remain valid, if we consider more general spherically symmetric distributions. For instance, if the underlying distribution has a regularly varying tail, cf. section 6 for a definition, (3.6-3.8) hold with constants depending additionally on F then.

In order to evaluate the stochastic integrals representing the variance in section 4 we need an asymptotic formula, which is based on Watson's lemma:

Lemma C: *Let $R \in C(0, \frac{1}{2}]$ fulfill $R(t) \sim Lt^{\beta}$ for $t \to 0+$ and constants $L > 0$ and $\beta > -1$. Then, for $d \ge 1$:*

$$(3.9) \qquad \binom{n}{d} \int_{0}^{1/2} (1-\tau)^{n-d} \tau^{d-1} R(\tau) d\tau \sim \frac{\Gamma(d+\beta)}{d!} R(\tfrac{1}{n}), \; n \to \infty.$$

Equation (3.9) is also valid if L will be replaced by an at zero slowly varying function, cf. section 6 for a precise definition. We will apply Lemma C to functions R of type $R(\tau) := \tau^{\sigma_1}(1-\check{G}(\tau))^{\sigma_2}$, where $\sigma_1, \sigma_2 \ge 0$ and $\check{G}$ is the inverse function of G. These functions fulfill the prepositions of Lemma C with $\beta = \sigma_1 + \sigma_2 \frac{2}{d+1}$ for uniformly distributed vectors in the ball, as is proven by inversion of (3.6). The funtions R satisfy the prepositions also in the more general case of spherically symmetric distribution with regularly varying tail in the ball.

Finally, we look at the distribution function $\hat{P}$ of an angle enclosed between two independently and spherically symmetrically distributed points in B^d, $d \ge 2$. $\hat{P}$ has a density function $\hat{p}$, given by Lemma D.

Lemma D: *Let a_1 and a_2 be independently distributed by a spherically symmetric distribution in B^d, $d \ge 2$, and let $\hat{P}(\varphi) = \Pr(\angle(a_1, a_2) \le \varphi)$ be the* *distribution function of the angle enclosed by* a_1 *and* a_2. *Then* P *has a density* p, *given by*

$$p(\varphi) = \frac{\kappa_{d-1}}{\kappa_d} \sin^{d-2} \varphi. \tag{3.10}$$

Proof: Without loss of generality let $a_1 = e_1$ and a_2 be uniformly distributed on the sphere S^{d-1}. In this case, we obviously have

$$P(\varphi) = \Pr(\angle(a_1, a_2) \le \varphi) = \Pr(a_2^{(1)} \ge \cos\varphi)$$

For the uniform distribution on the sphere S^{d-1}, $\Pr(a_2^{(1)} \ge \cos\varphi)$ equals the fraction of the area of the cap of S^{d-1} cut off by the hyperplane $a^{(1)} = \cos\varphi$. Thus,

$$P(\varphi) = \frac{\kappa_{d-1}}{\kappa_d} \int_0^{\varphi} \sin^{d-2} \psi d\psi.$$

Thus, (3.10) is immediate by taking derivatives.

4. The variance—Proof of Theorem 1

We consider the second moment of the polytope functional $\tilde{\Delta}$, cf. (3.1), instead of Δ's, which was motivated in section 3. By (3.3) we know, that

$$\mathrm{E}(\tilde{\Delta}^2(X_n)) = \sum_{I,J} \mathrm{E}(\Pr(S_I, S_J \text{ bd. simpls. of } \tilde{X}_n)\, \tilde{\delta}(A_I)\tilde{\delta}(A_J)). \tag{4.1}$$

Let us first overestimate the probability that S_I and S_J are jointly boundary simplices of $\tilde{X}_n$. If we introduce the probability function $G_{1,1}(A_I, A_J)$,

$$G_{1,1}(A_I, A_J) := \Pr(a \in H^{(1)}(A_I) \cap H^{(1)}(A_J)), \tag{4.2}$$

we obtain by use of the definition of χ_1, cf. section 3,

(4.3)

$$\Pr(S_I, S_J \text{ bd. simpls. of } \tilde{X}_n) = G_{1,1}(A_I, A_J)^{n-2d+|I\cap J|} \chi_1(A_J, A_I)\chi_1(A_I, A_J),$$

as all a_i are identically and independently distributed. (4.3) reduces the probability that S_I and S_J are jointly boundary simplices of $\tilde{X}_n$ to the probability $G_{1,1}(A_I, A_J)$ that a point a lies in both of the halfspaces $H^{(1)}(A_I)$ and $H^{(1)}(A_J)$ respectively. Hence, inserting (4.3) into (4.1) we obtain

(4.4)

$$\mathrm{E}(\tilde{\Delta}^2(X_n)) = \sum_{I,J} \mathrm{E}(G_{1,1}^{n-2d+|I\cap J|}(A_I, A_J)\chi_1(A_I, A_J)\chi_1(A_J, A_I)\, \tilde{\delta}(A_I)\tilde{\delta}(A_J)).$$

