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Summary. Let a_i, $i := 1, \ldots, m$, be an i.i.d. sequence taking values in $\mathbb{R}^n$, whose convex hull is interpreted as a stochastic polyhedron $\mathcal{P}$. For a special class of random variables, which decompose additively relative to their boundary simplices, eg. the volume of $\mathcal{P}$, integral representations of their first two moments are given which lead to asymptotic estimations of variances for special "additive variables" known from stochastic approximation theory in case of rotationally symmetric distributions.

1. Introduction

In their pioneering articles "Über die konvexe Hülle von n zufällig gewählten Punkten I-II"[10,11], Renyi and Sulanke analyzed the expectation values of the number of edges, the area and the circumference of random convex polytopes generated by the convex hull of i.i.d. vectors on $\mathbb{R}^2$ for the first time. Their ideas and their techniques while treating random variables on stochastic polyhedra led to a variety of generalizing publications in different fields of research like the stochastic approximation theory of bounded convex sets, cf. Gruber[6] or Buchta[3] for a survey, or the stochastic complexity theory of algorithms, cf. Borgwardt[2]. Most of these articles exclusively deal with expectation values and don't give information about the concentration of the random variable's values around the mean. In many cases experimental evidence suggests that the distribution of the random variable is highly concentrated. For that reason it is of interest to obtain higher moments, especially the variance, of the variable investigated in order to quantify large deviations from the mean.

The aim of this paper's first part is to give integral representations for the first two moments of a class of random variables on stochastic bounded polyhedra on $\mathbb{R}^n$, $n \geq 2$, suitable to deduce asymptotic estimations of variances, which is done in the second part.

To explain our stochastic model precisely, let $a_1, \ldots, a_m$, $m \geq n$, be an i.i.d. sequence taking values in $\mathbb{R}^n$ distributed by an arbitrary rotationally symmetric distribution not concentrated in 0 and let $F(t) := \mathrm{P}(\|x\|_2 \leq t)$ be the associated radial distribution function (RDF). If there is a function $f \in \mathcal{L}^1(\mathbb{R}^n)$ such that $F(r) = \int_{\|x\|_2 \leq r} f(x)\,dx$, $r > 0$, we call f the density function of the underlying distribution. The theorems of the present paper are partly formulated for distributions with density functions in order to avoid a too difficult notation. (This is no restriction of generality because the set of distributions with density functions is pointwise dense in the class of all rotationally symmetric distributions and so the general results are easily gained limiting cases, cf. Natanson[8].) In the above described respect

$$\mathcal{P} := \mathrm{convhull}\{a_1, \ldots, a_m\} \tag{1.1}$$

is a randomly generated bounded polyhedron.

For an axiomatic definition of the random variables we are dealing with we need some more notation. For any set of indices $I := \{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$ with $i_j < i_k$ for $i < k$, let $A_I := (a_{i_1}, \ldots, a_{i_n})$ and $S_I := \mathrm{convhull}\{a_i \mid i \in I\}$. Furthermore, if the vectors a_i, $i \in I$, are linearly independent, i.e. S_I is a simplex, $H(A_I) := \mathrm{affhull}\{S_I\}$ denotes the hyperplane containing S_I and if additionally $0 \notin H(A_I)$, $H^{(1)}(A_I)$ denotes the closed halfspace generated by the hyperplane $H(A_I)$ containing the origin whereas $H^{(2)}(A_I)$ represents its closed complement. (The special case $0 \in H(A_I)$ doesn't matter because $P(0 \in H(A_I)) = 0$ for all distributions of our model.) $h(A_I)$ denotes the distance of $H(S_I)$ to the origin.

We call a simplex S_I, whose associated hyperplane $H(A_I)$ doesn't contain the origin, a boundary simplex of $\mathcal{P}$ of the first kind, if $a_j \in H^{(1)}(A_I)$ for $j \in \overline{I}$, and we call it a boundary simplex of $\mathcal{P}$ of the second kind, if $a_j \in H^{(2)}(A_I)$ for $j \in \overline{I}$, $\overline{I}$ being complementary to I as a subset of $\{1, \ldots, m\}$. We relate the last definitions to the characteristic functions

$$\chi_i(A_I) := \begin{cases} 1 & \text{if } S_I \text{ is a boundary simplex of } \mathcal{P} \text{ of the } i\text{-th kind} \\ 0 & \text{else} \end{cases} . \tag{1.2}$$

Now we are able to state the definition of the class of random variables investigated:

Definition *A random variable $\tilde{Z}$ of a randomly generated bounded polyhedron $\mathcal{P}$ is of additive type iff for almost all $\mathcal{P}$:*

$$\tilde{Z}(\mathcal{P}) = \sum_{\substack{I \subset \{1, \ldots, m\} \\ |I| = n}} [\chi_1(A_I) + \sigma\chi_2(A_I)] Z(A_I). \tag{1.3}$$

where $\sigma \in \{-1, 1\}$ is fixed and $Z : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}_0^+$ is a continous function invariant under orthogonal transformations and under exchanging columns of the argument.

Roughly speaking, a random variable on a polyhedron is of additive type if it can be additively decomposed relative to its boundary simplices.

Most of the random variables on bounded stochastic polyhedra discussed in the literature are of type (1.3) as the following list of examples confirms:

The random variable:	with:	
number of boundary simplices	$\sigma = 1$.	$Z(A_I) = 1$
volume	$\sigma = -1$.	$Z(A_I) = \lvert \mathrm{ch}(0, S_I) \rvert$
surface area	$\sigma = 1$.	$Z(A_I) = \lvert S_I \rvert$
the weighted volume	$\sigma = -1$.	$Z(A_I) = \int_{\mathrm{ch}(0, S_I)} f(a)\, da$
$\chi(\mathrm{cone}(\mathcal{P}) = \mathbb{R}^n)$	$\sigma = -1$.	$Z(A_I) = \frac{\lvert \omega_n \cap \mathrm{cone}(S_I) \rvert}{\lvert \omega_n \rvert}$

Here, $|\cdot|$ represents Lebesgue-measure of appropriate dimension and ch abbreviates convex hull. The validity of (1.3) in case of the first three examples in our schedule was implicitly shown by Renyi and Sulanke in [10], whereas the same result for the latter special cases can be found in Ziezold[13] and Borgwardt[2] respectively. A further important example of interest included is the average number of boundary simplices, which are intersected by a randomly chosen plane. This random variable is closely related to the analysis of average complexity of the simplex algorithm, cf. Borgwardt[2]. It is the aim of definition (1.3) to simplify formulations and to enable an illustration of additive type variables' structural properties.

In section 2 we give an integral representation for the first moment of a random variable of type (1.3). Section 3 deals with the second moment. It is the main characteristic of these integral representations that they can be considered as generalized Laplace-type integrals. For concrete applications of the derived integral formulae we refer to section 4, where variances of an important subclass of additive type random variables are asymptotically estimated. We deal with additive type random variables under distributions with compact support (without loss of generality we take the n-dimensional unit ball Ω_n), which fulfill the asymptotic condition:

$$(1.4) \qquad Z(A_I) \sim C_{n,Z} |\det(A_I)|^p , \; h(A_I) \to 1 ,$$

where p is fix in $\mathbb{N}_0$ and $C_{n,Z}$ is a positive constant.

All examples introduced above except for the weighted volume are special cases of this subclass or can be reduced to it. The class of distributions we concentrate on in section 4 is Carnal's class of distributions with regular varying behaviour near the boundary of the distribution's domain Ω_n, cf. [4], that means in terms of radial distribution functions that for all $r \in (0,1)$ and a fix $\alpha \geq 0$:

$$(1.5) \qquad \frac{1 - F(1 - rt)}{1 - F(1 - r)} \sim t^{\alpha} , \; t \to 0+ .$$

For this class of distributions one can evaluate simple asymptotic equivalents for the expectation values as m tends to infinity. The asymptotic order of the equivalents' growth in m is intimately related to the distribution's behaviour near of its the boundary's domain. The main result of section 4 yields for additive random variables of type 1.4, distributions with property 1.5 and for positive a that:

$$(1.6) \quad \mathrm{P}\left(\left|\frac{\tilde{Z}}{\mathrm{E}(\tilde{Z})} - 1\right| > a\right) \leq \frac{\mathrm{Var}(\tilde{Z})}{a^2 \mathrm{E}^2(\tilde{Z})} = O((1 - \tilde{G}_2(\tfrac{1}{m}))^{(n-1)/2}) = o(1) , \; m \to \infty ,$$

where $\tilde{G}_2$ is the inverse function of $G_2(h) := \mathrm{P}(a^{(n)} > h)$. In the more special case of $1 - F(1-r) \sim Lr^\alpha$ with $\alpha > 0$ and $L > 0$, (1.6) simplifies. It holds:

$$\mathrm{P}\left(\left|\frac{\tilde{Z}}{\mathrm{E}(\tilde{Z})} - 1\right| > a\right) = O\left(m^{-(n-1)/(n-1+2\alpha)}\right),\ m \to \infty. \tag{1.6a}$$

This means that even small deviations of our random variables from the mean are very unlikely for big m. The variables' distribution is highly concentrated around the expectation value.

Of course, our integral representations facilitate similar results in case of distribution classes with non-compact support, too. The analysis of the resulting integrals, however, is widely different from the compact situation and is avoided for lack of space. It would be interesting to compare our methods with those of Groeneboom[5] and Aldous et al.[1], who derived asymptotic results about all moments of special cases among additive variables for a class of distributions in $\mathbb{R}^2$ with the aid of Markov-chains and limiting distributions respectively. Our techniques enable an explicit treatise of additive variable's second moments, which allows not only asymptotic results on the probability of large deviations but proper estimations, too. It is an easy matter to refine our asymptotic results slightly such that proper estimations are obtained. But these bounds are only meaningful in case of big m. For moderately chosen m a new very careful and lengthy analysis is required which we delay to a further article. The research done in this article was widely part of the author's dissertation[7] under the advise of Prof. Brakhage.