Next, we split up the expectation on the right of (4.4) in two parts. We distinguish between those pairs of subsets A_I and A_J, whose associated hyperplanes $H(A_I)$ and $H(A_J)$ have common points inside the unit ball B^d, and those, whose associated hyperplanes do not intersect inside the ball. Pairs (A_I, A_J), whose corresponding hyperplanes do not meet at all, form a set of measure zero. So, we do not care about them. Let $r(A_I, A_J)$,

$$r(A_I, A_J) := \min\{\|a\|_2 \mid a \in H(A_I) \cap H(A_J)\}, \tag{4.5}$$

be the distance from the origin to $H(A_I) \cap H(A_J)$. Then, $r(A_I, A_J) > 1$ if $H(A_I)$ and $H(A_J)$ have no common points inside B^d and $r(A_I, A_J) \leq 1$ otherwise. We define $e(n)$,

(4.6)

$$e(n) := \sum_{I,J} \mathrm{E}(G_{1,1}^{n-2d+|I\cap J|}(A_I, A_J)\chi_1(A_I, A_J)\chi_1(A_J, A_I)\tilde{\delta}(A_I)\tilde{\delta}(A_J)\,\chi(r > 1))$$

and the complement $\bar{e}(n)$ of $e(n)$ by

(4.7)

$$\bar{e}(n) := \sum_{I,J} \mathrm{E}(G_{1,1}^{n-2d+|I\cap J|}(A_I, A_J)\chi_1(A_I, A_J)\chi_1(A_J, A_I)\tilde{\delta}(A_I)\tilde{\delta}(A_J)\,\chi(r \leq 1)),$$

where $r = r(A_I, A_J)$ in (4.6) and (4.7). As obviously holds,

$$\mathrm{Var}(\tilde{\Delta}(X_n)) = e(n) - \mathrm{E}^2(\tilde{\Delta}(X_n)) + \bar{e}(n), \tag{4.8}$$

Theorem 1 will be established by the proof of two claims, which estimate $e(n)$ and $\bar{e}(n)$ from above. For uniformly distributed vectors a_i in the unit ball B_d, $d \geq 1$, holds:

Claim 1: There exists a constant C_d such that for $n \geq 2d$:

$$e(n) - \mathrm{E}^2(\tilde{\Delta}(X_n)) \leq C_d(1 - \tilde{G}(\tfrac{1}{n}))^{\frac{d+5}{2}} = \Theta(n^{-\frac{d+5}{d+1}}). \tag{4.9}$$

Claim 2:

$$\bar{e}(n) = \mathcal{O}((1 - \tilde{G}(\tfrac{1}{n}))^{\frac{d+3}{2}}) = \Theta(n^{-\frac{d+3}{d+1}}),\ n \to \infty. \tag{4.10}$$

$\tilde{G}$ denotes the inverse function of G, cf section 3. We notate (4.9) and (4.10) in terms of $\tilde{G}$ also as the left hand sides of the claims remain valid for spherically symmetric distributions with regularly varying tail, cf. section 6. The order of decadence of $\bar{e}(n)$ in line (4.10) cannot be improved. This is unfortunately not necessarily true for the estimate in line (4.9), as $e(n) - \mathrm{E}^2(\tilde{\Delta}(X_n))$ might be negative and could have the same order in modulus, which $\bar{e}(n)$ has. Thus, in order to determine the sharp order of decadence of $\mathrm{Var}(\tilde{\Delta}(X_n))$, we need a lower bound, which we cannot give for $d \geq 3$ so far. Furthermore, from (4.9-4.10) we learn, that $\mathrm{E}(\tilde{\Delta}^2(X_n)) \sim e(n) \sim \mathrm{E}^2(\tilde{\Delta}(X_n))$ for $n \to \infty$. This means the second moment of $\tilde{\Delta}(X_n)$ is dominated by the contribution of those pairs of boundary simplices S_I and S_J, whose supporting hyperplanes do not have common points inside B^d.

The proofs of both claims in the following subsections are done without making use of the fact that we investigate uniformly distributed vectors. We only assume that the functions G, p and g satisfy (3.6-3.8), that the functions of type $R(\tau) = \tau^{\sigma_1}(1 - \tilde{G}(\tau))^{\sigma_2}$ fulfill the prepositions of Lemma C and that $\mathrm{E}(\tilde{\Delta}(X_n)) = \Theta(1 - \tilde{G}(\frac{1}{n}))$ for $n \to \infty$.

4.1 Proof of Claim 1

If $r(A_I, A_J) > 1$, we know additionally that $I \cap J = \emptyset$. As there are $\binom{n}{d}\binom{n-d}{d}$ pairs of sets of indices (I, J), which do not intersect, we receive from definition (4.6) for any pair (I, J):

(4.1.1)

$$e(n) = \binom{n}{d}\binom{n-d}{d}\mathrm{E}_0(G_{1,1}^{n-2d}(A_I, A_J)\chi_1(A_I, A_J)\chi_1(A_J, A_I)\tilde{\delta}(A_I)\tilde{\delta}(A_J)\,\chi(r > 1))$$

with $r = r(A_I, A_J)$. The index 0 indicates that I and J fulfill $|I \cap J| = 0$. The function $G_{1,1}(A_I, A_J)$ can be represented in terms of G, if jointly $r(A_I, A_J) > 1$, $A_I \subset H^{(1)}(A_J)$ and $A_J \subset H^{(1)}(A_I)$.