2. Expectation values

The results in this short section are mainly due to Raynaud[9] and Borgwardt[2], who treated special cases of additive variables under rotationally symmetric distributions on $\mathbb{R}^n$. We cite them adapted to our notation. Let

$$G_1(h) := \mathrm{P}(x^{(n)} \le h),\ h \in \mathbb{R}, \tag{2.1}$$

$$G_2(h) := \mathrm{P}(x^{(n)} > h) = 1 - G_1(h),\ h \in \mathbb{R}, \tag{2.2}$$

$x^{(n)}$ being the n-th coordinate of the vector x, we state without proof, which can be given analogously to Raynaud's special case in [9]:

Theorem 2.1 *For random variables $\tilde{Z}$ of additive type on bounded stochastic polyhedra $\mathcal{P}$ generated by a rotationally symmetric distribution with density function f holds while $m \ge n \ge 2$:*

$$\mathrm{E}(\tilde{Z}) = \binom{m}{n} \int_0^\infty [G_1^{m-n}(h) + \sigma G_2^{m-n}(h)] A_Z(h)\, dh \tag{2.3}$$

with

$$A_Z(h) = |\omega_n| \int_{\mathbb{R}^{n-1}}^{(n)} Z\begin{pmatrix} \bar{b}_1 & \dots & \bar{b}_n \\ h & \dots & h \end{pmatrix} |\det(B)| \prod_{i=1}^n f((\bar{b}_i, h)^T)\, d\bar{b}_i, \tag{2.4}$$

where

$$B := \begin{pmatrix} \bar{b}_1 & \dots & \bar{b}_n \\ 1 & \dots & 1 \end{pmatrix}, \quad \bar{b}_i := (b_i^{(1)}, \dots, b_i^{(n-1)})^T, \; i = 1, \dots, n. \tag{2.5}$$

and

$$|\omega_n| := \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}. \tag{2.6}$$

Remarks on 2.1:

i) The functions $G_i(h)$ are representable by

$$G_2(h) = \int_h^{\infty} g_0(t)dt, \quad g_0(t) := \int_{\mathbb{R}^{n-1}} f((\bar{b}, t)^T)d\bar{b}. \tag{2.7}$$

for distributions with density function f.

ii) The function $A_Z(h)$ can be interpreted in the language of conditional expectation values by

$$A_Z(h) = \mathrm{E}(Z(A_I) \,|\, h(A_I) = h)\, p(h), \tag{2.8}$$

where

$$p(h) := |\omega_n| \int_{\mathbb{R}^{n-1}}^{(n)} |\det(B)| \prod_{i=1}^{n} f((\bar{b}_i, h)^T) d\bar{b}_i, \tag{2.9}$$

is the marginal density function of the distribution function $\mathrm{P}(h(A_I) \leq h)$ for some set of indices I.

iii) If one replaces $G_2(h) = e^{-t}$ in (2.3), $E(\tilde{Z})$ can be considered asymptotically to be a Laplace-type integral for big m, a fact we will make use of in section 4.

3. Second moments

Because of the nonlinearity of the functional $\mathrm{E}(\tilde{Z}^2)$ it is much more complicated to give an integral formula for second moments of an additive type random variable $\tilde{Z}$ corresponding to (2.4). So, we deduce our result step by step and introduce further notation when necessary.

Let $\tilde{Z}$ be a random variable of additive type and $m \geq 2n \geq 4$, then by (1.3):

$$\mathrm{E}(\tilde{Z}^2) = \sum_{i,j=1}^{2} \sigma^{i+j-2} \sum_{I,J \subset \{1,\dots,m\}} \tilde{e}_Z(I, J, i, j), \tag{3.1}$$

where $I = \{i_1, \dots, i_n\}$ and $J = \{j_1, \dots, j_n\}$ are arbitrary sets of indices $1, \dots, m$ with cardinality n and

$$\tilde{e}_Z(I,J,i,j) := \int\limits_{\mathbb{R}^n}^{(m)} \chi_i(A_I)\chi_j(A_J)Z(A_I)Z(A_J)\prod_{\ell=1}^{m} f(a_\ell)da_\ell, \tag{3.2}$$

f being the density function of the underlying rotationally symmetric distribution. For reasons of symmetry $\tilde{e}_Z(I,J,i,j)$ obviously depends only on the cardinality k of the set $I \setminus J$ concerning the variables I and J. Denoting q_k for the number of pairs (I,J), $I,J \subset \{1,\dots,m\}$, with $|I \setminus J| = k$, we define

$$e_Z(k,i,j) := \tilde{e}_Z(I,J,i,j) \text{ if } |I \setminus J| = k \tag{3.3}$$

for $i,j \in \{1,2\}$ and $k \in \{0,\dots,n\}$. Now, we get from (3.1) by changing summations:

$$\mathrm{E}(\tilde{Z}^2) = \sum_{k=0}^{n} q_k \sum_{i,j=1}^{2} \sigma^{i+j-2} e_Z(k,i,j), \tag{3.4}$$

where

$$q_k = \binom{m}{n}\binom{n}{k}\binom{m-n}{k}, \; k = 0,\dots,n, \tag{3.5}$$

which is obtained by simple combinatorial considerations.

In order to evaluate the terms $e_Z(k,i,j)$ further we define for $i,j \in \{1,2\}$ and $k = 0,\dots,n$:

$$p_{i,j}(A_0,A_k) := \int\limits_{\mathbb{R}^n}^{(m-n-k)} \chi_i(A_0)\chi_j(A_k) \prod_{\ell=n+k+1}^{m} f(a_\ell)da_\ell \tag{3.6}$$

with

$$A_\ell := (a_{\ell+1},\dots,a_{\ell+n}), \; \ell = 0,\dots,n. \tag{3.7}$$

$p_{i,j}(A_0,A_k)$ equals the probability of A_0 being a boundary simplex of the i-th kind of $\mathcal{P}$ and A_k being a boundary simplex of the j-th kind of $\mathcal{P}$ at the same time. Using definitions (3.3) and (3.6) we obtain

$$\begin{aligned} e_Z(k,i,j) &= \int\limits_{\mathbb{R}^n}^{(m)} \chi_i(A_0)\chi_j(A_k)Z(A_0)Z(A_k)\prod_{\ell=1}^{m} f(a_\ell)da_\ell \\ &= \int\limits_{\mathbb{R}^n}^{(n+k)} p_{i,j}(A_0,A_k)Z(A_0)Z(A_k)\prod_{\ell=1}^{n+k} f(a_\ell)da_\ell \end{aligned} \tag{3.8}$$

proving the

Proposition 3.1 *For random variables $\tilde{Z}$ of additive type, rotationally symmetric distributions with density function f and $m \geq 2n \geq 4$ holds:*

$$E(\tilde{Z}^2) = \sum_{k=0}^{n} q_k \sum_{i,j=1}^{2} \sigma^{i+j-2} e_Z(k,i,j) \tag{3.9}$$

with

$$q_k = \binom{m}{n}\binom{n}{k}\binom{m-n}{k} \tag{3.10}$$

and

$$e_Z(k,i,j) = \int\limits_{\mathbb{R}^n}^{(n+k)} p_{i,j}(A_0, A_k) Z(A_0) Z(A_k) \prod_{\ell=1}^{n+k} f(a_\ell) da_\ell, \tag{3.11}$$

where

(3.12) $p_{i,j}(A_0, A_k) = \mathrm{P}(A_0$ *is boundary simplex of* $\mathcal{P}$ *of the i-th kind* $\wedge$
A_k *is boundary simplex of* $\mathcal{P}$ *of the j-th kind*$)$.

After this first step of evaluation we look at $p_{i,j}$ more closely. By use of the obvious equations

$$\chi_i(A_\ell) = \prod_{j=0}^{\ell} \chi(a_j \in H^{(i)}(A_\ell)) \prod_{j=n+\ell+1}^{m} \chi(a_j \in H^{(i)}(A_\ell)) \tag{3.13}$$

for $i = 1, 2$ and $\ell = 0, \ldots, n$ we gain

$$p_{i,j}(A_0, A_k) = (\tilde{G}_{i,j}(A_0, A_k))^{m-n-k} \chi(S_k \subset H^{(i)}(A_0)) \chi(S_0 \subset H^{(j)}(A_k)) \tag{3.14}$$

with

$$\tilde{G}_{i,j}(A_0, A_k) := \mathrm{P}(a \in (H^{(i)}(A_0) \cap H^{(j)}(A_k))) \tag{3.15}$$

and S_ℓ defined analogously to (3.7).

The quantities $\tilde{G}_{i,j}$ can be illustrated geometrically as weighted areas dividing the measure space $(\mathbb{R}^n, \mathrm{P})$ in four parts, cf. figure 1.
More formally holds:

$$\sum_{i,j=1}^{2} \tilde{G}_{i,j}(A_0, A_k) = 1. \tag{3.16}$$

Elementary geometric insight makes clear that $G_{i,j}(A_0, A_k)$ for $k = 1, \ldots, n$ depends only on the distance $h_1 = h(A_0)$ of the hyperplane $H(A_0)$ to the origin, the distance $h_2 = h(A_k)$ of the hyperplane $H(A_k)$ to the origin and on the angle $\varphi(A_0, A_k)$ enclosed by the normal vectors n_1 of $H(A_0)$ and n_2 of $H(A_k)$, cf. figure 1. Clearly, $||n_1||_2 = h_1$ and $||n_2||_2 = h_2$. The natural objective now is to represent $e_Z(k, i, j)$ for $k = 1, \ldots, n$ by an integral of the form

$$e_Z(k,i,j) = \int_0^\infty \int_0^\infty \int_0^\pi G_{i,j}^{m-n-k}(h_1, h_2, \varphi) A_{i,j,k,Z}(h_1, h_2, \varphi) d\varphi dh_1 dh_2 \tag{3.17}$$

with appropriately chosen functions $A_{i,j,k,Z}$ and

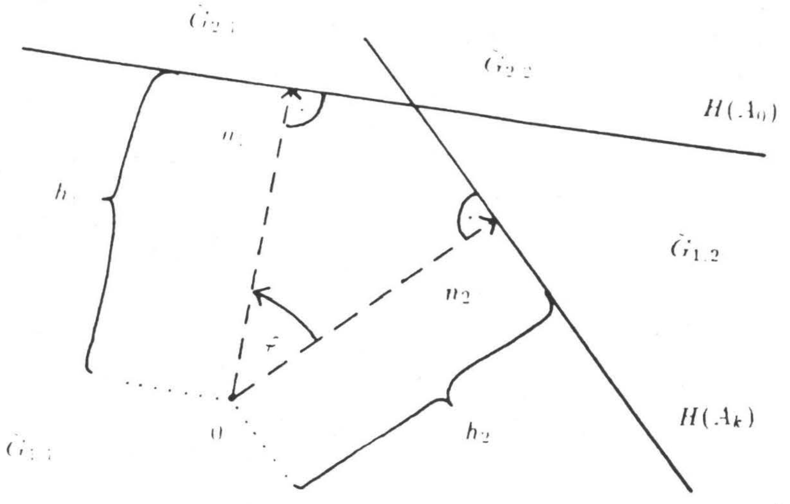

Fig. 1.

$$G_{i,j}(h_1, h_2, \varphi) := \tilde{G}_{i,j}(A_0, A_k). \tag{3.18}$$

For $k = 0$ we obtain for $i, j \in \{1, 2\}$:

$$\tilde{G}_{i,j}(A_0, A_0) = \begin{cases} G_i(h(A_0)) & i = j \\ 0 & i \neq j \end{cases}. \tag{3.19}$$

G_i being defined by (2.1) and (2.2) respectively, which leads to

$$e_Z(0, i, j) = \begin{cases} \int\limits_0^\infty G_i^{m-n}(h) A_{Z^2}(h) dh & i = j \\ 0 & i \neq j \end{cases}, \; i, j \in \{1, 2\}, \tag{3.20}$$

with A_{Z^2} as in (2.4).