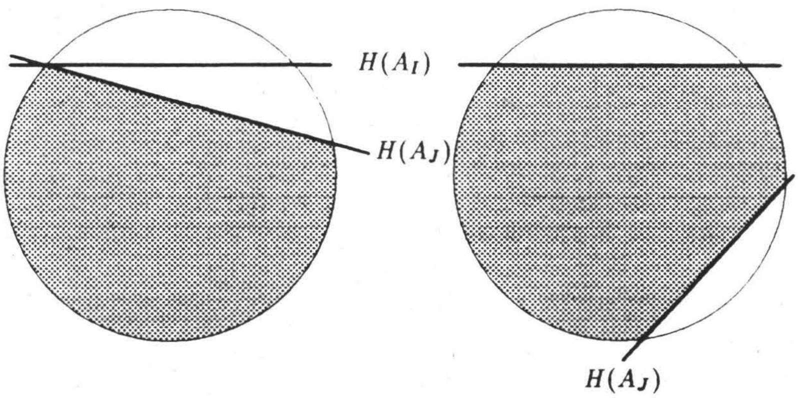

Fig. 1. Illustration of $G_{1,1}(A_I, A_J)$

By geometrical insight, cf. Figure 1, we have under these conditions on A_I and A_J:

$$G_{1,1}(A_I, A_J) = 1 - G(h(A_I)) - G(h(A_J)). \tag{4.1.2}$$

Thus, if we insert (4.1.2) into (4.1.1) and if we afterwards estimate the indicator functionals by one, we obtain an upper bound for $e(n)$. It holds:

$$e(n) \le \binom{n}{d}\binom{n-d}{d}\mathrm{E}_0((1 - G(h(A_I)) - G(h(A_J)))^{n-2d}\tilde{\delta}(A_I)\tilde{\delta}(A_J)). \tag{4.1.3}$$

On the right hand side of (4.1.3), the variables $h_1 = h(A_I)$ and $h_2 = h(A_J)$ and the variables $\tilde{\delta}(A_I)$ and $\tilde{\delta}(A_J)$ are independent, as I and J are disjoint sets. Therefore, by Bayes' theorem the right hand side of (4.1.3) is equal to

(4.1.4)

$$\binom{n}{d}\binom{n-d}{d}\int_0^1\int_0^1(1 - G(h_1) - G(h_2))^{n-2d}\mathrm{E}(\tilde{\delta}(A_I)|h_1)\mathrm{E}(\tilde{\delta}(A_J)|h_2)dP(h_1)dP(h_2)$$

with $P(h') = \Pr(h(A_I) \le h')$, cf. section 3. The condition $(\,\cdot\,|h')$ is an abbreviation for the condition $(\,\cdot\,|h(A_K) = h')$, $K \in \{I, J\}$. As P has a density function p, we are allowed to replace $dP(h_i)$ by $p(h_i)dh_i$ in (4.1.4). For ease of notation we introduce an auxiliary function Λ by the equation

$$\text{(4.1.5)} \qquad \mathrm{E}(\tilde{\delta}(A_K)|h)p(h) = \Lambda(h)g(h)$$

and substitute $G(h_i) = \tau_i$ in (4.1.4). We obtain from (4.1.3):

$$\text{(4.1.6)} \qquad e(n) \le \binom{n}{d}\binom{n-d}{d} \int_0^{1/2}\int_0^{1/2} (1-\tau_1-\tau_2)^{n-2d}\Lambda(\tilde{G}(\tau_1))\Lambda(\tilde{G}(\tau_2))d\tau_1 d\tau_2,$$

where $\tilde{G}$ denotes the inverse function of G.

It is our next objective to compare $e(n)$ with $\mathrm{E}^2(\tilde{\Delta}(X_n))$. For that reason we derive a representation of $\mathrm{E}(\tilde{\Delta}(X_n))$, which is similar to $e(n)$'s bound in (4.1.6). By definition (3.3) and the identical distribution of the vectors a_i we have for any set of indices I:

$$\text{(4.1.7)} \qquad \mathrm{E}(\tilde{\Delta}(X_n)) = \binom{n}{d}\mathrm{E}(\Pr(S_I \text{ bd. simpl. of } \tilde{X}_n)\tilde{\delta}(A_I)).$$

The probability that S_I is a boundary simplex of $\tilde{X}_n$ can be expressed in terms of G. We gain by the definition of χ_1, cf. section 3:

$$\text{(4.1.8)} \quad \Pr(S_I \text{ bd. simpl. of } \tilde{X}_n) = \Pr{}^{n-d}(a \in H^{(1)}(A_I)) = (1 - G(h(A_I)))^{n-d}.$$

Thus, with the same arguments as above we receive

$$\text{(4.1.9)} \qquad \mathrm{E}(\tilde{\Delta}(X_n)) = \binom{n}{d}\int_0^{1/2} (1-\tau)^{n-d}\Lambda(\tilde{G}(\tau))d\tau.$$

We reduce the discussion of the difference $e(n) - \mathrm{E}^2(\tilde{\Delta}(X_n))$ to the analysis of a bilinear form. If we define kernels $K^{(n)}(\tau_1, \tau_2)$,

$$\text{(4.1.10)} \qquad K^{(n)}(\tau_1,\tau_2) := \frac{(1-\tau_1-\tau_2)^{n-2d}}{(1-\tau_1)^{n-d}(1-\tau_2)^{n-d}} - 1,$$

for $\tau_i \in [0, \frac{1}{2}]$ and functions $\tilde{\Lambda}_n$,

$$\text{(4.1.11)} \qquad \tilde{\Lambda}_n(\tau) := \frac{(1-\tau)^{n-d}\Lambda(\tilde{G}(\tau))}{\int_0^{1/2} (1-\tau')^{n-d}\Lambda(\tilde{G}(\tau'))d\tau'},$$

for $\tau \in [0, \frac{1}{2}]$, we obtain from (4.1.6) and (4.1.9),

$$\text{(4.1.12)} \qquad \frac{e(n) - \mathrm{E}^2(\tilde{\Delta}(X_n))}{\mathrm{E}^2(\tilde{\Delta}(X_n))} \le \int_0^{1/2}\int_0^{1/2} K^{(n)}(\tau_1,\tau_2)\tilde{\Lambda}_n(\tau_1)\tilde{\Lambda}_n(\tau_2)d\tau_1 d\tau_2,$$

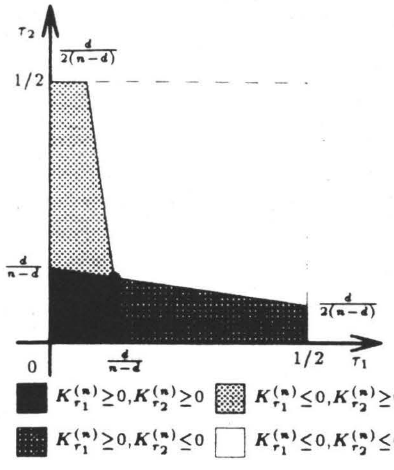

Fig. 2.1. Monotonicity of $K^{(n)}$

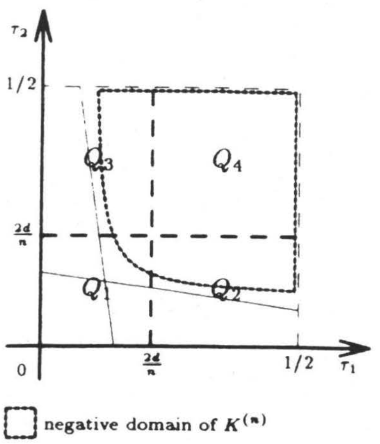

Fig. 2.2. Partition of the domain of $K^{(n)}$

as $\binom{n}{d}\binom{n-d}{d} \le \binom{n}{d}^2$. The only matter left is a careful estimation of the bilinearform on the right hand side of (4.1.12), which we do for $n \ge 4d$. We dissect the domain of integration $[0, \frac{1}{2}]^2$ into four parts $Q_1, \ldots, Q_4$, cf. Figure 2.2. It is not very hard to show that $K^{(n)}(\frac{2d}{n}, \frac{2d}{n}) < 0$. Hence, by the monotonicity of $K^{(n)}$, cf. Figure 2.1, we know $K^{(n)}(\tau_1, \tau_2) < 0$ for $(\tau_1, \tau_2) \in Q_4$. In the region Q_2 we have

$$(4.1.13) \qquad K^{(n)}(\tau_1, \tau_2) \le (1 - \tau_1)^{-d} - 1 \le d\,2^{d+1}\tau_1, \ (\tau_1, \tau_2) \in Q_2,$$

arguing again with the monotonicity of $K^{(n)}$. The same estimate holds in Q_3, if we replace τ_1 by τ_2 on the right hand side of (4.1.13). Finally, we have to analyze $K^{(n)}$ in Q_1. As $K^{(n)}$ has a local maximum in $(\frac{d}{n}, \frac{d}{n})$, cf. Figure 2.1, we obtain:

$$(4.1.14) \quad K^{(n)}(\tau_1, \tau_2) \le \max(K^{(n)}(0, \tfrac{2d}{n}), K^{(n)}(\tfrac{d}{n}, \tfrac{d}{n})) \le \tfrac{d^2}{n} 2^{d+2}, \ (\tau_1, \tau_2) \in Q_1.$$

Summarizing the discussion of the kernel function $K^{(n)}$, we receive from (4.1.12):

$$(4.1.15) \qquad \frac{e(n) - \mathrm{E}^2(\tilde{\Delta}(X_n))}{\mathrm{E}^2(\tilde{\Delta}(X_n))} \le d 2^{d+2} \int_0^{1/2} \tau \tilde{\Lambda}_n(\tau) d\tau + \frac{d^2}{n} 2^{d+2}.$$