In the following we will establish (3.17) by three successive simultaneous rotations of the vectors a_ℓ, $\ell = 1, \ldots, n + k$.

First Rotation:
The normal vector n_1 of the hyperplane $H(A_0)$ have the polar representation

$$(h_1, \psi_1, \ldots, \psi_{n-1})^T, \tag{3.21}$$

where $h_1 \in \mathbb{R}_0^+$, $\psi_1 \in [0, 2\pi)$, $\psi_\ell \in [0, \pi)$ for $\ell = 2, \ldots, n-1$.
Furthermore, let $\tilde{d}_1, \ldots, \tilde{d}_n$ be an orthonormal basis of $\mathbb{R}^n$, with $\tilde{d}_n = n_1^0$. This basis can be chosen as

$$\tilde{d}_n^{(\ell)} = \begin{cases} \sin\psi_1 \sin\psi_2 \ldots \sin\psi_{n-1} & \text{if } \ell = 1 \\ \cos\psi_{\ell-1} \sin\psi_\ell \ldots \sin\psi_{n-1} & \text{if } \ell = 2, \ldots, n-1 \\ \cos\psi_{n-1} & \text{if } \ell = n \end{cases}, \tag{3.22}$$

$$\tilde{d}_\ell = \frac{\sin\psi_1 \ldots \sin\psi_\ell}{\sin\psi_1 \ldots \sin\psi_{\ell-1}} \frac{\partial \tilde{d}_n}{\partial \psi_\ell}, \quad \ell = 1, \ldots, n-1 .$$

Now define the orthogonal matrix $R_1 := (\tilde{d}_1, \ldots, \tilde{d}_n)$ and the new variables $b_\ell := R_1^{-1} a_\ell$ for $\ell = 1, \ldots, n+k$ with $b_\ell := (\bar{b}_\ell, h_1)^T$.
The Jacobian Φ_1 of this transformation satisfies

$$(3.23) \qquad |\det(\Phi_1)| = \sin\psi_2 (\sin\psi_3)^2 \ldots (\sin\psi_{n-1})^{n-2} |\det(\mathrm{B})|.$$

as one easily proves like Raynaud[7] with B defined in (2.5).
Using the formula
(3.24)

$$\int_0^{2\pi}\int_0^{\pi} \ldots \int_0^{\pi} |\det(\Phi_1)| \sin\psi_2 \sin\psi_3^2 \ldots \sin\psi_{n-1}^{n-2} d\psi_{n-1} \ldots d\psi_2 d\psi_1 = |\omega_n| |\det(B)| ,$$

we obtain for $e_Z(k, i, j)$ the representation
(3.25)

$$e_Z(k, i, j) := \int_0^{\infty} \int_{\mathbb{R}^n}^{(k)} p_{i,j}(\tilde{A}_0, \tilde{A}_k) \tilde{A}_{k,Z}(\tilde{A}_0, \tilde{A}_k) \prod_{i=n+1}^{n+k} f((\bar{b}_i, h_1)^T) d\bar{b}_i \, dh_1 ,$$

where now defining $\tilde{A}_0 := R_1^{-1} A_0$ and $\tilde{A}_k := R_1^{-1} A_k$ holds:

$$(3.26) \quad p_{i,j}(\tilde{A}_0, \tilde{A}_k) = G_{i,j}^{m-n-k}(h_1, h_2, \varphi) \chi(\tilde{A}_k \subset H^{(i)}(\tilde{A}_0) \wedge \tilde{A}_0 \subset H^{(j)}(\tilde{A}_k))$$

with $h_2 = h_2(\tilde{A}_k)$, $\varphi = \varphi(\tilde{A}_0, \tilde{A}_k)$ and where

$$(3.27) \quad \tilde{A}_{k,Z}(\tilde{A}_0, \tilde{A}_k) = |\omega_n| \int_{\mathbb{R}^{n-1}}^{(n)} Z(\tilde{A}_0) Z(\tilde{A}_k) |\det(B)| \prod_{\ell=1}^{n} f((\bar{b}_\ell, h_1)^T) d\bar{b}_\ell .$$

Geometrically, by this first rotation the hyperplane $H(A_0)$ containing the boundary simplex A_0 was rotated into a hyperplane orthogonal to the unit vector e_n, cf. figure 2.
Next, by a second simultaneous rotation of the vectors b_ℓ we will establish h_2 und φ as free quantities. This transformation is only required for $n \geq 3$, for that reason let $n \geq 3$ in the following.

Second Rotation:
The normal vector $\tilde{n}_2$ of the hyperplane $H(\tilde{A}_k)$ may be written in the form (cf. figure 2):

$$(3.28) \qquad \tilde{n}_2 = (\bar{n}_2, \tilde{n}_2^{(n)})^T .$$

The vector $\bar{n}_2 \in \mathbb{R}^{n-1}$ have the polar coordinates $(\|\bar{n}_2\|_2, \psi_1, \ldots, \psi_{n-2})^T$ with $\psi_1 \in [0, 2\pi)$, $\psi_\ell \in [0, \pi)$ for $\ell = 2, \ldots, n-2$. Additionally, $d_1, \ldots, d_{n-1}$ be an orthonormal basis of $\mathbb{R}^{n-1}$, where $d_{n-1} = \bar{n}_2^0$. Like (3.22) we define

$$(3.29) \qquad \tilde{d}_{n-1}^{(\ell)} = \begin{cases} \sin\psi_1 \sin\psi_2 \ldots \sin\psi_{n-2} & \text{if } \ell = 1 \\ \cos\psi_{\ell-1} \sin\psi_\ell \ldots \sin\psi_{n-2} & \text{if } \ell = 2, \ldots, n-2 \\ \cos\psi_{n-2} & \text{if } \ell = n-1 \end{cases}$$

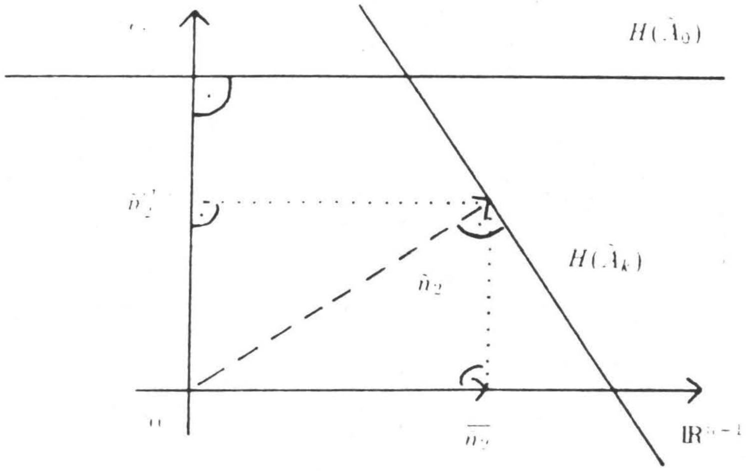

Fig. 2.

$$d_\ell = \frac{\sin\psi_1 \dots \sin\psi_\ell}{\sin\psi_1 \dots \sin\psi_{\ell-1}} \frac{\partial d_{n-1}}{\partial\psi_\ell}, \quad \ell = 1, \dots, n-2 .$$

We define $R_2 := (d_1, \dots, d_{n-1})$ and the new variables c_ℓ by $c_\ell := (\bar{c}_\ell, c_\ell^{(n)})^T$ with $\bar{c}_\ell := R_2^{-1}\bar{b}_\ell$ and by $c_\ell^{(n)} := b_\ell^{(n)}$ for $\ell = 1, \dots, n+k$. In the following c_ℓ will often be notated by $c_\ell = (\bar{\bar{c}}_\ell, c_\ell^{(n-1)}, c_\ell^{(n)})^T$.
For $\ell = k+1, \dots, n+k$ the vector c_ℓ satisfies the linear equation

$$\begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix} \begin{pmatrix} c_\ell^{(n-1)} \\ c_\ell^{(n)} \end{pmatrix} - h_2 = 0 \tag{3.30}$$

for $\varphi \in [0, \pi]$. Especially for $\varphi \in (0, \pi)$ holds, cf. figure 3:

$$c_\ell^{(n-1)} = \frac{h_2 - \cos\varphi c_\ell^{(n)}}{\sin\varphi}. \tag{3.31}$$

The evaluation of the second rotation's Jacobian Φ_2 requires lengthy calculations omitted here. Φ_2 satisfies the relation

$$|\det\Phi_2| = \frac{1}{\sin^2\varphi} \prod_{i=2}^{n-2} \sin^{i-1}\psi_i |\det(C)|, \tag{3.32}$$

which leads to

$$\int_0^{2\pi}\int_0^{\pi} \dots \int_0^{\pi} |\det(\Phi_2)| d\psi_1 \dots d\psi_{n-2} = \frac{|\omega_{n-1}|}{\sin^2\varphi} |\det(C)|, \tag{3.33}$$

corresponding to (3.24), where

$$(3.34)\qquad C' = \begin{pmatrix} \bar{\bar{c}}_{k+1} & \dots & \bar{\bar{c}}_{n+k} \\ \frac{c^{(n)}_{k+1} - \cos\varphi h_2}{\sin\varphi} & \dots & \frac{c^{(n)}_{n+k} - \cos\varphi h_2}{\sin\varphi} \\ 1 & \dots & 1 \end{pmatrix}.$$

For detailed proofs of (3.32) and (3.33) the interested reader is referred to Küfer[7]. The objective of the second rotation was to move both normal vectors n_1 and n_2 of the hyperplanes $H(\hat{A}_0)$ and $H(\hat{A}_k)$ into the (e_{n-1}, e_n)-plane, cf. figure 3.

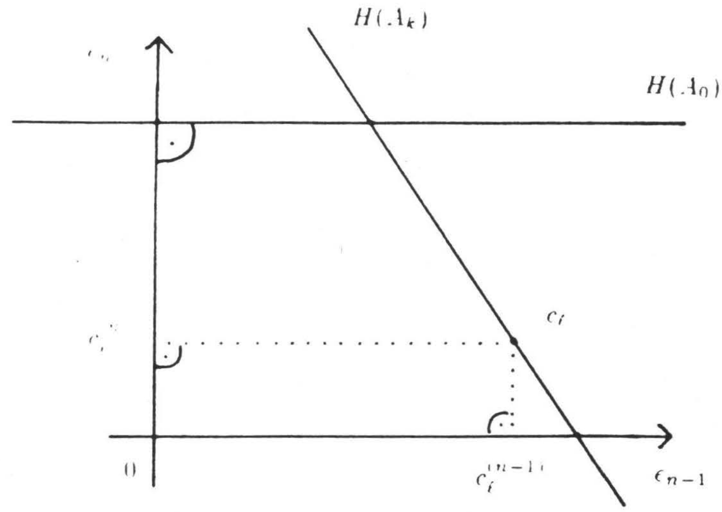

Fig. 3.

Having performed the second rotation we come to the desired form (3.17) for the quantities $e_Z(k, i, j)$. It holds for $k = 1, \dots, n$ and $i, j \in \{1, 2\}$:
(3.35)

$$A_{i,j,k,Z}(h_1, h_2, \varphi) = \frac{|\omega_n||\omega_{n-1}|}{\sin^2\varphi} \int\limits_{\mathbb{R}^{n-2}}^{(n+k)} \int\limits_{\hat{K}_{1,i,j}}^{(k)} \int\limits_{\hat{K}_{2,i,j}}^{(k)} \lambda_{k,Z}(c_1, \dots, c_{n+k})\, d\hat{\mu}_k\,.$$

where

$$(3.36)\qquad K_{1,i,j} := \begin{cases} (-\infty, h_1] & \text{if } i = 1 \\ [h_1, \infty) & \text{if } i = 2 \end{cases}, \quad K_{2,i,j} := \begin{cases} (-\infty, d_1] & \text{if } j = 1 \\ [d_1, \infty) & \text{if } j = 2 \end{cases},$$

$$(3.37)\qquad \lambda_{k,Z}(c_1, \dots, c_{n+k}) := |\det(\hat{C}_k)| Z(\hat{A}_k) |\det(\hat{C}_0)| Z(\hat{A}_0)$$

and

$$d\mu_k := \prod_{i=1}^{k+n} f(c_i) \prod_{i=1}^{k} dc_i^{(n-1)} \prod_{\ell=n+1}^{n+k} dc_\ell^{(n)} \prod_{\ell=1}^{n+k} d\bar{\bar{c}}_\ell. \tag{3.38}$$

the matrices C_0 and C_k taking forms

$$C_0 = \begin{pmatrix} \bar{\bar{c}}_1 & \cdots & \bar{\bar{c}}_k & \bar{\bar{c}}_{k+1} & \cdots & \bar{\bar{c}}_n \\ c_1^{(n-1)} & \cdots & c_k^{(n-1)} & d_1 & \cdots & d_1 \\ 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix} \tag{3.39}$$

and

$$C_k = \begin{pmatrix} \bar{\bar{c}}_{k+1} & \cdots & \bar{\bar{c}}_n & \bar{\bar{c}}_{n+1} & \cdots & \bar{\bar{c}}_{n+k} \\ d_2 & \cdots & d_2 & \frac{c_{n+1}^{(n)} - \cos\varphi h_2}{\sin\varphi} & \cdots & \frac{c_{n+k}^{(n)} - \cos\varphi h_2}{\sin\varphi} \\ 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{3.40}$$

Furthermore, $A_0 := R_2^{-1}\tilde{A}_0$ and $A_k := R_2^{-1}\tilde{A}_k$. The quantities d_ℓ, $\ell = 1, 2$, defined by

$$d_1 := \frac{h_2 - \cos\varphi h_1}{\sin\varphi}, \quad d_2 := \frac{h_1 - \cos\varphi h_2}{\sin\varphi} \tag{3.41}$$

can be interpreted geometrically. $|d_i|$ equals the distance between the intersection point of the hyperplanes $H(A_0)$ and $H(A_k)$ in the (e_{n-1}, e_n)-plane and the intersection point of the ray $\mathbb{R}^+ n_i$ and the hyperplane $H(\tilde{A}_0)$ for $i = 1$ and the hyperplane $H(A_k)$ respectively for $i = 2$, cf. figure 4.

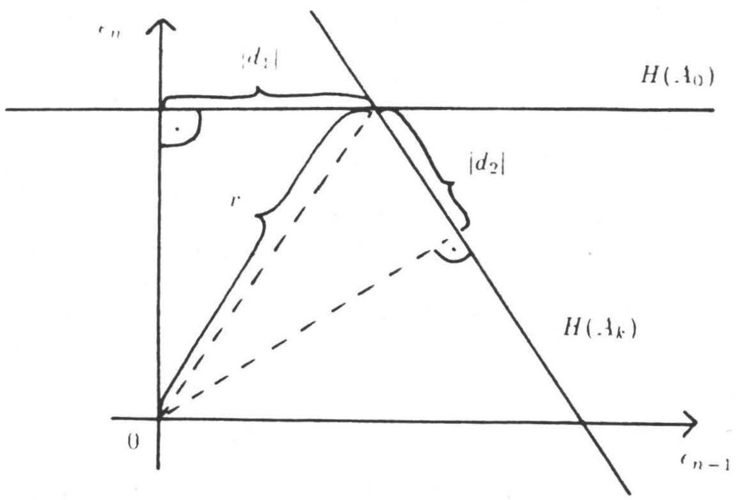

Fig. 4.

Third Rotation:
Intuitively it is clear that functions $\Lambda_{i,j,k,Z}(h_1, h_2, \varphi)$ are to be symmetric in the arguments h_1 und h_2. In order to realize this suggestion formally by the representation of the functions $\Lambda_{i,j,k,Z}(h_1, h_2, \varphi)$ we rotate the vectors $c_{k+1}, \ldots, c_{n+k}$ simultaneously concerning their last two coordinates by the angle $\pi - \varphi$ counterclockwise in the (e_{n-1}, e_n)-plane. Because of (1.3) $Z(A_k)$ remains unchanged while performing the described rotation. Defining the matrix

$$R_3 := \begin{pmatrix} Id_{n-2} & 0 \\ 0 & \tilde{R} \end{pmatrix}, \tag{3.42}$$

where Id_{n-2} represents the identity in $\mathbb{R}^{n-2}$ and while

$$\tilde{R} = \begin{pmatrix} -\cos\varphi & \sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \tag{3.43}$$

it holds:

$$R_3 c_\ell = \begin{cases} (\bar{\bar{c}}_\ell, d_2, h_2)^T & \text{for } \ell = k+1, \ldots, n \\ (\bar{\bar{c}}_\ell, \frac{\bar{c}_\ell^{(n)} - \cos\varphi h_2}{\sin\varphi}, h_2)^T & \text{for } \ell = n+1, \ldots, n+k \end{cases}. \tag{3.44}$$

We substitute

$$c_\ell^{(p)} = \begin{cases} c_\ell^{(p-1)} \sin\varphi + \cos\varphi h_2 & p = n;\ \ell = n+1, \ldots, n+k \\ c_\ell^{(p)} & p = 1, \ldots, n-2;\ \ell = n+1, \ldots, n+k \\ c_\ell^{(p)} & p = 1, \ldots, n;\ \ell = 1, \ldots, n \end{cases} \tag{3.45}$$

and finally reach a symmetric form of $\Lambda_{i,j,k,Z}(h_1, h_2, \varphi)$, which we summarize in

Theorem 3.2 *Under the prepositions of proposition* 3.1 $e_Z(k,i,j)$ *satisfies* (3.20) *for* $k = 0$ *and* (3.17) *for* $k = 1, \ldots, n$, *where in the latter case:*

(3.46)
$$\Lambda_{i,j,k,Z}(h_1, h_2, \varphi) = \frac{|\omega_n||\omega_{n-1}|}{\sin^{2-k}\varphi} \int\limits_{\mathbb{R}^{n-2}}^{(n+k)} \int\limits_{K_{1,i,j}}^{(k)} \int\limits_{K_{2,i,j}}^{(k)} \lambda_{k,Z}(c_1, \ldots, c_{n+k})\, d\mu_k$$

with

$$K_{1,i,j} := \begin{cases} (-\infty, d_2] & \text{if } i = 1 \\ [d_2, \infty) & \text{if } i = 2 \end{cases}, \quad K_{2,i,j} := \begin{cases} (-\infty, d_1] & \text{if } j = 1 \\ [d_1, \infty) & \text{if } j = 2 \end{cases}, \tag{3.47}$$

$$\lambda_{k,Z}(c_1, \ldots, c_{n+k}) := \left|\det\begin{pmatrix} \overline{C}_0 \\ \underline{e}^T \end{pmatrix}\right| Z\begin{pmatrix} \overline{C}_0 \\ h_1 \underline{e}^T \end{pmatrix} \left|\det\begin{pmatrix} \overline{C}_k \\ \underline{e}^T \end{pmatrix}\right| Z\begin{pmatrix} \overline{C}_k \\ h_2 \underline{e}^T \end{pmatrix} \tag{3.48}$$

and

$$d\mu_k := \prod_{\ell=1}^{k+n} f(c_\ell) \prod_{\ell=1}^{k} dc_\ell^{(n-1)} \prod_{\ell=n+1}^{n+k} dc_\ell^{(n-1)} \prod_{\ell=1}^{n+k} d\bar{\bar{c}}_\ell. \tag{3.49}$$

The matrices $\overline{C}_\ell$ take the form

$$
(3.50) \qquad \overline{C}_0 = \begin{pmatrix} \bar{c}_1 & \dots & \bar{c}_k & \bar{\bar{c}}_{k+1} & \dots & \bar{\bar{c}}_n \\ & & & d_1 & \dots & d_1 \end{pmatrix}
$$

and

$$
(3.51) \qquad \overline{C}_k = \begin{pmatrix} \bar{\bar{c}}_{k+1} & \dots & \bar{\bar{c}}_n & \bar{c}_{n+1} & \dots & \bar{c}_{n+k} \\ d_2 & \dots & d_2 & & & \end{pmatrix}
$$

respectively, whereas $\underline{e} = (1, \dots, 1)^T \in \mathbb{R}^n$.

Remarks on 3.2:

i) At first sight, theorem 3.2 is a perfect analogon to theorem 2.1, but there is an essential difference. Contrary to A_Z, cf. (2.4), the function $A_{i,j,k,Z}$ partly contains the condition for the simplices S_I and S_J to be boundary simplices of appropriate kind, more exactly $A_{i,j,k,Z}$ contains the χ-part of (3.26), which finds expression in the limits of integration (3.47). Similar to A_Z in (2.8) the sum $A_{k,Z} = \sum_{i,j=1}^{2} A_{i,j,k,Z}$, cf. (4.1), which is the same as $A_{i,j,k,Z}$ without the restrictive limits (3.47), can be interpreted as a conditional expectation value by

$$
(3.52) \qquad A_{k,Z}(h_1, h_2, \varphi) = \mathrm{E}(Z(A_I)Z(A_J) \mid \mathcal{B})\, A_{k,1}(h_1, h_2, \varphi),
$$

$$
\mathcal{B} = (h(A_I) = h_1) \wedge (h(A_J) = h_2) \wedge (\varphi(A_I, A_J) = \varphi) \wedge (|I \setminus J| = k),
$$

where the function $A_{k,1}$ has to be seen as marginal density of the distribution function $\mathrm{P}(h(A_I) \le h_1 \wedge h(A_J) \le h_2 \wedge \varphi(A_I, A_J) \le \varphi \wedge |I \setminus J| = k)$ for any sets of indices I and J.

ii) In contrast to (2.3) the representation (3.17) is not yet easily transformed into a Laplace-integral of one variable, which will be an essential point in the further evaluation of (3.17).