Let us look at the integral on the right hand side of (4.1.15). Invoking the definition of $\tilde{\Lambda}_n$ and formula (4.1.9), we get:

$$(4.1.16) \qquad \int_0^{1/2} \tau \tilde{\Lambda}_n(\tau) d\tau = \frac{\binom{n}{d} \int_0^{1/2} (1-\tau)^{n-d} \tau \Lambda(\tilde{G}(\tau)) d\tau}{\mathrm{E}(\tilde{\Delta}(X_n))}.$$

By the aid of (3.7) and Λ's definition (4.1.5), we know for $d \ge 2$ that

(4.1.17) $$\Lambda(h) \sim \beta_d G^{d-2}(h)\mathrm{E}(\tilde{\delta}(A_I)|h)F(h), h \to 1-.$$

For $d = 1$, $\Lambda(h) = 2\mathrm{E}(\tilde{\delta}(A_I)|h)$. By Lemma A, $\mathrm{E}(\tilde{\delta}(A_I)|h) = \mathcal{O}((1-h)^{\frac{d+1}{2}})$ for $h \to 1-$. Therefore, by (3.6) and (4.1.17) we get $\Lambda(h) = \mathcal{O}((1-h)G^d(h))$. Thus, as p/g is bounded, there exists a continous function R and a positive constant L such that

(4.1.18) $$\Lambda(\tilde{G}(\tau)) \leq \tau^{d-1}R(\tau),\ \tau \in [0, \tfrac{1}{2}],\ R(\tau) \sim L\tau(1-\tilde{G}(\tau)),\ \tau \to 0+.$$

As R satisfies the prepositions of Lemma C,

(4.1.19) $$\binom{n}{d}\int_0^{1/2}(1-\tau)^{n-d}\tau\Lambda(\tilde{G}(\tau))d\tau = \mathcal{O}(R(\tfrac{1}{n})) = \Theta(\tfrac{1}{n}(1-\tilde{G}(\tfrac{1}{n}))),\ n \to \infty.$$

As

(4.1.20) $$\mathrm{E}(\tilde{\Delta}(X_n)) = \Theta(1-\tilde{G}(\tfrac{1}{n})),\ n \to \infty.$$

Hence, by (4.1.16),

(4.1.21) $$\int_0^{1/2}\tau\tilde{\Lambda}_n(\tau)d\tau = \mathcal{O}(n^{-1}),\ n \to \infty.$$

The proof of Claim 1 is completed if we invoke (4.1.20) for the denominator on the left hand side of (4.1.16).

4.2 Proof of Claim 2

The purpose of the present subsection is an estimation of the sequence $\bar{e}(n)$ from above. We analyze the contribution of pairs of indices (I, J) with $|I \cap J| = k$, $k \in \{0, \ldots, d\}$, seperately. Using easy combinatorial arguments, we see that there are q_k,

(4.2.1) $$q_k := \binom{n}{2d-k}\binom{2d-k}{d-k}\binom{d}{k},$$

pairs of sets of indices I and J, which both have cardinality d and which have exactly k elements in common. Hence, if we define

(4.2.2)

$$\bar{e}_k(n) := q_k\mathrm{E}_k(G_{1,1}^{n-2d+k}(A_I, A_J)\chi_1(A_I, A_J)\chi_1(A_J, A_I)\tilde{\delta}(A_I)\tilde{\delta}(A_J)\chi(r \leq 1))$$

with $r = r(A_I, A_J)$ and where the index k indicates that $|I \cap J| = k$, we get from (4.7):

(4.2.3) $$\bar{e}(n) = \sum_{k=0}^{d}\bar{e}_k(n).$$

In the sequel, we estimate $\bar{e}_k(n)$ from above. First, we give a bound for $G_{1,1}(A_I, A_J)$ in terms of G. For all (A_I, A_J), (4.2.4) $$G_{1,1}(A_I, A_J) \leq \min\{1 - G(h(A_I)), 1 - G(h(A_J))\},$$

which is immediate by the definitions G and $G_{1,1}$. We introduce the conditioned probability function P_k,

(4.2.5) $$P_k(h_1, h_2) := \Pr_k(h(A_I) \leq h_1, h(A_J) \leq h_2),$$

and the conditioned expectation

(4.2.6)
$$\epsilon_k(h_1, h_2) := \mathrm{E}_k(\tilde{\delta}(A_I)\tilde{\delta}(A_J)\,\chi(r(A_I, A_J) \leq 1)\,|h(A_I) = h_1, h(A_J) = h_2),$$

where in both formulae the index k indicates that $|I \cap J| = k$. The condition $(\cdot|h_1, h_2)$ abbreviates the joint conditions $h(A_I) = h_1$ and $h(A_J) = h_2$ on A_I and A_J. If we estimate the indicator functionals $\chi_1(A_I, A_J)$ and $\chi_1(A_J, A_I)$ in (4.2.2) by one, we receive by Bayes' theorem and (4.2.4):