In the remaining part of the paragraph we concentrate on the quantities $G_{i,j}(h_1, h_2, \varphi)$ in order to receive integral formulae like (2.7). Obviously, because of (3.16) and the easily proved equations

$$
(3.53) \qquad G_{1,i}(h_1, h_2, \varphi) + G_{2,i}(h_1, h_2, \varphi) = G_i(h_2),
$$

$$
G_{i,1}(h_1, h_2, \varphi) + G_{i,2}(h_1, h_2, \varphi) = G_i(h_1)
$$

for $h_1, h_2 \in [0, \infty)$, $\varphi \in (0, \pi)$ and $i \in \{1, 2\}$ it is enough to investigate $G_{1,1}$. Before we formulate our result on this issue we introduce a new variable r by

$$
(3.54) \qquad r = \sqrt{d_1^2 + h_1^2} = \sqrt{d_2^2 + h_2^2} = \frac{\sqrt{h_1^2 + h_2^2 - 2h_1h_2\cos\varphi}}{\sin\varphi}.
$$

Geometrically, r equals the distance between the origin and the intersection point of the hyperplanes containing the boundary simplices we are dealing with, cf. figure 4.

Theorem 3.3 ***For $n \ge 2$, rotationally symmetric distributions with RDF F, $0 \le h_1, h_2 < \infty$ and $\varphi \in (0, \pi)$ holds:***

$$G_{1,1}(h_1, h_2, \varphi) = 1 - \tfrac{1}{2}G_2(h_1) - \tfrac{1}{2}G_2(h_2) - A_1(h_1, h_2, \varphi) - A_2(h_1, h_2, \varphi)\,, \tag{3.55}$$

where for $\ell \in \{1, 2\}$:

$$A_\ell(h_1, h_2, \varphi) := \operatorname{sign}(d_\ell) \int_{h_\ell}^{r} g_{0,2}(t) \frac{h_\ell dt}{t\sqrt{t^2 - h_\ell^2}} \tag{3.56}$$

with $r = r(h_1, h_2, \varphi)$ *and*

$$g_{0,2}(t) := \frac{|\omega_{n-2}|}{|\omega_n|(n-2)} \int_t^\infty \frac{(s^2 - t^2)^{(n-2)/2}}{s^{n-2}} dF(s)\,. \tag{3.57}$$

Proof:
We prove the theorem for $n \geq 3$ the result remaining valid for the easy special case $n = 2$.

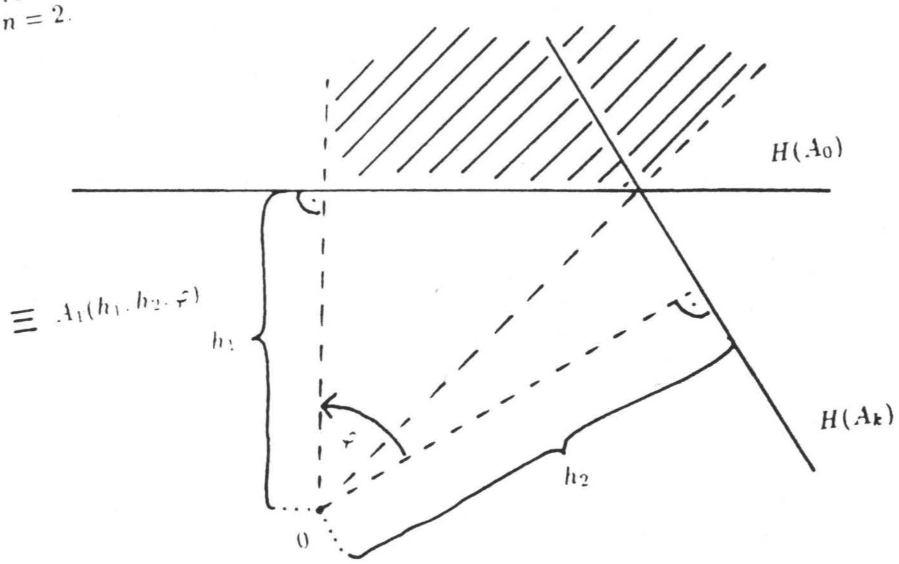

Fig. 5.

The weighted area $A_1(h_1, h_2, \varphi)$, cf. figure 5, satisfies:

$$A_1(h_1, h_2, \varphi) = \int_0^{\arcsin \frac{d_1}{r}} \int_{\frac{h_1}{\cos\psi}}^{\infty} g_{0,0}(t)\, t\, dt\, d\psi \tag{3.58}$$

with

$$g_{0,0}(t) := \frac{|\omega_{n-2}|}{|\omega_n|}\int_t^\infty \frac{(s^2-t^2)^{(n-4)/2}}{s^{n-2}}dF(s) \tag{3.59}$$

being the radial density of $G_{1,1}$ for a point with polar coordinates (t,ψ) in the (e_n, e_{n-1})-plane. The variable r is seen as a function of variables h_1, h_2, φ. Using

$$\frac{d}{dt}g_{0,2}(t) = -tg_{0,0}(t)\ ,\quad g_{0,2}(\infty) = 0\ , \tag{3.60}$$

we have

$$A_1(h_1,h_2,\varphi) = \int_0^{\arcsin\frac{d_1}{r}} g_{0,2}(\frac{h_1}{\cos\psi})d\psi\ . \tag{3.61}$$

By substituting

$$\frac{h_1}{\cos\psi} = t\ ,\quad d\psi = \frac{h_1}{t\sqrt{t^2-h_1^2}}dt\ . \tag{3.62}$$

we reach the desired

$$A_1(h_1,h_2,\varphi) := \operatorname{sign}(d_1)\int_{h_1}^{r} g_{0,2}(t)\frac{h_1 dt}{t\sqrt{t^2-h_1^2}}\ . \tag{3.63}$$

The corresponding representation for the weighted area $A_2(h_1,h_2,\varphi)$ is established in the same manner. By use of (3.53) the statement (3.54) follows.

□

Finally we give a useful

Corollary 3.4 *Under the prepositions of theorem 3.3 holds:*

$$1 - G_2(h_1) - G_2(h_2) \le G_{1,1}(h_1,h_2,\varphi) \le \min(G_1(h_1), G_1(h_2)) \tag{3.64}$$

and

$$G_{1,1}(h_1,h_2,\varphi_1) \le G_{1,1}(h_1,h_2,\varphi_2)\ \textit{for}\ \varphi_1 \ge \varphi_2 \tag{3.65}$$

Proof:
While (3.64) being geometrically obvious, we have:

$$\begin{aligned}\frac{d}{d\varphi}G_{1,1}(h_1,h_2,\varphi) &= -\frac{d}{d\varphi}\sum_{\ell\in\{1,2\}} A_\ell(h_1,h_2,\varphi) \\ &= -\frac{d}{d\varphi}(\arcsin\frac{d_1}{r} + \arcsin\frac{d_2}{r})g_{0,2}(r) = -g_{0,2}(r) \le 0\ .\end{aligned} \tag{3.66}$$

□

Corollary 3.4 is a tool, which can be used to estimate (3.17) by a similar Laplace-type integral of one variable, but for certain distributions or certain

$k \in \{0, \dots, n\}$ the upper bound in (3.64) is too rough to guarantee sharp estimations for $e_Z(k, i, j)$, which complicates the considerations as one can see in the following section.

4. Asymptotic estimations of variances

It is the objective of the section to exploit the integral representations of theorems 2.1 and 3.2 in order to classify asymptotically large relative deviations of additive random variables of type (1.4) from the mean for big m. This is done for the class of rotationally symmetric distributions with compact support, which vary regularly near the domain's boundary, cf. (1.5).

Before we can formulate our main result on this issue, we have to prove some preliminary lemmata, which open the way to the the moments' treatise as Laplace-type integrals.

4.1 Preliminaries

First we introduce some more notation. Let

$$(4.1) \qquad A_{k,Z}(h_1, h_2, \varphi) := \sum_{i,j \in \{1,2\}} A_{i,j,k,Z}(h_1, h_2, \varphi)$$

with $A_{i,j,k,Z}$ as in (3.46) and

$$(4.2) \quad e_{\tilde{Z}}(k) := q_k \int_0^1 \int_0^1 \min(G_1(h_1), G_1(h_2))^{m-n-k} \int_0^\pi \chi(r < 1) A_{k,Z} d\varphi dh_1 dh_2$$

for $k = 1, \dots, n$, where $r = r(h_1, h_2, \varphi)$, cf. (3.54). Furthermore, let

$$(4.3) \qquad e_{\tilde{Z}}(0) := q_0 \int_0^1 (G_1(h)^{m-n} + G_2(h)^{m-n}) A_{Z^2}(h) dh$$

and

$$(4.4) \qquad \bar{e}_{\tilde{Z}} := q_n \int_0^1 \int_0^1 (1 - G_2(h_1) - G_2(h_2))^{m-2n} A_Z(h_1) A_Z(h_2) dh_1 dh_2.$$

Under distributions with compact domain there are pairs of boundary simplices S_I and S_J, whose corresponding hyperplanes $H(A_I)$ and $H(A_J)$ intersect outside the distribution's support. This can, of course, only happen for $|I \setminus J| = n$. In this case the function $G_{1,1}$ satisfies $G_{1,1}(h_1, h_2, \varphi) = 1 - G_2(h_1) - G_2(h_2)$. An estimation of $G_{1,1}$ by the right hand side of (3.64) would be too weak. Lemma 4.1 separates the described special case from the others, where the bound of (3.64) is more appropriate.