(4.2.7)
$$\bar{e}_k(n) \leq q_k \int_{0 \leq h_1, h_2 \leq 1} (1 - G(\min\{h_1, h_2\}))^{n-2d+k} \epsilon_k(h_1, h_2) d\{P_k(h_1, h_2)\}.$$

For $d\{P_k(h_1, h_2)\}$ holds

(4.2.8) $$d\{P_k(h_1, h_2)\} = d\{\Pr_k(h(A_J) \leq h_2 | h(A_I) = h_1)\} p(h_1) dh_1$$

with p as in section 3. Thus, exploiting the symmetry of (4.2.7) we obtain

(4.2.9) $$\bar{e}_k(n) \leq 2q_k \int_0^1 (1 - G(h_1))^{n-2d+k} T_k(h_1) p(h_1) dh_1,$$

where the auxiliary function T_k is defined by

(4.2.10) $$T_k(h_1) := \int_{h_1}^1 \epsilon_k(h_1, h_2) d\{\Pr_k(h(A_J) \leq h_2 | h(A_I) = h_1)\}.$$

By Lemma A, we know that

(4.2.11)
$$\epsilon_k(h_1, h_2) = \mathcal{O}((1 - h_1)^{\frac{d+1}{2}}(1 - h_2)^{\frac{d+1}{2}}) \Pr_k(r(A_I, A_J) \leq 1 | h_1, h_2),\ h_1, h_2 \to 1-.$$

Hence, if we insert (4.2.11) into (4.2.10) we obtain an asymptotic bound for T_k:

(4.2.12)
$$T_k(h_1) = \mathcal{O}((1 - h_1)^{d+1}) \Pr_k(h(A_J) \geq h_1,\ r(A_I, A_J) \leq 1 | h(A_I) = h_1),\ h_1 \to 1-.$$

For $k \neq d$ the conditioned probability on the right hand side of (4.2.12) can be estimated by use of geometrical observations, cf. Figure 3. If $h(A_J) \geq h_1$ and $h_1 = h(A_I)$, each vector $a \in A_J \setminus A_I$ must lie outside a ball of radius h_1 centered at the origin. On the other hand, as $r(A_I, A_J) \leq 1$, the angle enclosed between the normal vector $w(A_I)$ of the hyperplane $H(A_I)$ and each vector $a \in A_J \setminus A_I$

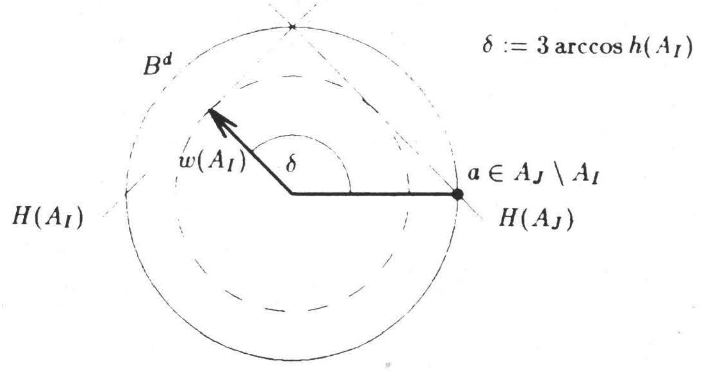

Fig. 3. **Maximal possible angle between $w(A_I)$ and a vector $a \in A_J \setminus A_I$**

cannot exceed $3 \arccos h_1$. Thus, as $w(A_I)$ and each $a \in A_J \setminus A_I$ are stochastically independent, we get

(4.2.13)
$$\Pr_k(h(A_J) \geq h_1,\ r(A_I, A_J) \leq 1 | h(A_I) = h_1) \leq (F(h_1)\hat{P}(3 \arccos h_1))^{d-k},$$

which is trivially valid in case of $d = k$ also. From Lemma D one easily derives that $\hat{P}(3 \arccos h_1) = \mathcal{O}((1 - h_1)^{\frac{d-1}{2}})$ for $h_1 \to 1-$. Thus, invoking (3.6) we obtain from (4.2.13) and (4.2.12):

$$T_k(h_1) = \mathcal{O}((1 - h_1)^{d+1} G(h_1)^{d-k}),\ h_1 \to 1- . \tag{4.2.14}$$

For simplicity of notation we introduce a function S_k by the equation

$$S_k(h_1) g(h_1) = T_k(h_1) p(h_1) \tag{4.2.15}$$

As p fulfills (3.7) we obtain

$$S_k(h_1) = \mathcal{O}(G^{2d-k-1}(h_1)(1 - h_1)^{\frac{d+3}{2}}),\ h_1 \to 1- . \tag{4.2.16}$$

Hence, as p/g is bounded there exists a continous function R_k and a positive constant L, such that
(4.2.17)
$$S_k(\tilde{G}(\tau)) \leq \tau^{2d-k-1} R_k(\tau),\ \tau \in [0, \tfrac{1}{2}],\ R_k(\tau) \sim L(1 - \tilde{G}(\tau))^{\frac{d+3}{2}},\ \tau \to 0+ .$$

If we substitute $G(h_1) = \tau$ in line (4.2.9) and use (4.2.15) and (4.2.17), we receive

$$\bar{e}_k(n) \leq 2\binom{2d-k}{d-k}\binom{d}{k}\binom{n}{2d-k} \int_0^{1/2} (1 - \tau)^{n-2d+k} \tau^{2d-k-1} R_k(\tau) d\tau. \tag{4.2.18}$$

As R_k satisfies the prepositions of Lemma C, we obtain the desired estimate of Claim 2.

5. The limiting distribution—Proof of Theorem 2

If we apply Chebychev's inequality to the random variable $\Delta(X_n) - \mathrm{E}(\Delta(X_n))$, we obtain for $n \geq d+1$ and $\alpha > 0$

$$\Pr(|\Delta(X_n) - \mathrm{E}(\Delta(X_n))| \geq \alpha \mathrm{E}(\Delta(X_n))) \leq \alpha^{-2} \frac{\mathrm{Var}(\Delta(X_n))}{\mathrm{E}^2(\Delta(X_n))}. \tag{5.1}$$

By (1.1) and Theorem 1, we know that for uniformly distributed a_i:

$$\eta_n := \frac{\mathrm{Var}(\Delta(X_n))}{\mathrm{E}^2(\Delta(X_n))} = \mathcal{O}(n^{-\frac{d-1}{d+1}}),\ n \to \infty. \tag{5.2}$$

Hence, the subsequence η_{m_k}, $m_k := (k+d)^4$, is summable for $d \geq 2$. This means

$$\sum_{k=1}^{\infty} \Pr(|\Delta(X_{m_k}) - \mathrm{E}(\Delta(X_{m_k}))| \geq \alpha \mathrm{E}(\Delta(X_{m_k}))) < \infty \tag{5.3}$$

for all $\alpha > 0$. Thus, by the lemma of Borel-Cantelli, we receive almost surely for $d \geq 2$:

$$\Delta(X_{m_k}) \sim E(\Delta(X_{m_k})),\ k \to \infty. \tag{5.4}$$

In the sequel, we exploit the monotonicity of the sequence $(\Delta(X_n))_{n\in\mathbb{N}}$ in order to show (5.4) for the sequence itself. For every $n \geq m_1$ there exists an index $k(n)$ such that $m_{k(n)} \leq n < m_{k(n)+1}$. By the definition of m_k, it is a simple matter to prove $m_{k(n)} \sim n$ for $n \to \infty$. Using the monotonicity of $(\Delta(X_n))_{n\in\mathbb{N}}$, we obtain

$$\frac{\Delta(X_{m_{k(n)+1}})}{\mathrm{E}(\Delta(X_{m_{k(n)}}))} \geq \frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_{m_{k(n)}}))} \geq \frac{\Delta(X_{m_{k(n)+1}})}{\mathrm{E}(\Delta(X_{m_{k(n)}}))}. \tag{5.5}$$

By (5.4) the left hand side of (5.5) tends to one almost surely for $n \to \infty$. For the quotient on the right hand side we receive almost surely for $n \to \infty$ by (5.4) and (1.1):

$$\frac{\Delta(X_{m_{k(n)+1}})}{\mathrm{E}(\Delta(X_{m_{k(n)}}))} \sim \frac{\mathrm{E}(\Delta(X_{m_{k(n)+1}}))}{\mathrm{E}(\Delta(X_{m_{k(n)}}))} \sim \left(1 + \frac{1}{m_{k(n)}}\right)^{-\frac{2}{d+1}} \sim 1. \tag{5.6}$$

Hence, $\Delta(X_n) \sim \mathrm{E}(\Delta(X_{m_{k(n)}}))$ almost surely. Therefore, we get almost surely for $n \to \infty$ and $d \geq 2$:

$$\frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_n))} \sim \frac{\mathrm{E}(\Delta(X_{m_{k(n)}}))}{\mathrm{E}(\Delta(X_n))} \sim \left(\frac{m_{k(n)}}{n}\right)^{-\frac{2}{d+1}} \sim 1, \tag{5.7}$$

which completes the proof of Theorem 2.

6. Generalizations and concluding remarks

As mentioned in the introduction, analogous results as in section 2 for the uniform distribution can be obtained for other distributions in B^d. Carnal [4] considered spherically symmetric distributions with regularly varying tail.

Let the polar decompositions of the vectors a_i, $a_i = \rho_i s_i$, consist of independent radial parts $\rho_i \in [0,1]$ and spherical parts $s_i \in S^{d-1}$. The radial distribution function F, cf. section 3, satisfy

$$F(1-t) \sim L(1/t)t^{\sigma}, t \to 0+, \tag{6.1}$$

for a $\sigma \geq 0$ and an at infinity slowly varying function L. A function L is called slowly varying at infinity if $\lim_{x\to\infty} \frac{L(x\rho)}{L(x)} = 1$ for all $\rho \in (0,\infty)$. If $\sigma = 0$, we assume moreover $\lim_{t\to 0+} L(1/t) = 0$. The spherical parts s_i be uniformly distributed on the unit sphere S^{d-1}. Typical examples for slowly varying functions are constants, logarithms, iterated logarithms et cetera. Spherically symmetric distributions, which satisfy (6.1), are called distributions with σ-regularly varying tail in the following. For instance, the uniform distribution in B^d satisfies (6.1) with $\sigma = 1$ and the constant function $L(1/t) = d$, cf. section 3. Carnal proved for $d = 2$ and F, which fulfill (6.1):