Lemma 4.1 *For rotationally symmetric distributions with support Ω_n and additive type random variables Z holds, while $m \geq 2n \geq 4$:*

(4.5) $$\mathrm{E}(\tilde{Z}^2) \leq \bar{\epsilon}_{\tilde{Z}} + \sum_{k=0}^{n} \epsilon_{\tilde{Z}}(k) .$$

Proof:
We estimate the quantities $\epsilon_Z(k,i,j)$, cf. (3.17), from above, looking first at the case $k = n$.
For $i,j \in \{0,1\}$ we devide the domain of integration $K := [0,\pi] \times [0,1] \times [0,1]$ into the regions $K_I := \{(\varphi,h_1,h_2) \in K \mid r(h_1,h_2,\varphi) < 1\}$ and $K_{II} := K \setminus K_I$. Using (3.47) it is immediate that

(4.6) $$A_{i,j,k,Z}(h_1,h_2,\varphi)\,\chi((h_1,h_2,\varphi) \in K_{II}) = 0 ,$$

if $i = 2$ or $j = 2$. Thus, introducing the abbreviations

(4.7) $$s_{\tilde{Z}}(k) := \sum_{i,j\in\{0,1\}} \sigma^{i+j-2}\epsilon_Z(k,i,j)\,, k = 0,\ldots,n ,$$

it follows with

(4.8) $$G_{1,1}(h_1,h_2,\varphi) = 1 - G_2(h_1) - G_2(h_2),$$

theorem 3.2 and corollary 3.4:
(4.9)
$$s_{\tilde{Z}}(n) \leq \epsilon_{\tilde{Z}}(n) + q_n \int_0^1\int_0^1 (1-G_2(h_1)-G_2(h_2))^{m-2n} \int_0^{\pi} \chi(r \geq 1)A_{1,1,n,Z}\,d\varphi dh_1 dh_2 .$$

Estimating

(4.10) $$\chi(r \geq 1)A_{1,1,n,Z}(h_1,h_2,\varphi) \leq A_{n,Z}(h_1,h_2,\varphi),$$

cf. (4.1) and (4.6), exploiting the identity

(4.11) $$A_{n,Z}(h_1,h_2,\varphi) = \frac{|\omega_{n-1}|}{|\omega_n|}\sin^{n-2}\varphi A_Z(h_1)A_Z(h_2)$$

and using

(4.12) $$|\omega_{n-1}| \int_0^{\pi} \sin^{n-2}\varphi d\varphi = |\omega_n|$$

we reach

(4.13) $$s_{\tilde{Z}}(n) \leq \epsilon_{\tilde{Z}}(n) + \bar{\epsilon}_{\tilde{Z}} .$$

For the simpler cases $k = 1,\ldots,n-1$ we notate the redundancy

(4.14) $$A_{i,j,k,Z}(h_1,h_2,\varphi) = A_{i,j,k,Z}(h_1,h_2,\varphi)\,\chi((h_1,h_2,\varphi) \in K_I) ,$$

because $r > 1$ implies the relations $d_1^2 \geq 1 - h_1^2$ and $d_2^2 \geq 1 - h_2^2$.

Again with theorem 3.2 and corollary 3.4 it follows for $k = 1, \dots, n-1$, that $s_{\tilde{Z}}(k) \leq \epsilon_{\tilde{Z}}(k)$, $k = 1, \dots, n-1$. With the obvious estimation $s_{\tilde{Z}}(0) \leq \epsilon_{\tilde{Z}}(0)$ the proof is complete.

□

For reasons of the integrand's symmetry concerning variables h_1 and h_2 the quantities $\epsilon_{\tilde{Z}}(k)$, $k = 1, \dots, n$, cf. (4.2), satisfy:

(4.15) $$\epsilon_{\tilde{Z}}(k) = 2q_k \int_0^1 G_1^{m-n-k}(h_1) R_{k,Z}(h_1) dh_1 ,$$

where we define:

(4.16) $$R_{k,Z}(h_1) := \int_{h_1}^{1} \int_0^{\pi} A_{k,Z}(h_1, h_2, \varphi) d\varphi dh_2 ,$$

which we are to estimate further. This is done by replacing the variable φ by the much more suitable variable r by the following lemma.

Lemma 4.2 *For $k \in \{1, \dots, n\}$, $n \geq 2$, and every function $S_{k,Z}$, satisfying the property*

(4.17) $$S_{k,Z}(h_1, h_2, r(h_1, h_2, \varphi)) \geq \sin^{-k+2} \varphi A_{k,Z}(h_1, h_2, \varphi)$$

for all $1 \geq h_2 \geq h_1 > 0$ and $\varphi \in [0, \pi]$, holds:

(4.18) $$R_{k,Z}(h_1) \leq \left(\frac{2\sqrt{1-h_1^2}}{h_1} \right)^{k-1} \int_{h_1}^{1} \int_{h_2}^{1} \frac{r S_{k,Z}(h_1, h_2, r)}{\sqrt{r^2 - h_1^2}\sqrt{r^2 - h_2^2}} dr dh_2 .$$

Proof:
By geometric insight, cf. figure 6, we have

(4.19) $$\{\varphi \in [0, \pi] | \chi(r < 1), 0 < h_1 < h_2 < 1\} =$$
$$(\arccos h_1 - \arccos h_2, \arccos h_1 + \arccos h_2) .$$

Therefore:

(4.20) $$R_{k,Z}(h_1) = \int_{h_1}^{1} \int_{\arccos h_1 - \arccos h_2}^{\arccos h_1 + \arccos h_2} A_{k,Z}(h_1, h_2, \varphi) d\varphi dh_2 .$$

In order to introduce r as new variable instead of φ we study the growth of r with respect to φ and find:

(4.21) $$r(h_1, h_2, \varphi) \begin{cases} \text{increases in } & I_1 := (\arccos h_1 - \arccos h_2, \arccos \frac{h_1}{h_2}) \\ \text{decreases in } & I_2 := (\arccos \frac{h_1}{h_2}, \arccos h_1 + \arccos h_2) \end{cases} .$$

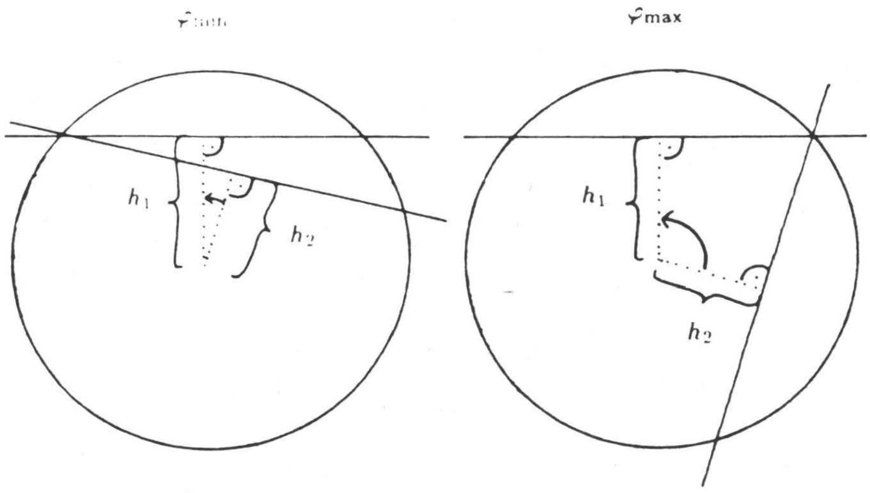

Fig. 6.

In the interval I_1 we substitute:

$$\varphi = \arccos\frac{h_1}{r} - \arccos\frac{h_2}{r}\,.\; d\varphi = \frac{1}{r}\Big(\frac{h_1}{|d_1|} - \frac{h_2}{|d_2|}\Big)dr\,, \tag{4.22}$$

$$\sin\varphi = \frac{1}{r^2}(|d_1|h_2 - |d_2|h_1)\,.\; \frac{d\varphi}{\sin\varphi} = \frac{-rdr}{|d_1||d_2|}\,,$$

whereas in the interval I_2 we substitute:

$$\varphi = \arccos\frac{h_1}{r} + \arccos\frac{h_2}{r}\,.\; d\varphi = \frac{1}{r}\Big(\frac{h_1}{|d_1|} + \frac{h_2}{|d_2|}\Big)dr\,. \tag{4.23}$$

$$\sin\varphi = \frac{1}{r^2}(|d_1|h_2 + |d_2|h_1)\,.\; \frac{d\varphi}{\sin\varphi} = \frac{rdr}{|d_1||d_2|}\,,$$

and recieve the upper bound
(4.24)

$$R_{k,Z}(h_1) \leq \int_{h_1}^{1}\int_{h_2}^{1} \frac{r^{-2k+3}}{|d_1||d_2|}[(|d_1|h_2 - |d_2|h_1)^{k-1} + (|d_1|h_2 + |d_2|h_1)^{k-1}]S_{k,Z}drdh_2\,,$$

if we estimate $A_{k,Z}$ as in proposition 4.2, (4.17). By use of the inequality

$$(x-y)^\gamma + (x+y)^\gamma \leq 2^\gamma x^\gamma,\ 0 \leq y \leq x,\ \gamma > 0, \tag{4.25}$$

and $\frac{h_2}{r} \leq 1$ we gain:

$$(4.26)\quad R_{k,Z}(h_1) \le 2^{k-1} \int_{h_1}^{1}\int_{h_2}^{1} \frac{r}{\sqrt{r^2-h_1^2}\sqrt{r^2-h_2^2}} \left(\frac{\sqrt{r^2-h_1^2}}{r}\right)^{k-1} S_{k,Z}\,dr\,dh_2 ,$$

from which the claim of the lemma is easily derived.

□

4.2 The main result

Let F be the radial distribution function (RDF) of a distribution with regularly varying behaviour near the boundary of Ω_n, cf. (1.5), implying the function F's tail $\overline{F}(t) := 1 - F(t)$ to have the asymptotic equivalence

$$(4.27)\qquad \overline{F}(\sqrt{1-r^2}) \sim L(1/r)r^{\alpha},\ r \to 0+ ,$$

where L is a slowly varying function at infinity, that means for all $r \in (0, \infty)$:

$$(4.28)\qquad \frac{L(rt)}{L(t)} \sim 1,\ t \to \infty .$$

Furthermore, let $\tilde{Z}$ be an additive variable acquainted with the additional asymptotic property (1.4), i.e. for a fixed $p \in \mathbb{N}_0$:

$$(4.29)\qquad Z(A_I) \sim C_{n,Z}|\det(A_I)|^p,\ h(A_I) \to 1 .$$

Theorem 4.3 *Let F be a RDF with property (4.27), $\tilde{Z}$ an additive variable of type (4.28) and $n \ge 2$ then:*
a)

$$(4.30)\qquad \mathrm{E}(\tilde{Z}) \sim C_{\alpha,n,Z}(1-\tilde{G}_2(\tfrac{1}{m}))^{(\gamma-n+1)/2},\ m \to \infty,$$

with $\gamma = (n-1)p$ and a positive constant $C_{\alpha,n,Z}$.
b)

$$(4.31)\qquad \frac{\mathrm{Var}(\tilde{Z})}{\mathrm{E}^2(\tilde{Z})} = O((1-\tilde{G}_2(\tfrac{1}{m}))^{(n-1)/2}),\ m \to \infty.$$

$\tilde{G}_2$ is the inverse of the function G_2 defined by (2.2). If in (4.27) $\alpha > 0$ and L is a constant function, the (4.31) simplifies. It holds:

$$(4.31a)\qquad \frac{\mathrm{Var}(\tilde{Z})}{\mathrm{E}^2(\tilde{Z})} = O(m^{(n-1)/(n-1+2\alpha)}),\ m \to \infty.$$

Proof:
The proof of theorem 4.3 is done in three stages for $n \ge 3$, the simpler special case $n = 2$ is omitted. Stage 1 proves claim a) of the theorem by showing that

$$(4.32)\qquad A_Z(h) \sim \tilde{C}_{\alpha,n,Z}\, g_0(h) G_2^{n-2}(h) F_{\gamma/2}(h),\ h \to 1,$$

where $\tilde{C}_{\alpha,n,Z}$ is a positive constant, g_0 is as in (2.7) and

$$F_\beta(h) := \int_h^1 (r^2 - h^2)^\beta dr, \ \beta > -1. \tag{4.33}$$

(4.31) and (2.3) imply, substituting $v = G_2(h)$:

$$\mathrm{E}(\tilde{Z}) \sim \tilde{C}_{\alpha,n,Z} \binom{m}{n} \int_0^{1/2} (1 - v)^{m-n} v^{n-2} F_{\gamma/2}(h(v)) dv, \ m \to \infty, \tag{4.34}$$

from which claim a) is easily derived by Watsons's lemma concerning Laplace-type integrals, cf. Küfer[7] for a detailed evaluation.
In order to establish the second claim of our theorem we show in stage 2 that

$$\frac{\bar{\epsilon}_{\tilde{Z}}}{\mathrm{E}^2(\tilde{Z})} - 1 \sim O(\tfrac{1}{m}), \ m \to \infty, \tag{4.35}$$

which means that $\mathrm{E}^2(\tilde{Z})$ is almost eliminated by simplex configurations whose supporting hyperplanes intersect outside the distribution's domain. Finally in stage 3 we show for $k = 0, \ldots, n$:

$$\frac{\epsilon_{\tilde{Z}}(k)}{\mathrm{E}^2(\tilde{Z})} = O((1 - \tilde{G}_2(\tfrac{1}{m}))^{(n-1)/2}) \tag{4.36}$$

with quantities $\bar{\epsilon}_{\tilde{Z}}$ and $\epsilon_{\tilde{Z}}(k)$ defined in (4.2-4). The asymptotic order on the right hand side of (4.36) is always bigger than $O(\frac{1}{m})$, which is shown at the end of the proof.

Stage 1:
The proof of (4.32) is done first for odd p, where we have

$$Z\begin{pmatrix} \overline{B} \\ h\underline{e}^T \end{pmatrix} |\det(B)| \sim C_{n,Z}(\det(B))^{p+1}, \ h \to 1, \tag{4.37}$$

with B as in (2.5) and $\overline{B}$ resulting from B by deleting the last row. The right hand side of (4.37) enables an expansion of the determinant with respect to its last row and a binomial representation by:

$$(\det(B))^{p+1} = \sum_{|k|=p+1} \binom{p+1}{k} \prod_{j=1}^{n} (-1)^{jk_j} \det(\overline{B}_j)^{k_j}, \tag{4.38}$$

from which we gain by (2.4), some easy linear integral transformations and (4.33):

$$A_Z(h) \sim |\omega_n| \sum_{|k|=p+1} C_{k,n,Z} \prod_{j=1}^{n} F_{(n-2+p-k_j)/2}(h), \ h \to 1, \tag{4.39}$$

with

$$(4.40)\quad C_{k,n,Z} := \binom{p+1}{k} C_{n,Z} \frac{|\omega_{n-1}|^n}{|\omega_n|^n} \int\limits_{\omega_{n-1}}^{(n)} \prod_{j=1}^{n} ((-1)^j \det(\overline{B}_j))^{k_j} d^0_{\omega_{n-1}}(\overline{b}_j).$$

The differential $d^0_{\omega_{n-1}}(\overline{b})$ is the normed differential on the unit sphere ω_{n-1} in direction of $\overline{b}$, i.e. $\int_{\omega_{n-1}} d^0_{\omega_{n-1}}(\overline{b}) = 1$.
With the asymptotic equations

$$(4.41)\qquad F_3(h) \sim \overline{F}(h)(1-h^2)^3(\alpha+\beta+1)\mathrm{B}(\alpha+1,\beta+1)$$

for $h \to 1$, following from definition (4.33) and from (4.27), and with

$$(4.42)\qquad G_2(h) \sim \frac{|\omega_{n-1}|}{(n-1)|\omega_n|} F_{(n-1)/2}(h),\ g_0(h) \sim \frac{|\omega_{n-1}|}{|\omega_n|} F_{(n-3)/2}(h)$$

for $h \to 1$, as consequences of their defintion (2.7), it is now a simple matter to show (4.32).
The situation is a bit more complicated for even p, for as (4.37) takes now the form

$$(4.43)\qquad Z\begin{pmatrix} \overline{B} \\ h\underline{c}^T \end{pmatrix} |\det(B)| \sim C_{n,Z}|\det(B)|^{p+1},\ h \to 1.$$

If we rotate the vectors $\overline{b}_1, \ldots, \overline{b}_n$ within the integral (2.4) simultaneously such that the $n-1$-th coordinates of $\overline{b}_1, \ldots, \overline{b}_{n-1}$ become equal, which can be done using similar methods as in section 3, we obtain
(4.44)

$$A_Z(h) \sim |\omega_n||\omega_{n-1}| \int\limits_0^{\sqrt{1-h^2}} \int\limits_0^{\sqrt{1-h^2}} (|r-s|^{p+1} + |r+s|^{p+1}) K(r,h) g_{h,0}(s)\,ds\,dr$$

for $h \to 1$ with

$$(4.45)\qquad K(r,h) \sim C_{n,Z} \int\limits_{\sqrt{1-r^2-h^2}\,\Omega_{n-2}}^{(n-1)} \det{}^{p+2}(C) \prod_{i=1}^{n-1} f((\overline{\overline{c}}_i, r, h)^T) d\overline{\overline{c}}_i$$

for $0 \le r \le \sqrt{1-h^2}$, $h \to 1$, and where

$$(4.46)\qquad C := \begin{pmatrix} \overline{\overline{c}}_1 & \ldots & \overline{\overline{c}}_{n-1} \\ 1 & \ldots & 1 \end{pmatrix}.$$

Furthermore,

$$(4.47)\qquad g_{h,0}(s) := \int\limits_{\sqrt{1-s^2-h^2}\,\Omega_{n-2}} f((\overline{\overline{c}}, s, h)^T) d\overline{\overline{c}}$$

for $0 \le s \le \sqrt{1-h^2}$.
Now we apply the same methods as in (4.38) and gain asymptotically:

$$(4.48)\qquad K(r,h) \sim \sum_{|k|=p+2} C_{k,n,Z} \prod_{j=1}^{n-1} F_{(n-2+p-k_j)/2}(\sqrt{r^2+h^2})$$

for $0 \le r \le \sqrt{1-h^2}$ and $h \to 1$ with
(4.49)

$$C_{k,n,Z} := \binom{p+2}{k} C'_{n,Z} \frac{|\omega_{n-2}|^{n-1}}{|\omega_n|^{n-1}} \int\limits_{\omega_{n-2}}^{(n-1)} \prod_{j=1}^{n-1} ((-1)^j \det(\overline{C}_j))^{k_j} d^0_{\omega_{n-2}}(\bar{\bar{c}}_j).$$

where $\overline{C}_j$ is analogously defined as $\overline{B}_j$ above, and

$$(4.50)\qquad g_{h,0}(s) \sim \frac{|\omega_{n-2}|}{|\omega_n|} F_{(n-4)/2}(\sqrt{s^2+h^2})$$

for $0 \le s \le \sqrt{1-h^2}$ and $h \to 1$.
Having inserted (4.48) and (4.50) in line (4.44) it is straightforward to show (4.32) in case of even p by application of standard results on functions of regular variation , cf. Seneta[12] for an extensive treatment, and equations (4.41) and (4.42) respectively.

□

Stage 2:
It is an easily established matter that under prepositions (4.27) and (4.29) the function:

$$(4.51)\qquad u_Z(\psi) := \frac{A_Z(h(\psi))}{g_0(h(\psi))}$$

is of regularly varying behaviour for $\psi \to 0+$. Especially holds:

$$(4.52)\qquad u_Z(\psi) \sim \psi^{n-2+p} \tilde{L}(\frac{1}{\psi}),\ \psi \to 0+,$$

where $\tilde{L}$ is slowly varying at infinity, cf. (4.32),(4.42). Furthermore, the function u_Z is in $\mathcal{L}^1[0,1/2]$ because of the continuity of Z.
By substitution $\psi = G_2(h)$ line (2.3) implies:

$$(4.53)\qquad \mathrm{E}(\tilde{Z}) = \binom{m}{n} \int\limits_0^{1/2} (1-\psi)^{m-n} u_Z(\psi) d\psi + O(m^n(\tfrac{1}{2})^m)$$

for $m \to \infty$. Analogously, (4.4) gives:

$$(4.54)\qquad \overline{e}_{\tilde{Z}} = \binom{m-n}{n}\binom{m}{n} \int\limits_0^{1/2}\int\limits_0^{1/2} (1-\psi_1-\psi_2)^{m-2n} u_Z(\psi_1) u_Z(\psi_2) d\psi_1 d\psi_2.$$

Let

$$(4.55)\qquad \tilde{u}_Z(\psi) := \frac{(1-\psi)^{m-n} u_Z(\psi)}{\int\limits_0^{1/2} (1-t)^{m-n} u_Z(t) dt}$$

for $\psi \in [0, 0.5]$ and

$$(4.56)\qquad \tilde{K}(\psi_1, \psi_2) := \frac{(1-\psi_1-\psi_2)^{m-2n}}{(1-\psi_1)^{m-n}(1-\psi_2)^{m-n}} - 1,$$

then by (4.53) and (4.52):
(4.57)

$$\frac{\sigma_{\hat{Z}}^2}{E^2(\hat{Z})} - 1 \le \int_0^{1/2}\int_0^{1/2} \tilde{K}(\psi_1, \psi_2)\tilde{u}_Z(\psi_1)\tilde{u}_Z(\psi_2)d\psi_1 d\psi_2 + O(m^{n-1+p}(\tfrac{1}{2})^m),\ m \to \infty.$$

We now concentrate on the bilinear form on the right hand side of (4.57) and discuss its kernel $\tilde{K}$ on the square $[0, 0.5] \times [0, 0.5]$ for $m \ge 2n$. Figure 7 illustrates the growth of the function $\tilde{K}$.

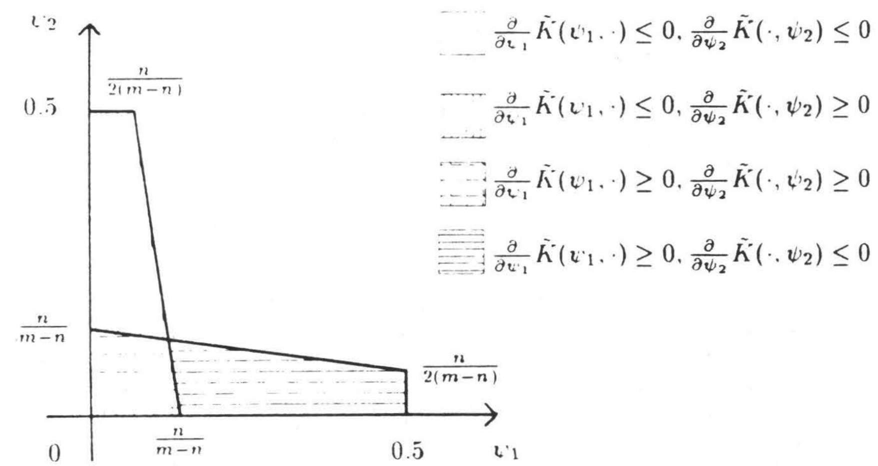

Fig. 7.