$$\mathrm{E}(\Delta(X_n)) \sim L_\sigma(n)\, n^{-\frac{2}{1+2\sigma}}, n \to \infty, \tag{6.2}$$

where L_σ is a certain at infinity slowly varying function related to L. Müller [9] studied uniformly distributed random vectors a_i on the sphere S^{d-1} and obtained for $d \geq 2$

$$\mathrm{E}(\Delta(X_n)) \sim C_d\, n^{-\frac{2}{d-1}}, n \to \infty. \tag{6.3}$$

The uniform distribution on the sphere can be considered a limiting case of Carnal's class taking a pointwise limit of σ-regularly varying tailed distributions with $\sigma \to 0+$ and appropriately chosen functions L.

Theorem 3 generalizes Carnal's result (6.2) on the expectation of $\Delta(X_n)$ to distributions with σ-regularly varying tail in B^d for $d \geq 1$. If $\tilde{G}$ is the inverse function of G, cf. section 3, then:

Theorem 3: *For any spherically symmetric distribution in B^d, $d \geq 1$, with σ-regularly varying tail, there is a positive constant $C_{d,\sigma}$ such that*

$$\mathrm{E}(\Delta(X_n)) \sim C_{d,\sigma}(1 - \tilde{G}(\tfrac{1}{n})), n \to \infty. \tag{6.4}$$

It should be worth noticing that (6.4) relates the rate of decadence of $\mathrm{E}(\Delta(X_n))$ to the intrinsic probabilistic quantity $\tilde{G}$, which can be interpreted geometrically: Let H be the boundary hyperplane of an halfspace, whose intersection with the unit ball has probability content $\tau \in [0, 1/2]$, then $h = \tilde{G}(\tau)$ is the distance from H to the origin. It is a disadvantage of the description (6.4) that the rate of $\mathrm{E}(\Delta(X_n))$'s decadence is not explicitly given. For this reason we remark that for any spherically symmetric distribution in B^d, $d \geq 1$, with σ-regularly varying tail, there is a positive, at infinity slowly varying function $L_{d,\sigma}$ such that

$$1 - \tilde{G}(\tfrac{1}{n}) \sim L_{d,\sigma}(n)\, n^{-\frac{2}{d-1+2\sigma}}, n \to \infty. \tag{6.5}$$

Formula (6.5) is obtained inverting (3.6). The constant $C_{d,\sigma}$ in (6.4) can be given exactly. We have

(6.6) $$C_{d,\sigma} = \kappa_d \left(1 - \frac{d-1}{d}\frac{2}{d+1+2\sigma}\right)\frac{\Gamma(d+\frac{2}{d-1+2\sigma})}{(d-1)!}.$$

Theorem 4: *For any spherically symmetric distribution in B^d, $d \geq 1$, with σ-regularly varying tail,*

(6.7) $$\mathrm{Var}(\Delta(X_n)) = \mathcal{O}((1-\tilde{G}(\tfrac{1}{n}))^{\frac{d+3}{2}}),\ n \to \infty.$$

Analogously to the special case of uniform distribution in Theorem 2, we have a 0-1-law for the limiting distributions:

Theorem 5: *For any spherically symmetric distribution in B^d, $d \geq 2$, with σ-regularly varying tail, almost everywhere holds:*

(6.8) $$\Delta(X_n) \sim \mathrm{E}(\Delta(X_n)),\ n \to \infty.$$

For a proof of Theorem 3, the interested reader might take Dwyer's [5] proof for the expected number of X_n's facets as a guide. Dwyer's article also provides asymptotic formulae for p and G in case of spherically symmetric distributions with σ-regularly varying tail, from which Lemma B can be derived in this more general situation. Theorem 4 is completely proven in section 4 as the left hand sides of the claims are valid for regularly tailed distributions also. It is an easy exercise to modify the proof of Theorem 2 in section 5 for this more general situation.

It is possible to give upper bounds for $\Delta(X_n)$'s variance for other classes of spherically symmetric distributions. For instance, if $F(1-t) \sim \exp(-t^{-\sigma})$, $t \to 0+$, for a positive σ, we can show that the expectation value and the variance of $\Delta(X_n)$ decade in n like a logarithm. But, though the quotient of variance and squared expectation tends to zero as well, we cannot argue along the line of section 5, as the decadence of the quotient is to slow. So, we conjecture that the limiting distribution of $\frac{\Delta(X_n)}{\mathrm{E}(\Delta(X_n))}$ does not fulfill a 0-1-law in this situation.

As a final remark we mention that all theorems of the article remain valid if $\Delta(X_n)$ is replaced by the difference of the surface area of S^{d-1} and X_n's surface area.

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