$\tilde{K}(0,0) = 0$, $\tilde{K}(0.5, 0.5) = -1$. $\tilde{K}(0.5, 0) = \tilde{K}(0, 0.5) = 2^n - 1$. $\tilde{K}$ has a relative maximum in $(\frac{n}{m}, \frac{n}{m})$. We devide the area of $\tilde{K}$'s definition in four regions $Q_1, \ldots, Q_4$, which is revealed by figure 8.
Obviously,

$$(4.58)\qquad Q_4 \subset \{(\psi_1, \psi_2) \in [0, 0.5]^2 \mid \tilde{K}(\psi_1, \psi_2) \le 0\},$$

because $\tilde{K}$'s monotonicity and $\tilde{K}(\frac{4n}{m+3n}, \frac{4n}{m+3n}) \le 0$.
In Q_2 holds:

$$(4.59)\qquad \tilde{K}(\psi_1, \psi_2) \le (1-\psi_1)^{-n} - 1 \le n2^{n+1}\psi_1,$$

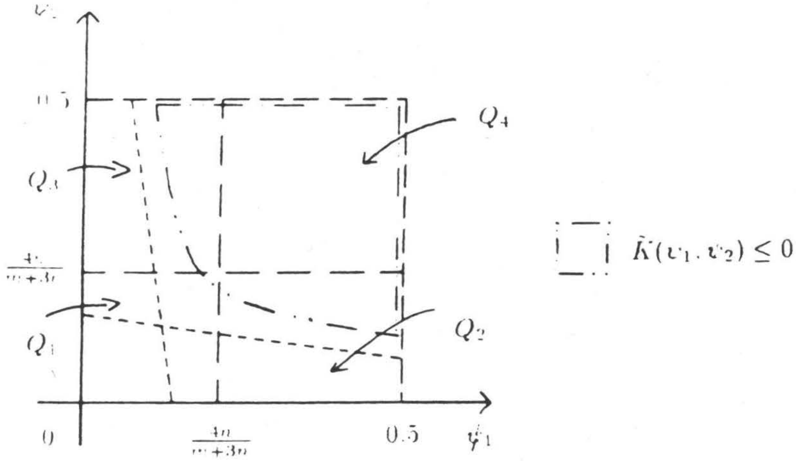

Fig. 8.

likewise in Q_3, where:

$$\tilde{K}(v_1, v_2) \leq (1 - v_2)^{-n} - 1 \leq n 2^{n+1} v_2. \tag{4.60}$$

In Q_1, we have:

$$\tilde{K}(v_1, v_2) \leq \max\{\tilde{K}(0, \frac{4n}{m+3n}), \tilde{K}(\frac{n}{m}, \frac{n}{m})\} = \tag{4.61}$$

$$= \tilde{K}(0, \frac{4n}{m+3n}) \leq \frac{n^2}{m+3n} 2^{n+3}.$$

Summarizing (4.59)-(4.61) we get by reasons of symmetry for $m \to \infty$:
(4.62)

$$\int_0^{1/2}\int_0^{1/2} \tilde{K}(v_1, v_2)\tilde{u}_Z(v_1)\tilde{u}_Z(v_2)dv_1 dv_2 \leq n 2^{n+2} \int_0^{1/2} v\tilde{u}_Z(v)dv + O(m^{-1}).$$

On the other hand, using preposition (4.51) we obtain

$$\int_0^{1/2} v\tilde{u}_Z(v)dv \sim \frac{n-1+p}{m} \tag{4.63}$$

for $m \to \infty$, which proves the claim of stage 2.

□

Stage 3:
A well known geometric result yields

$$(4.64)\qquad |\det\begin{pmatrix}\bar b_1 & \dots & \bar b_n\\ 1 & \dots & 1\end{pmatrix}| \le 2\sqrt{n}$$

for $\bar b_1,\dots,\bar b_n \in \Omega_{n-1}$, which implies

$$(4.65)\qquad \lambda_{k,Z}(c_1,\dots,c_{n+k}) \le n^{p+1}h_1^p h_2^p(1-h_1^2)^{(\gamma+n-1)/2}(1-h_2^2)^{(\gamma+n-1)/2},$$

for $k=1,\dots,n$ and $\lambda_{k,Z}$ as in theorem 3.2, line (3.48), for as the truncated vectors $\bar c_1,\dots,\bar c_n \in \sqrt{1-h_1^2}\Omega_{n-1}$ and $\bar c_{k+1},\dots,\bar c_{k+n} \in \sqrt{1-h_2^2}\Omega_{n-1}$.
Thus, for $k=1,\dots,n$ we obtain:

$$(4.66)\qquad \frac{\Lambda_{k,Z}(h_1,h_2,\varphi)}{\sin^{k-2}\varphi} \le S_{k,Z}(h_1,h_2,r),$$

with
(4.67)
$$S_{k,Z}(h_1,h_2,\varphi) := n^{p+1}(1-h_1^2)^{(\gamma+n-1)/2}(1-h_2^2)^{(\gamma+n-1)/2}g_0^k(h_1)g_0^k(h_2)g_{0,0}^{n-k}(r),$$

where $r=r(h_1,h_2,\varphi)$ and $g_{0,0}$ as in (3.59). With the aid of lemma 4.2 and by use of the asymptotic equivalences (4.41), (4.42) and

$$(4.68)\qquad g_{0,0}(h_1) \sim \frac{|\omega_{n-2}|}{|\omega_n|}F_{(n-4)/2}(h_1),\ h\to 1,$$

we obtain after some asymptotic transformations:

$$(4.69)\qquad R_{k,Z}(h_1) = O(g_0(h_2)G_2^{n+k-2}(h_1)F_\gamma(h_1)),\ h\to 1.$$

Therefore, analogously to (4.34), we gain

$$(4.70)\qquad \epsilon_{\tilde Z}(k) = O((1-\tilde G_2(\tfrac{1}{m}))^\gamma),\ m\to\infty,$$

for $k=1,\dots,n$. With respect to part a) of our theorem it is enough to establish (4.70) for $k=0$ in order to complete the proof of stage 3. This is done by estimating the function Λ_{Z^2}, cf. (3.20), from above by

$$(4.71)\qquad \Lambda_{Z^2}(h) \le n^{p+1/2}(1-h^2)^{(2\gamma+n-1)/2}g_0^n(h),$$

implying

$$(4.72)\qquad \Lambda_{Z^2}(h) = O(G_2^{n-2}(h)g_0(h)F_\gamma(h)),\ h\to\infty,$$

from which claim (4.69) of stage 3 follows.

□

Finally we remark that

$$(4.73)\qquad (1-\tilde G_2(\tfrac{1}{m}))^{(n-1)/2} \sim m^{-\beta}\hat L(m),\ m\to\infty,$$

with $\beta := (n-1)/(n-1+2\alpha)$. $\hat L$ is a function of slow variation at infinity, which has the additional property $\lim_{m\to\infty}\hat L(m)\to\infty$ in case $\alpha=0$. Thus the right hand side of (4.73) dominates $\frac{1}{m}$ asymptotically as m tends to infinity. If additionally in (4.27) $\alpha > 0$ and L is a constant function, L is constant too. So, (4.73) simplifies as indicated in the theorem, which now is completely proved.

□

Discussion of theorem 4.3:

i) The asymptotic estimation (4.31) is sharp in order, i.e. there are positive constants $K_n^{(1)}$ and $K_n^{(2)}$ such that for $m \geq M$, M being appropriately chosen:

$$(4.74) \qquad K_n^{(1)}(1 - \tilde{G}_2(\tfrac{1}{m}))^{(n-1)/2} \leq \frac{\mathrm{Var}(\tilde{Z})}{\mathrm{E}^2(\tilde{Z})} \leq K_n^{(2)}(1 - \tilde{G}_2(\tfrac{1}{m}))^{(n-1)/2}.$$

ii) Under the prepositions of 4.3 we have by (4.73) and Chebychev's inequality:

$$(4.75) \qquad \mathrm{P}(|\frac{\tilde{Z}}{\mathrm{E}(\tilde{Z})} - 1| > a) \leq \frac{\mathrm{Var}(\tilde{Z})}{a^2\mathrm{E}^2(\tilde{Z})} = o(1), \ m \to \infty,$$

which proves claim (1.6). (4.75) is also true for all rotationally symmetric distributions and our class (1.4) of additive type variables, as one can show with another technique of proof. But in general case neither the vague asymptotic estimation $o(1)$ seems to be improvable nor a distribution independent bound seems possible, as is suggested by (4.73).

iii) The examples of additive variables treated by Renyi and Sulanke and others are included in 4.3, e.g. the number of boundary simplices with $p = 0$, the volume, the surface area and the characteristic function $\chi(\mathrm{cone}(\mathcal{P}) = \mathbb{R}^n)$ with $p = 1$.

iv) It is not difficult to deduce a proper estimation under conservation of the m-order of growth in (4.31) by a slightly refined analysis as mentioned in the introduction, but the result is not of much worth for moderately chosen n and m, because the constant depending on n on the right hand side would be to big. In order to estimate the variance of additive type variables properly and meaningful for small n and m too, another type of proofs is required, which will be item of a further article being preparated.

In order to illustrate our main result we conclude the section with a familiar special case and a limiting case of distributions in the unit ball Ω_n included in 4.3:

Corollary 4.4

i) Let $a > 0$, $F(r) := r^n$ for $r \in [0,1]$ (the "uniform distribution in the unit ball Ω_n"), then for random variables $\tilde{Z}$ of type (1.4):

$$(4.76) \qquad \mathrm{P}(|\frac{\tilde{Z}}{\mathrm{E}(\tilde{Z})} - 1| > a) = O(a^{-2}m^{(n-1)/(n+1)}), \ m \to \infty.$$

ii) Let $a > 0$, $F(r) := 0$ for $r \in [0,1)$ and $F(r) := 1$ for $r \in [1,\infty)$ (the "uniform distribution on the unit sphere ω_n"), then for random variables $\tilde{Z}$ of type (1.4):

$$(4.77) \qquad \mathrm{P}(|\frac{\tilde{Z}}{\mathrm{E}(\tilde{Z})} - 1| > a) = O(a^{-2}m^{-1}), \ m \to \infty.$$

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